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Global wellposedness of the 3-D full water wave problem

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Abstract We consider the problem of global in time existence and uniqueness of solutions of the 3-D infinite depth full water wave problem, in the setting that the interface tends to the horizontal plane, the velocity and acceleration on the interface tend to zero at spatial infinity. We show that the nature of the nonlinearity of the water wave equation is essentially of cubic and higher orders. For any initial interface that is sufficiently small in its steepness and velocity, we show that there exists a unique smooth solution of the full water wave problem for all time, and the solution decays at the rate $1/t$.

1 Introduction

In this paper we continue our study of the global in time behaviors of the full water wave problem, in the setting that the interface tends to the horizontal plane, the velocity and acceleration on the interface tend to zero at spatial infinity.

The mathematical problem of n-dimensional water wave concerns the motion of the interface separating an inviscid, incompressible, irrotational fluid, under the influence of gravity, from a region of zero density (i.e. air) in n-dimensional space. It is assumed that the fluid region is below the air region. Assume that the density of the fluid is 1, the gravitational field is $-\mathbf{k}$, where $\mathbf{k}$ is the unit vector pointing in the upward vertical direction, and at time $t \geq 0$,

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the free interface is $\Sigma(t)$, and the fluid occupies region $\Omega(t)$. When surface tension is zero, the motion of the fluid is described by

$$\begin{cases} \mathbf{v}_t + \mathbf{v}\cdot\nabla\mathbf{v} = -\mathbf{k} - \nabla P & \text{on } \Omega(t),\ t\ge 0, \\ \operatorname{div}\mathbf{v} = 0, \qquad \operatorname{curl}\mathbf{v} = 0, & \text{on } \Omega(t),\ t\ge 0, \\ P = 0, & \text{on } \Sigma(t) \\ (1,\mathbf{v}) \text{ is tangent to the free surface } (t,\Sigma(t)), \end{cases} \tag{1.1}$$

where $\mathbf{v}$ is the fluid velocity, P is the fluid pressure. It is well-known that when surface tension is neglected, the water wave motion can be subject to the Taylor instability [2, 3, 29]. Assume that the free interface $\Sigma(t)$ is described by $\xi = \xi(\alpha,t)$, where $\alpha \in R^{n-1}$ is the Lagrangian coordinate, i.e. $\xi_t(\alpha,t) = \mathbf{v}(\xi(\alpha,t),t)$ is the fluid velocity on the interface, $\xi_{tt}(\alpha,t) = (\mathbf{v}_t + \mathbf{v}\cdot\nabla\mathbf{v})(\xi(\alpha,t),t)$ is the acceleration. Let $\mathbf{n}$ be the unit normal pointing out of $\Omega(t)$. The Taylor sign condition relating to Taylor instability is

$$-\frac{\partial P}{\partial \mathbf{n}} = (\xi_{tt} + \mathbf{k})\cdot\mathbf{n} \ge c_0 > 0, \tag{1.2}$$

point-wisely on the interface for some positive constant c_0. In previous works [30, 31], we showed that the Taylor sign condition (1.2) always holds for the n-dimensional infinite depth water wave problem (1.1), $n \ge 2$, as long as the interface is non-self-intersecting; and the initial value problem of the water wave system (1.1) is uniquely solvable **locally** in time in Sobolev spaces for arbitrary given data. Earlier work includes Nalimov [20], Yosihara [33] and Craig [10] on local existence and uniqueness for small data in 2D. We mention the following recent work on local wellposedness [1, 5, 6, 15, 18, 19, 21, 24, 34]. However the global in time behavior of the solutions remained open until last year.

In [32], we showed that for the 2D full water wave problem (1.1) ($n = 2$), the quantities $\Theta = (I - \mathfrak{H})y$, $(I - \mathfrak{H})\psi$, under an appropriate coordinate change $k = k(\alpha,t)$, satisfy equations of the type

$$\partial_t^2\Theta - i\partial_\alpha\Theta = G \tag{1.3}$$

with G consisting of nonlinear terms of only cubic and higher orders. Here $\mathfrak{H}$ is the Hilbert transform related to the water region $\Omega(t)$, y is the height function for the interface $\Sigma(t) : (x(\alpha,t), y(\alpha,t))$, and ψ is the trace on $\Sigma(t)$ of the velocity potential. Using this favorable structure, and the L^∞ time decay rate for the 2D water wave $1/t^{1/2}$, we showed that the full water wave equation (1.1) in two space dimensions has a unique smooth solution for a time period $[0, e^{c/\epsilon}]$ for initial data $\epsilon\Phi$, where Φ is arbitrary, c depends only on Φ, and ϵ is sufficiently small.

Briefly, the structural advantage of (1.3) can be explained as the following. We know the water wave equation (1.1) is equivalent to an equation on the interface of the form

$$\partial_t^2 u + |D|u = \text{nonlinear terms}, \tag{1.4}$$

where the nonlinear terms contain quadratic nonlinearity. For given smooth data, the free equation $\partial_t^2 u + |D|u = 0$ has a unique solution globally in time, with L^∞ norm decays at the rate $1/t^{\frac{n-1}{2}}$. However the nonlinear interaction can cause blow-up at finite time. The weaker the nonlinear interaction, the longer the solution stays smooth. For small data, quadratic interactions are in general stronger than the cubic and higher order interactions. In (1.3) there is no quadratic terms, using it we are able to prove a longer time existence of classical solutions for small initial data.

Naturally, we would like to know if the 3D water wave equation also posses such special structures. We find that indeed this is the case. A natural setting for 3D to utilize the ideas of 2D is the Clifford analysis. However deriving such equations (1.3) in 3D in the Clifford Algebra framework is not straightforward due to the non-availability of the Riemann mapping, the non-commutativity of the Clifford numbers, and the fact that the multiplication of two Clifford analytic functions is not necessarily analytic. Nevertheless we have overcome these difficulties.

Let $\Sigma(t) : \xi = (x(\alpha, \beta, t), y(\alpha, \beta, t), z(\alpha, \beta, t))$ be the interface in Lagrangian coordinates $(\alpha, \beta) \in \mathbb{R}^2$, and let $\mathfrak{H}$ be the Hilbert transform associated to the water region $\Omega(t)$, $N = \xi_\alpha \times \xi_\beta$ be the outward normal. In this paper, we show that the quantity $\theta = (I - \mathfrak{H})z$ satisfies such equation

$$\partial_t^2 \theta - \mathfrak{a} N \times \nabla \theta = G, \tag{1.5}$$

where G is a nonlinearity of cubic and higher orders in nature. We also find a coordinate change k that transforms (1.5) into an equation consisting of a linear part plus only cubic and higher order nonlinear terms.[1] For ψ the trace of the velocity potential, $(I - \mathfrak{H})\psi$ also satisfies a similar type equation. However we will not derive it since we do not need it in this paper.

Given that in 3D the L^∞ time decay rate is a faster $1/t$, it is not surprising that for small data, the water wave equation (1.1) ($n = 3$) is solvable globally in time. In fact we obtain better results than in 2D in terms of the initial data set. We show that if the steepness of the initial interface and the fluid velocity on the initial interface (and finitely many of their derivatives) are sufficiently small, then the solution of the 3D full water wave equation (1.1) remains smooth for all time and decays at a L^∞ rate of $1/t$. No assumptions

[1] We will explain more precisely the meaning of these statements in Sect. 1.2.

are made to the height of the initial interface and the velocity field in the fluid domain. In particular, this means that the amplitude of the initial interface can be arbitrary large, the initial kinetic energy $\frac{1}{2}\|\mathbf{v}\|^2_{L^2(\Omega(0))}$ can be infinite. This certainly makes sense physically. We note that the almost global wellposedness result we obtained for 2D water wave [32] requires the initial amplitude of the interface and the initial kinetic energy $\frac{1}{2}\|\mathbf{v}\|^2_{L^2(\Omega(0))}$ being small. One may view 2D water wave as a special case of 3D where the wave is constant in one direction. In 2D there is one less direction for the wave to disperse and the L^∞ time decay rate is a slower $1/t^{1/2}$. Technically our proof of the almost global wellposedness result in 2D [32] used to the full extend the decay rate and required the smallness in the amplitude and kinetic energy since we controlled the higher and lower regularity norms of the solution all at once (see Proposition 4.4 of [32]).[2] One may think the assumption on the smallness in amplitude and kinetic energy is to compensate the lack of decay in one direction. However this is merely a technical reason. In 3D assuming the wave tends to zero at spatial infinity, we have a faster L^∞ time decay rate $1/t$. This allows us a less elaborate proof and a global wellposedness result with less assumptions on the initial data.

1.1 Notations and Clifford analysis

We study the 3D water wave problem in the setting of the Clifford Algebra $\mathcal{C}(V_2)$, i.e. the algebra of quaternions. We refer to [14] for an in depth discussion of Clifford analysis.

Let $\{1, e_1, e_2, e_3\}$ be the basis of $\mathcal{C}(V_2)$ satisfying

$$e_i^2 = -1, \qquad e_i e_j = -e_j e_i, \quad i, j = 1, 2, 3, \ i \neq j, \qquad e_3 = e_1 e_2. \quad (1.6)$$

An element $\sigma \in \mathcal{C}(V_2)$ has a unique representation $\sigma = \sigma_0 + \sum_{i=1}^3 \sigma_i e_i$, with $\sigma_i \in \mathbb{R}$ for $0 \le i \le 3$. We call σ_0 the real part of σ and denote it by $\operatorname{Re}\sigma$ and $\sum_{i=1}^3 \sigma_i e_i$ the vector part of σ. We call σ_i the e_i component of σ. We denote $\overline{\sigma} = e_3 \sigma e_3$, $|\sigma|^2 = \sum_{i=0}^3 \sigma_i^2$. If not specified, we always assume in such an expression $\sigma = \sigma_0 + \sum_{i=1}^3 \sigma_i e_i$ that $\sigma_i \in \mathbb{R}$, for $0 \le i \le 3$. We define $\sigma \cdot \xi = \sum_{j=0}^3 \sigma_j \xi_j$. We call $\sigma \in \mathcal{C}(V_2)$ a vector if $\operatorname{Re}\sigma = 0$. We identify a point or vector $\xi = (x, y, z) \in \mathbb{R}^3$ with its $\mathcal{C}(V_2)$ counterpart $\xi = xe_1 + ye_2 + ze_3$. For

[2] A method such as the one used in this paper by combining two estimates: one is that a quantity which controls higher regularity norms grows very slowly in time as long as a low regularity norm remains small (Proposition 3.5), another is that a quantity that controls the low regularity norm remains small as long as the high regularity norms do not grow too fast (Proposition 3.6), does not give an existing time anywhere near $e^{c/\epsilon}$ for the two dimensional water wave due to the slow L^∞ time decay rate $1/t^{1/2}$.

vectors $\xi,\ \eta \in \mathcal{C}(V_2)$, we know

$$\xi\eta = -\xi\cdot\eta + \xi\times\eta, \tag{1.7}$$

where $\xi\cdot\eta$ is the dot product, $\xi\times\eta$ the cross product. For vectors ξ,ζ,η, $\xi(\zeta\times\eta)$ is obtained by first finding the cross product $\zeta\times\eta$, then regard it as a Clifford vector and calculating its multiplication with ξ by the rule (1.6). We write $\mathcal{D} = \partial_x e_1 + \partial_y e_2 + \partial_z e_3$, $\nabla = (\partial_x, \partial_y, \partial_z)$, $\Delta = \partial_x^2 + \partial_y^2 + \partial_z^2$. At times we also use the notation $\xi = (\xi_1, \xi_2, \xi_3)$ to indicate a point in $\mathbb{R}^3$. In this case $\nabla = (\partial_{\xi_1}, \partial_{\xi_2}, \partial_{\xi_3})$, $\mathcal{D} = \partial_{\xi_1} e_1 + \partial_{\xi_2} e_2 + \partial_{\xi_3} e_3$, $\Delta = \partial_{\xi_1}^2 + \partial_{\xi_2}^2 + \partial_{\xi_3}^2$.

Let Ω be an unbounded[3] C^2 domain in $\mathbb{R}^3$, $\Sigma = \partial\Omega$ be its boundary and Ω^c be its complement. A $\mathcal{C}(V_2)$ valued function F is Clifford analytic in Ω if $\mathcal{D}F = 0$ in Ω. In particular, if $F = \sum_{i=1}^3 f_i e_i$, we have by (1.7) $\mathcal{D}F = -\mathrm{div}F + \mathrm{curl}F$. So $F = \sum_{i=1}^3 f_i e_i$ is Clifford analytic in Ω if and only of $\mathrm{div}F = 0$ and $\mathrm{curl}F = 0$ in Ω.[4] Let

$$\begin{aligned} \Gamma(\xi) &= -\frac{1}{\omega_3}\frac{1}{|\xi|}, \\ K(\xi) &= -2\mathcal{D}\Gamma(\xi) = -\frac{2}{\omega_3}\frac{\xi}{|\xi|^3}, \quad \text{for } \xi = \sum_1^3 \xi_i e_i, \end{aligned} \tag{1.8}$$

where ω_3 is the surface area of the unit sphere in $\mathbb{R}^3$. Let $\xi = \xi(\alpha,\beta)$, $(\alpha,\beta)\in\mathbb{R}^2$ be a parameterization of Σ with $N = \xi_\alpha \times \xi_\beta$ pointing out of Ω. The Hilbert transform associated to the parameterization $\xi = \xi(\alpha,\beta)$, $(\alpha,\beta)\in\mathbb{R}^2$ is defined by

$$\begin{aligned} \mathfrak{H}_\Sigma f(\alpha,\beta) = p.v.\iint_{\mathbb{R}^2} K(\xi(\alpha',\beta') \\ - \xi(\alpha,\beta))(\xi'_{\alpha'}\times\xi'_{\beta'}) f(\alpha',\beta')\, d\alpha' d\beta'. \end{aligned} \tag{1.9}$$

We know a $\mathcal{C}(V_2)$ valued function F that decays at infinity is Clifford analytic in Ω if and only if its trace on Σ: $f(\alpha,\beta) = F(\xi(\alpha,\beta))$ satisfies

$$f = \mathfrak{H}_\Sigma f. \tag{1.10}$$

[3] Similar definitions and results exist for bounded domains, see [14]. For the purpose of this paper, we discuss only for unbounded domain Ω with a single boundary.

[4] The fluid velocity $\mathbf{v}$ satisfying (1.1) is a Clifford analytic function in $\Omega(t)$.

We know $\mathfrak{H}_\Sigma^2 = I$ in L^2. We use the convention $\mathfrak{H}_\Sigma 1 = 0$. We abbreviate

$$\begin{aligned}\mathfrak{H}_\Sigma f(\alpha,\beta) &= \iint K(\xi(\alpha',\beta') - \xi(\alpha,\beta))(\xi'_{\alpha'} \times \xi'_{\beta'})\, f(\alpha',\beta')\, d\alpha' d\beta' \\ &= \iint K(\xi' - \xi)(\xi'_{\alpha'} \times \xi'_{\beta'}) f'\, d\alpha' d\beta' = \iint K\, N'\, f'\, d\alpha' d\beta'.\end{aligned}$$

Let $f = f(\alpha,\beta)$ be defined for $(\alpha,\beta) \in \mathbb{R}^2$, and f decays at the infinity. We say $f^\hbar$ is the harmonic extension of f to Ω if $\Delta f^\hbar = 0$ in Ω, $f^\hbar(\xi(\alpha,\beta)) = f(\alpha,\beta)$ and $f^\hbar$ decays at the infinity. We denote by $\mathcal{D}_\xi f$ the trace of $\mathcal{D} f^\hbar$ on Σ, i.e.

$$\mathcal{D}_\xi f(\alpha,\beta) = \mathcal{D} f^\hbar(\xi(\alpha,\beta)). \tag{1.11}$$

Similarly $\nabla_\xi f(\alpha,\beta) = \nabla f^\hbar(\xi(\alpha,\beta))$, $\partial_x f(\alpha,\beta) = \partial_x f^\hbar(\xi(\alpha,\beta))$ etc. In the context of water wave where $\Omega(t)$ is the fluid domain, we denote by $\nabla_\xi^+ f$ (respectively $\nabla_\xi^- f$) the trace of $\nabla f^\hbar$ on $\Sigma(t)$, where $f^\hbar$ is the harmonic extension of f to $\Omega(t)$ (respectively $\Omega(t)^c$). We have

Lemma 1.1 1. *Let $f = f(\alpha,\beta)$, $(\alpha,\beta) \in \mathbb{R}^2$ be a real valued smooth function decays fast at infinity. We have*

$$\iint K(\xi(\alpha',\beta') - \xi(\alpha,\beta)) \cdot (N' \times \nabla' f)(\alpha',\beta')\, d\alpha' d\beta' = 0. \tag{1.12}$$

2. *For any function $f = \sum_1^3 f_i e_i$ satisfying $f = \mathfrak{H}_\Sigma f$ or $f = -\mathfrak{H}_\Sigma f$, we have*

$$\xi_\beta \cdot \partial_\alpha f - \xi_\alpha \cdot \partial_\beta f = 0. \tag{1.13}$$

Proof Let $f^\hbar$ be the harmonic extension of f to the domain Ω. We know $\mathcal{D} f^\hbar$ is Clifford analytic in Ω. Therefore the trace of $\mathcal{D} f^\hbar$ on Σ satisfies

$$\mathcal{D}_\xi f = \mathfrak{H}_\Sigma \mathcal{D}_\xi f. \tag{1.14}$$

Taking the real part of (1.14) gives us (1.12).

For (1.13), we only prove for the case $f = \mathfrak{H}_\Sigma f$. The proof for the case $f = -\mathfrak{H}_\Sigma f$ is similar, since $f = -\mathfrak{H}_\Sigma f$ is equivalent to the harmonic extension of f to Ω^c being analytic.

We have from $f = \mathfrak{H}_\Sigma f$ that $\mathcal{D}_\xi f = 0$. Therefore

$$\xi_\beta \cdot \partial_\alpha f - \xi_\alpha \cdot \partial_\beta f = \sum_{i,j=1}^3 \partial_\beta \xi_i \partial_\alpha \xi_j \partial_{\xi_j} f_i - \sum_{i,j=1}^3 \partial_\alpha \xi_j \partial_\beta \xi_i \partial_{\xi_i} f_j = 0.$$

□

Assume that for each $t \in [0, T]$, $\Omega(t)$ is a C^2 domain with boundary $\Sigma(t)$. Let $\Sigma(t): \xi = \xi(\alpha, \beta, t)$, $(\alpha, \beta) \in \mathbb{R}^2$; $\xi \in C^2(\mathbb{R}^2 \times [0, T])$, $N = \xi_\alpha \times \xi_\beta$. We know $N \times \nabla = \xi_\beta \partial_\alpha - \xi_\alpha \partial_\beta$. Denote $[A, B] = AB - BA$. We have

Lemma 1.2 *Let $f \in C^1(\mathbb{R}^2 \times [0, T])$ be a $\mathcal{C}(V_2)$ valued function vanishing at spatial infinity, and $\mathfrak{a}$ be real valued. Then*

$$[\partial_t, \mathfrak{H}_{\Sigma(t)}]f = \iint K(\xi' - \xi)(\xi_t - \xi'_t) \times (\xi'_{\beta'}\partial_{\alpha'} - \xi'_{\alpha'}\partial_{\beta'}) f' \, d\alpha' d\beta', \tag{1.15}$$

$$[\partial_\alpha, \mathfrak{H}_{\Sigma(t)}]f = \iint K(\xi' - \xi)(\xi_\alpha - \xi'_{\alpha'}) \times (\xi'_{\beta'}\partial_{\alpha'} - \xi'_{\alpha'}\partial_{\beta'}) f' \, d\alpha' d\beta', \tag{1.16}$$

$$[\partial_\beta, \mathfrak{H}_{\Sigma(t)}]f = \iint K(\xi' - \xi)(\xi_\beta - \xi'_{\beta'}) \times (\xi'_{\beta'}\partial_{\alpha'} - \xi'_{\alpha'}\partial_{\beta'}) f' \, d\alpha' d\beta', \tag{1.17}$$

$$[\mathfrak{a}N \times \nabla, \mathfrak{H}_{\Sigma(t)}]f = \iint K(\xi' - \xi)(\mathfrak{a}N - \mathfrak{a}'N') \times (\xi'_{\beta'}\partial_{\alpha'} - \xi'_{\alpha'}\partial_{\beta'}) f' \, d\alpha' d\beta', \tag{1.18}$$

$$\begin{aligned}[\partial_t^2, \mathfrak{H}_{\Sigma(t)}]f = &\iint K(\xi' - \xi)(\xi_{tt} - \xi'_{tt}) \times (\xi'_{\beta'}\partial_{\alpha'} - \xi'_{\alpha'}\partial_{\beta'}) f' \, d\alpha' d\beta' \\ &+ \iint K(\xi' - \xi)(\xi_t - \xi'_t) \\ &\times (\xi'_{t\beta'}\partial_{\alpha'} - \xi'_{t\alpha'}\partial_{\beta'}) f' \, d\alpha' d\beta' \\ &+ \iint \partial_t K(\xi' - \xi)(\xi_t - \xi'_t) \\ &\times (\xi'_{\beta'}\partial_{\alpha'} - \xi'_{\alpha'}\partial_{\beta'}) f' \, d\alpha' d\beta' \\ &+ 2\iint K(\xi' - \xi)(\xi_t - \xi'_t) \\ &\times (\xi'_{\beta'}\partial_{\alpha'} - \xi'_{\alpha'}\partial_{\beta'}) f'_t \, d\alpha' d\beta'.\end{aligned} \tag{1.19}$$

Proof Applying Lemma 3.1 of [31] component-wisely to f gives (1.15), (1.16), (1.17). (1.19) is a direct consequence of (1.15) and the fact $[\partial_t^2, \mathfrak{H}_{\Sigma(t)}] = \partial_t[\partial_t, \mathfrak{H}_{\Sigma(t)}] + [\partial_t, \mathfrak{H}_{\Sigma(t)}]\partial_t$. We now prove (1.18) for f real valued. Notice that $\mathfrak{a}N \times \nabla \mathfrak{H}_{\Sigma(t)} f = \mathfrak{a}\xi_\beta \partial_\alpha \mathfrak{H}_{\Sigma(t)} f - \mathfrak{a}\xi_\alpha \partial_\beta \mathfrak{H}_{\Sigma(t)} f$. From (1.16), we have

$$\begin{aligned}
\mathfrak{a}\xi_\beta \partial_\alpha \mathfrak{H}_{\Sigma(t)} f &= \mathfrak{a}\xi_\beta [\partial_\alpha, \mathfrak{H}_{\Sigma(t)}] f + \mathfrak{a}\xi_\beta \mathfrak{H}_{\Sigma(t)} \partial_\alpha f \\
&= \iint \mathfrak{a}\xi_\beta K(\xi' - \xi)(\xi_\alpha - \xi'_{\alpha'}) \times (\xi'_{\beta'}\partial_{\alpha'} - \xi'_{\alpha'}\partial_{\beta'}) f' \, d\alpha' d\beta' \\
&\quad + \mathfrak{a}\xi_\beta \iint K N' \partial_{\alpha'} f' \, d\alpha' \, d\beta' \\
&= \iint \mathfrak{a}\xi_\beta K(\xi' - \xi)\, \xi_\alpha \times (\xi'_{\beta'}\partial_{\alpha'} - \xi'_{\alpha'}\partial_{\beta'}) f' \, d\alpha' d\beta'. \qquad (1.20)
\end{aligned}$$

Similarly

$$\mathfrak{a}\xi_\alpha \partial_\beta \mathfrak{H}_{\Sigma(t)} f = \iint \mathfrak{a}\xi_\alpha K(\xi' - \xi)\, \xi_\beta \times (\xi'_{\beta'}\partial_{\alpha'} - \xi'_{\alpha'}\partial_{\beta'}) f' \, d\alpha' d\beta'. \qquad (1.21)$$

Now for any vectors K, η,

$$\begin{aligned}
&\xi_\beta K \xi_\alpha \times \eta - \xi_\alpha K \xi_\beta \times \eta \\
&\quad = -K\xi_\beta\, \xi_\alpha \times \eta + K\xi_\alpha\, \xi_\beta \times \eta - 2\xi_\beta \cdot K\, \xi_\alpha \times \eta + 2\xi_\alpha \cdot K\, \xi_\beta \times \eta \\
&\quad = -K\xi_\beta \times (\xi_\alpha \times \eta) + K\xi_\alpha \times (\xi_\beta \times \eta) \\
&\qquad + K\xi_\beta \cdot (\xi_\alpha \times \eta) - K\xi_\alpha \cdot (\xi_\beta \times \eta) - 2(K \times (\xi_\alpha \times \xi_\beta)) \times \eta \\
&\quad = K(-\xi_\alpha \xi_\beta \cdot \eta + \xi_\beta \xi_\alpha \cdot \eta) - 2K(\xi_\alpha \times \xi_\beta) \cdot \eta \\
&\qquad + 2K(\xi_\alpha \times \xi_\beta) \cdot \eta - 2K \cdot \eta(\xi_\alpha \times \xi_\beta) \\
&\quad = K(\xi_\alpha \times \xi_\beta) \times \eta - 2K \cdot \eta(\xi_\alpha \times \xi_\beta). \qquad (1.22)
\end{aligned}$$

In the above calculation, we used repeatedly the identities (1.7) and $a \times (b \times c) = b\, a \cdot c - c\, a \cdot b$. Combining (1.20), (1.21) and applying (1.12) and (1.22) with $\eta = (\xi'_{\beta'}\partial_{\alpha'} - \xi'_{\alpha'}\partial_{\beta'}) f' = N' \times \nabla' f'$, we get

$$\mathfrak{a}N \times \nabla \mathfrak{H}_{\Sigma(t)} f = \iint K(\xi' - \xi)\mathfrak{a}(\xi_\alpha \times \xi_\beta) \times (N' \times \nabla' f') \, d\alpha' \, d\beta'.$$

Notice that

$$\mathfrak{H}_{\Sigma(t)}(\mathfrak{a}N \times \nabla f) = \iint K(\xi' - \xi)\mathfrak{a}' N' \times (N' \times \nabla' f') \, d\alpha' \, d\beta'.$$

(1.18) therefore holds for real valued f. (1.18) for $\mathcal{C}(V_2)$ valued f directly follows. □

1.2 The main equations and main results

We now discuss the 3D water wave. Let $\Sigma(t) : \xi(\alpha,\beta,t) = x(\alpha,\beta,t)e_1 + y(\alpha,\beta,t)e_2 + z(\alpha,\beta,t)e_3$, $(\alpha,\beta) \in \mathbb{R}^2$ be the parameterization of the interface at time t in Lagrangian coordinates (α,β) with $N = \xi_\alpha \times \xi_\beta = (N_1, N_2, N_3)$ pointing out of the fluid domain $\Omega(t)$. Let $\mathfrak{H} = \mathfrak{H}_{\Sigma(t)}$, and

$$\mathfrak{a} = -\frac{1}{|N|}\frac{\partial P}{\partial \mathbf{n}}.$$

We know from [31] that $\mathfrak{a} > 0$ and (1.1) is equivalent to the following nonlinear system defined on the interface $\Sigma(t)$:

$$\xi_{tt} + e_3 = \mathfrak{a}N, \tag{1.23}$$

$$\xi_t = \mathfrak{H}\xi_t. \tag{1.24}$$

Motivated by [32], we would like to know whether in 3-D, the quantity $\pi = (I - \mathfrak{H})ze_3$ under an appropriate coordinate change satisfies an equation with nonlinearities containing no quadratic terms. We first derive the equation for π in Lagrangian coordinates.

Proposition 1.3 *We have*

$$\begin{aligned}
&(\partial_t^2 - \mathfrak{a}N \times \nabla)\pi \\
&\quad = \iint K(\xi' - \xi)(\xi_t - \xi_t') \times (\xi_{\beta'}'\partial_{\alpha'} - \xi_{\alpha'}'\partial_{\beta'})\overline{\xi_t'}\, d\alpha' d\beta' \\
&\qquad - \iint K(\xi' - \xi)(\xi_t - \xi_t') \times (\xi_{t\beta'}'\partial_{\alpha'} - \xi_{t\alpha'}'\partial_{\beta'})z'\, d\alpha' d\beta' e_3 \\
&\qquad - \iint \partial_t K(\xi' - \xi)(\xi_t - \xi_t') \times (\xi_{\beta'}'\partial_{\alpha'} - \xi_{\alpha'}'\partial_{\beta'})z'\, d\alpha' d\beta' e_3.
\end{aligned} \tag{1.25}$$

Proof Notice from (1.23)

$$(\partial_t^2 - \mathfrak{a}N \times \nabla)ze_3 = z_{tt}e_3 + \mathfrak{a}N_1 e_1 + \mathfrak{a}N_2 e_2 = \xi_{tt} \tag{1.26}$$

and from (1.24) that

$$(I - \mathfrak{H})\xi_{tt} = [\partial_t, \mathfrak{H}]\xi_t \tag{1.27}$$

(1.25) is an easy consequence of (1.15), (1.18) and (1.19) and (1.23), (1.26), (1.27):

$$\begin{aligned}(\partial_t^2 - \mathfrak{a}N \times \nabla)\pi &= (I - \mathfrak{H})(\partial_t^2 - \mathfrak{a}N \times \nabla)ze_3 - [\partial_t^2 - \mathfrak{a}N \times \nabla, \mathfrak{H}]ze_3\\
&= [\partial_t, \mathfrak{H}]\xi_t - [\partial_t^2 - \mathfrak{a}N \times \nabla, \mathfrak{H}]ze_3\\
&= \iint K(\xi' - \xi)(\xi_t - \xi_t') \times (\xi'_{\beta'}\partial_{\alpha'} - \xi'_{\alpha'}\partial_{\beta'})\overline{\xi_t'}\, d\alpha' d\beta'\\
&\quad - \iint K(\xi' - \xi)(\xi_t - \xi_t') \times (\xi'_{t\beta'}\partial_{\alpha'} - \xi'_{t\alpha'}\partial_{\beta'})z'\, d\alpha' d\beta' e_3\\
&\quad - \iint \partial_t K(\xi' - \xi)(\xi_t - \xi_t') \times (\xi'_{\beta'}\partial_{\alpha'} - \xi'_{\alpha'}\partial_{\beta'})z'\, d\alpha' d\beta' e_3. \qquad \square\end{aligned}$$

We see that the second and third terms in the right hand side of (1.25) are consisting of terms of cubic and higher orders, while the first term contains quadratic terms. Unlike the 2D case, multiplications of Clifford analytic functions are not necessarily analytic, so we cannot reduce the first term at the right hand side of (1.25) into a cubic form. However we notice that $\overline{\xi}_t = x_t e_1 + y_t e_2 - z_t e_3$ is almost analytic[5] in the air region $\Omega(t)^c$, and this implies that the first term is almost analytic in the fluid domain $\Omega(t)$. Notice that the left hand side of (1.25) is almost analytic in the air region. The orthogonality of the projections $(I - \mathfrak{H})$ and $(I + \mathfrak{H})$ allows us to reduce the first term to cubic in energy estimates.

Notice that the left hand side of (1.25) still contains quadratic terms and (1.25) is invariant under a change of coordinates. We now want to see if in 3D, there is a coordinate change k, such that under which the left hand side of (1.25) becomes a linear part plus only cubic and higher order terms. In 2D, such a coordinate change exists (see (2.18) in [32]). However it is defined by the Riemann mapping. Although there is no Riemann mapping in 3D, we realize that the Riemann mapping used in 2-D is just a holomorphic function in the fluid region with its imaginary part equal to zero on $\Sigma(t)$. This motivates us to define

$$k = k(\alpha, \beta, t) = \xi(\alpha, \beta, t) - (I + \mathfrak{H})z(\alpha, \beta, t)e_3 + \mathfrak{K}z(\alpha, \beta, t)e_3. \tag{1.28}$$

Here $\mathfrak{K} = \operatorname{Re}\mathfrak{H}$:

$$\mathfrak{K}f(\alpha, \beta, t) = -\iint K(\xi(\alpha', \beta', t) - \xi(\alpha, \beta, t)) \cdot N' f(\alpha', \beta', t)\, d\alpha' d\beta' \tag{1.29}$$

[5] Since the order of smallness is what matters in this paper, here a quantity X of size $O(\epsilon^N)$, in other words of order N, is said to be almost analytic in the fluid region $\Omega(t)$ (respectively in the air region $\Omega(t)^c$), if $(I - \mathfrak{H})X$ (respectively $(I + \mathfrak{H})X$) is at most of size $O(\epsilon^{N+1})$, or in other words at least of order $N + 1$.

is the double layered potential operator. It is clear that the e_3 component of k as defined in (1.28) is zero. In fact, the real part of k is also zero. This is because

$$\begin{aligned}
&\iint K(\xi'-\xi)\times(\xi'_{\alpha'}\times\xi'_{\beta'})z'e_3\,d\alpha'\,d\beta'\\
&=\iint(\xi'_{\alpha'}\xi'_{\beta'}\cdot K-\xi'_{\beta'}\xi'_{\alpha'}\cdot K)z'e_3\,d\alpha'\,d\beta'\\
&=-2\iint(\xi'_{\alpha'}\partial_{\beta'}\Gamma(\xi'-\xi)-\xi'_{\beta'}\partial_{\alpha'}\Gamma(\xi'-\xi))z'e_3\,d\alpha'\,d\beta'\\
&=2\iint\Gamma(\xi'-\xi)(\xi'_{\alpha'}z_{\beta'}-\xi'_{\beta'}z_{\alpha'})e_3\,d\alpha'\,d\beta'\\
&=2\iint\Gamma(\xi'-\xi)(N'_1e_1+N'_2e_2)\,d\alpha'\,d\beta'.
\end{aligned}$$

So

$$\mathfrak{H}ze_3=\mathfrak{K}ze_3+2\iint\Gamma(\xi'-\xi)(N'_1e_1+N'_2e_2)\,d\alpha'\,d\beta'. \tag{1.30}$$

This shows that the mapping k defined in (1.28) has only the e_1 and e_2 components $k=(k_1,k_2)=k_1e_1+k_2e_2$. If $\Sigma(t)$ is a graph of small steepness, i.e. if z_α and z_β are small, then the Jacobian of $k=k(\cdot,t)$: $J(k)=J(k(t))=\partial_\alpha k_1\partial_\beta k_2-\partial_\alpha k_2\partial_\beta k_1>0$ and $k(\cdot,t):\mathbb{R}^2\to\mathbb{R}^2$ defines a valid coordinate change. We will make this point more precise in Lemma 4.1.

Denote $\nabla_\perp=(\partial_\alpha,\partial_\beta)$, $U_gf(\alpha,\beta,t)=f(g(\alpha,\beta,t),t)=f\circ g(\alpha,\beta,t)$. Assume that $k=k(\cdot,t):\mathbb{R}^2\to\mathbb{R}^2$ defined in (1.28) is a diffeomorphism satisfying $J(k(t))>0$. Let k^{-1} be such that $k\circ k^{-1}(\alpha,\beta,t)=\alpha e_1+\beta e_2$. Define

$$\begin{aligned}
&\zeta=\xi\circ k^{-1}=\mathfrak{x}e_1+\mathfrak{y}e_2+\mathfrak{z}e_3,\\
&u=\xi_t\circ k^{-1},\quad\text{and}\quad w=\xi_{tt}\circ k^{-1}.
\end{aligned}\tag{1.31}$$

Let

$$\begin{aligned}
&b=k_t\circ k^{-1},\\
&A\circ ke_3=\mathfrak{a}J(k)e_3=\mathfrak{a}k_\alpha\times k_\beta,\quad\text{and}\quad\mathcal{N}=\zeta_\alpha\times\zeta_\beta.
\end{aligned}\tag{1.32}$$

By a simple application of the chain rule, we have

$$\begin{aligned}
&U_k^{-1}\partial_tU_k=\partial_t+b\cdot\nabla_\perp,\quad\text{and}\\
&U_k^{-1}(\mathfrak{a}N\times\nabla)U_k=A\mathcal{N}\times\nabla=A(\zeta_\beta\partial_\alpha-\zeta_\alpha\partial_\beta),
\end{aligned}\tag{1.33}$$

and $U_k^{-1}\mathfrak{H}U_k = \mathcal{H}$, with

$$\mathcal{H}f(\alpha,\beta,t) = \iint K(\zeta(\alpha',\beta',t) - \zeta(\alpha,\beta,t))(\zeta'_{\alpha'}\times\zeta'_{\beta'})f(\alpha',\beta',t)\,d\alpha'\,d\beta'. \quad (1.34)$$

Let $\chi = \pi\circ k^{-1}$. Applying coordinate change U_k^{-1} to (1.25). We get

$$\begin{aligned}
&((\partial_t + b\cdot\nabla_\perp)^2 - A\mathcal{N}\times\nabla)\chi \\
&\quad = \iint K(\zeta'-\zeta)(u-u')\times(\zeta'_{\beta'}\partial_{\alpha'} - \zeta'_{\alpha'}\partial_{\beta'})\overline{u'}\,d\alpha' d\beta' \\
&\qquad - \iint K(\zeta'-\zeta)(u-u')\times(u'_{\beta'}\partial_{\alpha'} - u'_{\alpha'}\partial_{\beta'})\mathfrak{z}'\,d\alpha' d\beta' e_3 \\
&\qquad - \iint((u'-u)\cdot\nabla)K(\zeta'-\zeta)(u-u') \\
&\qquad \times(\zeta'_{\beta'}\partial_{\alpha'} - \zeta'_{\alpha'}\partial_{\beta'})\mathfrak{z}'\,d\alpha' d\beta' e_3. \qquad (1.35)
\end{aligned}$$

We show in the following proposition that b, $A-1$ are consisting of only quadratic and higher order terms. Let $\mathcal{K} = \operatorname{Re}\mathcal{H} = U_k^{-1}\mathfrak{K}U_k$, $P = \alpha e_1 + \beta e_2$, and

$$\begin{aligned}
&\Lambda^* = (I+\mathfrak{H})ze_3, \qquad \Lambda = (I+\mathfrak{H})ze_3 - \mathfrak{K}ze_3, \\
&\lambda^* = (I+\mathcal{H})\mathfrak{z}e_3, \qquad \lambda = \lambda^* - \mathcal{K}\mathfrak{z}\mathfrak{e}_3.
\end{aligned} \qquad (1.36)$$

Therefore

$$\zeta = P + \lambda. \qquad (1.37)$$

Let the velocity $u = u_1e_1 + u_2e_2 + u_3e_3$.

Proposition 1.4 *Let $b = k_t\circ k^{-1}$ and $A\circ k = \mathfrak{a}J(k)$. We have*[6]

$$\begin{aligned}
b = &\frac{1}{2}(\mathcal{H}-\overline{\mathcal{H}})\overline{u} - [\partial_t + b\cdot\nabla_\perp, \mathcal{H}]\mathfrak{z}e_3 \\
&+ [\partial_t + b\cdot\nabla_\perp, \mathcal{K}]\mathfrak{z}e_3 + \mathcal{K}u_3e_3, \qquad (1.38)
\end{aligned}$$

[6] Formulas for b and A similar to those in 2D [32] are also available and can be obtained in a similar way:

$$(I-\mathcal{H})b = (I-\mathcal{H})([\partial_t + b\cdot\nabla_\perp, \mathcal{K}]\mathfrak{z}e_3 - [\partial_t + b\cdot\nabla_\perp, \mathcal{H}]\mathfrak{z}e_3 + \mathcal{K}u_3e_3),$$

$$\begin{aligned}
(I-\mathcal{H})(Ae_3) = &e_3 + [\partial_t + b\cdot\nabla_\perp, \mathcal{H}]u + [A\mathcal{N}\times\nabla, \mathcal{H}]\lambda^* \\
&+ (I-\mathcal{H})(-A\zeta_\beta\times(\partial_\alpha\mathcal{K}\mathfrak{z}e_3) + A\zeta_\alpha\times(\partial_\beta\mathcal{K}\mathfrak{z}e_3) + A\partial_\alpha\lambda\times\partial_\beta\lambda).
\end{aligned}$$

However we choose to use those in Proposition 1.4.

$$(A-1)e_3 = \frac{1}{2}(-\mathcal{H}+\overline{\mathcal{H}})\overline{w} + \frac{1}{2}([\partial_t + b\cdot\nabla_\perp, \mathcal{H}]u - \overline{[\partial_t + b\cdot\nabla_\perp, \mathcal{H}]u})$$
$$+ [A\mathcal{N}\times\nabla, \mathcal{H}]\mathfrak{z}e_3 - A\zeta_\beta \times (\partial_\alpha \mathcal{K}\mathfrak{z}e_3)$$
$$+ A\zeta_\alpha \times (\partial_\beta \mathcal{K}\mathfrak{z}e_3) + A\partial_\alpha\lambda \times \partial_\beta\lambda. \tag{1.39}$$

Here $\overline{\mathcal{H}}f = e_3\mathcal{H}(e_3 f) = \iint e_3 K\mathcal{N}' e_3 f'$.

Proof Taking derivative to t to (1.28), we get

$$k_t = \xi_t - \partial_t(I+\mathfrak{H})ze_3 + \partial_t\mathfrak{K}ze_3$$
$$= \xi_t - z_t e_3 - \mathfrak{H}z_t e_3 - [\partial_t, \mathfrak{H}]ze_3 + \partial_t\mathfrak{K}ze_3. \tag{1.40}$$

Now

$$\xi_t - z_t e_3 - \mathfrak{H}z_t e_3 = \frac{1}{2}(\xi_t + \overline{\xi}_t) - \frac{1}{2}\mathfrak{H}(\xi_t - \overline{\xi}_t)$$
$$= \frac{1}{2}\overline{\xi}_t + \frac{1}{2}\mathfrak{H}\overline{\xi}_t = \frac{1}{2}(\mathfrak{H} - \overline{\mathfrak{H}})\overline{\xi}_t. \tag{1.41}$$

Combining (1.40), (1.41) we get

$$k_t = \frac{1}{2}(\mathfrak{H} - \overline{\mathfrak{H}})\overline{\xi}_t - [\partial_t, \mathfrak{H}]ze_3 + [\partial_t, \mathfrak{K}]ze_3 + \mathfrak{K}z_t e_3. \tag{1.42}$$

Making the change of coordinate U_k^{-1}, we get (1.38).

Notice that $A \circ ke_3 = \mathfrak{a}k_\alpha \times k_\beta$. From the definition $k = \xi - \Lambda^* + \mathfrak{K}ze_3 = \xi - \Lambda$, we get

$$k_\alpha \times k_\beta = \xi_\alpha \times \xi_\beta + \xi_\beta \times \partial_\alpha\Lambda^* - \xi_\alpha \times \partial_\beta\Lambda^*$$
$$- \xi_\beta \times (\partial_\alpha\mathfrak{K}ze_3) + \xi_\alpha \times (\partial_\beta\mathfrak{K}ze_3) + \partial_\alpha\Lambda \times \partial_\beta\Lambda.$$

Using (1.30) and (1.13), we have

$$\xi_\beta \times \partial_\alpha\Lambda^* - \xi_\alpha \times \partial_\beta\Lambda^* = \xi_\beta\partial_\alpha\Lambda^* - \xi_\alpha\partial_\beta\Lambda^* = (N\times\nabla)\Lambda^*.$$

From (1.23), and the fact that $\mathfrak{a}N \times \nabla ze_3 = -\mathfrak{a}N_1 e_1 - \mathfrak{a}N_2 e_2$, we obtain

$$\mathfrak{a}\xi_\alpha \times \xi_\beta + \mathfrak{a}(N\times\nabla)\Lambda^*$$
$$= \xi_{tt} + e_3 + (I+\mathfrak{H})(\mathfrak{a}N\times\nabla)ze_3 + [\mathfrak{a}N\times\nabla, \mathfrak{H}]ze_3$$
$$= \xi_{tt} + e_3 - \frac{1}{2}(I+\mathfrak{H})(\xi_{tt} + \overline{\xi}_{tt}) + [\mathfrak{a}N\times\nabla, \mathfrak{H}]ze_3$$

and furthermore from (1.24),

$$\begin{aligned}
&\xi_{tt}-\frac{1}{2}(I+\mathfrak{H})(\xi_{tt}+\overline{\xi}_{tt})\\
&\quad=\frac{1}{2}(\xi_{tt}-\mathfrak{H}\xi_{tt})-\frac{1}{2}(\overline{\xi}_{tt}+\mathfrak{H}\overline{\xi}_{tt})\\
&\quad=\frac{1}{2}[\partial_t,\mathfrak{H}]\xi_t-\frac{1}{2}(\overline{\xi}_{tt}-\overline{\mathfrak{H}\xi_{tt}})-\frac{1}{2}(\mathfrak{H}\overline{\xi}_{tt}+\overline{\mathfrak{H}\xi_{tt}})\\
&\quad=\frac{1}{2}[\partial_t,\mathfrak{H}]\xi_t-\frac{1}{2}\overline{[\partial_t,\mathfrak{H}]\xi_t}+\frac{1}{2}(\overline{\mathfrak{H}}-\mathfrak{H})\overline{\xi}_{tt}.
\end{aligned}$$

Combining the above calculations and make the change of coordinates U_k^{-1}, we obtain (1.39). □

From Proposition 1.4, we see that b and $A-1$ are consisting of terms of quadratic and higher orders. Therefore the left hand side of (1.35) is

$$(\partial_t^2-e_2\partial_\alpha+e_1\partial_\beta)\chi-\partial_\beta\lambda\partial_\alpha\chi+\partial_\alpha\lambda\partial_\beta\chi+\text{cubic and higher order terms.}$$

The quadratic term $\partial_\beta\lambda\partial_\alpha\chi-\partial_\alpha\lambda\partial_\beta\chi$ is new in 3D. We notice that this is one of the null forms studied in [17] and we find that it is also null for our equation and can be written as the factor $1/t$ times a quadratic expression involving some "invariant vector fields" for $\partial_t^2-e_2\partial_\alpha+e_1\partial_\beta$. Therefore this term is cubic in nature and equation (1.35) is of the type "linear + cubic and higher order perturbations".

We can now study the global in time behavior of small solutions of the water wave equations using (1.35) and the method of invariant vector fields. In fact, in Sect. 3, we will see that it is more natural to treat $(\partial_t+b\cdot\nabla_\perp)^2-A\mathcal{N}\times\nabla$ as the main operator for the water wave equation than treating it as a perturbation of the linear operator $\partial_t^2-e_2\partial_\alpha+e_1\partial_\beta$. We obtain a uniform bound for all time of a properly constructed energy that involves invariant vector fields of $\partial_t^2-e_2\partial_\alpha+e_1\partial_\beta$ by combining energy estimates for the (1.35) and a generalized Sobolev inequality that gives a $L^2\to L^\infty$ estimate with the decay rate $1/t$. We point out that not only does the projection $(I-\mathfrak{H})$ give us the quantity $(I-\mathfrak{H})ze_3$, but it is also used in various ways to project away "quadratic noises" in the course of deriving the energy estimates. The global in time existence follows from a local well-posedness result, the uniform boundedness of the energy and a continuity argument. We state our main theorem.

Let $|D|=\sqrt{-\partial_\alpha^2-\partial_\beta^2}$, $H^s(\mathbb{R}^2)=\{f\,|\,(I+|D|)^sf\in L^2(\mathbb{R}^2)\}$, with $\|f\|_{H^s}=\|f\|_{H^s(\mathbb{R}^2)}=\|(I+|D|)^sf\|_{L^2(\mathbb{R}^2)}$.

Let $s \geq 27$, $\max\{[\frac{s}{2}] + 1, 17\} \leq l \leq s - 10$. Assume that initially

$$\begin{aligned} &\xi(\alpha, \beta, 0) = \xi^0 = (\alpha, \beta, z^0(\alpha, \beta)), \qquad \xi_t(\alpha, \beta, 0) = \mathfrak{u}^0(\alpha, \beta), \\ &\xi_{tt}(\alpha, \beta, 0) = \mathfrak{w}^0(\alpha, \beta), \end{aligned} \tag{1.43}$$

and the data in (1.43) satisfy the compatibility condition (5.29)–(5.30) of [31].[7] Let $\Gamma = \partial_\alpha,\ \partial_\beta,\ \alpha\partial_\alpha + \beta\partial_\beta,\ \alpha\partial_\beta - \beta\partial_\alpha$. Assume that

$$\begin{aligned} \sum_{\substack{|j| \leq s-1 \\ \partial = \partial_\alpha, \partial_\beta}} &\|\Gamma^j |D|^{1/2} z^0\|_{L^2(\mathbb{R}^2)} + \|\Gamma^j \partial z^0\|_{H^{1/2}(\mathbb{R}^2)} + \|\Gamma^j \mathfrak{u}^0\|_{H^{3/2}(\mathbb{R}^2)} \\ &+ \|\Gamma^j \mathfrak{w}^0\|_{H^1(\mathbb{R}^2)} < \infty. \end{aligned} \tag{1.44}$$

Let

$$\begin{aligned} \epsilon = \sum_{\substack{|j| \leq l+3 \\ \partial = \partial_\alpha, \partial_\beta}} &\|\Gamma^j |D|^{1/2} z^0\|_{L^2(\mathbb{R}^2)} + \|\Gamma^j \partial z^0\|_{L^2(\mathbb{R}^2)} \\ &+ \|\Gamma^j \mathfrak{u}^0\|_{H^{1/2}(\mathbb{R}^2)} + \|\Gamma^j \mathfrak{w}^0\|_{L^2(\mathbb{R}^2)}. \end{aligned} \tag{1.45}$$

Theorem 1.5 (Main Theorem) *There exists $\epsilon_0 > 0$, such that for $0 \leq \epsilon \leq \epsilon_0$, the initial value problem (1.23)–(1.24)–(1.43) has a unique classical solution globally in time. For each time $0 \leq t < \infty$, the interface is a graph, the solution has the same regularity as the initial data and remains small. Moreover the L^∞ norm of the steepness and the acceleration of the interface, the derivative of the velocity on the interface decay at the rate $\frac{1}{t}$.*

A more detailed and precise statement of the main Theorem is given in Theorem 4.6 and the remarks at the end of this paper.

Remark 1.6 In this paper, we consider only the case that the velocity $\mathbf{v} \to 0$ at the spatial infinity. One can also treat water waves with $\mathbf{v} \to c = (c', 0)$ a constant velocity at spatial infinity. Exact analogous computations lead to an analogue of (1.35). This yields a result like Theorem 1.5 with $\mathbf{v} \to c$ at spatial infinity. In Appendix B, we will indicate how to obtain an equation such as (1.35) for 3D water waves with $\mathbf{v} \to c$ at spatial infinity.

Remark 1.7 Theorem 1.5 and its analogue for $\mathbf{v} \to c$ at spatial infinity show that there is a lower bound on the size of possible solitary waves in 3D since

[7] The compatibility condition basically requires that ξ^0, $\mathfrak{u}^0$ $\mathfrak{w}^0$ satisfy the water wave equations (1.23), (1.24) initially. See (4.6) and the nearby paragraphs in this paper for more details.

the L^∞ decay of small solutions rules out small solitary waves. The same statement holds for 2D water waves (see Proposition 4.6 and Theorem 5.5 of [32]).[8]

In Sect. 2 we will give a set of invariant vector fields to the operator $\partial_t^2 - e_2\partial_\alpha + e_1\partial_\beta$ and study their commutation properties with various operators appeared in the equations (1.35), (2.41), and (2.38). We will then prove a generalized Sobolev inequality (in terms of the invariant vector fields) that yields a dispersion estimate suitable for the energy method. Finally, relations between various quantities introduced in the transformation of the systems will be studied. In Sect. 3, two energy estimates will be proven. One shows that a quantity which controls higher regularity norms grows very slowly as long as a low regularity norm remains small (Proposition 3.5. Using Proposition 2.4 or Lemma 3.3), another shows that a quantity that controls the low regularity norm remains small as long as the high regularity norm do not grow too fast (Proposition 3.6). Finally these two estimates are put together and a properly constructed energy is shown to be uniformly bounded for all the time when an a-priori assumption holds (Theorem 3.7). In the last section, we prove the global well-posedness of the 3D full water wave problem by using a local well-posedness result (Theorem 4.3), Theorem 3.7 and a continuity argument.

Three appendixes are added at the revision: Appendix A contains a partial list of various quantities introduced in this paper; Appendix B concerns water waves with velocity $\mathbf{v} \to c$ at the spatial infinity; Appendix C discusses normal forms for the water wave system.

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2 Basic analysis preparations

For a function $f = f(\alpha, \beta, t)$, we use the notation $f = f(\cdot, t) = f(t)$,

$$\|f(t)\|_2 = \|f(t)\|_{L^2} = \|f(\cdot, t)\|_{L^2(\mathbb{R}^2)},$$

$$|f(t)|_\infty = \|f(t)\|_{L^\infty} = \|f(\cdot, t)\|_{L^\infty(\mathbb{R}^2)}.$$

2.1 Vector fields and a generalized Sobolev inequality

As in [32], we will use the method of invariant vector fields. We know the linear part of the operator $\mathcal{P} = (\partial_t + b \cdot \nabla_\perp)^2 - A\mathcal{N} \times \nabla$ is $\mathfrak{P} = \partial_t^2 - e_2\partial_\alpha + e_1\partial_\beta$.

[8] Non-existence of positively or negatively elevated solitary waves for 2-D infinite depth water wave with or without surface tension was established in [28]; for 3-D finite or infinite depth positively elevated water wave without surface tension it was established in [11].

Although the invariant vector fields of $\mathfrak{P}$ was not known, it is not difficult to find them.[9] Using a combined method as that in [4, 26], we find that the set of operators

$$\Gamma = \Big\{\partial_t,\ \partial_\alpha,\ \partial_\beta,\ L_0 = \frac{1}{2}t\partial_t + \alpha\partial_\alpha + \beta\partial_\beta,\ \text{and}$$
$$\varpi = \alpha\partial_\beta - \beta\partial_\alpha - \frac{1}{2}e_3\Big\} \tag{2.1}$$

satisfy

$$[\partial_t, \mathfrak{P}] = [\partial_\alpha, \mathfrak{P}] = [\partial_\beta, \mathfrak{P}] = [\varpi, \mathfrak{P}] = 0, \qquad [L_0, \mathfrak{P}] = -\mathfrak{P}. \tag{2.2}$$

Let $\Upsilon = \alpha\partial_\beta - \beta\partial_\alpha$. So $\varpi = \Upsilon - \frac{1}{2}e_3$. We have

$$\begin{aligned}
&[\partial_t, \partial_\alpha] = [\partial_t, \partial_\beta] = [\partial_t, \Upsilon] = [\partial_\alpha, \partial_\beta] = [L_0, \Upsilon] = 0,\\
&[\partial_t, L_0] = \frac{1}{2}\partial_t, \qquad [\partial_\alpha, L_0] = [\Upsilon, \partial_\beta] = \partial_\alpha,\\
&[\partial_\beta, L_0] = [\partial_\alpha, \Upsilon] = \partial_\beta.
\end{aligned} \tag{2.3}$$

Furthermore, we have

$$\begin{aligned}
&[\partial_t, \partial_t + b\cdot\nabla_\perp] = b_t\cdot\nabla_\perp,\\
&[\partial, \partial_t + b\cdot\nabla_\perp] = (\partial b)\cdot\nabla_\perp, \quad \text{for } \partial = \partial_\alpha, \partial_\beta,\\
&[L_0, \partial_t + b\cdot\nabla_\perp] = \left(L_0 b - \frac{1}{2}b\right)\cdot\nabla_\perp - \frac{1}{2}(\partial_t + b\cdot\nabla_\perp),\\
&[\varpi, \partial_t + b\cdot\nabla_\perp] = \left(\varpi b - \frac{1}{2}e_3 b\right)\cdot\nabla_\perp.
\end{aligned} \tag{2.4}$$

Let $\mathcal{P}^\pm = (\partial_t + b\cdot\nabla_\perp)^2 \pm A\mathcal{N}\times\nabla$. Notice that $\mathcal{P} = \mathcal{P}^-$. We have

$$\begin{aligned}
[\partial_t, \mathcal{P}^\pm] &= \pm\{(A\zeta_\beta)_t\partial_\alpha - (A\zeta_\alpha)_t\partial_\beta\} + \{\partial_t(\partial_t + b\cdot\nabla_\perp)b - b_t\cdot\nabla_\perp b\}\cdot\nabla_\perp\\
&\quad + b_t\cdot\{(\partial_t + b\cdot\nabla_\perp)\nabla_\perp + \nabla_\perp(\partial_t + b\cdot\nabla_\perp)\},\\
[\partial, \mathcal{P}^\pm] &= \pm\{(\partial(A\zeta_\beta))\partial_\alpha - (\partial(A\zeta_\alpha))\partial_\beta\}
\end{aligned}$$

[9] One may find using the method in [4] that for the scaler operator $\partial_t^2 + |D|$, the following are invariants: $\partial_t,\ \partial_\alpha,\ \partial_\beta,\ L_0,\ \Upsilon,\ \alpha\partial_t + \frac{1}{2}t\partial_\alpha|D|^{-1},\ \beta\partial_t + \frac{1}{2}t\partial_\beta|D|^{-1}$. Those for $\mathfrak{P}$ are then obtained by properly modifying this set.

$$
\begin{aligned}
&+\{\partial(\partial_t+b\cdot\nabla_\perp)b-(\partial b)\cdot\nabla_\perp b\}\cdot\nabla_\perp\\
&+(\partial b)\cdot\{(\partial_t+b\cdot\nabla_\perp)\nabla_\perp+\nabla_\perp(\partial_t+b\cdot\nabla_\perp)\},\quad \text{for } \partial=\partial_\alpha,\partial_\beta,\\
[L_0,\mathcal{P}^\pm]&=-\mathcal{P}^\pm\pm\{L_0(A\zeta_\beta)\partial_\alpha-L_0(A\zeta_\alpha)\partial_\beta\},\\
&+\left\{(\partial_t+b\cdot\nabla_\perp)\left(L_0b-\frac{1}{2}b\right)\right\}\cdot\nabla_\perp\\
&+\left(L_0b-\frac{1}{2}b\right)\cdot\{(\partial_t+b\cdot\nabla_\perp)\nabla_\perp+\nabla_\perp(\partial_t+b\cdot\nabla_\perp)\},\\
[\varpi,\mathcal{P}^\pm]&=\pm(\Upsilon A)(\zeta_\beta\partial_\alpha-\zeta_\alpha\partial_\beta)\\
&\pm A\left(\partial_\beta\left(\varpi\lambda+\frac{1}{2}\lambda e_3\right)\partial_\alpha-\partial_\alpha\left(\varpi\lambda+\frac{1}{2}\lambda e_3\right)\partial_\beta\right)\\
&+\left(\varpi b-\frac{1}{2}e_3b\right)\cdot\{(\partial_t+b\cdot\nabla_\perp)\nabla_\perp+\nabla_\perp(\partial_t+b\cdot\nabla_\perp)\}\\
&+\left\{(\partial_t+b\cdot\nabla_\perp)\left(\varpi-\frac{1}{2}e_3\right)b\right\}\cdot\nabla_\perp.
\end{aligned}
\tag{2.5}
$$

For any positive integer m, and any operator P,

$$
[\Gamma^m,P]=\sum_{j=1}^{m}\Gamma^{m-j}[\Gamma,P]\Gamma^{j-1}. \tag{2.6}
$$

Let

$$
\mathbf{K}f(\alpha,\beta,t)=p.v.\iint k(\alpha,\beta,\alpha',\beta';t)f(\alpha',\beta',t)\,d\alpha'\,d\beta'
$$

where for some $\iota=0$, 1, or 2, $|(\alpha,\beta)-(\alpha',\beta')|^\iota k(\alpha,\beta,\alpha',\beta';t)$ is bounded, and k is smooth away from the diagonal $\Delta=\{(\alpha,\beta)=(\alpha',\beta')\}$. We have for f vanish fast at spatial infinity,

$$
\begin{aligned}
&[\partial_t,\mathbf{K}]f(\alpha,\beta,t)=\iint\partial_t k(\alpha,\beta,\alpha',\beta';t)f(\alpha',\beta',t)\,d\alpha'\,d\beta',\\
&[\partial,\mathbf{K}]f(\alpha,\beta,t)=\iint(\partial+\partial')k(\alpha,\beta,\alpha',\beta';t)f(\alpha',\beta',t)\,d\alpha'\,d\beta',\\
&\quad\partial=\partial_\alpha,\partial_\beta,\\
&[L_0,\mathbf{K}]f(\alpha,\beta,t)\\
&\quad=2\mathbf{K}f(\alpha,\beta,t)
\end{aligned}
$$

$$+\iint\left(\alpha\partial_\alpha+\beta\partial_\beta+\alpha'\partial_\alpha'+\beta'\partial_\beta'+\frac{1}{2}t\partial_t\right)k(\alpha,\beta,\alpha',\beta';t) \tag{2.7}$$
$$\times f(\alpha',\beta',t)\,d\alpha'\,d\beta',$$

$$[\Upsilon,\mathbf{K}]f(\alpha,\beta,t)=\iint(\Upsilon+\Upsilon')k(\alpha,\beta,\alpha',\beta';t)f(\alpha',\beta',t)\,d\alpha'\,d\beta',$$

$$\begin{aligned}[\partial_t+b\cdot\nabla_\perp,\mathbf{K}]f=&\iint(\partial_t+b\cdot\nabla_\perp+b'\cdot\nabla_\perp')k(\alpha,\beta,\alpha',\beta';t)\\&\times f(\alpha',\beta',t)\,d\alpha'\,d\beta'\\&+\iint k(\alpha,\beta,\alpha',\beta';t)\,\mathrm{div}'b'\,f(\alpha',\beta',t)\,d\alpha'\,d\beta'.\end{aligned}$$

One of the operators in (1.35), (2.38) and (2.40) is of the following type:

$$\mathbf{B}(g,f)=p.v.\iint K(\zeta'-\zeta)(g-g')\times(\zeta'_{\beta'}\partial_{\alpha'}-\zeta'_{\alpha'}\partial_{\beta'})f(\alpha',\beta',t)\,d\alpha'\,d\beta',$$

where $\mathrm{Re}\,g=0$. We have for $\Gamma=\partial_t,\partial_\alpha,\partial_\beta,L_0,\varpi$,

$$\begin{aligned}\Gamma\mathbf{B}(g,f)=&\iint K(\zeta'-\zeta)(\dot\Gamma g-\dot\Gamma' g')\\&\times(\zeta'_{\beta'}\partial_{\alpha'}-\zeta'_{\alpha'}\partial_{\beta'})f(\alpha',\beta',t)\,d\alpha'\,d\beta'\\&+\iint K(\zeta'-\zeta)(g-g')\\&\times(\zeta'_{\beta'}\partial_{\alpha'}-\zeta'_{\alpha'}\partial_{\beta'})\Gamma' f(\alpha',\beta',t)\,d\alpha'\,d\beta'\\&+\iint K(\zeta'-\zeta)(g-g')\\&\times(\partial_{\beta'}\dot\Gamma'\lambda'\partial_{\alpha'}-\partial_{\alpha'}\dot\Gamma'\lambda'\partial_{\beta'})f(\alpha',\beta',t)\,d\alpha'\,d\beta'\\&+\iint((\dot\Gamma'\lambda'-\dot\Gamma\lambda)\cdot\nabla)K(\zeta'-\zeta)(g-g')\\&\times(\zeta'_{\beta'}\partial_{\alpha'}-\zeta'_{\alpha'}\partial_{\beta'})f(\alpha',\beta',t)\,d\alpha'\,d\beta',\end{aligned} \tag{2.8}$$

where $\dot\Gamma g=\partial_t g,\ \partial_\alpha g,\ \partial_\beta g,\ (L_0-I)g,\ \varpi g+\frac{1}{2}ge_3$ respectively. (2.8) is straightforward with an application of (2.7), the definition $\zeta=P+\lambda$, and in the case $\Gamma=L_0$, the fact $(\xi\cdot\nabla)K(\xi)=-2K(\xi)$ and (2.3); in the case $\Gamma=\varpi$, the fact $((e_3\times\xi)\cdot\nabla)K(\xi)=\frac{1}{2}(e_3K(\xi)-K(\xi)e_3)$, (2.3) and $-e_3\,a\times b+a\times b\,e_3+2(e_3\times a)\times b+2a\times(e_3\times b)=0$, for $a,b\in\mathbb{R}^3$.

Before we derive the commutativity relations between L_0, ϖ and $\mathcal{H}$, we record

Lemma 2.1 *Let Ω be a C^2 domain in $\mathbb{R}^2$, with its boundary $\Sigma = \partial\Omega$ being parametrized by $\xi = \xi(\alpha, \beta)$, $(\alpha, \beta) \in \mathbb{R}^2$. For any vector η, and function f on $\mathbb{R}^2$, we have*

$$(\eta \times \xi_\beta) f_\alpha - (\eta \times \xi_\alpha) f_\beta = (\xi_\alpha \times \xi_\beta)(\eta \cdot \nabla_\xi) f - (\eta \cdot (\xi_\alpha \times \xi_\beta)) \mathcal{D}_\xi f, \tag{2.9}$$

$$-(\eta \cdot \nabla) K(\xi)(\xi'_{\alpha'} \times \xi'_{\beta'}) + (\xi'_{\alpha'} \cdot \nabla) K(\xi)(\eta \times \xi'_{\beta'}) + (\xi'_{\beta'} \cdot \nabla) K(\xi)(\xi'_{\alpha'} \times \eta) = 0, \tag{2.10}$$

for $\xi \neq 0$.

(2.9) is proved in the same way as the identity (5.17) in [31]. We omit the details. (2.10) is the identity (3.5) in [31].

We have the following commutativity relations between L_0, ϖ and $\mathcal{H}$.

Proposition 2.2 *Let $f \in C^1(\mathbb{R}^2 \times [0, T])$ be a $\mathcal{C}(V_2)$ valued function vanishing at spatial infinity. Then*

$$[L_0, \mathcal{H}] f = \iint K(\zeta' - \zeta)((L_0 - I)\lambda - (L'_0 - I)\lambda') \times (\zeta'_{\beta'}\partial_{\alpha'} - \zeta'_{\alpha'}\partial_{\beta'}) f' \, d\alpha' d\beta', \tag{2.11}$$

$$[\varpi, \mathcal{H}] f = \iint K(\zeta' - \zeta)\left(\varpi\lambda + \frac{1}{2}\lambda e_3 - \varpi'\lambda' - \frac{1}{2}\lambda' e_3\right) \times (\zeta'_{\beta'}\partial_{\alpha'} - \zeta'_{\alpha'}\partial_{\beta'}) f' \, d\alpha' d\beta'. \tag{2.12}$$

Proof Using (1.15), (1.16), (1.17) and argue similarly as in the proof of (1.18), we can show that

$$[L_0, \mathcal{H}] f = \iint K(\zeta' - \zeta)(L_0\zeta - L'_0\zeta') \times (\zeta'_{\beta'}\partial_{\alpha'} - \zeta'_{\alpha'}\partial_{\beta'}) f' \, d\alpha' d\beta'.$$

Using integration by parts and (2.10), we can check the following identity:

$$\iint K(\zeta' - \zeta)(\zeta - \zeta') \times (\zeta'_{\beta'}\partial_{\alpha'} - \zeta'_{\alpha'}\partial_{\beta'}) f' \, d\alpha' d\beta' = 0.$$

(2.11) then follows from the fact that $(L_0 - I)\zeta = (L_0 - I)\lambda$.

(2.12) is obtained similarly. First we have by using (1.16), (1.17) that

$$[\Upsilon, \mathcal{H}] f = \iint K(\zeta' - \zeta)(\Upsilon\zeta - \Upsilon'\zeta') \times (\zeta'_{\beta'}\partial_{\alpha'} - \zeta'_{\alpha'}\partial_{\beta'}) f' \, d\alpha' d\beta'.$$

We now check the identity

$$\frac{1}{2}[e_3, \mathcal{H}]f = \iint K(\zeta' - \zeta)(e_3 \times (\zeta - \zeta')) \times (\zeta'_{\beta'}\partial_{\alpha'} - \zeta'_{\alpha'}\partial_{\beta'}) f' \, d\alpha' d\beta'.$$

Using integration by parts, we have

$$\begin{aligned}
&\iint K(\zeta' - \zeta)(e_3 \times (\zeta - \zeta')) \times (\zeta'_{\beta'}\partial_{\alpha'} - \zeta'_{\alpha'}\partial_{\beta'}) f' \, d\alpha' \, d\beta' \\
&\quad = - \iint (\partial'_{\alpha} K \, (e_3 \times (\zeta - \zeta')) \\
&\qquad \times \zeta'_{\beta'} - \partial_{\beta'} K (e_3 \times (\zeta - \zeta')) \times \zeta'_{\alpha'}) f' \, d\alpha' \, d\beta' \\
&\qquad + \iint K(\zeta' - \zeta)((e_3 \times \zeta'_{\alpha'}) \times \zeta'_{\beta'} - (e_3 \times \zeta'_{\beta'}) \times \zeta'_{\alpha'}) f' \, d\alpha' \, d\beta' \\
&\quad = \iint ((e_3 \times (\zeta' - \zeta)) \cdot \nabla) K(\zeta' - \zeta) \mathcal{N}' f' \, d\alpha' \, d\beta' \\
&\qquad + \iint K(\zeta' - \zeta) e_3 \times \mathcal{N}' f' \, d\alpha' \, d\beta' \\
&\quad = \frac{1}{2} \iint (e_3 K \mathcal{N}' - K \mathcal{N}' e_3) f' \, d\alpha' \, d\beta' = \frac{1}{2}[e_3, \mathcal{H}] f.
\end{aligned}$$

Here in the second step we used (2.10), and the identity $(a \times b) \times c = b\, a \cdot c - a\, b \cdot c$. In the last step we used the fact $((e_3 \times \xi) \cdot \nabla) K(\xi) = \frac{1}{2}(e_3 K(\xi) - K(\xi) e_3)$ and $e_3 \times \mathcal{N} = \frac{1}{2}(e_3 \mathcal{N} - \mathcal{N} e_3)$. (2.12) therefore follows since $\Upsilon \zeta - e_3 \times \zeta = \Upsilon \lambda - e_3 \times \lambda = \varpi \lambda + \frac{1}{2} \lambda e_3$. □

In what follows, we denote the vector fields in (2.1) by Γ_i $i = 1, \ldots, 5$, or simply suppress the subscript and write as Γ. We shall write

$$\Gamma^k = \Gamma_1^{k_1} \Gamma_2^{k_2} \Gamma_3^{k_3} \Gamma_4^{k_4} \Gamma_5^{k_5}$$

for $k = (k_1, k_2, k_3, k_4, k_5)$. For a nonnegative integer k, we shall also use Γ^k to indicate a k-product of Γ_i. $i = 1, \ldots, 5$.

We now develop a generalized Sobolev inequality. Let $\alpha_1 = \alpha$, $\alpha_2 = \beta$. We introduce

$$\Omega_{0j}^{\pm} = \pm \alpha_j \partial_t + \frac{1}{2} t \partial_{\alpha_j} |D|^{-1} H, \quad j = 1, 2, \tag{2.13}$$

where $H=(e_2\partial_{\alpha_1}-e_1\partial_{\alpha_2})|D|^{-1}$. Therefore $H^2=I$. We also denote $\Omega_{0j}^-=\Omega_{0j}$.[10] Let $\mathfrak{P}^\pm=\partial_t^2\pm(e_2\partial_{\alpha_1}-e_1\partial_{\alpha_2})$. Notice that $\mathfrak{P}=\mathfrak{P}^-$. We know

$$\begin{aligned}\mathcal{P}^\pm=\mathfrak{P}^\pm&+\partial_t(b\cdot\nabla_\perp)+b\cdot\nabla_\perp(\partial_t+b\cdot\nabla_\perp)\\&\pm A(\lambda_\beta\partial_\alpha-\lambda_\alpha\partial_\beta)\pm(A-1)(e_2\partial_\alpha-e_1\partial_\beta).\end{aligned}\tag{2.14}$$

Let $P_d(\partial)$ be a polynomial of ∂_{α_j}, $j=1,2$, homogeneous of degree d, with coefficients in $\mathbb{R}$. We have

Lemma 2.3 1.

$$\begin{aligned}(\partial_{\alpha_1}^2+\partial_{\alpha_2}^2)\Omega_{01}^\pm&=\pm\left(\partial_{\alpha_1}\left(2\partial_t+L_0\partial_t-\frac{1}{2}t\mathfrak{P}^\pm\right)+\partial_{\alpha_2}\Upsilon\partial_t\right),\\(\partial_{\alpha_1}^2+\partial_{\alpha_2}^2)\Omega_{02}^\pm&=\pm\left(\partial_{\alpha_2}\left(2\partial_t+L_0\partial_t-\frac{1}{2}t\mathfrak{P}^\pm\right)-\partial_{\alpha_1}\Upsilon\partial_t\right).\end{aligned}\tag{2.15}$$

2.

$$\||D|\Omega_{0j}^\pm F(t)\|_{L^2}\le 2\sum_{k\le 1}\|\partial_t\Gamma^k F(t)\|_{L^2}+t\|\mathfrak{P}^\pm F(t)\|_{L^2}.\tag{2.16}$$

3.

$$\begin{aligned}&[\hat\Gamma,P_{d+l}(\partial)|D|^{-d}]=R|D|^l,\quad \textit{for } \hat\Gamma=L_0,\Upsilon,\ l=0,1,\\&[\Omega_{0j},P_{d+1}(\partial)|D|^{-d}]=R\partial_t,\end{aligned}\tag{2.17}$$

where R is a finite sum of operators of the type $P_k(\partial)|D|^{-k}$, and need not be the same for different $\hat\Gamma$, Ω_{0j}, $j=1,2$ or $l=0,1$.

Proof (2.17) is straightforward using Fourier transform. We prove (2.15) for $\Omega_{01}=\Omega_{01}^-$, the other cases follows similarly. We have

$$\begin{aligned}&\partial_{\alpha_1}\Omega_{02}-\partial_{\alpha_2}\Omega_{01}=\Upsilon\partial_t,\\&\partial_{\alpha_1}\Omega_{01}+\partial_{\alpha_2}\Omega_{02}=-2\partial_t-L_0\partial_t+\frac{1}{2}t\mathfrak{P}.\end{aligned}$$

Therefore

$$(\partial_{\alpha_1}^2+\partial_{\alpha_2}^2)\Omega_{01}=\partial_{\alpha_1}\left(-2\partial_t-L_0\partial_t+\frac{1}{2}t\mathfrak{P}\right)-\partial_{\alpha_2}\Upsilon\partial_t.$$

(2.16) is straightforward from here. □

[10] One may check that $[\Omega_{01}(e_2\partial_{\alpha_1}-e_1\partial_{\alpha_2})-\frac{1}{2}\partial_t e_2,\mathfrak{P}]=0$, $[\Omega_{02}(e_2\partial_{\alpha_1}-e_1\partial_{\alpha_2})+\frac{1}{2}\partial_t e_1,\mathfrak{P}]=0$. These are some of the invariant vector fields for $\mathfrak{P}$, not included in (2.1).

Proposition 2.4 (Generalized Sobolev inequality) *Let* $f \in C^\infty(\mathbb{R}^{2+1})$ *be a* $\mathcal{C}(V_2)$ *valued function, vanishing at spatial infinity. We have for* $l = 1, 2$,

$$(1 + t + |\alpha_1| + |\alpha_2|)|\partial_{\alpha_l} f(\alpha_1, \alpha_2, t)| \lesssim \sum_{k\le 4, j=1,2} (\|\Gamma^k \partial_t f(t)\|_{L^2} + \|\Gamma^k \partial_{\alpha_j} f(t)\|_{L^2}) + t \sum_{k\le 3} \|\mathfrak{P}\Gamma^k f(t)\|_{L^2}. \tag{2.18}$$

Here $a \lesssim b$ *means that there is a universal constant* c, *such that* $a \le cb$.

Proof Let $r^2 = \alpha_1^2 + \alpha_2^2$, $r\partial_r = \alpha_1\partial_{\alpha_1} + \alpha_2\partial_{\alpha_2}$. We have

$$\sum_1^2 \frac{\alpha_j}{r}\Omega_{0j} = -r\partial_t + \frac{1}{2}t\partial_r|D|^{-1}H, \qquad L_0 = \frac{1}{2}t\partial_t + r\partial_r$$

therefore

$$rL_0 + \frac{t}{2}\sum_1^2 \frac{\alpha_j}{r}\Omega_{0j} = r^2\partial_r + \frac{t^2}{4}\partial_r|D|^{-1}H. \tag{2.19}$$

Also

$$\sum_1^2 \Omega_{0j}\partial_{\alpha_j} = -r\partial_r\partial_t - \frac{1}{2}t|D|H, \qquad \sum_1^2 \alpha_j L_0\partial_{\alpha_j} = \frac{1}{2}tr\partial_t\partial_r + r^2\partial_r^2$$

gives

$$\frac{1}{2}t\sum_1^2 \Omega_{0j}\partial_{\alpha_j}|D|^{-1}H + \sum_1^2 \alpha_j L_0\partial_{\alpha_j}|D|^{-1}H = r^2\partial_r^2|D|^{-1}H - \frac{1}{4}t^2. \tag{2.20}$$

Let g be a $\mathcal{C}(V_2)$ valued function, $h = \partial_r|D|^{-1}Hg$. From (2.19), (2.20) we have (i is the complex number in this proof)

$$r^2\partial_r(g + ih) - \frac{1}{4}t^2 i(g + ih) = F, \tag{2.21}$$

where

$$F = rL_0 g + \frac{t}{2}\sum_1^2 \frac{\alpha_j}{r}\Omega_{0j} g$$

$$+ i\left(\frac{1}{2}t\sum_1^2 \Omega_{0j}\partial_{\alpha_j}|D|^{-1}Hg + \sum_1^2 \alpha_j L_0\partial_{\alpha_j}|D|^{-1}Hg\right).$$

Using (2.21) we get $r^2\partial_r(e^{\frac{it^2}{4r}}(g+i\,h)) = e^{\frac{it^2}{4r}}F$, therefore. (Recall by definition, components of a $\mathcal{C}(V_2)$ valued function are real valued.)

$$|g(re^{i\theta},t)| \le |(g+i\,h)(re^{i\theta},t)| \le \int_r^\infty \frac{1}{s^2}|F(se^{i\theta},t)|\,ds \tag{2.22}$$

and this implies that

$$\begin{aligned}|g(re^{i\theta},t)|^2 &\lesssim \frac{1}{r^2}\int_r^\infty s(|L_0 g|^2 + \sum_1^2 |L_0\partial_{\alpha_j}|D|^{-1}Hg|^2)\,ds\\ &\quad + \frac{t^2}{r^4}\sum_{j=1}^2\int_r^\infty s(|\Omega_{0j}g|^2 + |\Omega_{0j}\partial_{\alpha_j}|D|^{-1}Hg|^2)\,ds.\end{aligned} \tag{2.23}$$

Now in (2.23) we let $g = \Upsilon^k\partial_{\alpha_l}f$. Using Lemma 1.2 on p. 40 of [22], and (2.17), (2.3), we obtain

$$\begin{aligned}|\partial_{\alpha_l}f(re^{i\theta_0},t)|^2 &\lesssim \sum_{k\le 2}\int_0^{2\pi}|\Upsilon^k\partial_{\alpha_l}f(re^{i\theta},t)|^2\,d\theta\\ &\lesssim \frac{1}{r^2}\sum_{j=1}^2\sum_{k\le 2,m\le 1}\|L_0^m\Upsilon^k\partial_{\alpha_j}f(t)\|_{L^2}^2\\ &\quad + \frac{t^2}{r^4}\left(\sum_{j=1}^2\sum_{k\le 2}\||D|\Omega_{0j}\Upsilon^k f(t)\|_{L^2}^2 + \|\partial_t\Upsilon^k f(t)\|_{L^2}^2\right).\end{aligned} \tag{2.24}$$

A further application of (2.16) gives

$$\begin{aligned}|\partial_{\alpha_l}f(re^{i\theta_0},t)|^2 &\lesssim \frac{1}{r^2}\sum_{k\le 3}\sum_{j=1}^2\|\Gamma^k\partial_{\alpha_j}f(t)\|_2^2 + \frac{t^2}{r^4}\sum_{k\le 3}\|\Gamma^k\partial_t f(t)\|_2^2\\ &\quad + \frac{t^4}{r^4}\sum_{k\le 2}\|\mathfrak{P}\Gamma^k f(t)\|_2^2.\end{aligned} \tag{2.25}$$

From a similar argument we also have

$$
\begin{aligned}
&|\partial_{\alpha_m}|D|^{-1}H\partial_{\alpha_l}f(re^{i\theta_0},t)|^2\\
&\quad\lesssim \frac{1}{r^2}\sum_{k\le 3}\sum_{j=1}^{2}\|\Gamma^k\partial_{\alpha_j}f(t)\|_2^2+\frac{t^2}{r^4}\sum_{k\le 3}\|\Gamma^k\partial_t f(t)\|_2^2\\
&+\frac{t^4}{r^4}\sum_{k\le 2}\|\mathfrak{P}\Gamma^k f(t)\|_2^2.
\end{aligned}
\tag{2.26}
$$

Case 0: $|t|+r\le 1$. (2.18) follows from the standard Sobolev embedding.
Case 1: $t\le r$ and $|t|+r\ge 1$. (2.18) follows from (2.25).
Case 2: $r\le t$ and $|t|+r\ge 1$. We use (2.20). We have

$$
\begin{aligned}
\frac{1}{4}t^2\partial_{\alpha_l}f&=r^2\partial_r^2|D|^{-1}H\partial_{\alpha_l}f-\frac{1}{2}t\sum_{j=1}^{2}\Omega_{0j}\partial_{\alpha_j}|D|^{-1}H\partial_{\alpha_l}f\\
&\quad-\sum_{j=1}^{2}\alpha_jL_0\partial_{\alpha_j}|D|^{-1}H\partial_{\alpha_l}f.
\end{aligned}
\tag{2.27}
$$

Using (2.26) to estimate the first term, the standard Sobolev embedding and Lemma 2.3 to estimate the second and third term on the right hand side of (2.27). We obtain (2.18). □

Finally, the following Proposition shows that the term $\partial_\beta\lambda\partial_\alpha\chi-\partial_\alpha\lambda\partial_\beta\chi$ in (1.35) posses the desired null structure.

Proposition 2.5 *Let f, g be real valued functions. We have*

$$
\begin{aligned}
\partial_{\alpha_1}f\partial_{\alpha_2}g-\partial_{\alpha_2}f\partial_{\alpha_1}g=\frac{2}{t}\{&\mp\partial_t(e_2\partial_{\alpha_1}-e_1\partial_{\alpha_2})f\,\Upsilon g\\
&+\Omega_{01}^{\pm}(e_2\partial_{\alpha_1}-e_1\partial_{\alpha_2})f\,\partial_{\alpha_2}g\\
&-\Omega_{02}^{\pm}(e_2\partial_{\alpha_1}-e_1\partial_{\alpha_2})f\,\partial_{\alpha_1}g\}.
\end{aligned}
\tag{2.28}
$$

The proof is straightforward from definition. We omit the details.

2.2 Estimates of the Cauchy type integral operators

Let $J \in C^1(\mathbb{R}^d; \mathbb{R}^l),\ A_i \in C^1(\mathbb{R}^d), i = 1, \dots, m,\ F \in C^\infty(\mathbb{R}^l)$. Define (for $x, y \in \mathbb{R}^d$)

$$C_1(J, A, f)(x) = p.v. \int F\left(\frac{J(x) - J(y)}{|x - y|}\right) \frac{\Pi_{i=1}^m (A_i(x) - A_i(y))}{|x - y|^{d+m}} f(y)\, dy. \tag{2.29}$$

Assume that $k_1(x, y) = F(\frac{J(x)-J(y)}{|x-y|}) \frac{\Pi_{i=1}^m (A_i(x)-A_i(y))}{|x-y|^{d+m}}$ is odd, i.e. $k_1(x, y) = -k_1(y, x)$.

Proposition 2.6 *There exist constants* $c_1 = c_1(F, \|\nabla J\|_{L^\infty}),\ c_2 = c_2(F, \|\nabla J\|_{L^\infty})$, *such that* 1. *For any* $f \in L^2(\mathbb{R}^d),\ \nabla A_i \in L^\infty(\mathbb{R}^d),\ 1 \le i \le m$,

$$\|C_1(J, A, f)\|_{L^2(\mathbb{R}^d)} \le c_1 \|\nabla A_1\|_{L^\infty(\mathbb{R}^d)} \cdots \|\nabla A_m\|_{L^\infty(\mathbb{R}^d)} \|f\|_{L^2(\mathbb{R}^d)}. \tag{2.30}$$

2. *For any* $f \in L^\infty(\mathbb{R}^d),\ \nabla A_i \in L^\infty(\mathbb{R}^d),\ 2 \le i \le m,\ \nabla A_1 \in L^2(\mathbb{R}^d)$,

$$\begin{aligned} \|C_1(J, A, f)\|_{L^2(\mathbb{R}^d)} \le c_2 &\|\nabla A_1\|_{L^2(\mathbb{R}^d)} \|\nabla A_2\|_{L^\infty(\mathbb{R}^d)} \cdots \\ &\times \|\nabla A_m\|_{L^\infty(\mathbb{R}^d)} \|f\|_{L^\infty(\mathbb{R}^d)}. \end{aligned} \tag{2.31}$$

Proof (2.30) is a result of Coifman, McIntosh and Meyer [8, 9, 16].

We prove (2.31) by the method of rotations and the inequality (3.17) in [32]. We only write for $d = 2$, the same argument applies to general cases. Let $R_\theta f(x) = f(e^{i\theta} x)$, $x = (x_1, x_2) = x_1 + i\, x_2 \in \mathbb{R}^2$,

$$\begin{aligned} K(J, A, f)(x) = p.v. \int_{\mathbb{R}^1} &F\left(\frac{J(x) - J(x + r)}{r}\right) \\ &\times \frac{\Pi_{i=1}^m (A_i(x) - A_i(x + r))}{r^{m+1}} f(x + r)\, dr. \end{aligned}$$

We have, from the change of coordinate formula: $\int_{\mathbb{R}^2} g(y)\, dy = \int_{\mathbb{R}} \int_0^\pi g(re^{i\theta}) |r|\, dr\, d\theta$ and the assumption that $k_1(x, y)$ is odd, that

$$C_1(J, A, f)(x) = \int_0^\pi R_\theta^{-1} K(R_\theta J, R_\theta A, R_\theta f)(x)\, d\theta$$

where for $A = (A_1, \dots, A_m)$, $R_\theta A = (R_\theta A_1, \dots, R_\theta A_m)$. (2.31) now follows from the inequality (3.17) in [32]. □

Let J, A_i, F be as above, define (for $x, y \in \mathbb{R}^d$)

$$C_2(J, A, f)(x) = p.v. \int F\left(\frac{J(x) - J(y)}{|x - y|}\right) \times \frac{\Pi_{i=1}^m (A_i(x) - A_i(y))}{|x - y|^{d+m-1}} \partial_{y_k} f(y)\, dy. \quad (2.32)$$

Assume that $k_2(x, y) = F(\frac{J(x)-J(y)}{|x-y|}) \frac{\Pi_{i=1}^m (A_i(x)-A_i(y))}{|x-y|^{d+m-1}}$ is even, i.e. $k_2(x, y) = k_2(y, x)$.

Proposition 2.7 *There exist constants* $c_1 = c_1(F, \|\nabla J\|_{L^\infty})$, $c_2 = c_2(F, \|\nabla J\|_{L^\infty})$, *such that* 1. *For any* $f \in L^2(\mathbb{R}^d)$, $\nabla A_i \in L^\infty(\mathbb{R}^d)$, $1 \le i \le m$,

$$\|C_2(J, A, f)\|_{L^2(\mathbb{R}^d)} \le c_1 \|\nabla A_1\|_{L^\infty(\mathbb{R}^d)} \cdots \|\nabla A_m\|_{L^\infty(\mathbb{R}^d)} \|f\|_{L^2(\mathbb{R}^d)}. \quad (2.33)$$

2. *For any* $f \in L^\infty(\mathbb{R}^d)$, $\nabla A_i \in L^\infty(\mathbb{R}^d)$, $2 \le i \le m$, $\nabla A_1 \in L^2(\mathbb{R}^d)$,

$$\|C_1(J, A, f)\|_{L^2(\mathbb{R}^d)} \le c_2 \|\nabla A_1\|_{L^2(\mathbb{R}^d)} \|\nabla A_2\|_{L^\infty(\mathbb{R}^d)} \cdots \times \|\nabla A_m\|_{L^\infty(\mathbb{R}^d)} \|f\|_{L^\infty(\mathbb{R}^d)}. \quad (2.34)$$

Proposition 2.7 follows from Proposition 2.6 and integration by parts. We also have the following L^∞ estimate for $C_1(J, A, f)$ as defined in (2.29).

Proposition 2.8 *There exists a constant* $c = c(F, \|\nabla J\|_{L^\infty}, \|\nabla^2 J\|_{L^\infty})$, *such that for any real number* $r > 0$,

$$\|C_1(J, A, f)\|_{L^\infty} \le c\left(\prod_{i=1}^m (\|\nabla A_i\|_{L^\infty} + \|\nabla^2 A_i\|_{L^\infty})(\|f\|_{L^\infty} + \|\nabla f\|_{L^\infty}) + \prod_{i=1}^m \|\nabla A_i\|_{L^\infty} \|f\|_{L^\infty} \ln r + \prod_{i=1}^m \|\nabla A_i\|_{L^\infty} \|f\|_{L^2} \frac{1}{r^{d/2}}\right). \quad (2.35)$$

The proof of Proposition 2.8 is an easy modification of that of Proposition 3.4 in [32]. We omit.

At last, we record the standard Sobolev embedding.

Proposition 2.9 *For any* $f \in C^\infty(\mathbb{R}^2)$,

$$\|f\|_{L^\infty} \lesssim \|f\|_{L^2} + \|\nabla f\|_{L^2} + \|\nabla^2 f\|_{L^2}. \tag{2.36}$$

2.3 Regularities and relations among various quantities

Let $\mathfrak{v} = (\partial_t + b \cdot \nabla_\perp)\chi$,

$$E_m(t) = \sum_{|j| \le m} (\|(\partial_t + b \cdot \nabla_\perp)\Gamma^j \chi(t)\|_2^2 + \|(\partial_t + b \cdot \nabla_\perp)\Gamma^j \mathfrak{v}(t)\|_2^2). \tag{2.37}$$

In Sect. 3, we will use (1.35) and (2.41) for χ and $\mathfrak{v}$ to obtain energy estimates, and the estimates concern the quantity $E_m(t)$. So in this subsection, we give estimates of the L^2 norms of derivatives of various quantities involved in (1.35), (2.41) in terms of $E_m(t)$, and the L^∞ norms of derivatives of various quantities in terms of that of $\partial\chi$, $\partial\mathfrak{v}$, $\partial = \partial_\alpha, \partial_\beta$. Eventually the generalized Sobolev inequality will be used to estimate the L^∞ norms of the derivatives of $\partial\chi$ and $\partial\mathfrak{v}$ in terms of $E_m(t)$ (Lemma 3.3).

We first present the quasi-linear equation for $u = \xi_t \circ k^{-1}$ and a formula for $\mathfrak{a}_t$. These are very much the same as those derived in [31] and will be used to get a local wellposedness result. We also give the equations for $\lambda^* = (I + \mathcal{H})\mathfrak{z} e_3$ and $\mathfrak{v}$. The equation for λ^* is an auxiliary equation and will be used in the proof of Proposition 3.6.

Proposition 2.10 *We have* 1.

$$((\partial_t + b \cdot \nabla_\perp)^2 + A\mathcal{N} \times \nabla)u = U_k^{-1}(\mathfrak{a}_t N), \tag{2.38}$$

where

$$\begin{aligned}
&(I - \mathcal{H})(U_k^{-1}(\mathfrak{a}_t N)) \\
&\quad = 2\iint K(\zeta' - \zeta)(w - w') \times (\zeta'_{\beta'}\partial_{\alpha'} - \zeta'_{\alpha'}\partial_{\beta'})u'\, d\alpha' d\beta' \\
&\qquad + \iint K(\zeta' - \zeta)\{((u - u') \times u'_{\beta'})u'_{\alpha'} - ((u - u') \times u'_{\alpha'})u'_{\beta'}\}\, d\alpha' d\beta' \\
&\qquad + 2\iint K(\zeta' - \zeta)(u - u') \times (\zeta'_{\beta'}\partial_{\alpha'} - \zeta'_{\alpha'}\partial_{\beta'})w'\, d\alpha' d\beta' \\
&\qquad + \iint ((u' - u) \cdot \nabla)K(\zeta' - \zeta)(u - u') \times (\zeta'_{\beta'}\partial'_{\alpha} - \zeta'_{\alpha'}\partial'_{\beta})u'\, d\alpha' d\beta'.
\end{aligned} \tag{2.39}$$

2.

$$
\begin{aligned}
&((\partial_t + b\cdot\nabla_\perp)^2 + A\mathcal{N}\times\nabla)\lambda^* \\
&\quad = -(\mathcal{H}-\overline{\mathcal{H}})\overline{w} - e_3\iint K(\zeta'-\zeta)(u-u')\times(\zeta'_{\beta'}\partial_{\alpha'}-\zeta'_{\alpha'}\partial_{\beta'})u'\,d\alpha' d\beta' e_3 \\
&\qquad + 2\iint K(\zeta'-\zeta)(w-w')\times(\zeta'_{\beta'}\partial_{\alpha'}-\zeta'_{\alpha'}\partial_{\beta'})\mathfrak{z}'\,d\alpha' d\beta' e_3 \\
&\qquad + \iint K(\zeta'-\zeta)\{((u-u')\times u'_{\beta'})\mathfrak{z}'_{\alpha'} - ((u-u')\times u'_{\alpha'})\mathfrak{z}'_{\beta'}\}\,d\alpha' d\beta' e_3 \\
&\qquad + 2\iint K(\zeta'-\zeta)(u-u')\times(\zeta'_{\beta'}\partial_{\alpha'}-\zeta'_{\alpha'}\partial_{\beta'})u'_3\,d\alpha' d\beta' e_3 \\
&\qquad + \iint((u'-u)\cdot\nabla)K(\zeta'-\zeta)(u-u')\times(\zeta'_{\beta'}\partial'_{\alpha}-\zeta'_{\alpha'}\partial'_{\beta})\mathfrak{z}'\,d\alpha' d\beta' e_3.
\end{aligned}
\tag{2.40}
$$

3.

$$
\begin{aligned}
&\big((\partial_t + b\cdot\nabla_\perp)^2 - A\mathcal{N}\times\nabla\big)\mathfrak{v} \\
&\quad = \frac{\mathfrak{a}_t}{\mathfrak{a}}\circ k^{-1}A\mathcal{N}\times\nabla\chi + A(u_\beta\chi_\alpha - u_\alpha\chi_\beta) \\
&\qquad + (\partial_t + b\cdot\nabla_\perp)((\partial_t+b\cdot\nabla_\perp)^2 - A\mathcal{N}\times\nabla)\chi.
\end{aligned}
\tag{2.41}
$$

Proof (2.38) is derived from (1.23), (1.24). Taking derivative to t to (1.23), we have $\xi_{ttt} - \mathfrak{a}N_t = \mathfrak{a}_t N$. Using (1.13), (1.24), we derive

$$N_t = -\xi_\beta\times\xi_{t\alpha} + \xi_\alpha\times\xi_{t\beta} = -\xi_\beta\xi_{t\alpha} + \xi_\alpha\xi_{t\beta} = -N\times\nabla\xi_t.$$

Therefore

$$\xi_{ttt} + \mathfrak{a}N\times\nabla\xi_t = \mathfrak{a}_t N. \tag{2.42}$$

Now to derive an equation for $\mathfrak{a}_t N$, we apply $(I-\mathfrak{H})$ to both sides of (2.42). We get

$$(I-\mathfrak{H})(\mathfrak{a}_t N) = (I-\mathfrak{H})(\xi_{ttt} + (\mathfrak{a}N\times\nabla)\xi_t) = [\partial_t^2 + \mathfrak{a}N\times\nabla, \mathfrak{H}]\xi_t$$

(2.39) then follows from (1.19), (1.18) and an application of the coordinate change U_k^{-1}. An application of the coordinate change U_k^{-1} to (2.42) gives (2.38).

We can derive the equation for λ^* in a similar way as that for χ. We have

$$(\partial_t^2 + \mathfrak{a}N\times\nabla)\Lambda^* = (I+\mathfrak{H})(\partial_t^2 + \mathfrak{a}N\times\nabla)ze_3 + [\partial_t^2 + \mathfrak{a}N\times\nabla, \mathfrak{H}]ze_3. \tag{2.43}$$

Notice that $(\partial_t^2 + \mathfrak{a}N \times \nabla)ze_3 = -\overline{\xi_{tt}}$ and

$$(I+\mathfrak{H})\overline{\xi_{tt}} = e_3(\xi_{tt} - \mathfrak{H}\xi_{tt})e_3 + (\mathfrak{H} - e_3\mathfrak{H}e_3)\overline{\xi_{tt}} = e_3[\partial_t, \mathfrak{H}]\xi_t e_3 + (\mathfrak{H} - \overline{\mathfrak{H}})\overline{\xi_{tt}}$$

(2.40) again follows from Lemma 1.2 and then an application of the change of coordinate U_k^{-1} to (2.43). We remark that the right hand sides of both (2.38), (2.40) are of terms that are at least quadratic.

(2.41) is obtained by taking derivative ∂_t to (1.25), then make the change of variable U_k^{-1}. □

We present some useful identities in the following. Proposition 2.11 combined with Proposition 2.12 for example, will be used to put the first term on the right hand side of (1.35) into cubic in the energy estimates.

Proposition 2.11 *For $f^\pm$ satisfying $f^\pm = \pm\mathcal{H}f^\pm$, and g being vector valued, we have*

$$\iint K(\zeta'-\zeta)(g-g') \times (\zeta'_\beta \partial_{\alpha'} - \zeta'_{\alpha'}\partial_{\beta'}) f'^{\pm}\, d\alpha' d\beta' = (\pm I - \mathcal{H})(g \cdot \nabla_\xi^\pm f). \tag{2.44}$$

Proof We only prove for f satisfying $f = \mathcal{H}f$. We know $f(\alpha,\beta,t) = F(\zeta(\alpha,\beta,t),t)$ for some F analytic in $\Omega(t)$. From (2.9), we have

$$\begin{aligned}\iint K(g-g') &\times (\zeta'_\beta \partial_{\alpha'} - \zeta'_{\alpha'}\partial_{\beta'}) f'\, d\alpha' d\beta' \\ &= \iint K\mathcal{N}'(g-g') \cdot \nabla_\xi^+ f' = (I-\mathcal{H})(g \cdot \nabla_\xi^+ f),\end{aligned}$$

where in the last step we used the fact that $\partial_{\xi_i}^+ f = \mathcal{H}\partial_{\xi_i}^+ f$, since $\partial_{\xi_i}^+ f$ is the trace on $\Sigma(t)$ of the analytic function $\partial_{\xi_i} F$, $i = 1, 2, 3$. □

Define

$$\mathcal{H}^* f = -\iint \zeta_\alpha \times \zeta_\beta K(\zeta'-\zeta) f(\alpha',\beta',t)\, d\alpha'\, d\beta' = -\iint \mathcal{N} K f'\, d\alpha'\, d\beta'. \tag{2.45}$$

Proposition 2.12 *For $\mathcal{C}(V_2)$ valued smooth functions f and g, we have*

$$\iint f \cdot \{\mathcal{H}g\} = \iint \{\mathcal{H}^* f\} \cdot g \quad \textit{and} \tag{2.46}$$

$$(\mathcal{H}^* - \mathcal{H}) f = \iint \{K(\zeta'-\zeta) \cdot (\mathcal{N} + \mathcal{N}') + K(\zeta'-\zeta) \times (\mathcal{N} - \mathcal{N}')\} f'. \tag{2.47}$$

Proof Both identities are straightforward from definition. We omit the details. □

Let $\sigma_i = \{\sigma\}_i$ denote the e_i component of σ.

Lemma 2.13 *Let Ω be a C^2 domain in $\mathbb{R}^3$ with $\partial\Omega = \Sigma$ being parametrized by $\xi = \xi(\alpha, \beta)$, $(\alpha, \beta) \in \mathbb{R}^2$, and $N = \xi_\alpha \times \xi_\beta$, $\mathbf{n} = \frac{N}{|N|}$. Assume that F is a Clifford analytic function in Ω. Then the trace of ∇F_i : $\nabla_\xi F_i = \nabla F_i(\xi(\alpha, \beta))$ satisfies*

$$\nabla_\xi F_i = \mathbf{n}\left(-\frac{1}{|N|}(\xi_\beta\partial_\alpha - \xi_\alpha\partial_\beta)F_i + \left\{\frac{1}{|N|}(\xi_\beta\partial_\alpha - \xi_\alpha\partial_\beta)F\right\}_i\right), \quad i = 1, 2, 3. \tag{2.48}$$

Proof We know $\mathcal{D}F = 0$ in Ω. Therefore $\mathbf{n}\mathcal{D}_\xi F = -\mathbf{n}\cdot\mathcal{D}_\xi F + \mathbf{n}\times\mathcal{D}_\xi F = 0$. This implies

$$\mathbf{n}\cdot\nabla_\xi F_i = \{\mathbf{n}\times\nabla_\xi F\}_i = \left\{\frac{1}{|N|}(\xi_\beta\partial_\alpha - \xi_\alpha\partial_\beta)F\right\}_i.$$

Therefore

$$\begin{aligned}\nabla_\xi F_i &= -\mathbf{n}\mathbf{n}\nabla_\xi F_i = \mathbf{n}(\mathbf{n}\cdot\nabla_\xi F_i - \mathbf{n}\times\nabla_\xi F_i)\\ &= \mathbf{n}\left(\left\{\frac{1}{|N|}(\xi_\beta\partial_\alpha - \xi_\alpha\partial_\beta)F\right\}_i - \frac{1}{|N|}(\xi_\beta\partial_\alpha - \xi_\alpha\partial_\beta)F_i\right). \end{aligned}\tag{2.49}$$

□

The following identities give relations among various quantities. We note that the right hand side of each of the identities is of quadratic or higher orders.

Lemma 2.14 *We have*

$$\overline{\lambda} + \chi = (\overline{\mathcal{H}} - \mathcal{H})\mathfrak{z}e_3 + \mathcal{K}\mathfrak{z}e_3, \qquad \overline{\lambda^*} + \chi = (\overline{\mathcal{H}} - \mathcal{H})\mathfrak{z}e_3, \tag{2.50}$$

$$\begin{aligned}\partial_\alpha\mathfrak{z} + \mathcal{N}\cdot e_1 &= (\partial_\alpha\lambda\times\partial_\beta\lambda)\cdot e_1,\\ \partial_\beta\mathfrak{z} + \mathcal{N}\cdot e_2 &= (\partial_\alpha\lambda\times\partial_\beta\lambda)\cdot e_2,\end{aligned}\tag{2.51}$$

$$\mathcal{N} - e_3 - \partial_\alpha\lambda\times e_2 + \partial_\beta\lambda\times e_1 = \partial_\alpha\lambda\times\partial_\beta\lambda, \tag{2.52}$$

$$w - (\mathcal{N} - e_3) = (A-1)\mathcal{N}, \tag{2.53}$$

$$2(\overline{u} + (\partial_t + b\cdot\nabla_\perp)\chi) = (\mathcal{H} - \overline{\mathcal{H}})\overline{u} - 2[\partial_t + b\cdot\nabla_\perp, \mathcal{H}]\mathfrak{z}e_3, \tag{2.54}$$

$$2(\overline{w} + (\partial_t + b \cdot \nabla_\perp)\mathfrak{v}) = (\mathcal{H} - \overline{\mathcal{H}})\overline{w} + [\partial_t + b \cdot \nabla_\perp, \mathcal{H} - \overline{\mathcal{H}}]\overline{u} \\ - 2(\partial_t + b \cdot \nabla_\perp)[\partial_t + b \cdot \nabla_\perp, \mathcal{H}]\mathfrak{z} e_3, \quad (2.55)$$

$$(\mathcal{H} - \overline{\mathcal{H}}) f = -2 \iint K \cdot \mathcal{N}' f' + 2 \iint (K_1 \mathcal{N}_2' - K_2 \mathcal{N}_1') e_3 f', \quad (2.56)$$

where $K = K_1 e_1 + K_2 e_2 + K_3 e_3$, $\mathcal{N} = \mathcal{N}_1 e_1 + \mathcal{N}_2 e_2 + \mathcal{N}_3 e_3$, *and* f *is a function.*

Proof (2.50), (2.52), (2.56) are straightforward from definition, (2.53) is (1.23) with a change of coordinate U_k^{-1}. Notice that the e_3 component of λ is $\mathfrak{z}$, therefore (2.51) follows straightforwardly from (2.52).

We now derive (2.54) from the definition of $\pi = (I - \mathfrak{H}) z e_3$. We have

$$\begin{aligned} 2\partial_t \pi &= 2(I - \mathfrak{H}) z_t e_3 - 2[\partial_t, \mathfrak{H}] z e_3 = (I - \mathfrak{H})(\xi_t - \overline{\xi}_t) - 2[\partial_t, \mathfrak{H}] z e_3 \\ &= -(I - \mathfrak{H})\overline{\xi}_t - 2[\partial_t, \mathfrak{H}] z e_3 = -2\overline{\xi}_t + (\mathfrak{H} - \overline{\mathfrak{H}})\overline{\xi}_t - 2[\partial_t, \mathfrak{H}] z e_3. \end{aligned} \quad (2.57)$$

Here in the last step we used (1.24). (2.54) follows from (2.57) with a change of coordinate U_k^{-1}. (2.55) is obtained by taking derivative $\partial_t + b \cdot \nabla_\perp$ to (2.54). □

In what follows, we let $l \geq 4$, $l + 2 \leq q \leq 2l$, $\xi = \xi(\alpha, \beta, t)$, $t \in [0, T]$ be a solution of the water wave system (1.23)–(1.24). Assume that the mapping $k(\cdot, t) : \mathbb{R}^2 \to \mathbb{R}^2$ defined in (1.28) is a diffeomorphism and its Jacobian $J(k(t)) > 0$, for $t \in [0, T]$. Assume for $\partial = \partial_\alpha, \ \partial_\beta$,

$$\Gamma^j \partial\lambda, \ \Gamma^j \partial\mathfrak{z}, \ \Gamma^j (\partial_t + b \cdot \nabla_\perp)\chi, \Gamma^j (\partial_t + b \cdot \nabla_\perp)\mathfrak{v} \in C([0, T], L^2(\mathbb{R}^2)), \\ \text{for } |j| \leq q. \quad (2.58)$$

Let $t \in [0, T]$ be fixed. Assume that at this time t,

$$\sum_{\substack{|j| \leq l+2 \\ \partial = \partial_\alpha, \partial_\beta}} (\|\Gamma^j \partial\lambda(t)\|_2 + \|\Gamma^j \partial\mathfrak{z}(t)\|_2 \\ + \|\Gamma^j \mathfrak{v}(t)\|_2 + \|\Gamma^j (\partial_t + b \cdot \nabla_\perp)\mathfrak{v}(t)\|_2) \leq M, \quad (2.59)$$

$$|\zeta(\alpha, \beta, t) - \zeta(\alpha', \beta', t)| \geq \frac{1}{4}(|\alpha - \alpha'| + |\beta - \beta'|) \\ \text{for } \alpha, \beta, \alpha', \beta' \in \mathbb{R}.$$

We will take $M \leq M_0$, where $M_0 > 0$ will be made clear in the following analysis. For the rest of this paper, the inequality $a \lesssim b$ means that there is a constant $c = c(M_0)$ depending on M_0, or a universal constant c, such that $a \le c\,b$. $a \simeq b$ means $a \lesssim b$ and $b \lesssim a$.

Lemma 2.15 *We have for $m \le 2l$, and any function $\phi \in C_0^\infty(\mathbb{R}^2 \times [0, T])$,*

$$\begin{aligned}
&\|[\partial_t + b\cdot\nabla_\perp, \Gamma^m]\phi(t)\|_2 \\
&\quad \lesssim \sum_{j\le l+2} \|\Gamma^j b(t)\|_2 \sum_{\substack{|j|\le m-1\\ \partial=\partial_\alpha,\partial_\beta}} \|\partial\Gamma^j\phi(t)\|_2 \\
&\qquad + \sum_{|j|\le m} \|\Gamma^j b(t)\|_2 \sum_{\substack{|j|\le l+1\\ \partial=\partial_\alpha,\partial_\beta}} \|\partial\Gamma^j\phi(t)\|_2 \\
&\qquad + \sum_{|j|\le m-1} \|(\partial_t + b\cdot\nabla_\perp)\Gamma^j\phi(t)\|_2,
\end{aligned} \tag{2.60}$$

$$\begin{aligned}
&\|[\partial_t + b\cdot\nabla_\perp, \Gamma^m]\phi(t)\|_2 \\
&\quad \lesssim \sum_{j\le l+2} \|\Gamma^j b(t)\|_2 \sum_{\substack{|j|\le m-1\\ \partial=\partial_\alpha,\partial_\beta}} \|\partial\Gamma^j\phi(t)\|_2 \\
&\qquad + \sum_{|j|\le m} \|\Gamma^j b(t)\|_2 \sum_{\substack{|j|\le l+1\\ \partial=\partial_\alpha,\partial_\beta}} \|\partial\Gamma^j\phi(t)\|_2 \\
&\qquad + \sum_{|j|\le m-1} \|\Gamma^j(\partial_t + b\cdot\nabla_\perp)\phi(t)\|_2.
\end{aligned}$$

Proof (2.60) is an easy consequence of the identities (2.4), (2.6) and Proposition 2.9:

$$[\partial_t + b\cdot\nabla_\perp, \Gamma^m]\phi = \sum_{j=1}^m \Gamma^{m-j}[\partial_t + b\cdot\nabla_\perp, \Gamma]\Gamma^{j-1}\phi.$$

□

The following proposition gives the L^2 estimates of various quantities in terms of that of χ and $\mathfrak{v}$. Let $t \in [0, T]$ be the time when (2.59) holds.

Proposition 2.16 *Let $m \le q$. There is a $M_0 > 0$, sufficiently small, such that for $M \le M_0$,*

$$\begin{aligned}
&\sum_{\partial=\partial_\alpha,\partial_\beta} (\|\Gamma^m\partial\lambda(t)\|_2 + \|\Gamma^m\partial\lambda^*(t)\|_2 + \|\Gamma^m\partial\chi(t)\|_2 + \|\Gamma^m\partial\mathfrak{z}(t)\|_2) \\
&\qquad + \|\Gamma^m u(t)\|_2 + \|\Gamma^m w(t)\|_2 + \|\Gamma^m(\partial_t + b\cdot\nabla_\perp)\lambda(t)\|_2
\end{aligned}$$

$$+ \|\Gamma^m(\partial_t + b\cdot\nabla_\perp)\lambda^*(t)\|_2$$
$$\lesssim \sum_{|j|\le m}(\|(\partial_t + b\cdot\nabla_\perp)\Gamma^j\chi(t)\|_2 + \|(\partial_t + b\cdot\nabla_\perp)\Gamma^j\mathfrak{v}(t)\|_2), \quad (2.61)$$

$$\|\Gamma^m b(t)\|_2 + \|\Gamma^m(\partial_t + b\cdot\nabla_\perp)b(t)\|_2 + \|\Gamma^m(A-1)(t)\|_2$$
$$\lesssim M_0\sum_{|j|\le m}(\|(\partial_t + b\cdot\nabla_\perp)\Gamma^j\chi(t)\|_2 + \|(\partial_t + b\cdot\nabla_\perp)\Gamma^j\mathfrak{v}(t)\|_2), \quad (2.62)$$

$$\sum_{|j|\le m}\|\Gamma^j(\partial_t + b\cdot\nabla_\perp)\chi(t)\|_2 + \|\Gamma^j(\partial_t + b\cdot\nabla_\perp)\mathfrak{v}(t)\|_2$$
$$\simeq \sum_{|j|\le m}(\|(\partial_t + b\cdot\nabla_\perp)\Gamma^j\chi(t)\|_2 + \|(\partial_t + b\cdot\nabla_\perp)\Gamma^j\mathfrak{v}(t)\|_2). \quad (2.63)$$

Proof We prove Proposition 2.16 in five steps. Notice that

$$[\Gamma, \overline{\mathcal{H}}] = e_3[\Gamma, \mathcal{H}]e_3, \qquad [\Gamma, \mathcal{K}] = Re[\Gamma, \mathcal{H}]. \quad (2.64)$$

Step 1. We first show that for $m \le q$, there is a M_0 sufficiently small, such that if $M \le M_0$,

$$\sum_{\substack{|j|\le m\\ \partial=\partial_\alpha,\partial_\beta}}\|\Gamma^j\partial\lambda(t)\|_2 + \|\Gamma^j\partial\chi(t)\|_2 \lesssim \sum_{\substack{|j|\le m\\ \partial=\partial_\alpha,\partial_\beta}}\|\Gamma^j\partial\mathfrak{z}(t)\|_2. \quad (2.65)$$

Let $\partial = \partial_\alpha$ or ∂_β. From the definition $\lambda = (I+\mathcal{H})\mathfrak{z}e_3 - \mathcal{K}\mathfrak{z}e_3$, we have

$$\partial\lambda = (I+\mathcal{H})\partial\mathfrak{z}e_3 + [\partial, \mathcal{H}]\mathfrak{z}e_3 - [\partial, \mathcal{K}]\mathfrak{z}e_3 - \mathcal{K}\partial\mathfrak{z}e_3.$$

Using (2.6) we get

$$\Gamma^m\partial\lambda = \sum_{j=1}^m \Gamma^{m-j}[\Gamma, \mathcal{H}]\Gamma^{j-1}\partial\mathfrak{z}e_3 + (I+\mathcal{H})\Gamma^m\partial\mathfrak{z}e_3 + \Gamma^m[\partial, \mathcal{H}]\mathfrak{z}e_3$$
$$- \sum_{j=1}^m \Gamma^{m-j}[\Gamma, \mathcal{K}]\Gamma^{j-1}\partial\mathfrak{z}e_3 - \mathcal{K}\Gamma^m\partial\mathfrak{z}e_3 - \Gamma^m[\partial, \mathcal{K}]\mathfrak{z}e_3.$$

Therefore from (2.7), Lemma 1.2, (2.64), Propositions 2.2, 2.6, 2.7, 2.9, we have

$$\|\Gamma^m \partial\lambda(t)\|_2 \lesssim \sum_{\substack{|j|\le m\\ \partial=\partial_\alpha,\partial_\beta}} \|\Gamma^j \partial\lambda(t)\|_2 \sum_{|j|\le l+2} \|\Gamma^j \partial\mathfrak{z}(t)\|_2 + \Bigg(1 + \sum_{\substack{|j|\le l+2\\ \partial=\partial_\alpha,\partial_\beta}} \|\Gamma^j \partial\lambda(t)\|_2\Bigg) \sum_{|j|\le m} \|\Gamma^j \partial\mathfrak{z}(t)\|_2.$$

This gives us

$$\sum_{\substack{|j|\le m\\ \partial=\partial_\alpha,\partial_\beta}} \|\Gamma^j \partial\lambda(t)\|_2 \lesssim \sum_{\substack{|j|\le m\\ \partial=\partial_\alpha,\partial_\beta}} \|\Gamma^j \partial\mathfrak{z}(t)\|_2 \tag{2.66}$$

when M_0 is sufficiently small. The proof for the part of estimate for χ in (2.65) follows from a similar calculation and an application of (2.66). We therefore obtain (2.65).

Step 2. We show that for $m \le q$, there is a sufficiently small M_0, such that if $M \le M_0$,

$$\sum_{|j|\le m} \|\Gamma^j u(t)\|_2 \lesssim M_0 \sum_{\substack{|j|\le m\\ \partial=\partial_\alpha,\partial_\beta}} \|\Gamma^j \partial\mathfrak{z}(t)\|_2 + \sum_{|j|\le m} \|\Gamma^j (\partial_t + b\cdot\nabla_\perp)\chi(t)\|_2. \tag{2.67}$$

$$\sum_{|j|\le m} \|\Gamma^j w(t)\|_2 \lesssim M_0 \sum_{\substack{|j|\le m\\ \partial=\partial_\alpha,\partial_\beta}} \|\Gamma^j \partial\mathfrak{z}(t)\|_2 + \sum_{|j|\le m} (\|\Gamma^j (\partial_t + b\cdot\nabla_\perp)\chi(t)\|_2 + \|\Gamma^j (\partial_t + b\cdot\nabla_\perp)\mathfrak{v}(t)\|_2). \tag{2.68}$$

We first prove (2.67). From (2.54), similar to Step 1 by using (2.6), then apply (2.7), Lemma 1.2, (2.64), Propositions 2.2, 2.6, 2.7, 2.9, and furthermore (2.65), we have for M_0 sufficiently small,

$$\|\Gamma^m (\overline{u} + (\partial_t + b\cdot\nabla_\perp)\chi)(t)\|_2 \lesssim \sum_{|j|\le m} \|\Gamma^j u(t)\|_2 \sum_{\substack{|j|\le l+2\\ \partial=\partial_\alpha,\partial_\beta}} \|\Gamma^j \partial\mathfrak{z}(t)\|_2$$

$$+\sum_{|j|\le l+2}\|\Gamma^j u(t)\|_2 \sum_{\substack{|j|\le m\\ \partial=\partial_\alpha,\partial_\beta}}\|\Gamma^j\partial\mathfrak{z}(t)\|_2. \tag{2.69}$$

Now in (2.69) let $m=l+2$. We get for M_0 sufficiently small,

$$\sum_{|j|\le l+2}\|\Gamma^j u(t)\|_2 \lesssim \sum_{|j|\le l+2}\|\Gamma^j(\partial_t+b\cdot\nabla_\perp)\chi(t)\|_2. \tag{2.70}$$

Applying (2.70) to the right hand side of (2.69) and we obtain (2.67).

Similar to the proof of (2.67), we start from (2.55), and use furthermore the estimates (2.65), (2.67), and the fact that $(\partial_t+b\cdot\nabla_\perp)\mathfrak{z}=u_3$, we have (2.68).

Step 3. We have for $m\le q$, there is a sufficiently small M_0, such that if $M\le M_0$,

$$\begin{aligned}&\sum_{|j|\le m}(\|\Gamma^j(A-1)(t)\|_2+\|\Gamma^j(\partial_t+b\cdot\nabla_\perp)b(t)\|_2+\|\Gamma^j b(t)\|_2)\\ &\quad\lesssim M_0\sum_{\substack{|j|\le m\\ \partial=\partial_\alpha,\partial_\beta}}(\|\Gamma^j\partial\mathfrak{z}(t)\|_2+\|\Gamma^j(\partial_t+b\cdot\nabla_\perp)\chi(t)\|_2\\ &\qquad+\|\Gamma^j(\partial_t+b\cdot\nabla_\perp)\mathfrak{v}(t)\|_2).\end{aligned} \tag{2.71}$$

Starting from Proposition 1.4, the proof of (2.71) is similar to that in Steps 1 and 2, and uses the results in Steps 1 & 2. We omit the details.

We have

Step 4. There is M_0 sufficiently small, such that for $m\le q$, $M\le M_0$,

$$\begin{aligned}\sum_{\substack{|j|\le m\\ \partial=\partial_\alpha,\partial_\beta}}\|\Gamma^j\partial\mathfrak{z}(t)\|_2 &\lesssim \sum_{|j|\le m}(\|\Gamma^j(\partial_t+b\cdot\nabla_\perp)\chi(t)\|_2\\ &\quad+\|\Gamma^j(\partial_t+b\cdot\nabla_\perp)\mathfrak{v}(t)\|_2).\end{aligned} \tag{2.72}$$

(2.72) is obtained by using (2.51), (2.53). We have

$$\partial_\alpha\mathfrak{z}=-w\cdot e_1+(A-1)\mathcal{N}\cdot e_1+(\partial_\alpha\lambda\times\partial_\beta\lambda)\cdot e_1$$

therefore

$$\begin{aligned}&\|\Gamma^m\partial_\alpha\mathfrak{z}(t)\|_2\\ &\quad\lesssim\|\Gamma^m w(t)\|_2+\sum_{|j|\le m}\|\Gamma^j(A-1)(t)\|_2\end{aligned}$$

$$+\sum_{\substack{|j|\le m\\ \partial=\partial_\alpha,\partial_\beta}}\|\Gamma^j\partial\lambda(t)\|_2\sum_{\substack{|j|\le l+2\\ \partial=\partial_\alpha,\partial_\beta}}(\|\Gamma^j(A-1)(t)\|_2+\|\Gamma^j\partial\lambda(t)\|_2).$$

Now we apply estimates in Steps 1–3, we get (2.72). Finally

Step 5. Apply Lemma 2.15 to $\phi=\chi$ and $\phi=(\partial_t+b\cdot\nabla_\perp)\chi$, and use results in Steps 1–4, we obtain (2.63). From definition we know $\lambda^*=2\mathfrak{z}e_3-\chi=2\lambda_3e_3-\chi$ and $(\partial_t+b\cdot\nabla_\perp)\lambda=u-b$. Combine Steps 1–4 and apply (2.63), we obtain (2.61), (2.62). This finishes the proof of Proposition 2.16. □

We now give the L^∞ estimates for various quantities in terms of that of $\nabla_\perp\chi$, and $\nabla_\perp\mathfrak{v}$. Let $t\in[0,T]$ be the time when (2.59) holds.

Proposition 2.17 *There exist a $M_0>0$ small enough, such that if $M\le M_0$,*

1. *for $2\le m\le l$ we have*

$$\sum_{\substack{|j|\le m\\ \partial=\partial_\alpha,\partial_\beta}}(|\Gamma^j\partial\lambda(t)|_\infty+|\Gamma^j\partial\lambda^*(t)|_\infty+|\Gamma^j\partial\mathfrak{z}(t)|_\infty)\lesssim\sum_{\substack{|j|\le m\\ \partial=\partial_\alpha,\partial_\beta}}|\Gamma^j\partial\chi(t)|_\infty;\tag{2.73}$$

2. *for $2\le m\le l-1$, we have*

$$\sum_{\substack{|j|\le m\\ \partial=\partial_\alpha,\partial_\beta}}|\Gamma^j\partial u(t)|_\infty\lesssim\sum_{\substack{|j|\le m\\ \partial=\partial_\alpha,\partial_\beta}}(|\Gamma^j\partial\chi(t)|_\infty+|\Gamma^j\partial\mathfrak{v}(t)|_\infty);\tag{2.74}$$

3. *for $2\le m\le l-2$, we have*

$$\sum_{|j|\le m}|\Gamma^j w(t)|_\infty\lesssim\sum_{\substack{|j|\le m+1\\ \partial=\partial_\alpha,\partial_\beta}}(|\Gamma^j\partial\chi(t)|_\infty+|\Gamma^j\partial\mathfrak{v}(t)|_\infty),\tag{2.75}$$

$$\sum_{|j|\le m}(|\Gamma^j(A-1)(t)|_\infty+|\Gamma^j(\partial_t+b\cdot\nabla_\perp)b(t)|_\infty)\lesssim E_{m+2}^{1/2}(t)\sum_{\substack{|j|\le m+1\\ \partial=\partial_\alpha,\partial_\beta}}(|\Gamma^j\partial\chi(t)|_\infty+|\Gamma^j\partial\mathfrak{v}(t)|_\infty)\quad\textit{and}\tag{2.76}$$

$$\sum_{|j|\le m}|\Gamma^j b(t)|_\infty\lesssim E_{m+2}^{1/2}(t)\sum_{\substack{|j|\le m+2\\ \partial=\partial_\alpha,\partial_\beta}}|\Gamma^j\partial\chi(t)|_\infty;\tag{2.77}$$

4. *for* $l+1\le m\le q-2$, *we have*

$$\sum_{\substack{|j|\le m\\ \partial=\partial_\alpha,\partial_\beta}} |\Gamma^j\partial\lambda(t)|_\infty+|\Gamma^j\partial\lambda^*(t)|_\infty+|\Gamma^j\partial\mathfrak{z}(t)|_\infty$$

$$\lesssim \sum_{\substack{|j|\le m\\ \partial=\partial_\alpha,\partial_\beta}} |\Gamma^j\partial\chi(t)|_\infty$$

$$+E_{m+2}^{1/2}(t)\left\{\frac{1}{t}+\sum_{\substack{|j|\le[\frac{m+2}{2}]+1\\ \partial=\partial_\alpha,\partial_\beta}} |\Gamma^j\partial\chi(t)|_\infty(1+\ln t)\right\}; \quad (2.78)$$

5. *for* $l\le m\le q-4$, *we have*

$$\sum_{\substack{|j|\le m\\ \partial=\partial_\alpha,\partial_\beta}} |\Gamma^j\partial u(t)|_\infty$$

$$\lesssim \sum_{\substack{|j|\le m\\ \partial=\partial_\alpha,\partial_\beta}} |\Gamma^j\partial\mathfrak{v}(t)|_\infty+E_{m+3}^{1/2}(t)\left\{\frac{1}{t}+\sum_{\substack{|j|\le[\frac{m+2}{2}]+1\\ \partial=\partial_\alpha,\partial_\beta}} (|\Gamma^j\partial\chi(t)|_\infty\right.$$

$$\left.+|\Gamma^j\partial\mathfrak{v}(t)|_\infty)(1+\ln t)\right\}, \quad (2.79)$$

here $[s]$ *is the largest integer* $< s$.

Proof We will again use the identities in Lemma 2.14.

Step 1. We use (2.50) to prove (2.73) for $2\le m\le l$. Taking derivative ∂ to (2.50), $\partial=\partial_\alpha$ or ∂_β, we get

$$\partial\bar\lambda+\partial\chi=[\partial,\overline{\mathcal{H}}-\mathcal{H}]\mathfrak{z}e_3+(\overline{\mathcal{H}}-\mathcal{H})\partial\mathfrak{z}e_3+[\partial,\mathcal{K}]\mathfrak{z}e_3+\mathcal{K}\partial\mathfrak{z}e_3. \quad (2.80)$$

Using (1.16), (1.17), (2.64), (2.6), (2.7) and Propositions 2.6, 2.9, we obtain

$$\sum_{\substack{|j|\le m\\ \partial=\partial_\alpha,\partial_\beta}} |\Gamma^j(\partial\bar\lambda+\partial\chi)(t)|_\infty$$

$$\lesssim \sum_{\substack{|j|\le m\\ \partial=\partial_\alpha,\partial_\beta}} |\Gamma^j\partial\lambda(t)|_\infty \sum_{\substack{|j|\le m+2\\ \partial=\partial_\alpha,\partial_\beta}} \|\Gamma^j\partial\mathfrak{z}(t)\|_2$$

$$+ \sum_{\substack{|j|\le m\\ \partial=\partial_\alpha,\partial_\beta}} |\Gamma^j \partial \mathfrak{z}(t)|_\infty \sum_{\substack{|j|\le m+2\\ \partial=\partial_\alpha,\partial_\beta}} (\|\Gamma^j \partial \lambda(t)\|_2 + \|\Gamma^j \partial \mathfrak{z}(t)\|_2).$$

Using Proposition 2.16, we have that for $M_0 > 0$ small enough,

$$\sum_{\substack{|j|\le m\\ \partial=\partial_\alpha,\partial_\beta}} |\Gamma^j \partial \lambda(t)|_\infty \lesssim \sum_{\substack{|j|\le m\\ \partial=\partial_\alpha,\partial_\beta}} |\Gamma^j \partial \chi(t)|_\infty + M_0 \sum_{\substack{|j|\le m\\ \partial=\partial_\alpha,\partial_\beta}} |\Gamma^j \partial \mathfrak{z}(t)|_\infty. \tag{2.81}$$

Similar argument also gives that

$$\begin{aligned}&\sum_{\substack{|j|\le m\\ \partial=\partial_\alpha,\partial_\beta}} |\Gamma^j \partial \lambda^*(t)|_\infty \\ &\quad\lesssim \sum_{\substack{|j|\le m\\ \partial=\partial_\alpha,\partial_\beta}} |\Gamma^j \partial \chi(t)|_\infty + M_0 \sum_{\substack{|j|\le m\\ \partial=\partial_\alpha,\partial_\beta}} |\Gamma^j \partial \mathfrak{z}(t)|_\infty.\end{aligned} \tag{2.82}$$

On the other hand, from the definition we have $2\mathfrak{z}e_3 = \lambda^* + \chi$, this implies

$$\sum_{\substack{|j|\le m\\ \partial=\partial_\alpha,\partial_\beta}} |\Gamma^j \partial \mathfrak{z}(t)|_\infty \lesssim \sum_{\substack{|j|\le m\\ \partial=\partial_\alpha,\partial_\beta}} |\Gamma^j \partial \chi(t)|_\infty + \sum_{\substack{|j|\le m\\ \partial=\partial_\alpha,\partial_\beta}} |\Gamma^j \partial \lambda^*(t)|_\infty. \tag{2.83}$$

Combine (2.82), (2.83), we have for M_0 small enough,

$$\sum_{\substack{|j|\le m\\ \partial=\partial_\alpha,\partial_\beta}} |\Gamma^j \partial \lambda^*(t)|_\infty \lesssim \sum_{\substack{|j|\le m\\ \partial=\partial_\alpha,\partial_\beta}} |\Gamma^j \partial \chi(t)|_\infty \quad \text{and}$$

$$\sum_{\substack{|j|\le m\\ \partial=\partial_\alpha,\partial_\beta}} |\Gamma^j \partial \mathfrak{z}(t)|_\infty \lesssim \sum_{\substack{|j|\le m\\ \partial=\partial_\alpha,\partial_\beta}} |\Gamma^j \partial \chi(t)|_\infty.$$

Applying to (2.81), we obtain (2.73).

Step 2. We prove (2.74) for $2 \le m \le l-1$. The argument is similar to Step 1.

Starting from (2.54), using (1.15), (2.64), (2.6), (2.7) and Propositions 2.6, 2.7, 2.9, we have

$$\begin{aligned}&|\Gamma^m (\partial_\alpha \overline{u} + \partial_\alpha \mathfrak{v})(t)|_\infty \\ &\quad\lesssim \sum_{\substack{|j|\le m+2\\ \partial=\partial_\alpha,\partial_\beta}} (\|\Gamma^j \partial \lambda(t)\|_2 + \|\Gamma^j \partial \mathfrak{z}(t)\|_2) \sum_{\substack{|j|\le m\\ \partial=\partial_\alpha,\partial_\beta}} |\Gamma^j \partial u(t)|_\infty\end{aligned}$$

$$+\sum_{\substack{|j|\le m\\ \partial=\partial_\alpha,\partial_\beta}}(|\Gamma^j\partial\lambda(t)|_\infty+|\Gamma^j\partial\mathfrak{z}(t)|_\infty)\sum_{\substack{|j|\le m+2\\ \partial=\partial_\alpha,\partial_\beta}}\|\Gamma^j\partial u(t)\|_2.$$

Argue similarly for $\partial_\beta u$ and using (2.73) and Proposition 2.16, we obtain for $2\le m\le l-1$ and M_0 small enough,

$$\sum_{\substack{|j|\le m\\ \partial=\partial_\alpha,\partial_\beta}}|\Gamma^j\partial u(t)|_\infty\lesssim\sum_{\substack{|j|\le m\\ \partial=\partial_\alpha,\partial_\beta}}(|\Gamma^j\partial\chi(t)|_\infty+|\Gamma^j\partial\mathfrak{v}(t)|_\infty).$$

Step 3. We prove (2.75) and (2.76) for $2\le m\le l-2$.
From (2.53), using (2.52), (2.73) and Proposition 2.16, we get

$$\begin{aligned}\sum_{|j|\le m}|\Gamma^j w(t)|_\infty&\lesssim\sum_{\substack{|j|\le m\\ \partial=\partial_\alpha,\partial_\beta}}(|\Gamma^j(A-1)(t)|_\infty+|\Gamma^j\partial\lambda(t)|_\infty)\\&\lesssim\sum_{\substack{|j|\le m\\ \partial=\partial_\alpha,\partial_\beta}}(|\Gamma^j(A-1)(t)|_\infty+|\Gamma^j\partial\chi(t)|_\infty).\end{aligned}\tag{2.84}$$

On the other hand, from (1.39), using similar argument as in Steps 1 and 2 and using (2.73), (2.74), Proposition 2.16, we have for M_0 small enough, $2\le m\le l-2$,

$$\begin{aligned}&\sum_{|j|\le m}|\Gamma^j(A-1)(t)|_\infty\\&\quad\lesssim E_{m+2}^{1/2}(t)\sum_{|j|\le m}|\Gamma^j w(t)|_\infty\\&\qquad+E_{m+2}^{1/2}(t)\Bigg(\sum_{\substack{|j|\le m+1\\ \partial=\partial_\alpha,\partial_\beta}}(|\Gamma^j\partial\chi(t)|_\infty+|\Gamma^j\partial\mathfrak{v}(t)|_\infty)\Bigg).\end{aligned}\tag{2.85}$$

Combining (2.84), (2.85), we obtain for M_0 small enough,

$$\sum_{|j|\le m}|\Gamma^j(A-1)(t)|_\infty\lesssim E_{m+2}^{1/2}(t)\Bigg(\sum_{\substack{|j|\le m+1\\ \partial=\partial_\alpha,\partial_\beta}}(|\Gamma^j\partial\chi(t)|_\infty+|\Gamma^j\partial\mathfrak{v}(t)|_\infty)\Bigg)\tag{2.86}$$

(2.75) therefore follows from (2.84), (2.86). Using (1.38), the estimate for $\sum_{|j|\le m}|\Gamma^j(\partial_t+b\cdot\nabla_\perp)b(t)|_\infty$ can be obtained similarly. We omit the details.

Step 4. We prove (2.77). We first put the terms $[\partial_t+b\cdot\nabla_\perp,\mathcal{H}]\mathfrak{z}$, $[\partial_t+b\cdot\nabla_\perp,\mathcal{K}]\mathfrak{z}$ in (1.38) in an appropriate form for carrying out our estimate. We know $2\mathfrak{z}e_3 = \lambda^* + \chi$ and λ^* (or χ) is the trace of an analytic function in $\Omega(t)$ (or $\Omega(t)^c$) respectively. Using (1.15), and Proposition 2.11, We have

$$\begin{aligned} 2[\partial_t + b\cdot\nabla_\perp, \mathcal{H}]\mathfrak{z}e_3 &= [\partial_t + b\cdot\nabla_\perp, \mathcal{H}]\lambda^* + [\partial_t + b\cdot\nabla_\perp, \mathcal{H}]\chi \\ &= u\cdot\nabla_\xi^+\lambda^* - \mathcal{H}(u\cdot\nabla_\xi^+\lambda^*) - u\cdot\nabla_\xi^-\chi - \mathcal{H}(u\cdot\nabla_\xi^-\chi). \end{aligned} \tag{2.87}$$

Notice that $[\partial_t + b\cdot\nabla_\perp, \mathcal{K}]\mathfrak{z} = \mathrm{Re}[\partial_t + b\cdot\nabla_\perp, \mathcal{H}]\mathfrak{z}$. Now using (1.38), and Proposition 2.9, Lemma 2.13, Proposition 2.16, (2.73), we obtain

$$\sum_{|j|\le m} |\Gamma^j b(t)|_\infty \lesssim E_{m+2}^{1/2}(t) \sum_{\substack{|j|\le m+2\\ \partial=\partial_\alpha,\partial_\beta}} |\Gamma^j \partial\chi(t)|_\infty.$$

Step 5. We prove (2.78).

Let $l+1 \le m \le q-2$ and $\partial = \partial_\alpha, \partial_\beta$. Applying Propositions 2.9, 2.6, 2.8 with $r = t$, and (2.61), (2.73) to (2.80), we get the estimate for $\partial\lambda$:

$$\begin{aligned} |\Gamma^m \partial\lambda(t)|_\infty &\lesssim |\Gamma^m \partial\chi(t)|_\infty \\ &\quad + E_{m+2}^{1/2}(t)\left\{\frac{1}{t} + \sum_{\substack{|j|\le[\frac{m+2}{2}]+1\\ \partial=\partial_\alpha,\partial_\beta}} |\Gamma^j\partial\chi(t)|_\infty(1+\ln t)\right\}. \end{aligned}$$

Similar argument gives the estimate for $\partial\lambda^*$. The estimate for $\partial\mathfrak{z}$ follows since $\mathfrak{z} = \lambda^* + \chi$.

Step 6. (2.79) is obtained similarly by using (2.54). We omit the details. □

For the L^2, L^∞ estimates of $\frac{\mathfrak{a}_t}{\mathfrak{a}} \circ k^{-1}$, we have the following Lemma.

Lemma 2.18 *Let f be real valued such that*

$$(I - \mathcal{H})(f\mathcal{N}) = g, \tag{2.88}$$

$t \in [0, T]$ be the time when (2.59) holds. There exists a $M_0 > 0$, such that if $M \le M_0$,

1. *for $0 \le m \le l+2$, we have*

$$\sum_{|j|\le m} \|\Gamma^j f(t)\|_2 \lesssim \sum_{|j|\le m} \|\Gamma^j g(t)\|_2; \tag{2.89}$$

2. *for* $l+2<m\le q$,

$$\sum_{|j|\le m}\|\Gamma^j f(t)\|_2\lesssim\sum_{|j|\le l+2}\|\Gamma^j g(t)\|_2\sum_{\substack{|j|\le m\\ \partial=\partial_\alpha,\partial_\beta}}\|\Gamma^j\partial\lambda(t)\|_2 + \sum_{|j|\le m}\|\Gamma^j g(t)\|_2;$$

3. *for* $0\le m\le l$,

$$\sum_{|j|\le m}|\Gamma^j f(t)|_\infty\lesssim\sum_{|j|\le m}|\Gamma^k g(t)|_\infty + \sum_{\substack{|j|\le 1\\ \partial=\partial_\alpha,\partial_\beta}}|\Gamma^j\partial\lambda(t)|_\infty\sum_{|j|\le m+2}\|\Gamma^j g(t)\|_2. \tag{2.90}$$

Proof The proof follows similar idea as that of Lemma 3.8 in [32]. From (2.88), we have

$$(I-\mathcal{H})((\Gamma^j f)\mathcal{N})=\Gamma^j g+[\Gamma^j,\mathcal{H}](f\mathcal{N})-(I-\mathcal{H})(\Gamma^j(f\mathcal{N}) - (\Gamma^j f)\mathcal{N}). \tag{2.91}$$

Let $R=\Gamma^j g+[\Gamma^j,\mathcal{H}](f\mathcal{N})-(I-\mathcal{H})(\Gamma^j(f\mathcal{N})-(\Gamma^j f)\mathcal{N})$. Multiplying e_3 both left and right to both sides of (2.91), we obtain

$$(I+\overline{\mathcal{H}})((\Gamma^j f)\overline{\mathcal{N}})=\overline{R}. \tag{2.92}$$

Here we used the fact that f is real valued. Therefore

$$2\Gamma^j f e_3=\Gamma^j f(\overline{\mathcal{N}}+e_3)+\Gamma^j f(-\mathcal{N}+e_3)+\mathcal{H}(\Gamma^j f(\overline{\mathcal{N}}+\mathcal{N})) + (\overline{\mathcal{H}}-\mathcal{H})(\Gamma^j f\overline{\mathcal{N}})+R-\overline{R}. \tag{2.93}$$

Lemma 2.18 is then obtained by applying (2.6), Lemma 1.2, Proposition 2.2, (2.56), Propositions 2.7, 2.9, 2.16 to (2.93) and by an inductive argument. We omit the details. □

Let $\mathcal{K}^*$ be the adjoint of the double layered potential operator $\mathcal{K}$:

$$\mathcal{K}^* f(\alpha,\beta,t)=\iint\mathcal{N}\cdot K(\zeta'-\zeta)f(\alpha',\beta',t)\,d\alpha'\,d\beta'. \tag{2.94}$$

Proposition 2.19 *Let $f(\cdot,t)$ be a real valued function on $\mathbb{R}^2$. Then*

1.

$$(I \pm \mathcal{K}^*)(\mathcal{N}\cdot\nabla_\xi^\pm f) = \pm\iint (\mathcal{N}\times K(\zeta'-\zeta))\cdot(\zeta'_{\beta'}\partial_{\alpha'}f' - \zeta'_{\alpha'}\partial_{\beta'}f')\,d\alpha' d\beta'. \quad (2.95)$$

2. *At $t\in[0,T]$ when (2.59) holds,*

$$\|\mathcal{N}\cdot\nabla_\xi^+ f(t) + \mathcal{N}\cdot\nabla_\xi^- f(t)\|_2 \lesssim \sum_{\partial=\partial_\alpha,\partial_\beta}(|\partial\lambda(t)|_\infty + |\partial\mathfrak{z}(t)|_\infty)\sum_{\partial=\partial_\alpha,\partial_\beta}\|\partial f(t)\|_2. \quad (2.96)$$

3. *At $t\in[0,T]$ when (2.59) holds,*

$$\|\mathcal{N}\cdot\nabla_\xi^+ f(t) + \mathcal{N}\cdot\nabla_\xi^- f(t)\|_2 \lesssim \sum_{\substack{1\le j\le 3\\ \partial=\partial_\alpha,\partial_\beta}}(|\partial^j\lambda(t)_\infty + |\partial^j\mathfrak{z}(t)|_\infty)\|f(t)\|_2. \quad (2.97)$$

Proof From definition, we know $N\cdot\nabla_\xi^+ f$ and $N\cdot\nabla_\xi^- f$ are the normal derivatives of the harmonic extensions of f into $\Omega(t)$ and $\Omega(t)^c$ respectively. (2.95) is basically the equality (3.13) in [31]. Therefore

$$\mathcal{N}\cdot\nabla_\xi^+ f + \mathcal{N}\cdot\nabla_\xi^- f = \mathcal{K}^*(-\mathcal{N}\cdot\nabla_\xi^+ f + \mathcal{N}\cdot\nabla_\xi^- f) \quad \text{and}$$

$$-\mathcal{N}\cdot\nabla_\xi^+ f + \mathcal{N}\cdot\nabla_\xi^- f = \mathcal{K}^*(\mathcal{N}\cdot\nabla_\xi^+ f + \mathcal{N}\cdot\nabla_\xi^- f) - 2\iint (\mathcal{N}\times K(\zeta'-\zeta))\cdot(\zeta'_{\beta'}\partial_{\alpha'}f' - \zeta'_{\alpha'}\partial_{\beta'}f')\,d\alpha' d\beta'.$$

This implies

$$\mathcal{N}\cdot\nabla_\xi^+ f + \mathcal{N}\cdot\nabla_\xi^- f = \mathcal{K}^{*2}(\mathcal{N}\cdot\nabla_\xi^+ f + \mathcal{N}\cdot\nabla_\xi^- f) - 2\mathcal{K}^*\iint (\mathcal{N}\times K(\zeta'-\zeta))\cdot(\zeta'_{\beta'}\partial_{\alpha'}f' - \zeta'_{\alpha'}\partial_{\beta'}f')\,d\alpha' d\beta'. \quad (2.98)$$

From (2.98), (2.96) is straightforward with an application of Proposition 2.6.

To prove (2.97), we rewrite

$$\iint (\mathcal{N} \times K(\zeta' - \zeta)) \cdot (\zeta'_{\beta'} \partial_{\alpha'} f' - \zeta'_{\alpha'} \partial_{\beta'} f')\, d\alpha' d\beta'$$
$$= \partial_\alpha \iint (\mathcal{N} \times K) \cdot \zeta'_{\beta'} f'\, d\alpha' d\beta' - \partial_\beta \iint (\mathcal{N} \times K) \cdot \zeta'_{\alpha'} f'\, d\alpha' d\beta'$$
$$- \iint (\partial_\alpha + \partial_{\alpha'})(\mathcal{N} \times K(\zeta' - \zeta)) \cdot \zeta'_{\beta'} f'\, d\alpha' d\beta'$$
$$+ \iint (\partial_\beta + \partial_{\beta'})(\mathcal{N} \times K(\zeta' - \zeta)) \cdot \zeta'_{\alpha'} f'\, d\alpha' d\beta'. \tag{2.99}$$

Here we just used integration by parts. (2.97) now follows from (2.98), (2.99) with a further application of integration by parts and an application of Proposition 2.6. (Notice that $\mathcal{N} \cdot \zeta_\alpha = \mathcal{N} \cdot \zeta_\beta = 0$.) □

3 Energy estimates

In this section, we use the expanded set of vector fields $\Gamma = \{\partial_t,\ \partial_\alpha,\ \partial_\beta,\ L_0 = \frac{1}{2} t\partial_t + \alpha\partial_\alpha + \beta\partial_\beta,\ \varpi = \alpha\partial_\beta - \beta\partial_\alpha - \frac{1}{2} e_3\}$ to construct energy functional and derive energy estimates for the water wave system (1.23)–(1.24). Our strategy is to construct two energy estimates, the first one is for a quantity involving a full range of derivatives of $(\partial_t + b \cdot \nabla_\perp)\chi$ and $(\partial_t + b \cdot \nabla_\perp)\mathfrak{v}$ and we will show using Lemma 3.3 that it grows no faster than $(1+t)^\epsilon$ provided the energy involving some lower orders of derivatives of $(\partial_t + b \cdot \nabla_\perp)\chi$ and $(\partial_t + b \cdot \nabla_\perp)\mathfrak{v}$ is bounded by $c\epsilon^2$. The second one concerns the aforementioned energy involving the lower orders of derivatives of $(\partial_t + b \cdot \nabla_\perp)\chi$ and $(\partial_t + b \cdot \nabla_\perp)\mathfrak{v}$, and we will show it stays bounded by $c\epsilon^2$ for all time provided initially it is bounded by $\frac{c}{2}\epsilon^2$ and the quantity involving the full range of derivatives of $(\partial_t + b \cdot \nabla_\perp)\chi$ and $(\partial_t + b \cdot \nabla_\perp)\mathfrak{v}$ does not grow faster than $(1+t)^\delta$ for some $\delta < 1$. Together these two estimates imply a uniform boundness result (see Theorem 3.7).

We first present the following basic energy estimates. The first one will be used to derive the estimate for the full range of derivatives, and second one the lower orders of derivatives.

Lemma 3.1 (Basic energy inequality I) *Assume that θ is real valued and satisfying*

$$(\partial_t + b \cdot \nabla_\perp)^2 \theta + A\mathcal{N} \cdot \nabla_\xi^+ \theta = \mathbf{G} \tag{3.1}$$

and θ is smooth and decays fast at spatial infinity. Let

$$E(t) = \iint \frac{1}{A} |(\partial_t + b \cdot \nabla_\perp)\theta(\alpha, \beta, t)|^2 + \theta(\mathcal{N} \cdot \nabla_\xi^+)\theta(\alpha, \beta, t)\, d\alpha\, d\beta. \tag{3.2}$$

Then

$$\frac{dE}{dt} \le \iint \frac{2}{A}\mathbf{G}(\partial_t + b\cdot\nabla_\perp)\theta\, d\alpha d\beta + \left(\left\|\frac{\mathfrak{a}_t}{\mathfrak{a}}\circ k^{-1}\right\|_{L^\infty} + 2\|\nabla\mathbf{v}(t)\|_{L^\infty(\Omega(t))}\right)E(t). \tag{3.3}$$

Proof Let $\theta^\hbar$ be the harmonic extension of θ to $\Omega(t)$. Make a change of coordinate to (3.1) and (3.2), and use the Green's identity, we have

$$(\partial_t^2 + \mathfrak{a}N\cdot\nabla_\xi^+)(\theta\circ k) = \mathbf{G}\circ k \quad \text{and}$$
$$E(t) = \iint \frac{1}{\mathfrak{a}}|\partial_t(\theta\circ k)|^2\, d\alpha\, d\beta + \int_{\Omega(t)} |\nabla\theta^\hbar|^2\, dV.$$

We know

$$\frac{d}{dt}\iint \frac{1}{\mathfrak{a}}|\partial_t(\theta\circ k)|^2\, d\alpha\, d\beta = \iint \frac{2}{\mathfrak{a}}\partial_t(\theta\circ k)\,\partial_t^2(\theta\circ k) - \frac{\mathfrak{a}_t}{\mathfrak{a}^2}|\partial_t(\Theta\circ k)|^2\, d\alpha\, d\beta.$$

To calculate $\frac{d}{dt}\int_{\Omega(t)}|\nabla\theta^\hbar|^2\, dV$, we introduce the fluid map $X(\cdot,t):\Omega(0)\to\Omega(t)$ satisfying $\partial_t X(\cdot,t) = \mathbf{v}(\cdot,t)$, $X(\cdot,0) = I$. From the incompressibility of $\mathbf{v}$ we know the Jacobian of $X(\cdot,t)$: $J(X(t)) = 1$. Let $D_t = \partial_t + \mathbf{v}\cdot\nabla$, we know

$$D_t\nabla\theta^\hbar - \nabla D_t\theta^\hbar = -\sum_{j=1}^{3}\nabla\mathbf{v}_j\,\partial_{\xi_j}\theta^\hbar. \tag{3.4}$$

Now applying the above calculation, we have

$$\begin{aligned}
&\frac{d}{dt}\int_{\Omega(t)}|\nabla\theta^\hbar|^2\, dV\\
&= \frac{d}{dt}\int_{\Omega(0)}|\nabla\theta^\hbar(X(\cdot,t),t)|^2\, dV\\
&= 2\int_{\Omega(0)} D_t\nabla\theta^\hbar\cdot\nabla\theta^\hbar(X(\cdot,t),t)\, dV = 2\int_{\Omega(t)} D_t\nabla\theta^\hbar\cdot\nabla\theta^\hbar\, dV\\
&= 2\int_{\Omega(t)}\nabla D_t\theta^\hbar\cdot\nabla\theta^\hbar\, dV - 2\sum_{j=1}^{3}\int_{\Omega(t)}\nabla\mathbf{v}_j\,\partial_{\xi_j}\theta^\hbar\cdot\nabla\theta^\hbar\, dV\\
&= 2\iint \partial_t(\theta\circ k)(N\cdot\nabla_\xi^+)(\theta\circ k)\, d\alpha\, d\beta
\end{aligned}$$

$$-2\sum_{j=1}^{3}\int_{\Omega(t)}\nabla \mathbf{v}_j\partial_{\xi_j}\theta^{\hbar}\cdot\nabla\theta^{\hbar}\,dV. \tag{3.5}$$

In the last step we used the divergence Theorem. So

$$\frac{dE}{dt}=\int\frac{2}{\mathfrak{a}}\partial_t(\theta\circ k)\mathbf{G}\circ k-\frac{\mathfrak{a}_t}{\mathfrak{a}^2}|\partial_t(\Theta\circ k)|^2\,d\alpha\,d\beta$$
$$-2\sum_{j=1}^{3}\int_{\Omega(t)}\nabla \mathbf{v}_j\partial_{\xi_j}\theta^{\hbar}\cdot\nabla\theta^{\hbar}\,dV. \tag{3.6}$$

Making a change of variable U_k^{-1} and using the Green's identity gives us (3.3). □

Lemma 3.2 (Basic energy equality II) *Assume that* Θ *is a smooth* $\mathcal{C}(V_2)$ *valued function satisfying* $\Theta=-\mathcal{H}\Theta$, *and*

$$((\partial_t+b\cdot\nabla_\perp)^2-A\mathcal{N}\times\nabla)\Theta=G. \tag{3.7}$$

Let

$$\mathbf{E}(t)=\iint\frac{1}{A}|(\partial_t+b\cdot\nabla_\perp)\Theta|^2-\Theta\cdot\{(\mathcal{N}\times\nabla)\Theta\}(\alpha,\beta,t)\,d\alpha\,d\beta. \tag{3.8}$$

Then

$$\frac{d\mathbf{E}}{dt}=\iint\left\{\frac{2}{A}G\cdot\{(\partial_t+b\cdot\nabla_\perp)\Theta\}-\frac{\mathfrak{a}_t}{\mathfrak{a}}\circ k^{-1}\frac{1}{A}|(\partial_t+b\cdot\nabla_\perp)\Theta|^2\right\}d\alpha d\beta$$
$$-\iint\{(\Theta\cdot(u_\beta\Theta_\alpha)-\Theta\cdot(u_\alpha\Theta_\beta))$$
$$+\mathcal{N}\times\nabla\Theta\cdot[\partial_t+b\cdot\nabla_\perp,\mathcal{H}]\Theta\}\,d\alpha\,d\beta$$
$$+\frac{1}{2}\iint\{(\mathcal{N}\cdot\nabla_\xi^{+}+\mathcal{N}\cdot\nabla_\xi^{-})\Theta\}\cdot[\partial_t+b\cdot\nabla_\perp,\mathcal{H}]\Theta\,d\alpha\,d\beta. \tag{3.9}$$

Proof By making a change of coordinates U_k we know Θ satisfies

$$(\partial_t^2-\mathfrak{a}N\times\nabla)\Theta\circ k=G\circ k\quad\text{and}$$
$$\mathbf{E}(t)=\iint\frac{1}{\mathfrak{a}}(\Theta\circ k)_t\cdot(\Theta\circ k)_t-\Theta\circ k\cdot\{(\xi_\beta\partial_\alpha-\xi_\alpha\partial_\beta)(\Theta\circ k)\}\,d\alpha\,d\beta.$$

Therefore

$$\frac{d\mathbf{E}}{dt}=\iint\left\{\frac{2}{\mathfrak{a}}(\Theta\circ k)_t\cdot(\Theta\circ k)_{tt}-\frac{\mathfrak{a}_t}{\mathfrak{a}^2}|(\Theta\circ k)_t|^2\right.$$

$$
\left.\begin{aligned}
&- \Theta \circ k \cdot \{(\xi_{t\beta}\partial_\alpha - \xi_{t\alpha}\partial_\beta)(\Theta \circ k)\} \\
&- (\Theta \circ k)_t \cdot \{(N \times \nabla)(\Theta \circ k)\} \\
&- \Theta \circ k \cdot \{(N \times \nabla)(\Theta \circ k)_t\}
\end{aligned}\right\} d\alpha\, d\beta. \tag{3.10}
$$

Now from the assumption $\Theta = -\mathcal{H}\Theta = \frac{1}{2}(I - \mathcal{H})\Theta$, we have

$$
(\Theta \circ k)_t = \frac{1}{2}(I - \mathfrak{H})(\Theta \circ k)_t - \frac{1}{2}[\partial_t, \mathfrak{H}]\Theta \circ k
$$

and

$$
[\partial_t, \mathfrak{H}]\Theta \circ k = \mathfrak{H}([\partial_t, \mathfrak{H}]\Theta \circ k). \tag{3.11}
$$

Using integration by parts, and the fact that for Φ satisfying $\Phi = \pm\mathfrak{H}\Phi$, $N \times \nabla\Phi = N \cdot \nabla_\xi^\pm \Phi$,[11] and $N \cdot \nabla_\xi^\pm$ is symmetric, we have

$$
\begin{aligned}
&\iint \Theta \circ k \cdot \{(N \times \nabla)(\Theta \circ k)_t\}\, d\alpha\, d\beta \\
&\quad = \frac{1}{2}\iint \Theta \circ k \cdot \{(N \times \nabla)(I - \mathfrak{H})(\Theta \circ k)_t\}\, d\alpha\, d\beta \\
&\qquad - \frac{1}{2}\iint \Theta \circ k \cdot \{(N \times \nabla)[\partial_t, \mathfrak{H}]\Theta \circ k\}\, d\alpha\, d\beta \\
&\quad = \frac{1}{2}\iint (N \times \nabla)\Theta \circ k \cdot \{(I - \mathfrak{H})(\Theta \circ k)_t\}\, d\alpha\, d\beta \\
&\qquad - \frac{1}{2}\iint (N \cdot \nabla_\xi^+)\Theta \circ k \cdot \{[\partial_t, \mathfrak{H}]\Theta \circ k\}\, d\alpha\, d\beta \\
&\quad = \iint (N \times \nabla)\Theta \circ k \cdot \{(\Theta \circ k)_t\}\, d\alpha\, d\beta \\
&\qquad + \iint (N \times \nabla)\Theta \circ k \cdot \{[\partial_t, \mathfrak{H}]\Theta \circ k\}\, d\alpha\, d\beta \\
&\qquad - \frac{1}{2}\iint (N \cdot \nabla_\xi^+ + N \cdot \nabla_\xi^-)\Theta \circ k \cdot \{[\partial_t, \mathfrak{H}]\Theta \circ k\}\, d\alpha\, d\beta.
\end{aligned}
$$

Sum up the above calculation and make a change of variable U_k^{-1} gives us (3.9). □

[11] We know $\Phi = \pm\mathfrak{H}\Phi$ implies Φ is analytic in $\Omega(t)$ or $\Omega(t)^c$, i.e. $\mathcal{D}_\xi \Phi = 0$, therefore $N \times \nabla\Phi = N \cdot \nabla_\xi^\pm \Phi$.

In what follows in this section we make the following assumptions on the solution. Let $l \ge 6$, $l+2 \le q \le 2l$, $\xi = \xi(\alpha, \beta, t)$, $t \in [0, T]$ be a solution of the water wave system (1.23)–(1.24). Assume that the mapping $k(\cdot, t) : \mathbb{R}^2 \to \mathbb{R}^2$ defined in (1.28) is a diffeomorphism and its Jacobian $J(k(t)) > 0$, for $t \in [0, T]$. Assume for $\partial = \partial_\alpha,\ \partial_\beta$,

$$\Gamma^j \partial\lambda,\ \Gamma^j \partial\mathfrak{z},\ \Gamma^j(\partial_t + b\cdot\nabla_\perp)\chi,\ \Gamma^j(\partial_t + b\cdot\nabla_\perp)\mathfrak{v} \in C([0,T], L^2(\mathbb{R}^2)), \quad \text{for } |j| \le q. \tag{3.12}$$

and

$$\begin{aligned} &\sup_{[0,T]} \sum_{\substack{|j|\le l+2 \\ \partial=\partial_\alpha,\partial_\beta}} (\|\Gamma^j\partial\lambda(t)\|_2 + \|\Gamma^j\partial\mathfrak{z}(t)\|_2 + \|\Gamma^j\mathfrak{v}(t)\|_2 \\ &\quad + \|\Gamma^j(\partial_t + b\cdot\nabla_\perp)\mathfrak{v}(t)\|_2) \le M, \\ &|\zeta(\alpha,\beta,t) - \zeta(\alpha',\beta',t)| \\ &\quad \ge \frac{1}{4}(|\alpha-\alpha'| + |\beta-\beta'|) \quad \text{for } \alpha,\beta,\alpha'\beta' \in \mathbb{R}, \end{aligned} \tag{3.13}$$

where $0 < M \le M_0$, M_0 is the constant such that all the estimates derived in Sect. 2.3 holds and such that $|A-1| \le \frac{1}{2}$.

We have from the generalized Sobolev inequality Proposition 2.4 the following

Lemma 3.3 *Let* $2 \le m \le \min\{2l-7, q-5\}$, $t \in [0,T]$. *There exists* $M_0 > 0$ *sufficiently small, such that for* $M \le M_0$,

$$\begin{aligned} &t \sum_{\substack{|i|\le m \\ \partial=\partial_\alpha,\partial_\beta}} (|\partial\Gamma^i\chi(t)|_\infty + |\partial\Gamma^i\mathfrak{v}(t)|_\infty) \\ &\quad \lesssim E_{m+5}^{1/2}(t)\Bigg(1 + t \sum_{\substack{|i|\le[\frac{m+3}{2}]+2 \\ \partial=\partial_\alpha,\partial_\beta}} |\partial\Gamma^i\chi(t)|_\infty + |\partial\Gamma^i\mathfrak{v}(t)|_\infty\Bigg). \end{aligned} \tag{3.14}$$

In particular, for $5 \le m \le \min\{2l-11, q-5\}$, *we have*

$$\sum_{\substack{|i|\le m \\ \partial=\partial_\alpha,\partial_\beta}} (|\partial\Gamma^i\chi(t)|_\infty + |\partial\Gamma^i\mathfrak{v}(t)|_\infty) \lesssim \frac{1}{1+t} E_{m+5}^{1/2}(t). \tag{3.15}$$

Proof Let $t \in [0, T]$, $i \le m \le \min\{2l-7, q-5\}$. From Propositions 2.4 and 2.16, we have

$$t(|\partial\Gamma^i\chi(t)|_\infty + |\partial\Gamma^i\mathfrak{v}(t)|_\infty) \lesssim E_{i+5}^{1/2}(t) + t\sum_{|k|\le 3+i}(\|\mathfrak{P}\Gamma^k\chi(t)\|_2 + \|\mathfrak{P}\Gamma^k\mathfrak{v}(t)\|_2). \tag{3.16}$$

We estimate $\|\mathfrak{P}\Gamma^k\chi(t)\|_2$ and $\|\mathfrak{P}\Gamma^k\mathfrak{v}(t)\|_2$ using (1.35), (2.41). We know for $\phi = \chi,\ \mathfrak{v}$,

$$\mathfrak{P}\Gamma^k\phi = \Gamma^k\mathcal{P}\phi + [\mathcal{P}, \Gamma^k]\phi + (\mathfrak{P} - \mathcal{P})\Gamma^k\phi.$$

Let $k \le 3 + m \le \min\{2l-4, q-2\}$. We know $E_{l+2}(t) \lesssim M_0^2$. Using (2.6), (2.5), (2.14), Propositions 2.16, 2.17, we have for $\phi = \chi,\ \mathfrak{v}$,

$$\|[\mathcal{P}, \Gamma^k]\phi(t)\|_2 \lesssim E_k^{1/2}(t)\sum_{\substack{|i|\le[\frac{k}{2}]+2\\ \partial=\partial_\alpha,\partial_\beta}}(|\partial\Gamma^i\chi(t)|_\infty + |\partial\Gamma^i\mathfrak{v}(t)|_\infty), \quad \text{and} \tag{3.17}$$

$$\begin{aligned}\|(\mathfrak{P} - \mathcal{P})\Gamma^k\chi(t)\|_2 &\lesssim E_{k+1}^{1/2}(t)\sum_{\substack{|i|\le 4\\ \partial=\partial_\alpha,\partial_\beta}}(|\partial\Gamma^i\chi(t)|_\infty + |\partial\Gamma^i\mathfrak{v}(t)|_\infty),\\ \|(\mathfrak{P} - \mathcal{P})\Gamma^k\mathfrak{v}(t)\|_2 &\lesssim E_{k+2}^{1/2}(t)\sum_{\substack{|i|\le 4\\ \partial=\partial_\alpha,\partial_\beta}}(|\partial\Gamma^i\chi(t)|_\infty + |\partial\Gamma^i\mathfrak{v}(t)|_\infty).\end{aligned} \tag{3.18}$$

We now estimate $\|\Gamma^k\mathcal{P}\chi(t)\|_2$. From (1.35), (2.8), we know there are two types of terms in $\Gamma^k\mathcal{P}\chi$. One are terms of cubic and higher orders. Collectively, we name such terms as C. Another type are quadratic terms of the following form:

$$Q_j = \iint K(\zeta' - \zeta)(\dot{\Gamma}^j u - \dot{\Gamma}'^j u') \times (\zeta'_{\beta'}\partial_{\alpha'} - \zeta'_{\alpha'}\partial_{\beta'})\Gamma'^{k-j}\bar{u}'\,d\alpha'\,d\beta'.$$

For the cubic terms C, we use Propositions 2.9, 2.6, 2.7, 2.16, 2.17. We have

$$\|C(t)\|_2 \lesssim E_k^{1/2}(t)\sum_{\substack{|i|\le[\frac{k}{2}]+2\\ \partial=\partial_\alpha,\partial_\beta}}(|\partial\Gamma^i\chi(t)|_\infty + |\partial\Gamma^i\mathfrak{v}(t)|_\infty). \tag{3.19}$$

For the quadratic terms Q_j with $j \le [\frac{k}{2}] + 1$, we use Proposition 2.7, and 2.16, 2.17. We have

$$\|Q_j(t)\|_2 \lesssim E_k^{1/2}(t) \sum_{\substack{|i|\le[\frac{k}{2}]+1 \\ \partial=\partial_\alpha,\partial_\beta}} (|\partial\Gamma^i \chi(t)|_\infty + |\partial\Gamma^i \mathfrak{v}(t)|_\infty). \tag{3.20}$$

To estimate $\|Q_j(t)\|_2$ for $k \ge j > [\frac{k}{2}] + 1$, we rewrite it by using Proposition 2.11:

$$\begin{aligned} Q_j &= \frac{1}{2} \iint K(\zeta' - \zeta)(\dot{\Gamma}^j u - \dot{\Gamma}'^j u') \\ &\quad \times (\zeta'_{\beta'}\partial_{\alpha'} - \zeta'_{\alpha'}\partial_{\beta'})((I + \mathcal{H}') + (I - \mathcal{H}'))\Gamma'^{k-j} \bar{u}' \, d\alpha' \, d\beta' \\ &= \frac{1}{2}(I - \mathcal{H})((\dot{\Gamma}^j u \cdot \nabla_\xi^+)(I + \mathcal{H})\Gamma^{k-j} \bar{u}) \\ &\quad - \frac{1}{2}(I + \mathcal{H})((\dot{\Gamma}^j u \cdot \nabla_\xi^-)(I - \mathcal{H})\Gamma^{k-j} \bar{u}). \end{aligned}$$

Applying Proposition 2.6, Lemma 2.13, we obtain

$$\begin{aligned} \|Q_j(t)\|_2 &\lesssim \|\dot{\Gamma}^j u(t)\|_2 \sum_{\partial=\partial_\alpha,\partial_\beta} (|\partial(I + \mathcal{H})\Gamma^{k-j} \bar{u}(t)|_\infty \\ &\qquad + |\partial(I - \mathcal{H})\Gamma^{k-j} \bar{u}(t)|_\infty) \\ &\lesssim \|\dot{\Gamma}^j u(t)\|_2 \sum_{\partial=\partial_\alpha,\partial_\beta} (|\partial(I + \mathcal{H})\Gamma^{k-j} \bar{u}(t)|_\infty + |\partial\Gamma^{k-j} \bar{u}(t)|_\infty). \end{aligned}$$

We know from the fact $-\overline{\mathcal{H}}\bar{u} = \bar{u}$[12] that

$$(I + \mathcal{H})\Gamma^{k-j} \bar{u} = \Gamma^{k-j}(-\bar{\mathcal{H}} + \mathcal{H})\bar{u} - [\Gamma^{k-j}, \mathcal{H}]\bar{u}.$$

Applying (2.56), (2.6), Propositions 2.9, 2.6, 2.7, 2.16, 2.17, we get

$$\begin{aligned} |\partial(I + \mathcal{H})\Gamma^{k-j} \bar{u}(t)|_\infty &\lesssim E_{k-j+3}^{1/2}(t) \sum_{\substack{|i|\le k-j+2 \\ \partial=\partial_\alpha,\partial_\beta}} |\partial\Gamma^i \chi(t)|_\infty \\ &\lesssim \sum_{\substack{|i|\le k-j+2 \\ \partial=\partial_\alpha,\partial_\beta}} |\partial\Gamma^i \chi(t)|_\infty. \end{aligned}$$

[12](1.24) gives $\mathcal{H}u = u$, therefore $-\overline{\mathcal{H}}\bar{u} = \bar{u}$.

Here we used the fact $E_{l+2}(t) \lesssim M_0^2$. Therefore for $k \geq j > [\frac{k}{2}] + 1$,

$$\|Q_j(t)\|_2 \lesssim E_k^{1/2}(t) \sum_{\substack{|i|\leq[\frac{k}{2}]+2 \\ \partial=\partial_\alpha,\partial_\beta}} (|\partial\Gamma^i \chi(t)|_\infty + |\partial\Gamma^i \mathfrak{v}(t)|_\infty). \tag{3.21}$$

Sum up (3.19)–(3.21), we have

$$\|\Gamma^k \mathcal{P}\chi(t)\|_2 \lesssim E_k^{1/2}(t) \sum_{\substack{|i|\leq[\frac{k}{2}]+2 \\ \partial=\partial_\alpha,\partial_\beta}} (|\partial\Gamma^i \chi(t)|_\infty + |\partial\Gamma^i \mathfrak{v}(t)|_\infty). \tag{3.22}$$

A similar argument gives that

$$\|\Gamma^k \mathcal{P}\mathfrak{v}(t)\|_2 \lesssim E_{k+1}^{1/2}(t) \sum_{\substack{|i|\leq[\frac{k}{2}]+2 \\ \partial=\partial_\alpha,\partial_\beta}} (|\partial\Gamma^i \chi(t)|_\infty + |\partial\Gamma^i \mathfrak{v}(t)|_\infty). \tag{3.23}$$

Combine (3.17)–(3.23), we obtain

$$\begin{aligned} \|\mathfrak{P}\Gamma^k \chi(t)\|_2 &\lesssim E_{k+1}^{1/2}(t) \sum_{\substack{|i|\leq\max\{[\frac{k}{2}]+2,4\} \\ \partial=\partial_\alpha,\partial_\beta}} (|\partial\Gamma^i \chi(t)|_\infty + |\partial\Gamma^i \mathfrak{v}(t)|_\infty), \\ \|\mathfrak{P}\Gamma^k \mathfrak{v}(t)\|_2 &\lesssim E_{k+2}^{1/2}(t) \sum_{\substack{|i|\leq\max\{[\frac{k}{2}]+2,4\} \\ \partial=\partial_\alpha,\partial_\beta}} (|\partial\Gamma^i \chi(t)|_\infty + |\partial\Gamma^i \mathfrak{v}(t)|_\infty). \end{aligned} \tag{3.24}$$

This gives us (3.14). For $t \leq 1$ (3.15) can be obtained from the Sobolev embedding and Proposition 2.16. For $t \geq 1$, (3.15) is obtained by first applying (3.14) to the case $5 \leq m \leq l - 3$, i.e. when $[\frac{m+3}{2}] + 2 \leq m$ and $m + 5 \leq l + 2$ and using the fact $E_{l+2}(t) \lesssim M_0^2$; then applying (3.14) to the case $m \leq \min\{2l - 11, q - 5\}$. We know in this case $[\frac{m+3}{2}] + 2 \leq l - 3$. □

In what follows we establish two energy estimates.

3.1 The first energy estimate

The first energy estimate concerns a full range of derivatives. We use Lemma 3.1 and (1.35), (2.41).

Assume that ϕ is a $\mathcal{C}(V_2)$ valued function satisfying the equation

$$((\partial_t + b \cdot \nabla_\perp)^2 - A\mathcal{N} \times \nabla)\phi = G^\phi. \tag{3.25}$$

Let $\Phi^j = (I - \mathcal{H})\Gamma^j \phi$. We know for $\mathcal{P} = (\partial_t + b \cdot \nabla_\perp)^2 - A\mathcal{N} \times \nabla$,

$$((\partial_t + b \cdot \nabla_\perp)^2 - A\mathcal{N} \times \nabla)\Phi^j = -[\mathcal{P}, \mathcal{H}]\Gamma^j \phi + (I - \mathcal{H})[\mathcal{P}, \Gamma^j]\phi + (I - \mathcal{H})\Gamma^j G^\phi. \tag{3.26}$$

Notice that $\Phi^j = -\mathcal{H}\Phi^j$ implies $\mathcal{N} \times \nabla \Phi^j = \mathcal{N} \cdot \nabla_\xi^- \Phi^j$. Therefore

$$((\partial_t + b \cdot \nabla_\perp)^2 + A\mathcal{N} \cdot \nabla_\xi^+)\Phi^j = \mathbf{G}_j^\phi, \tag{3.27}$$

where

$$\mathbf{G}_j^\phi = -[\mathcal{P}, \mathcal{H}]\Gamma^j \phi + (I - \mathcal{H})[\mathcal{P}, \Gamma^j]\phi + (I - \mathcal{H})\Gamma^j G^\phi + A(\mathcal{N} \cdot \nabla_\xi^+ + \mathcal{N} \cdot \nabla_\xi^-)\Phi^j.$$

Define

$$F_j^\phi(t) = \iint \frac{1}{A}|(\partial_t + b \cdot \nabla_\perp)\Phi^j(\alpha, \beta, t)|^2 + \Phi^j \cdot (\mathcal{N} \cdot \nabla_\xi^+)\Phi^j(\alpha, \beta, t)\, d\alpha\, d\beta. \tag{3.28}$$

We know $\iint \Phi^j \cdot (\mathcal{N} \cdot \nabla_\xi^+)\Phi^j(\alpha, \beta, t)\, d\alpha\, d\beta = \int_{\Omega(t)} |\nabla\{\Phi^j\}^\hbar|^2\, dV \geq 0$. Let

$$\mathcal{F}_n(t) = \sum_{|j| \leq n} (F_j^{\mathfrak{v}}(t) + F_j^\chi(t)) \tag{3.29}$$

and $V^j = (I - \mathcal{H})\Gamma^j \mathfrak{v}$, $\Pi^j = (I - \mathcal{H})\Gamma^j \chi$. We have

Lemma 3.4 *Let $n \leq q$, $t \in [0, T]$. There exists $M_0 > 0$ small enough, such that for $M \leq M_0$,*

$$\sum_{|j| \leq n} \iint \frac{1}{A}(|(\partial_t + b \cdot \nabla_\perp)V^j(\alpha, \beta, t)|^2 + |(\partial_t + b \cdot \nabla_\perp)\Pi^j(\alpha, \beta, t)|^2)\, d\alpha\, d\beta \simeq E_n(t). \tag{3.30}$$

Proof Notice that $\Gamma^j \phi = \frac{1}{2}(I + \mathcal{H})\Gamma^j \phi + \frac{1}{2}\Phi^j$. We know $\mathcal{H}\chi = -\chi$. So for $\phi = \chi,\ \mathfrak{v}$,

$$\begin{aligned} &(I + \mathcal{H})\Gamma^j \chi = -[\Gamma^j, \mathcal{H}]\chi, \\ &(I + \mathcal{H})\Gamma^j \mathfrak{v} = -[\Gamma^j, \mathcal{H}]\mathfrak{v} - \Gamma^j[\partial_t + b \cdot \nabla_\perp, \mathcal{H}]\chi. \end{aligned} \tag{3.31}$$

(3.30) follows by applying Lemma 1.2, Proposition 2.2, (2.6), Propositions 2.6, 2.7, 2.9, and (2.61) to (3.31). □

We now state the following energy estimate.

Proposition 3.5 *Let* $3 \le n \le \min\{2l-4, q\}$, $t \in [0, T]$. *There exists* $M_0 > 0$ *sufficiently small, such that for* $M \le M_0$,

$$\frac{d\mathcal{F}_n(t)}{dt} \lesssim \sum_{\substack{|i| \le [\frac{n}{2}]+2 \\ \partial=\partial_\alpha,\partial_\beta}} (|\Gamma^i \partial \chi(t)|_\infty + |\Gamma^i \partial \mathfrak{v}(t)|_\infty)\mathcal{F}_n(t). \tag{3.32}$$

Proof Let $\phi = \chi,\ \mathfrak{v}$, $|j| \le n$. From (3.27), applying Lemma 3.1 to each component of Φ^j then sum up, we get

$$\frac{dF_j^\phi(t)}{dt} \lesssim \|\mathbf{G}_j^\phi(t)\|_2 \{F_j^\phi(t)\}^{1/2} + \left(\left\| \frac{\mathfrak{a}_t}{\mathfrak{a}} \circ k^{-1} \right\|_{L^\infty} + 2\|\nabla \mathbf{v}(t)\|_{L^\infty(\Omega(t))} \right) F_j^\phi(t). \tag{3.33}$$

Notice that $\mathbf{v}$ is Clifford analytic in $\Omega(t)$. From Lemma 2.13, and the maximum principle, we have

$$\|\nabla \mathbf{v}(t)\|_{L^\infty(\Omega(t))} \lesssim |\partial_\alpha u(t)|_\infty + |\partial_\beta u(t)|_\infty. \tag{3.34}$$

Applying Lemma 2.18 (2.90), Proposition 2.9, 2.6, 2.7, 2.16, 2.17 to (2.39), we obtain

$$\left| \frac{\mathfrak{a}_t}{\mathfrak{a}} \circ k^{-1} \right|_\infty \lesssim \sum_{\substack{|i| \le 3 \\ \partial=\partial_\alpha,\partial_\beta}} (|\Gamma^i \partial \chi(t)|_\infty + |\Gamma^i \partial \mathfrak{v}(t)|_\infty). \tag{3.35}$$

We now estimate $\|\mathbf{G}_j^\phi(t)\|_2$ for $\phi = \chi,\ \mathfrak{v}$. We carry it out in four steps. Let

$$\mathbf{G}_j^\phi = G_{j,1}^\phi + G_{j,2}^\phi + G_{j,3}^\phi + G_{j,4}^\phi, \tag{3.36}$$

where $G_{j,1}^\phi = -[\mathcal{P}, \mathcal{H}]\Gamma^j \phi$, $G_{j,2}^\phi = (I-\mathcal{H})[\mathcal{P}, \Gamma^j]\phi$, $G_{j,3}^\phi = (I-\mathcal{H})\Gamma^j G^\phi$, and $G_{j,4}^\phi = A(\mathcal{N} \cdot \nabla_\xi^+ + \mathcal{N} \cdot \nabla_\xi^-)\Phi^j$.

Step 1. We have

$$\|G_{j,1}^\chi(t)\|_2 \lesssim \sum_{\partial=\partial_\alpha,\partial_\beta} |\partial u(t)|_\infty (\|\partial \Gamma^j \chi(t)\|_2 + \|(\partial_t + b \cdot \nabla_\perp)\Gamma^j \chi(t)\|_2),$$

$$\|G_{j,1}^{\mathfrak{v}}(t)\|_2 \lesssim \sum_{\partial=\partial_\alpha,\partial_\beta} |\partial u(t)|_\infty (\|\Gamma^j \mathfrak{v}(t)\|_2 + \|(\partial_t + b \cdot \nabla_\perp)\Gamma^j \mathfrak{v}(t)\|_2). \tag{3.37}$$

This is obtained by using Lemma 1.2 and applying Propositions 2.6, 2.7, 2.9.

Step 2. We have that for $\phi = \chi, \mathfrak{v}$,

$$\|G^{\phi}_{j,2}(t)\|_2 \lesssim \sum_{\substack{|i|\le[\frac{n}{2}]+2\\ \partial=\partial_\alpha,\partial_\beta}} (|\Gamma^i\partial\chi(t)|_\infty + |\Gamma^i\partial\mathfrak{v}(t)|_\infty)E_{|j|}(t)^{1/2}. \tag{3.38}$$

This is basically (3.17).[13] The estimate for the operator $(I-\mathcal{H})$ can be obtained by applying Proposition 2.6.

Step 3. From Propositions 2.19, 2.6, we have

$$\begin{aligned} \|G^{\chi}_{j,4}(t)\|_2 &\lesssim \sum_{\partial=\partial_\alpha,\partial_\beta} (|\partial\lambda(t)_\infty + |\partial\mathfrak{z}(t)|_\infty) \sum_{\partial=\partial_\alpha,\partial_\beta} \|\partial\Gamma^j\chi(t)\|_2, \\ \|G^{\mathfrak{v}}_{j,4}(t)\|_2 &\lesssim \sum_{\substack{1\le i\le 3\\ \partial=\partial_\alpha,\partial_\beta}} (|\partial^i\lambda(t)_\infty + |\partial^i\mathfrak{z}(t)|_\infty)\|\Gamma^j\mathfrak{v}(t)\|_2. \end{aligned} \tag{3.39}$$

Step 4. We have that

$$\|G^{\chi}_{j,3}(t)\|_2 \lesssim \sum_{\substack{|i|\le[\frac{n}{2}]+2\\ \partial=\partial_\alpha,\partial_\beta}} (|\Gamma^i\partial\chi(t)|_\infty + |\Gamma^i\partial\mathfrak{v}(t)|_\infty)E^{1/2}_{|j|}(t), \tag{3.40}$$

$$\|G^{\mathfrak{v}}_{j,3}(t)\|_2 \lesssim \sum_{\substack{|i|\le[\frac{n}{2}]+2\\ \partial=\partial_\alpha,\partial_\beta}} (|\Gamma^i\partial\chi(t)|_\infty + |\Gamma^i\partial\mathfrak{v}(t)|_\infty)E^{1/2}_{|j|}(t). \tag{3.41}$$

(3.40) is obtained by using Proposition 2.6 and (3.22).

However we cannot derive (3.41) from (3.23), since there is a loss of derivative in (3.23). This "loss of derivative" is due to the term $Au_\beta\chi_\alpha - Au_\alpha\chi_\beta$ in $G^{\mathfrak{v}}$. To obtain (3.41) we need to take advantage of the projection operator $I-\mathcal{H}$. We rewrite the term

$$\begin{aligned} &(I-\mathcal{H})\Gamma^j(Au_\beta\chi_\alpha - Au_\alpha\chi_\beta) \\ &\quad = (I-\mathcal{H})(A\Gamma^j u_\beta\,\chi_\alpha - A\Gamma^j u_\alpha\,\chi_\beta) + (I-\mathcal{H})(\Gamma^j(Au_\beta\chi_\alpha - Au_\alpha\chi_\beta) \\ &\qquad - A\Gamma^j u_\beta\,\chi_\alpha + A\Gamma^j u_\alpha\,\chi_\beta), \end{aligned} \tag{3.42}$$

in which we further rewrite, using the fact $u = \mathcal{H}u$,

$$(I-\mathcal{H})(A\Gamma^j u_\beta\,\chi_\alpha - A\Gamma^j u_\alpha\,\chi_\beta)$$

[13] Notice that (3.17), (3.22) (used in Step 4.) in fact hold for $k \le \min\{2l-4, q\}$.

$$= [\Gamma^j \partial_\beta, \mathcal{H}]u\, A\chi_\alpha - [\Gamma^j \partial_\alpha, \mathcal{H}]u\, A\chi_\beta$$
$$+ \sum_{i=1}^{3} ([A\partial_\alpha \chi_i, \mathcal{H}]\Gamma^j \partial_\beta u - [A\partial_\beta \chi_i, \mathcal{H}]\Gamma^j \partial_\alpha u) e_i. \quad (3.43)$$

Here χ_i is the e_i component of χ. Now with all the terms in appropriate forms, (3.41) results by applying Propositions 2.6, 2.7, 2.9, and Proposition 2.16, 2.17.

Sum up Steps 1–4 and (3.34), (3.35), and applying Propositions 2.16, 2.17, Lemma 3.4, we get (3.32). □

3.2 The second energy estimate

We now give an estimate that involves some lower orders of derivatives. We use Lemma 3.2 and (1.35), (2.41).

Assume that ϕ satisfies (3.25) and let $\Phi^j = (I - \mathcal{H})\Gamma^j \phi$. We know Φ^j satisfies (3.26), i.e.

$$\mathcal{P}\Phi^j = G_j^\phi,$$

where

$$G_j^\phi = -[\mathcal{P}, \mathcal{H}]\Gamma^j \phi + (I - \mathcal{H})[\mathcal{P}, \Gamma^j]\phi + (I - \mathcal{H})\Gamma^j G^\phi. \quad (3.44)$$

Define

$$\mathbf{F}_j^\phi(t) = \iint \frac{1}{A} |(\partial_t + b \cdot \nabla_\perp)\Phi^j|^2 - \Phi^j \cdot (\mathcal{N} \times \nabla)\Phi^j(\alpha, \beta, t)\, d\alpha\, d\beta. \quad (3.45)$$

We know $-\iint \Phi^j \cdot (\mathcal{N} \times \nabla)\Phi^j(\alpha, \beta, t)\, d\alpha\, d\beta = \int_{\Omega(t)^c} |\nabla\{\Phi^j\}^\hbar|^2\, dV \geq 0$. Let

$$\mathfrak{F}_n(t) = \sum_{|j| \leq n} (\mathbf{F}_j^{\mathfrak{v}}(t) + \mathbf{F}_j^\chi(t)). \quad (3.46)$$

We have

Proposition 3.6 *Let $l \geq 15$, $q \geq l + 9$, $t \in [0, T]$. There exists $M_0 > 0$ small enough, such that for $M \leq M_0$,*

$$\frac{d\mathfrak{F}_{l+2}(t)}{dt} \lesssim E_{l+2}^{1/2}(t) E_{l+3}^{1/2}(t) E_{l+9}^{1/2}(t) \left(\frac{1 + \ln(t+1)}{t+1}\right)^2. \quad (3.47)$$

Proof Let $\phi = \chi, \mathfrak{v}$, $|j| \leq l + 2$. We also use j to indicate $|j|$ in this proof. Assume $t \geq 1$. The argument can be easily modified for $t \leq 1$. Applying

Lemma 3.2 to Φ^j, we have

$$\begin{aligned}\frac{d\mathbf{F}_j^{\phi}(t)}{dt} = \iint \Big\{ \frac{2}{A} G_j^{\phi} \cdot \{(\partial_t + b\cdot\nabla_\perp)\Phi^j\} \\ - \frac{\mathfrak{a}_t}{\mathfrak{a}} \circ k^{-1} \frac{1}{A} |(\partial_t + b\cdot\nabla_\perp)\Phi^j|^2 \Big\} d\alpha d\beta \\ - \iint \{(\Phi^j \cdot (u_\beta \Phi^j_\alpha) - \Phi^j \cdot (u_\alpha \Phi^j_\beta)) \\ + \mathcal{N} \times \nabla \Phi^j \cdot [\partial_t + b\cdot\nabla_\perp, \mathcal{H}]\Phi^j\}\, d\alpha\, d\beta \\ + \frac{1}{2} \iint \{(\mathcal{N}\cdot\nabla_\xi^+ + \mathcal{N}\cdot\nabla_\xi^-)\Phi^j\} \cdot [\partial_t + b\cdot\nabla_\perp, \mathcal{H}]\Phi^j \, d\alpha\, d\beta.\end{aligned} \tag{3.48}$$

Using Lemma 2.18, (2.39), Propositions 2.6, 2.7, 2.9, 2.8 with $r = t$, and Propositions 2.16, 2.17, we obtain

$$\begin{aligned}\left|\frac{\mathfrak{a}_t}{\mathfrak{a}} \circ k^{-1} A(t)\right|_\infty &\lesssim \sum_{\substack{|i|\le 1 \\ \partial=\partial_\alpha,\partial_\beta}} |\Gamma^i \partial u(t)|_\infty \sum_{|i|\le 1} |\Gamma^i w(t)|_\infty (1+\ln t) \\ &\quad + \sum_{\substack{|i|\le 1 \\ \partial=\partial_\alpha,\partial_\beta}} (|\Gamma^i \partial u(t)|_\infty + |\Gamma^i w(t)|_\infty)(\|\partial u(t)\|_2 \\ &\quad + \|w(t)\|_2)\frac{1}{t} \\ &\quad + \sum_{\substack{|i|\le 2 \\ \partial=\partial_\alpha,\partial_\beta}} |\Gamma^i \partial u(t)|_\infty^2 + \sum_{\substack{|i|\le 1 \\ \partial=\partial_\alpha,\partial_\beta}} |\Gamma^i \partial \lambda(t)|_\infty \\ &\quad \times \Bigg(\sum_{|i|\le 2} |\Gamma^i w(t)|_\infty + \sum_{\substack{|i|\le 2 \\ \partial=\partial_\alpha,\partial_\beta}} |\Gamma^i \partial u(t)|_\infty^2 \Bigg) \\ &\lesssim \sum_{\substack{|i|\le 3 \\ \partial=\partial_\alpha,\partial_\beta}} (|\Gamma^i \partial \chi(t)|_\infty + |\Gamma^i \partial \mathfrak{v}(t)|_\infty) \Big\{ (|\Gamma^i \partial \chi(t)|_\infty \\ &\quad + |\Gamma^i \partial \mathfrak{v}(t)|_\infty)(1+\ln t) + E_1(t)^{1/2} \frac{1}{t} \Big\}.\end{aligned} \tag{3.49}$$

We now estimate $\iint \mathcal{N} \times \nabla \Phi^j \cdot [\partial_t + b \cdot \nabla_\perp, \mathcal{H}]\Phi^j \, d\alpha \, d\beta$. We know $[\partial_t + b \cdot \nabla_\perp, \mathcal{H}]\Phi^j = (I + \mathcal{H})[\partial_t + b \cdot \nabla_\perp, \mathcal{H}]\Gamma^j \phi$. Therefore using Proposition 2.12, we have

$$\begin{aligned} &\iint \mathcal{N} \times \nabla \Phi^j \cdot [\partial_t + b \cdot \nabla_\perp, \mathcal{H}]\Phi^j \, d\alpha \, d\beta \\ &\quad = \iint \{(I + \mathcal{H}^*)\mathcal{N} \times \nabla \Phi^j\} \cdot [\partial_t + b \cdot \nabla_\perp, \mathcal{H}]\Gamma^j \phi \, d\alpha \, d\beta. \end{aligned} \tag{3.50}$$

Now

$$\begin{aligned} (I + \mathcal{H}^*)\mathcal{N} \times \nabla \Phi^j &= (\mathcal{H}^* - \mathcal{H})\mathcal{N} \times \nabla \Phi^j + (I + \mathcal{H})\mathcal{N} \times \nabla \Phi^j \\ &= (\mathcal{H}^* - \mathcal{H})\mathcal{N} \times \nabla \Phi^j - [\mathcal{N} \times \nabla, \mathcal{H}]\Phi^j. \end{aligned}$$

Using (1.16), Proposition 2.6, we get

$$\|(I + \mathcal{H}^*)\mathcal{N} \times \nabla \Phi^j(t)\|_2 \lesssim \sum_{\partial = \partial_\alpha, \partial_\beta} (|\partial \lambda(t)|_\infty + |\partial \mathfrak{z}(t)|_\infty) \|\partial \Gamma^j \phi(t)\|_2. \tag{3.51}$$

On the other hand, we have from (1.15), Proposition 2.8 with $r = t$,

$$\begin{aligned} \|[\partial_t + b \cdot \nabla_\perp, \mathcal{H}]\Gamma^j \phi(t)\|_2 \lesssim \|u(t)\|_2 \Bigg(&\sum_{\substack{|i| \le j+1 \\ \partial = \partial_\alpha, \partial_\beta}} |\partial \Gamma^i \phi(t)|_\infty (1 + \ln t) \\ &+ \sum_{\substack{|i| \le j \\ \partial = \partial_\alpha, \partial_\beta}} \|\partial \Gamma^i \phi(t)\|_2 \frac{1}{t} \Bigg). \end{aligned} \tag{3.52}$$

Combining (3.50)–(3.52), and further use Propositions 2.16, 2.17, we obtain

$$\begin{aligned} &\left| \iint \mathcal{N} \times \nabla \Phi^j \cdot [\partial_t + b \cdot \nabla_\perp, \mathcal{H}]\Phi^j \, d\alpha \, d\beta \right| \\ &\quad \lesssim E_{j+1}^{1/2}(t) E_1^{1/2}(t) \sum_{\substack{|i| \le 2 \\ \partial = \partial_\alpha, \partial_\beta}} |\Gamma^i \partial \chi(t)|_\infty \\ &\qquad \times \Bigg(\sum_{\substack{|i| \le j+1 \\ \partial = \partial_\alpha, \partial_\beta}} \big(|\Gamma^i \partial \chi(t)|_\infty + |\Gamma^i \partial \mathfrak{v}(t)|_\infty\big)(1 + \ln t) + E_{j+1}^{1/2}(t) \frac{1}{t} \Bigg). \end{aligned} \tag{3.53}$$

The estimate of the term $\iint\{(\mathcal{N}\cdot\nabla_\xi^+ + \mathcal{N}\cdot\nabla_\xi^-)\Phi^j\}\cdot[\partial_t + b\cdot\nabla_\perp, \mathcal{H}]\Phi^j\,d\alpha\,d\beta$ can be obtained from Proposition 2.19 and (3.52). We have

$$\begin{aligned}&\left|\iint\{(\mathcal{N}\cdot\nabla_\xi^+ + \mathcal{N}\cdot\nabla_\xi^-)\Phi^j\}\cdot[\partial_t + b\cdot\nabla_\perp, \mathcal{H}]\Phi^j\,d\alpha\,d\beta\right|\\ &\quad\lesssim E_j^{1/2}(t)E_1^{1/2}(t)\sum_{\substack{|i|\le 2\\ \partial=\partial_\alpha,\partial_\beta}}|\Gamma^i\partial\chi(t)|_\infty\Bigg(\sum_{\substack{|i|\le j+1\\ \partial=\partial_\alpha,\partial_\beta}}(|\Gamma^i\partial\chi(t)|_\infty\\ &\qquad + |\Gamma^i\partial\mathfrak{v}(t)|_\infty)(1+\ln t) + E_{j+1}^{1/2}(t)\frac{1}{t}\Bigg).\end{aligned}\tag{3.54}$$

Now

$$\begin{aligned}&\iint\frac{1}{A}G_j^\phi\cdot\{(\partial_t + b\cdot\nabla_\perp)\Phi^j\}d\alpha d\beta\\ &\quad=\iint\left(\frac{1}{A}-1\right)G_j^\phi\cdot\{(\partial_t + b\cdot\nabla_\perp)\Phi^j\}d\alpha d\beta\\ &\qquad+\iint G_j^\phi\cdot\{(\partial_t + b\cdot\nabla_\perp)\Phi^j\}d\alpha d\beta.\end{aligned}\tag{3.55}$$

The term $\iint(\frac{1}{A}-1)G_j^\phi\cdot\{(\partial_t + b\cdot\nabla_\perp)\Phi^j\}d\alpha d\beta$ can be estimated as the following:

$$\begin{aligned}&\left|\iint\left(\frac{1}{A}-1\right)G_j^\phi\cdot\{(\partial_t + b\cdot\nabla_\perp)\Phi^j\}d\alpha\,d\beta\right|\\ &\quad\lesssim|A-1|_\infty\|G_j^\phi(t)\|_2\|(\partial_t + b\cdot\nabla_\perp)\Phi^j(t)\|_2.\end{aligned}$$

We know $G_j^\phi = G_{j,1}^\phi + G_{j,2}^\phi + G_{j,3}^\phi$, where $G_{j,i}^\phi$ $i=1,2,3$ are as defined in (3.36). Using (3.37), (3.38), (3.40), (3.41), and notice that the n in these inequalities can be replaced by j. We have by further applying Proposition 2.17 that

$$\begin{aligned}&\left|\iint\left(\frac{1}{A}-1\right)G_j^\phi\cdot\{(\partial_t + b\cdot\nabla_\perp)\Phi^j\}d\alpha d\beta\right|\\ &\quad\lesssim E_j(t)\sum_{\substack{|i|\le 3\\ \partial=\partial_\alpha,\partial_\beta}}(|\Gamma^i\partial\chi(t)|_\infty + |\Gamma^i\partial\mathfrak{v}(t)|_\infty)\\ &\qquad\times\sum_{\substack{|i|\le[\frac{j}{2}]+2\\ \partial=\partial_\alpha,\partial_\beta}}(|\Gamma^i\partial\chi(t)|_\infty + |\Gamma^i\partial\mathfrak{v}(t)|_\infty).\end{aligned}\tag{3.56}$$

We now estimate the terms $\iint G_j^\phi \cdot \{(\partial_t + b \cdot \nabla_\perp)\Phi^j\} d\alpha d\beta$ and $\iint \{(\Phi^j \cdot (u_\beta \Phi_\alpha^j) - \Phi^j \cdot (u_\alpha \Phi_\beta^j))\} d\alpha\, d\beta$ for $\phi = \chi,\ \mathfrak{v}$. We carry out the estimates through six steps.

Step 1. We consider the term $\iint G_{j,1}^\phi \cdot \{(\partial_t + b \cdot \nabla_\perp)\Phi^j\} d\alpha d\beta$ for $\phi = \chi,\ \mathfrak{v}$.

We first put the term $\iint G_{j,1}^\phi \cdot \{(\partial_t + b \cdot \nabla_\perp)\Phi^j\} d\alpha d\beta$ in the right form for estimates. We know $(\partial_t + b \cdot \nabla_\perp)\Phi^j = (I - \mathcal{H})(\partial_t + b \cdot \nabla_\perp)\Gamma^j \phi - [\partial_t + b \cdot \nabla_\perp, \mathcal{H}]\Gamma^j \phi$. Using Proposition 2.12, we have

$$\begin{aligned}
&\iint G_{j,1}^\phi \cdot \{(\partial_t + b \cdot \nabla_\perp)\Phi^j\} d\alpha d\beta \\
&\quad = \iint \{(I - \mathcal{H}) G_{j,1}^\phi\} \cdot \{(\partial_t + b \cdot \nabla_\perp)\Gamma^j \phi\} d\alpha d\beta \\
&\qquad + \iint \{(\mathcal{H} - \mathcal{H}^*) G_{j,1}^\phi\} \cdot \{(\partial_t + b \cdot \nabla_\perp)\Gamma^j \phi\} d\alpha d\beta \\
&\qquad - \iint G_{j,1}^\phi \cdot \{[\partial_t + b \cdot \nabla_\perp, \mathcal{H}]\Gamma^j \phi\} d\alpha d\beta,
\end{aligned} \tag{3.57}$$

where by applying (1.19), (1.18), and the change of variable U_k^{-1}, we know

$$\begin{aligned}
-G_{j,1}^\phi &= 2 \iint K(\zeta' - \zeta)(u - u') \\
&\quad \times (\zeta'_{\beta'} \partial_{\alpha'} - \zeta'_{\alpha'} \partial_{\beta'})(\partial'_t + b' \cdot \nabla'_\perp)\Gamma'^j \phi'\, d\alpha' d\beta' \\
&\quad + \iint K(\zeta' - \zeta)\{(u - u') \times (u'_{\beta'} \partial_{\alpha'} - u'_{\alpha'} \partial_{\beta'})\Gamma'^j \phi'\}\, d\alpha' d\beta' \\
&\quad + \iint ((u' - u) \cdot \nabla) K(\zeta' - \zeta)(u - u') \\
&\quad \times (\zeta'_{\beta'} \partial'_\alpha - \zeta'_{\alpha'} \partial'_\beta)\Gamma'^j \phi'\, d\alpha' d\beta'.
\end{aligned} \tag{3.58}$$

Let $J_1^\phi = 2 \iint K(\zeta' - \zeta)(u - u') \times (\zeta'_{\beta'} \partial_{\alpha'} - \zeta'_{\alpha'} \partial_{\beta'})(\partial'_t + b' \cdot \nabla'_\perp)\Gamma'^j \phi'\, d\alpha' d\beta'$. To estimate the term $\iint \{(I - \mathcal{H}) G_{j,1}^\phi\} \cdot \{(\partial_t + b \cdot \nabla_\perp)\Gamma^j \phi\} d\alpha d\beta$, we use Proposition 2.11 to further rewrite

$$\begin{aligned}
(I - \mathcal{H}) J_1^\phi &= (I - \mathcal{H}) \iint K\, (u - u') \\
&\quad \times (\zeta'_{\beta'} \partial_{\alpha'} - \zeta'_{\alpha'} \partial_{\beta'})(I + \mathcal{H}')(\partial'_t + b' \cdot \nabla'_\perp)\Gamma'^j \phi'\, d\alpha' d\beta',
\end{aligned} \tag{3.59}$$

and notice that $\mathcal{H}\chi = -\chi$, so for $\phi = \chi, \mathfrak{v}$,

$$\begin{aligned}(I+\mathcal{H})(\partial_t + b\cdot\nabla_\perp)\Gamma^j\chi =& -[\partial_t + b\cdot\nabla_\perp, \mathcal{H}]\Gamma^j\chi\\ &-(\partial_t + b\cdot\nabla_\perp)[\Gamma^j, \mathcal{H}]\chi,\\ (I+\mathcal{H})(\partial_t + b\cdot\nabla_\perp)\Gamma^j\mathfrak{v} =& (I+\mathcal{H})[\partial_t + b\cdot\nabla_\perp, \Gamma^j]\mathfrak{v}\\ &+[\mathcal{H}, \Gamma^j](\partial_t + b\cdot\nabla_\perp)\mathfrak{v}\\ &-\Gamma^j[(\partial_t + b\cdot\nabla_\perp)^2, \mathcal{H}]\chi.\end{aligned} \tag{3.60}$$

With (3.57)–(3.60), $\iint G_{j,1}^{\phi}\cdot\{(\partial_t + b\cdot\nabla_\perp)\Phi^j\}d\alpha d\beta$ is in the right form for estimates. Using Lemma 1.2, Proposition 2.2, (2.6), Propositions 2.6, 2.7, 2.8 with $r = t$, and 2.16, we get

$$\begin{aligned}\|(I-\mathcal{H})J_1^{\chi}(t)\|_2 \lesssim& \sum_{\partial=\partial_\alpha,\partial_\beta} |\partial u(t)|_\infty \|(I+\mathcal{H})(\partial_t + b\cdot\nabla_\perp)\Gamma^j\chi(t)\|_2\\ \lesssim& \sum_{\partial=\partial_\alpha,\partial_\beta} |\partial u(t)|_\infty E_j^{1/2}(t) \sum_{\substack{|i|\le j\\ \partial=\partial_\alpha,\partial_\beta}} |\Gamma^i\partial\lambda(t)|_\infty\\ &+\sum_{\partial=\partial_\alpha,\partial_\beta} |\partial u(t)|_\infty E_j^{1/2}(t)\\ &\times\left(\sum_{\substack{|i|\le j+1\\ \partial=\partial_\alpha,\partial_\beta}} |\partial\Gamma^i\chi(t)|_\infty(1+\ln t) + E_j^{1/2}(t)\frac{1}{t}\right)\end{aligned} \tag{3.61}$$

and

$$\begin{aligned}\|(I-\mathcal{H})J_1^{\mathfrak{v}}(t)\|_2 \lesssim& \sum_{\partial=\partial_\alpha,\partial_\beta} |\partial u(t)|_\infty E_j^{1/2}(t) \sum_{\substack{|i|\le j\\ \partial=\partial_\alpha,\partial_\beta}} |\partial\Gamma^i\lambda(t)|_\infty\\ &+\sum_{\partial=\partial_\alpha,\partial_\beta} |\partial u(t)|_\infty E_j^{1/2}(t)\Bigg(\sum_{\substack{|i|\le[\frac{j}{2}]+2\\ \partial=\partial_\alpha,\partial_\beta}} (|\partial\Gamma^i\chi(t)|_\infty\\ &+|\partial\Gamma^i\mathfrak{v}(t)|_\infty)(1+\ln t) + E_j^{1/2}(t)\frac{1}{t}\Bigg).\end{aligned} \tag{3.62}$$

Applying Propositions 2.6, 2.7, 2.8 with $r = t$, and 2.16, 2.17 to other terms in (3.57) and using (3.37), (3.52), we obtain for $\phi = \chi,\ \mathfrak{v}$,

$$\begin{aligned}&\left|\iint G^{\phi}_{j,1}\cdot\{(\partial_t + b\cdot\nabla_\perp)\Phi^j\}d\alpha d\beta\right|\\ &\quad\lesssim E_j(t)\sum_{\substack{|i|\le 2\\ \partial=\partial_\alpha,\partial_\beta}}(|\partial\Gamma^i\chi(t)|_\infty + |\partial\Gamma^i\mathfrak{v}(t)|_\infty)\\ &\qquad\times\left\{(E^{1/2}_{j+2}(t)\sum_{\substack{|i|\le[\frac{j}{2}]+2\\ \partial=\partial_\alpha,\partial_\beta}}|\partial\Gamma^i\chi(t)|_\infty\right.\\ &\qquad\left.+\sum_{\substack{|i|\le j+1\\ \partial=\partial_\alpha,\partial_\beta}}(|\partial\Gamma^i\chi(t)|_\infty + |\partial\Gamma^i\mathfrak{v}(t)|_\infty))(1+\ln t) + E^{1/2}_{j+2}(t)\frac{1}{t}\right\}.\end{aligned} \tag{3.63}$$

Step 2. We consider the term $\iint G^{\phi}_{j,3}\cdot\{(\partial_t + b\cdot\nabla_\perp)\Phi^j\}d\alpha d\beta$ for $\phi = \chi$.

From (1.35), we know G^χ consists of three terms $G^\chi = I_1 + I_2 + I_3$. In particular, the first term

$$\begin{aligned}I_1 &= \iint K(\zeta'-\zeta)(u-u')\times(\zeta'_{\beta'}\partial_{\alpha'} - \zeta'_{\alpha'}\partial_{\beta'})\overline{u'}\,d\alpha' d\beta'\\ &= \frac{1}{2}\iint K(\zeta'-\zeta)(u-u')\\ &\qquad\times(\zeta'_{\beta'}\partial_{\alpha'} - \zeta'_{\alpha'}\partial_{\beta'})\{(I+\mathcal{H}')\overline{u'} + (I-\mathcal{H}')\overline{u'}\}\,d\alpha' d\beta'.\end{aligned} \tag{3.64}$$

Rewriting

$$(I-\mathcal{H})\Gamma^j I_1 = [\Gamma^j,\mathcal{H}]I_1 + \Gamma^j(I-\mathcal{H})I_1, \tag{3.65}$$

where using Proposition 2.11, we deduce

$$\begin{aligned}(I-\mathcal{H})I_1 &= (I-\mathcal{H})\frac{1}{2}\iint K(\zeta'-\zeta)(u-u')\\ &\qquad\times(\zeta'_{\beta'}\partial_{\alpha'} - \zeta'_{\alpha'}\partial_{\beta'})\{(I+\mathcal{H}')\overline{u'}\}\,d\alpha' d\beta'\\ &= \iint K(\zeta'-\zeta)(u-u')\\ &\qquad\times(\zeta'_{\beta'}\partial_{\alpha'} - \zeta'_{\alpha'}\partial_{\beta'})\{(I+\mathcal{H}')\overline{u'}\}\,d\alpha' d\beta';\end{aligned} \tag{3.66}$$

furthermore from (1.24), we know $(I+\mathcal{H})\bar{u}=(-\overline{\mathcal{H}}+\mathcal{H})\bar{u}$. Now with (3.64)–(3.66), all the terms in $G^{\chi}_{j,3}(t)=(I-\mathcal{H})\Gamma^j G^{\chi}$ are in appropriate forms for carrying out estimates. Using Propositions 2.6, 2.7, 2.8, and 2.16, we obtain

$$\|G^{\chi}_{j,3}(t)\|_2 \lesssim E_j^{1/2}(t) \sum_{\substack{|i|\le j\\ \partial=\partial_\alpha,\partial_\beta}} (|\Gamma^i\partial\lambda(t)|_\infty + |\Gamma^i\partial\mathfrak{z}(t)|_\infty) \times\left(\sum_{\substack{|i|\le[\frac{j}{2}]+2\\ \partial=\partial_\alpha,\partial_\beta}} |\Gamma^i\partial u(t)|_\infty(1+\ln t)+E_j^{1/2}(t)\frac{1}{t}\right).$$

Further using Proposition 2.17, we get for $\phi=\chi$,

$$\begin{aligned}\left|\iint G^{\phi}_{j,3}\cdot\{(\partial_t+b\cdot\nabla_\perp)\Phi^j\}d\alpha d\beta\right| &\lesssim E_j(t)\left\{\sum_{\substack{|i|\le j\\ \partial=\partial_\alpha,\partial_\beta}} |\Gamma^i\partial\chi(t)|_\infty + E_{j+2}^{1/2}(t)\right.\\ &\quad\left.\times\left(\sum_{\substack{|i|\le[\frac{j}{2}]+2\\ \partial=\partial_\alpha,\partial_\beta}} |\Gamma^i\partial\chi(t)|_\infty(1+\ln t)+\frac{1}{t}\right)\right\}\\ &\quad\times\left(\sum_{\substack{|i|\le[\frac{j}{2}]+2\\ \partial=\partial_\alpha,\partial_\beta}} (|\Gamma^i\partial\chi(t)|_\infty\right.\\ &\quad\left.+|\Gamma^i\partial\mathfrak{v}(t)|_\infty)(1+\ln t)+E_j^{1/2}(t)\frac{1}{t}\right).\end{aligned} \tag{3.67}$$

Step 3. We consider the term $\iint G^{\phi}_{j,3}\cdot\{(\partial_t+b\cdot\nabla_\perp)\Phi^j\}d\alpha d\beta$ for $\phi=\mathfrak{v}$. From (2.41), we know

$$G^{\mathfrak{v}}_{j,3}=(I-\mathcal{H})\Gamma^j\left(\frac{\mathfrak{a}_t}{\mathfrak{a}}\circ k^{-1}A\mathcal{N}\times\nabla\chi+A(u_\beta\chi_\alpha-u_\alpha\chi_\beta) + (\partial_t+b\cdot\nabla_\perp)G^{\chi}\right). \tag{3.68}$$

In this step, we will only consider the estimates of $\|(I-\mathcal{H})\Gamma^j(\frac{\mathfrak{a}_t}{\mathfrak{a}}\circ k^{-1}A\mathcal{N}\times\nabla\chi)(t)\|_2$, $\|(I-\mathcal{H})\Gamma^j((\partial_t+b\cdot\nabla_\perp)G^{\chi})(t)\|_2$ and $\|(I-\mathcal{H})\Gamma^j((A-1)(u_\beta\chi_\alpha-u_\alpha\chi_\beta))(t)\|_2$. We will leave the estimate of $\|(I-\mathcal{H})\Gamma^j(u_\beta\chi_\alpha-u_\alpha\chi_\beta)(t)\|_2$ to Step 5.

First, we have by using (2.39), Lemma 2.18, and Propositions 2.6, 2.7, 2.8, 2.9, and 2.16, 2.17 that

$$\left\|(I-\mathcal{H})\Gamma^j\left(\frac{\mathfrak{a}_t}{\mathfrak{a}}\circ k^{-1}A\mathcal{N}\times\nabla\chi\right)(t)\right\|_2$$
$$\lesssim E_j^{1/2}(t)\sum_{\substack{|i|\le[\frac{j}{2}]+2\\ \partial=\partial_\alpha,\partial_\beta}}(|\Gamma^i\partial\chi(t)|_\infty+|\Gamma^i\partial\mathfrak{v}(t)|_\infty)\Bigg(\sum_{\substack{|i|\le[\frac{j}{2}]+2\\ \partial=\partial_\alpha,\partial_\beta}}(|\Gamma^i\partial\chi(t)|_\infty$$
$$+|\Gamma^i\partial\mathfrak{v}(t)|_\infty)(1+\ln t)+E_j^{1/2}(t)\frac{1}{t}\Bigg). \tag{3.69}$$

And using Propositions 2.16, 2.17, we have

$$\|(I-\mathcal{H})\Gamma^j((A-1)(u_\beta\chi_\alpha-u_\alpha\chi_\beta))(t)\|_2$$
$$\lesssim E_{j+1}^{1/2}(t)\sum_{\substack{|i|\le[\frac{j}{2}]+2\\ \partial=\partial_\alpha,\partial_\beta}}(|\Gamma^i\partial\chi(t)|_\infty+|\Gamma^i\partial\mathfrak{v}(t)|_\infty)^2. \tag{3.70}$$

We handle the estimate of $\|(I-\mathcal{H})\Gamma^j((\partial_t+b\cdot\nabla_\perp)G^\chi)(t)\|_2$ similar to Step 2 by rewriting the term $(I-\mathcal{H})\Gamma^j(\partial_t+b\cdot\nabla_\perp)I_1$, where I_1 is as defined in (3.64), as the following:

$$(I-\mathcal{H})\Gamma^j(\partial_t+b\cdot\nabla_\perp)I_1=[\Gamma^j,\mathcal{H}](\partial_t+b\cdot\nabla_\perp)I_1+\Gamma^j[\partial_t+b\cdot\nabla_\perp,\mathcal{H}]I_1$$
$$+\Gamma^j(\partial_t+b\cdot\nabla_\perp)(I-\mathcal{H})I_1 \tag{3.71}$$

and use (3.66) to calculate $(I-\mathcal{H})I_1$. We get, by using Propositions 2.6, 2.7, 2.8 with $r=t$, and 2.16, 2.17 that

$$\|(I-\mathcal{H})\Gamma^j((\partial_t+b\cdot\nabla_\perp)G^\chi)(t)\|_2$$
$$\lesssim E_{j+1}^{1/2}(t)\Bigg(\sum_{\substack{|i|\le[\frac{j}{2}]+2\\ \partial=\partial_\alpha,\partial_\beta}}(|\Gamma^i\partial\chi(t)|_\infty$$
$$+|\Gamma^i\partial\mathfrak{v}(t)|_\infty)(1+\ln t)+E_j^{1/2}(t)\frac{1}{t}\Bigg)^2. \tag{3.72}$$

Step 4. We consider the term $\iint G_{j,2}^\phi\cdot\{(\partial_t+b\cdot\nabla_\perp)\Phi^j\}d\alpha d\beta$ for $\phi=\chi$ and $\mathfrak{v}$.

We know

$$G^{\phi}_{j,2} = (I - \mathcal{H})[\mathcal{P}, \Gamma^j]\phi = \sum_{k=1}^{j} (I - \mathcal{H})\Gamma^{j-k}[\mathcal{P}, \Gamma]\Gamma^{k-1}\phi.$$

A further expansion of (2.5) gives that for $\Gamma = \partial_t, \partial_\alpha, \partial_\beta, \varpi$,

$$\begin{aligned}[\Gamma, \mathcal{P}] = &-\{(\dot{\Gamma}(A-1)(\zeta_\beta\partial_\alpha - \zeta_\alpha\partial_\beta) + A(\partial_\beta\dot{\Gamma}\lambda\partial_\alpha - \partial_\alpha\dot{\Gamma}\lambda\partial_\beta)\} \\ &+ \{\ddot{\Gamma}(\partial_t + b\cdot\nabla_\perp)b - \ddot{\Gamma}b\cdot\nabla_\perp b\}\cdot\nabla_\perp \\ &+ \ddot{\Gamma}b\cdot\{(\partial_t + b\cdot\nabla_\perp)\nabla_\perp + \nabla_\perp(\partial_t + b\cdot\nabla_\perp)\},\end{aligned} \tag{3.73}$$

where $\dot{\Gamma}f = \partial_t f, \partial_\alpha f, \partial_\beta f, \varpi f + \frac{1}{2}fe_3$, $\ddot{\Gamma}f = \partial_t f, \partial_\alpha f, \partial_\beta f, (\varpi - \frac{1}{2}e_3)f$ respectively. Also

$$\begin{aligned}[L_0, \mathcal{P}] = &-\mathcal{P} - \{L_0(A-1)(\zeta_\beta\partial_\alpha - \zeta_\alpha\partial_\beta) \\ &+ A(\partial_\beta(L_0 - I)\lambda\partial_\alpha - \partial_\alpha(L_0 - I)\lambda\partial_\beta)\} \\ &+ \left\{L_0(\partial_t + b\cdot\nabla_\perp)b - \left(L_0 b - \frac{1}{2}b\right)\cdot\nabla_\perp b\right\}\cdot\nabla_\perp \\ &+ \left(L_0 b - \frac{1}{2}b\right)\cdot\left\{(\partial_t + b\cdot\nabla_\perp)\nabla_\perp + \nabla_\perp(\partial_t + b\cdot\nabla_\perp)\right\}.\end{aligned} \tag{3.74}$$

Therefore typically there are three types of terms in $[\mathcal{P}, \Gamma^j]\phi = \sum_{k=1}^{j}\Gamma^{j-k}[\mathcal{P}, \Gamma]\Gamma^{k-1}\phi$. The first type (C) are of cubic and higher orders and are consists of the following:

$$\begin{aligned}&\Gamma^{j-k}\{\Gamma^i(A-1)(\zeta_\beta\partial_\alpha - \zeta_\alpha\partial_\beta)\}\Gamma^{k-1}\phi, \quad i = 0, 1,\ k = 1, \dots, j, \\ &\Gamma^{j-k}\{\Gamma^i(\partial_t + b\cdot\nabla_\perp)b\}\cdot\nabla_\perp\Gamma^{k-1}\phi, \quad \Gamma^{j-k}\{\Gamma^i b\cdot\nabla_\perp b\}\cdot\nabla_\perp\Gamma^{k-1}\phi, \\ &\Gamma^{j-k}\{\Gamma^i b\cdot((\partial_t + b\cdot\nabla_\perp)\nabla_\perp + \nabla_\perp(\partial_t + b\cdot\nabla_\perp))\}\Gamma^{k-1}\phi.\end{aligned} \tag{3.75}$$

The second type (Q) are quadratic and are consists of the following:

$$\begin{aligned}&\Gamma^{j-k}\{A(\partial_\beta\Gamma^i\lambda\partial_\alpha\Gamma^{k-1}\phi - \partial_\alpha\Gamma^i\lambda\partial_\beta\Gamma^{k-1}\phi)\}, \quad i = 0, 1,\ k = 1, \dots, j, \\ &\Gamma^{j-k}\{A(\partial_\beta\lambda e_3\partial_\alpha\Gamma^{k-1}\phi - \partial_\alpha\lambda e_3\partial_\beta\Gamma^{k-1}\phi)\}.\end{aligned} \tag{3.76}$$

And the third type is of the form $\Gamma^{j-k}\mathcal{P}\Gamma^{k-1}\phi$ for some $1 \le k \le j$, which can be treated in the same way as in Steps 2–6. We first consider the terms of the first type (C) and let the sum of these terms be $C(t)$. We have, by using

Propositions 1.4, 2.6, 2.7, 2.8 with $r = t$, 2.16, 2.17 that

$$
\begin{aligned}
&\|(I-\mathcal{H})C(t)\|_2 \\
&\lesssim \sum_{|i|\le j}(\|\Gamma^i(A-1)(t)\|_2 + \|\Gamma^i(\partial_t + b\cdot\nabla_\perp)b(t)\|_2 + \|\Gamma^i b(t)\|_2) \\
&\quad\times \sum_{\substack{|i|\le[\frac{j}{2}]+2\\ \partial=\partial_\alpha,\partial_\beta}} |\partial\Gamma^i\phi(t)|_\infty \\
&\quad+ \sum_{|i|\le[\frac{j}{2}]}(\|\Gamma^i(A-1)(t)\|_2 + \|\Gamma^i(\partial_t + b\cdot\nabla_\perp)b(t)\|_2 + \|\Gamma^i b(t)\|_2) \\
&\quad\times \sum_{\substack{|i|\le j\\ \partial=\partial_\alpha,\partial_\beta}} |\partial\Gamma^i\phi(t)|_\infty \\
&\quad+ \sum_{|i|\le[\frac{j}{2}]+1}\|\Gamma^i b(t)\|_\infty\Bigg(\sum_{|i|\le j}\|\Gamma^i b(t)\|_2 \sum_{\substack{|i|\le[\frac{j}{2}]+2\\ \partial=\partial_\alpha,\partial_\beta}} |\partial\Gamma^i\phi(t)|_\infty \\
&\quad+ \sum_{|i|\le[\frac{j}{2}]}\|\Gamma^i b(t)\|_2 \sum_{\substack{|i|\le j\\ \partial=\partial_\alpha,\partial_\beta}} |\partial\Gamma^i\phi(t)|_\infty\Bigg) \\
&\lesssim E_j^{1/2}(t)\Bigg\{(1+\ln t)\sum_{\substack{|i|\le[\frac{j}{2}]+2\\ \partial=\partial_\alpha,\partial_\beta}}(|\Gamma^i\partial\chi(t)|_\infty + |\Gamma^i\partial\mathfrak{v}(t)|_\infty) + E_j^{1/2}(t)\frac{1}{t}\Bigg\} \\
&\quad\times\Bigg\{\sum_{\substack{|i|\le j+1\\ \partial=\partial_\alpha,\partial_\beta}} |\Gamma^i\partial\chi(t)|_\infty + |\Gamma^i\partial\mathfrak{v}(t)|_\infty + E_{j+3}^{1/2}(t) \\
&\quad\times\Bigg(\sum_{\substack{|i|\le[\frac{j+1}{2}]+2\\ \partial=\partial_\alpha,\partial_\beta}} |\Gamma^i\partial\chi(t)|_\infty(1+\ln t) + \frac{1}{t}\Bigg)\Bigg\}.
\end{aligned}
\tag{3.77}
$$

We also give the estimates of the following two cubic and higher order terms in (3.76). First we have for $\phi = \chi,\ \mathfrak{v}, k = 1, \dots, j, i = 0, 1,$

$$\|(I-\mathcal{H})\Gamma^{j-k}\{(A-1)(\partial_\beta\Gamma^i\lambda\partial_\alpha\Gamma^{k-1}\phi - \partial_\alpha\Gamma^i\lambda\partial_\beta\Gamma^{k-1}\phi)(t)\}\|_2$$

$$\lesssim E_j^{1/2}(t) \sum_{\substack{|i|\le[\frac{j}{2}]+2\\ \partial=\partial_\alpha,\partial_\beta}} (|\Gamma^i\partial\chi(t)|_\infty + |\Gamma^i\partial\mathfrak{v}(t)|_\infty)^2. \tag{3.78}$$

Recall definition (1.36): $\lambda = \lambda^* - \mathcal{K}\mathfrak{z}e_3$. We have for $\phi = \chi,\ \mathfrak{v}$, $k = 1, \ldots, j$, $i = 0, 1$,

$$\begin{aligned} &\|(I-\mathcal{H})\Gamma^{j-k}\{\partial_\beta\Gamma^i\mathcal{K}\mathfrak{z}e_3\partial_\alpha\Gamma^{k-1}\phi - \partial_\alpha\Gamma^i\mathcal{K}\mathfrak{z}e_3\partial_\beta\Gamma^{k-1}\phi)(t)\}\|_2 \\ &\quad\lesssim E_j^{1/2}(t) \sum_{\substack{|i|\le[\frac{j}{2}]+2\\ \partial=\partial_\alpha,\partial_\beta}} (|\Gamma^i\partial\chi(t)|_\infty + |\Gamma^i\partial\mathfrak{v}(t)|_\infty)^2(1+\ln t) \\ &\qquad + E_j(t)\frac{1}{t} \sum_{\substack{|i|\le[\frac{j}{2}]+2\\ \partial=\partial_\alpha,\partial_\beta}} (|\Gamma^i\partial\chi(t)|_\infty + |\Gamma^i\partial\mathfrak{v}(t)|_\infty). \end{aligned} \tag{3.79}$$

Therefore the only terms in (3.76) that are left to be estimated are the following

$$\begin{aligned} &\Gamma^{j-k}\{\partial_\beta\Gamma^i\lambda^*\partial_\alpha\Gamma^{k-1}\phi - \partial_\alpha\Gamma^i\lambda^*\partial_\beta\Gamma^{k-1}\phi\}, \\ &\Gamma^{j-k}\{\partial_\beta\lambda^*e_3\partial_\alpha\Gamma^{k-1}\phi - \partial_\alpha\lambda^*e_3\partial_\beta\Gamma^{k-1}\phi\}, \quad i = 0, 1,\ k = 1, \ldots, j. \end{aligned} \tag{3.80}$$

Step 5. We consider the term $(I-\mathcal{H})\Gamma^j(u_\beta\chi_\alpha - u_\alpha\chi_\beta)$ in $G^{\mathfrak{v}}_{j,3}$, the term $\iint\{\Phi^j\cdot(u_\beta\Phi^j_\alpha - u_\alpha\Phi^j_\beta)\}\,d\alpha\,d\beta$ for $\phi = \mathfrak{v}$ in (3.48), and those terms in (3.80) for $\phi = \mathfrak{v}$. Without loss of generality, for terms in (3.80) with $\phi = \mathfrak{v}$, we will only write for

$$\Gamma^{j-k}\{\partial_\beta\Gamma\lambda^*\partial_\alpha\Gamma^{k-1}\mathfrak{v} - \partial_\alpha\Gamma\lambda^*\partial_\beta\Gamma^{k-1}\mathfrak{v}\}.$$

Using (2.28), we rewrite

$$\begin{aligned} \partial_\beta u\partial_\alpha\chi - \partial_\alpha u\partial_\beta\chi = &\frac{2}{t}\{\Upsilon u\,\partial_t(e_2\partial_\alpha - e_1\partial_\beta)\chi \\ &+ \partial_\beta u\,\Omega^-_{01}(e_2\partial_\alpha - e_1\partial_\beta)\chi \\ &- \partial_\alpha u\,\Omega^-_{02}(e_2\partial_\alpha - e_1\partial_\beta)\chi\}, \end{aligned} \tag{3.81}$$

$$\begin{aligned} \partial_\beta u\partial_\alpha\Phi^j - \partial_\alpha u\partial_\beta\Phi^j = &\frac{2}{t}\{\Upsilon u\,\partial_t(e_2\partial_\alpha - e_1\partial_\beta)\Phi^j \\ &+ \partial_\beta u\,\Omega^-_{01}(e_2\partial_\alpha - e_1\partial_\beta)\Phi^j \\ &- \partial_\alpha u\,\Omega^-_{02}(e_2\partial_\alpha - e_1\partial_\beta)\Phi^j\}, \end{aligned} \tag{3.82}$$

and

$$
\begin{aligned}
&\partial_\alpha \Gamma \lambda^* \partial_\beta \Gamma^{k-1} \mathfrak{v} - \partial_\beta \Gamma \lambda^* \partial_\alpha \Gamma^{k-1} \mathfrak{v} \\
&\quad = \frac{2}{t} \{ -\partial_t (e_2 \partial_\alpha - e_1 \partial_\beta) \Gamma \lambda^* \Upsilon \Gamma^{k-1} \mathfrak{v} \\
&\qquad + \Omega_{01}^{+} (e_2 \partial_\alpha - e_1 \partial_\beta) \Gamma \lambda^* \partial_\beta \Gamma^{k-1} \mathfrak{v} \\
&\qquad - \Omega_{02}^{+} (e_2 \partial_\alpha - e_1 \partial_\beta) \Gamma \lambda^* \partial_\alpha \Gamma^{k-1} \mathfrak{v} \}.
\end{aligned} \tag{3.83}
$$

Notice that $[\Omega_{01}^{\pm}, e_2\partial_\alpha - e_1\partial_\beta] = \mp e_2 \partial_t$, $[\Omega_{02}^{\pm}, e_2\partial_\alpha - e_1\partial_\beta] = \pm e_1 \partial_t$. Using Propositions 2.16, 2.17 and (2.15), (2.17), we get

$$
\begin{aligned}
&\|(I - \mathcal{H}) \Gamma^j (\partial_\beta u \partial_\alpha \chi - \partial_\alpha u \partial_\beta \chi)(t)\|_2 \\
&\quad \lesssim \frac{1}{t} \left(E_{j+1}^{1/2}(t) + t \sum_{|i| \le j} \|\Gamma^i \mathfrak{P}^- \chi(t)\|_2 \right) \\
&\qquad \times \sum_{\substack{|i| \le [\frac{j}{2}]+2 \\ \partial = \partial_\alpha, \partial_\beta}} (|\partial \Gamma^i \chi(t)|_\infty + |\partial \Gamma^i \mathfrak{v}(t)|_\infty) \\
&\qquad + \frac{1}{t} \left(E_j^{1/2}(t) + t \sum_{|i| \le [\frac{j}{2}]} \|\Gamma^i \mathfrak{P}^- \chi(t)\|_2 \right) \\
&\qquad \times \sum_{\substack{|i| \le j \\ \partial = \partial_\alpha, \partial_\beta}} (|\Gamma^i \partial_t \partial \chi(t)|_\infty + |\Gamma^i \partial u(t)|_\infty).
\end{aligned}
$$

Further applying (3.24) and (2.2), we obtain

$$
\begin{aligned}
&\left\| (I - \mathcal{H}) \Gamma^j (\partial_\beta u \partial_\alpha \chi - \partial_\alpha u \partial_\beta \chi)(t) \right\|_2 \\
&\quad \lesssim E_{j+1}^{1/2}(t) \left(\sum_{\substack{|i| \le [\frac{j}{2}]+2 \\ \partial = \partial_\alpha, \partial_\beta}} (|\partial \Gamma^i \chi(t)|_\infty + |\partial \Gamma^i \mathfrak{v}(t)|_\infty) + \frac{1}{t} \right) \\
&\qquad \times \sum_{\substack{|i| \le [\frac{j}{2}]+2 \\ \partial = \partial_\alpha, \partial_\beta}} (|\partial \Gamma^i \chi(t)|_\infty + |\partial \Gamma^i \mathfrak{v}(t)|_\infty) \\
&\qquad + E_j^{1/2}(t) \left(\sum_{\substack{|i| \le [\frac{j}{2}]+2 \\ \partial = \partial_\alpha, \partial_\beta}} |\partial \Gamma^i \chi(t)|_\infty + |\partial \Gamma^i \mathfrak{v}(t)|_\infty + \frac{1}{t} \right)
\end{aligned}
$$

$$\times \left(\sum_{\substack{|i|\le j+1\\ \partial=\partial_\alpha,\partial_\beta}} |\partial\Gamma^i\chi(t)|_\infty + |\partial\Gamma^i\mathfrak{v}(t)|_\infty + E_{j+3}^{1/2}(t) \right.$$

$$\left. \times \left(\sum_{\substack{|i|\le[\frac{j}{2}]+2\\ \partial=\partial_\alpha,\partial_\beta}} (|\partial\Gamma^i\chi(t)|_\infty + |\partial\Gamma^i\mathfrak{v}(t)|_\infty)(1+\ln t) + \frac{1}{t} \right) \right). \tag{3.84}$$

Similarly,

$$\begin{aligned} &\left\| (\partial_\beta u \partial_\alpha \Phi^j - \partial_\alpha u \partial_\beta \Phi^j)(t) \right\|_2 \\ &\quad \lesssim \sum_{\substack{i=1,2\\ \partial=\partial_\alpha,\partial_\beta}} \frac{1}{t} (\|\Upsilon u(t)\|_2 |\partial_t \partial \Phi^j(t)|_\infty \\ &\qquad + |\partial u(t)|_\infty \|\Omega_{0i}(e_2\partial_\alpha - e_1\partial_\beta)\Phi^j(t)\|_2). \end{aligned}$$

For $\phi = \mathfrak{v}$, $\partial = \partial_\alpha, \partial_\beta$, we know

$$\begin{aligned} \partial_t \partial \Phi^j &= \partial_t \partial (I-\mathcal{H})\Gamma^j \mathfrak{v} = \partial_t (I-\mathcal{H})\partial\Gamma^j\mathfrak{v} - \partial_t[\partial,\mathcal{H}]\Gamma^j\mathfrak{v} \\ &= (I-\mathcal{H})\partial_t\partial\Gamma^j\mathfrak{v} - [\partial_t,\mathcal{H}]\partial\Gamma^j\mathfrak{v} - \partial_t[\partial,\mathcal{H}]\Gamma^j\mathfrak{v}. \end{aligned} \tag{3.85}$$

Therefore using Propositions 2.8, 2.9, 2.16, we have

$$|\partial_t\partial\Phi^j(t)|_\infty \lesssim \sum_{\substack{i\le j+2\\ \partial=\partial_\alpha,\partial_\beta}} \left(|\Gamma^i\partial v(t)|_\infty (1+\ln t) + E_{j+2}^{1/2}(t)\frac{1}{t} \right). \tag{3.86}$$

Using further Propositions 2.16, 2.17, 2.3, we obtain

$$\begin{aligned} &\|(\partial_\beta u \partial_\alpha \Phi^j - \partial_\alpha u \partial_\beta \Phi^j)(t)\|_2 \\ &\quad \lesssim \frac{1}{t} E_j^{1/2}(t) \sum_{\substack{i\le j+2\\ \partial=\partial_\alpha,\partial_\beta}} \left(|\Gamma^i\partial v(t)|_\infty(1+\ln t) + E_{j+2}^{1/2}(t)\frac{1}{t} \right) \\ &\qquad + \sum_{\substack{i\le 2\\ \partial=\partial_\alpha,\partial_\beta}} (|\Gamma^i\partial\chi(t)|_\infty + |\Gamma^i\partial v(t)|_\infty) \\ &\qquad \times \left(\frac{1}{t} E_{j+2}^{1/2}(t) + \|\mathfrak{P}^-\Phi^j(t)\|_2 \right). \end{aligned} \tag{3.87}$$

We know from (3.37), (3.38), (3.41) the estimate of $\|\mathcal{P}^-\Phi^j(t)\|_2 = \|G_j^{\mathfrak{v}}(t)\|_2$. Using further (2.14), Propositions 2.16, 2.17, we have

$$\|\mathfrak{P}^-\Phi^j(t)\|_2 \lesssim E_j^{1/2}(t) \sum_{\substack{i\le[\frac{j}{2}]+2\\ \partial=\partial_\alpha,\partial_\beta}} (|\Gamma^i\partial\chi(t)|_\infty + |\Gamma^i\partial v(t)|_\infty) + \sum_{\substack{i\le 4\\ \partial=\partial_\alpha,\partial_\beta}} (|\Gamma^i\partial\chi(t)|_\infty + |\Gamma^i\partial v(t)|_\infty) E_{j+2}^{1/2}(t). \quad (3.88)$$

Therefore

$$\begin{aligned}&\|(\partial_\beta u\partial_\alpha\Phi^j - \partial_\alpha u\partial_\beta\Phi^j)(t)\|_2\\ &\lesssim \frac{1}{t}E_j^{1/2}(t) \sum_{\substack{i\le j+2\\ \partial=\partial_\alpha,\partial_\beta}} \left(|\Gamma^i\partial v(t)|_\infty(1+\ln t) + E_{j+2}^{1/2}(t)\frac{1}{t}\right)\\ &\quad + \sum_{\substack{i\le 2\\ \partial=\partial_\alpha,\partial_\beta}} (|\Gamma^i\partial\chi(t)|_\infty + |\Gamma^i\partial v(t)|_\infty) E_{j+2}^{1/2}(t)\\ &\quad \times \left(\frac{1}{t} + \sum_{\substack{i\le[\frac{j}{2}]+2\\ \partial=\partial_\alpha,\partial_\beta}} |\Gamma^i\partial\chi(t)|_\infty + |\Gamma^i\partial v(t)|_\infty\right). \end{aligned}\quad (3.89)$$

The estimate for $\|(I-\mathcal{H})\Gamma^{j-k}\{\partial_\beta\Gamma\lambda^*\partial_\alpha\Gamma^{k-1}\mathfrak{v} - \partial_\alpha\Gamma\lambda^*\partial_\beta\Gamma^{k-1}\mathfrak{v}\}(t)\|_2$ is similar: we have from (3.83) that for $k=1,\dots,j$,

$$\begin{aligned}&\|(I-\mathcal{H})\Gamma^{j-k}\{\partial_\beta\Gamma\lambda^*\partial_\alpha\Gamma^{k-1}\mathfrak{v} - \partial_\alpha\Gamma\lambda^*\partial_\beta\Gamma^{k-1}\mathfrak{v}\}(t)\|_2\\ &\lesssim \frac{1}{t}E_j^{1/2}(t) \sum_{\substack{i\le j+1\\ \partial=\partial_\alpha,\partial_\beta}} |\Gamma^i\partial\lambda^*(t)|_\infty\\ &\quad + \frac{1}{t}\left(E_{j+1}^{1/2}(t) + t\sum_{|i|\le j-1}\|\Gamma^i\mathfrak{P}^-\Gamma\lambda^*(t)\|_2\right) \sum_{\substack{|i|\le[\frac{j}{2}]+2\\ \partial=\partial_\alpha,\partial_\beta}} |\partial\Gamma^i\mathfrak{v}(t)|_\infty\\ &\quad + \frac{1}{t}\left(E_j^{1/2}(t) + t\sum_{|i|\le[\frac{j}{2}]-1}\|\Gamma^i\mathfrak{P}^-\Gamma\lambda^*(t)\|_2\right) \sum_{\substack{|i|\le j-1\\ \partial=\partial_\alpha,\partial_\beta}} |\Gamma^i\partial\mathfrak{v}(t)|_\infty.\end{aligned}$$

Notice that

$$\mathfrak{P}^+\Gamma\lambda^* = (\mathfrak{P}^+ - \mathcal{P}^+)\Gamma\lambda^* + [\mathcal{P}^+,\Gamma]\lambda^* + \Gamma\mathcal{P}^+\lambda^*.$$

Using (2.40), (2.14), (2.5), Propositions 2.6, 2.7, 2.8, 2.9, 2.16, 2.17, we get for $k \le j-1$,

$$\begin{aligned} &\|\Gamma^k \mathfrak{P}^+ \Gamma\lambda^*(t)\|_2 \\ &\quad\lesssim E_{k+2}^{1/2}(t)\Bigg(\sum_{\substack{i\le[\frac{j}{2}]+2\\ \partial=\partial_\alpha,\partial_\beta}} (|\Gamma^i\partial\chi(t)|_\infty + |\Gamma^i\partial v(t)|_\infty)(1+\ln t) + E_j^{1/2}(t)\frac{1}{t}\Bigg). \end{aligned} \tag{3.90}$$

Therefore

$$\begin{aligned} &\|(I-\mathcal{H})\Gamma^{j-k}\{\partial_\beta\Gamma\lambda^*\partial_\alpha\Gamma^{k-1}\mathfrak{v} - \partial_\alpha\Gamma\lambda^*\partial_\beta\Gamma^{k-1}\mathfrak{v}\}(t)\|_2 \\ &\quad\lesssim \frac{1}{t}E_j^{1/2}(t)\Bigg\{\sum_{\substack{i\le j+1\\ \partial=\partial_\alpha,\partial_\beta}} |\Gamma^i\partial\chi(t)|_\infty + E_{j+3}^{1/2}(t) \\ &\qquad\times\Bigg(\sum_{\substack{i\le[\frac{j+1}{2}]+2\\ \partial=\partial_\alpha,\partial_\beta}} |\Gamma^i\partial\chi(t)|_\infty(1+\ln t) + \frac{1}{t}\Bigg)\Bigg\} \\ &\qquad+\Bigg(\sum_{\substack{i\le[\frac{j}{2}]+2\\ \partial=\partial_\alpha,\partial_\beta}} (|\Gamma^i\partial\chi(t)|_\infty + |\Gamma^i\partial v(t)|_\infty)(1+\ln t) + \frac{1}{t}\Bigg) \\ &\qquad\times\Bigg(E_j^{1/2}\sum_{\substack{|i|\le j-1\\ \partial=\partial_\alpha,\partial_\beta}} |\Gamma^i\partial\mathfrak{v}(t)|_\infty + E_{j+1}^{1/2}\sum_{\substack{|i|\le[\frac{j}{2}]+2\\ \partial=\partial_\alpha,\partial_\beta}} |\Gamma^i\partial\mathfrak{v}(t)|_\infty\Bigg). \end{aligned} \tag{3.91}$$

Step 6. Finally, we consider the term $\iint\{\Phi^j\cdot(u_\beta\Phi_\alpha^j - u_\alpha\Phi_\beta^j)\}\,d\alpha\,d\beta$ for $\phi=\chi$ in (3.48), and the remaining terms

$$\begin{aligned} &\Gamma^{j-k}\{\partial_\beta\Gamma^i\lambda^*\partial_\alpha\Gamma^{k-1}\chi - \partial_\alpha\Gamma^i\lambda^*\partial_\beta\Gamma^{k-1}\chi\}, \\ &\Gamma^{j-k}\{\partial_\beta\lambda^* e_3\partial_\alpha\Gamma^{k-1}\chi - \partial_\alpha\lambda^* e_3\partial_\beta\Gamma^{k-1}\chi\}, \quad i=0,1,\ k=1,\dots,j \end{aligned} \tag{3.92}$$

in (3.80). Without loss of generality, among terms in (3.92), we will only write for

$$\Gamma^{j-k}\{\partial_\beta\Gamma\lambda^*\partial_\alpha\Gamma^{k-1}\chi - \partial_\alpha\Gamma\lambda^*\partial_\beta\Gamma^{k-1}\chi\}.$$

Notice that the ideas as that in (3.82), (3.83) doesn't work here, since we do not have estimates for $\|\Phi^j(t)\|_2$ for $\phi=\chi$, and for $\|\Upsilon\Gamma^{k-1}\chi(t)\|_2$. We resolve these issue by using commutators.

We first consider $\iint\{\Phi^j\cdot(u_\beta\Phi^j_\alpha-u_\alpha\Phi^j_\beta)\}\,d\alpha\,d\beta$ for $\phi=\chi$. Using integration by parts, we have

$$\iint\{\Phi^j\cdot(u_\beta\Phi^j_\alpha-u_\alpha\Phi^j_\beta)\}\,d\alpha\,d\beta=-\iint\{\Phi^j_\beta\cdot(u\Phi^j_\alpha)-\Phi^j_\alpha\cdot(u\Phi^j_\beta)\}\,d\alpha\,d\beta.$$

Now for $\phi=\chi$, we have, by using Proposition 2.12,

$$\begin{aligned}&\iint\{\partial_\alpha(I-\mathcal{H})\Gamma^j\chi\cdot(u\Phi^j_\beta)-\partial_\beta(I-\mathcal{H})\Gamma^j\chi\cdot(u\Phi^j_\alpha)\}\,d\alpha\,d\beta\\&\quad=-\iint\{[\partial_\alpha,\mathcal{H}]\Gamma^j\chi\cdot(u\Phi^j_\beta)-[\partial_\beta,\mathcal{H}]\Gamma^j\chi\cdot(u\Phi^j_\alpha)\}\,d\alpha\,d\beta\\&\qquad+\iint\{(\mathcal{H}^*-\mathcal{H})\partial_\alpha\Gamma^j\chi\cdot(u\Phi^j_\beta)-(\mathcal{H}^*-\mathcal{H})\partial_\beta\Gamma^j\chi\cdot(u\Phi^j_\alpha)\}\,d\alpha\,d\beta\\&\qquad+\iint\{\partial_\alpha\Gamma^j\chi\cdot((I-\mathcal{H})(u\Phi^j_\beta))-\partial_\beta\Gamma^j\chi\cdot((I-\mathcal{H})(u\Phi^j_\alpha))\}\,d\alpha\,d\beta.\end{aligned}\tag{3.93}$$

We further rewrite the term $\iint\{\partial_\alpha\Gamma^j\chi\cdot((I-\mathcal{H})(u\Phi^j_\beta))-\partial_\beta\Gamma^j\chi\cdot((I-\mathcal{H})(u\Phi^j_\alpha))\}\,d\alpha\,d\beta$. We know $\mathcal{H}u=u$. Let Φ^j_i be the e_i component of Φ^j, for $i=0,\ldots,3$. We have

$$\begin{aligned}&\partial_\alpha\Gamma^j\chi\cdot\{(I-\mathcal{H})(u\,\partial_\beta\Phi^j_i e_i)\}-\partial_\beta\Gamma^j\chi\cdot\{(I-\mathcal{H})(u\,\partial_\alpha\Phi^j_i e_i)\}\\&\quad=\partial_\alpha\Gamma^j\chi\cdot([\partial_\beta\Phi^j_i,\mathcal{H}]ue_i)-\partial_\beta\Gamma^j\chi\cdot([\partial_\alpha\Phi^j_i,\mathcal{H}]ue_i)\\&\quad=\frac{2}{t}\partial_t(e_2\partial_\alpha-e_1\partial_\beta)\Gamma^j\chi\cdot\{[\Upsilon\Phi^j_i,\mathcal{H}]ue_i-[\alpha,\mathcal{H}](u\partial_\beta\Phi^j_i e_i)\\&\qquad+[\beta,\mathcal{H}](u\partial_\alpha\Phi^j_i e_i)\}+\frac{2}{t}\{\Omega^-_{01}(e_2\partial_\alpha-e_1\partial_\beta)\Gamma^j\chi\cdot[\partial_\beta\Phi^j_i,\mathcal{H}]ue_i\\&\qquad-\Omega^-_{02}(e_2\partial_\alpha-e_1\partial_\beta)\Gamma^j\chi\cdot[\partial_\alpha\Phi^j_i,\mathcal{H}]ue_i\}.\end{aligned}\tag{3.94}$$

Further applying integration by parts gives us:

$$\begin{aligned}&\iint\partial_t(e_2\partial_\alpha-e_1\partial_\beta)\Gamma^j\chi\cdot\{[\Upsilon\Phi^j_i,\mathcal{H}]ue_i-[\alpha,\mathcal{H}](u\partial_\beta\Phi^j_i e_i)\\&\qquad+[\beta,\mathcal{H}](u\partial_\alpha\Phi^j_i e_i)\}\,d\alpha\,d\beta\\&\quad=-\iint e_2\partial_t\Gamma^j\chi\cdot\partial_\alpha\{[\Upsilon\Phi^j_i,\mathcal{H}]ue_i-[\alpha,\mathcal{H}](u\partial_\beta\Phi^j_i e_i)\\&\qquad+[\beta,\mathcal{H}](u\partial_\alpha\Phi^j_i e_i)\}\,d\alpha\,d\beta\end{aligned}$$

$$+\iint e_1\partial_t\Gamma^j\chi\cdot\partial_\beta\{[\Upsilon\Phi_i^j,\mathcal{H}]ue_i-[\alpha,\mathcal{H}](u\partial_\beta\Phi_i^j e_i)$$
$$+[\beta,\mathcal{H}](u\partial_\alpha\Phi_i^j e_i)\}\,d\alpha\,d\beta. \tag{3.95}$$

Through (3.93)–(3.95) we have put $\iint\{\Phi^j\cdot(u_\beta\Phi_\alpha^j-u_\alpha\Phi_\beta^j)\}\,d\alpha\,d\beta$ for $\phi=\chi$ in the right form for estimates. Using Propositions 2.6, 2.16, Lemma 2.3, we obtain

$$\left|\iint\{\Phi^j\cdot(u_\beta\Phi_\alpha^j-u_\alpha\Phi_\beta^j)\}\,d\alpha\,d\beta\right|$$
$$\lesssim\sum_{\partial=\partial_\alpha,\partial_\beta}\{(|\partial\lambda(t)|_\infty+|\partial\mathfrak{z}(t)|_\infty)E_j(t)|\partial\Phi^j(t)|_\infty$$
$$+\frac{1}{t}E_j^{1/2}(t)E_{j+1}^{1/2}(t)|\partial\Phi^j(t)|_\infty+\|\mathfrak{P}^-\Gamma^j\chi(t)\|_2E_j^{1/2}(t)|\partial\Phi^j(t)|_\infty$$
$$+\frac{1}{t}E_j(t)|\partial\Upsilon\Phi^j(t)|_\infty\}.$$

We know for $\phi=\chi$, $\partial=\partial_\alpha,\partial_\beta$, $\partial\Phi^j=(I-\mathcal{H})\partial\Gamma^j\chi-[\partial,\mathcal{H}]\Gamma^j\chi$. Using Proposition 2.8, 2.9, 2.16, we have

$$|\partial\Phi^j(t)|_\infty+|\Upsilon\partial\Phi^j(t)|_\infty\lesssim\sum_{i\le j+2}|\Gamma^i\partial\chi(t)|_\infty(1+\ln t)+E_{j+1}^{1/2}(t)\frac{1}{t}.$$

Therefore by further using (3.24), we arrive at

$$\left|\iint\{\Phi^j\cdot(u_\beta\Phi_\alpha^j-u_\alpha\Phi_\beta^j)\}\,d\alpha\,d\beta\right|$$
$$\lesssim\left(\sum_{i\le j+2}|\Gamma^i\partial\chi(t)|_\infty(1+\ln t)+E_{j+1}^{1/2}(t)\frac{1}{t}\right)$$
$$\times E_j^{1/2}E_{j+1}^{1/2}\left(\sum_{\substack{|i|\le[\frac{j}{2}]+2\\ \partial=\partial_\alpha,\partial_\beta}}(|\Gamma^i\partial\chi(t)|_\infty+|\Gamma^j\partial\mathfrak{v}(t)|_\infty)+\frac{1}{t}\right). \tag{3.96}$$

At last we give the estimate of $\|(I-\mathcal{H})(\Gamma^{j-k}\{\partial_\beta\Gamma\lambda^*\partial_\alpha\Gamma^{k-1}\chi-\partial_\alpha\Gamma\lambda^*\partial_\beta\Gamma^{k-1}\chi\})(t)\|_2$ for $k=1,\dots,j$. We first rewrite,

$$(I-\mathcal{H})(\Gamma^{j-k}\{\partial_\beta\Gamma\lambda^*\partial_\alpha\Gamma^{k-1}\chi-\partial_\alpha\Gamma\lambda^*\partial_\beta\Gamma^{k-1}\chi\})$$
$$=\Gamma^{j-k}(I-\mathcal{H})\{\partial_\beta\Gamma\lambda^*\partial_\alpha\Gamma^{k-1}\chi-\partial_\alpha\Gamma\lambda^*\partial_\beta\Gamma^{k-1}\chi\}$$
$$+[\Gamma^{j-k},\mathcal{H}]\{\partial_\beta\Gamma\lambda^*\partial_\alpha\Gamma^{k-1}\chi-\partial_\alpha\Gamma\lambda^*\partial_\beta\Gamma^{k-1}\chi\}. \tag{3.97}$$

Let $\Gamma^{k-1}\chi_i$ be the e_i component of $\Gamma^{k-1}\chi$. Using the fact $\mathcal{H}\lambda^* = \lambda^*$, we rewrite further

$$\begin{aligned}
&(I-\mathcal{H})\{\partial_\beta\Gamma\lambda^*\partial_\alpha\Gamma^{k-1}\chi_i - \partial_\alpha\Gamma\lambda^*\partial_\beta\Gamma^{k-1}\chi_i\}\\
&\quad = \partial_\alpha\Gamma^{k-1}\chi_i[\partial_\beta\Gamma,\mathcal{H}]\lambda^* - \partial_\beta\Gamma^{k-1}\chi_i[\partial_\alpha\Gamma,\mathcal{H}]\lambda^*\\
&\qquad + [\partial_\alpha\Gamma^{k-1}\chi_i,\mathcal{H}-\mathcal{H}^*]\partial_\beta\Gamma\lambda^* - [\partial_\beta\Gamma^{k-1}\chi_i,\mathcal{H}-\mathcal{H}^*]\partial_\alpha\Gamma\lambda^*\\
&\qquad + [\partial_\alpha\Gamma^{k-1}\chi_i,\mathcal{H}^*]\partial_\beta\Gamma\lambda^* - [\partial_\beta\Gamma^{k-1}\chi_i,\mathcal{H}^*]\partial_\alpha\Gamma\lambda^* \qquad (3.98)
\end{aligned}$$

and in which we rewrite $[\partial_\alpha\Gamma^{k-1}\chi_i,\mathcal{H}^*]\partial_\beta\Gamma\lambda^* - [\partial_\beta\Gamma^{k-1}\chi_i,\mathcal{H}^*]\partial_\alpha\Gamma\lambda^*$ using the idea of (2.28):

$$\begin{aligned}
&-\frac{t}{2}\{[\partial_\alpha\Gamma^{k-1}\chi_i,\mathcal{H}^*]\partial_\beta\Gamma\lambda^* - [\partial_\beta\Gamma^{k-1}\chi_i,\mathcal{H}^*]\partial_\alpha\Gamma\lambda^*\}\\
&\quad = (-[\Upsilon\Gamma^{k-1}\chi_i,\mathcal{H}^*] + \partial_\beta\Gamma^{k-1}\chi_i[\alpha,\mathcal{H}^*]\\
&\qquad - \partial_\alpha\Gamma^{k-1}\chi_i[\beta,\mathcal{H}^*])\partial_t(e_2\partial_\alpha - e_1\partial_\beta)\Gamma\lambda^*\\
&\qquad + [\partial_\beta\Gamma^{k-1}\chi_i,\mathcal{H}^*]\Omega_{01}^+(e_2\partial_\alpha - e_1\partial_\beta)\Gamma\lambda^*\\
&\qquad - [\partial_\alpha\Gamma^{k-1}\chi_i,\mathcal{H}^*]\Omega_{02}^+(e_2\partial_\alpha - e_1\partial_\beta)\Gamma\lambda^*\}. \qquad (3.99)
\end{aligned}$$

Using (3.97)–(3.99), and Propositions 2.6, 2.7, 2.16, 2.17, we have

$$\begin{aligned}
&\|(I-\mathcal{H})(\Gamma^{j-k}\{\partial_\beta\Gamma\lambda^*\partial_\alpha\Gamma^{k-1}\chi - \partial_\alpha\Gamma\lambda^*\partial_\beta\Gamma^{k-1}\chi\})(t)\|_2\\
&\lesssim E_j^{1/2}(t)\sum_{|i|\le[\frac{j}{2}]+2}\ \sum_{\partial=\partial_\alpha,\partial_\beta}|\Gamma^i\partial\chi(t)|_\infty\\
&\quad\times\Bigg(\sum_{\substack{|i|\le[\frac{j}{2}]+2\\ \partial=\partial_\alpha,\partial_\beta}}|\Gamma^i\partial\chi(t)|_\infty(1+\ln t) + E_j^{1/2}(t)\frac{1}{t}\Bigg)\\
&\quad+\frac{1}{t}E_j^{1/2}(t)\sum_{\substack{i\le j\\ \partial=\partial_\alpha,\partial_\beta}}|\Gamma^i\partial\chi(t)|_\infty\\
&\quad+\frac{1}{t}\Bigg(E_{j+1}^{1/2}(t) + t\sum_{|i|\le j-1}\|\Gamma^i\mathfrak{P}^-\Gamma\lambda^*(t)\|_2\Bigg)\sum_{\substack{|i|\le[\frac{j}{2}]+2\\ \partial=\partial_\alpha,\partial_\beta}}|\partial\Gamma^i\chi(t)|_\infty\\
&\quad+\frac{1}{t}\Bigg(E_j^{1/2}(t) + t\sum_{|i|\le[\frac{j}{2}]-1}\|\Gamma^i\mathfrak{P}^-\Gamma\lambda^*(t)\|_2\Bigg)\sum_{\substack{|i|\le j-1\\ \partial=\partial_\alpha,\partial_\beta}}|\Gamma^i\partial\chi(t)|_\infty.
\end{aligned}$$

Further using (3.90), we arrive at

$$\begin{aligned}
&\|(I-\mathcal{H})(\Gamma^{j-k}\{\partial_\beta\Gamma\lambda^*\partial_\alpha\Gamma^{k-1}\chi-\partial_\alpha\Gamma\lambda^*\partial_\beta\Gamma^{k-1}\chi\})(t)\|_2\\
&\lesssim\Bigg(\sum_{\substack{i\le[\frac{j}{2}]+2\\ \partial=\partial_\alpha,\partial_\beta}}(|\Gamma^i\partial\chi(t)|_\infty+|\Gamma^i\partial v(t)|_\infty)(1+\ln t)+\frac{1}{t}\Bigg)\\
&\quad\times\Bigg(E_j^{1/2}\sum_{\substack{|i|\le j\\ \partial=\partial_\alpha,\partial_\beta}}|\Gamma^i\partial\chi(t)|_\infty+E_{j+1}^{1/2}\sum_{\substack{|i|\le[\frac{j}{2}]+2\\ \partial=\partial_\alpha,\partial_\beta}}|\Gamma^i\partial\chi(t)|_\infty\Bigg).
\end{aligned}\tag{3.100}$$

Combine (3.48), (3.49), (3.53), (3.54), (3.56), (3.63), (3.67), (3.69), (3.70), (3.72), (3.77), (3.78), (3.79), (3.89), (3.91), (3.96), (3.100), notice that for $l\ge 15$, $q\ge l+9$, $[\frac{l}{2}]+3+5\le l+2$ and $l+4\le\min\{2l-11,q-5\}$. Applying (3.15), we obtain (3.47). □

3.3 A conclusive estimate

We now sum up the results in Lemma 3.3, Propositions 3.5, 3.6.

Let $[0,T]$ be the time period when the a-priori assumption 3.13 holds, M_0 be such that furthermore Lemmas 3.3, 3.4, Propositions 3.5, 3.6 hold. Let ϵ, $L>0$. Assume that $\mathfrak{F}_{l+2}(0)\le\epsilon^2$, $\mathcal{F}_{l+3}(0)\le\epsilon^2$, and $\mathcal{F}_{l+9}(0)\le L^2$.

Theorem 3.7 *Let $l\ge 17$, $q\ge l+9$, $l\le n\le l+9$. There exists $\varepsilon_0>0$, depending on M_0, L, such that for $\epsilon\le\varepsilon_0$, we have* 1.

$$\mathcal{F}_n(t)\le\mathcal{F}_n(0)(1+t)^{1/2},\quad \textit{for } t\in[0,T];\tag{3.101}$$

2.

$$\mathfrak{F}_{l+2}^{1/2}(t)\le(cL+1)\epsilon,\quad \textit{for } t\in[0,T],\tag{3.102}$$

where c is a constant depending on M_0.

Proof We know for $l\ge 17$, $l\le n\le l+9\le q$, $5\le[\frac{n}{2}]+2\le\min\{2l-11,q-5\}$ and $[\frac{n}{2}]+7\le l+2$. From Lemmas 3.3, 3.4, Proposition 3.5, we get for $t\in[0,T]$,

$$\frac{d\mathcal{F}_n(t)}{dt}\le c_0(M_0)\frac{E_{l+2}^{1/2}(t)}{1+t}\mathcal{F}_n(t)\le c_1(M_0)\frac{\mathfrak{F}_{l+2}^{1/2}(t)}{1+t}\mathcal{F}_n(t),$$

where $c_0(M_0)$, $c_1(M_0)$ are constants depending on M_0. Therefore

$$\mathcal{F}_n(t)\le\mathcal{F}_n(0)(1+t)^{M_1(\tau)}\quad \text{for } t\in[0,\tau],\ \tau\le T,\tag{3.103}$$

where $M_1(\tau) = c_1(M_0)\sup_{[0,\tau]} \mathfrak{F}_{l+2}^{1/2}(t)$. Applying (3.103), Lemma 3.4 to Proposition 3.6, we obtain,

$$\frac{d\mathfrak{F}_{l+2}(t)}{dt} \le c_2(M_0)\,\epsilon L \mathfrak{F}_{l+2}^{1/2}(t)(1+t)^{M_1(\tau)}\left(\frac{1+\ln(1+t)}{t+1}\right)^2 \quad \text{for } t \in [0,\tau], \tag{3.104}$$

where $c_2(M_0)$ is a constant depending on M_0. Let $\varepsilon_0 = \min\{\frac{1}{4c_1(M_0)}, \frac{1}{2c_1(M_0)(Lc+1)}\}$, where $c = c_2(M_0)\int_0^\infty (1+t)^{-3/2}(1+\ln(1+t))^2\,dt$, and $\epsilon \le \varepsilon_0$. Therefore $c_1(M_0)\mathfrak{F}_{l+2}^{1/2}(0) \le \frac{1}{4}$. Let $0 < T_1 \le T$ be the largest such that $M_1(T_1) \le \frac{1}{2}$. From (3.104) we get

$$\mathfrak{F}_{l+2}^{1/2}(t) \le \frac{1}{2}\epsilon Lc + \epsilon \quad \text{for } t \in [0, T_1].$$

This implies $M_1(T_1) \le c_1(M_0)(\frac{1}{2}Lc+1)\varepsilon_0 < \frac{1}{2}$. So $T_1 = T$ or otherwise T_1 is not the largest. Therefore (3.101), (3.102) holds for $t \in [0, T]$. □

4 Global wellposedness of the 3D full water wave equation

In this section we prove that the 3D full water wave equation (1.1), or equivalently (1.23)–(1.24) is uniquely solvable globally in time for small data. This is achieved by combining a local wellposedness result for the quasilinear system (2.38)–(2.39)–(1.38) and Theorem 3.7.

In what follows all the constants $c(p)$, $c_i(p)$ etc. satisfy $c(p) \le c(p_0)$, $c_i(p) \le c_i(p_0)$ for some $p_0 > 0$ and all $0 \le p \le p_0$.

We first present two lemmas. The first shows that for interface that is a graph small in its steepness (and two more derivatives), the change of coordinate k defined in (1.28) is a diffeomorphism. The second gives the regularity relation on quantities before and after change of coordinates.

Lemma 4.1 *Let $\xi = (\alpha, \beta, z(\alpha,\beta))$, $k = \xi - (I+\mathfrak{H})ze_3 + \mathfrak{K}ze_3$ be defined as in (1.28), $P = (\alpha, \beta)$.*

1. *Assume that $N = \sum_{\substack{|i|\le 2\\ \partial=\partial_\alpha,\partial_\beta}} \|\partial^i \partial z\|_2 < \infty$. Then for $\partial = \partial_\alpha, \partial_\beta$,*

$$|\partial(k-P)|_\infty \le c(N)N \tag{4.1}$$

for some constant $c(N)$ depending on N. In particular, there exists a $N_0 > 0$, such that for $N \le N_0$, we have $|\partial(k-P)|_\infty \le \frac{1}{4}$, $\frac{1}{4} \le J(k) \le 2$, $k: \mathbb{R}^2 \to \mathbb{R}^2$ is a diffeomorphism and

$$\frac{1}{4}(|\alpha-\alpha'| + |\beta-\beta'|) \le |k(\alpha,\beta) - k(\alpha',\beta')| \le 2(|\alpha-\alpha'| + |\beta-\beta'|).$$

2. *Let* $q \geq 5$ *be an integer, and* $\Gamma = \{\partial_\alpha, \partial_\beta, L_0, \varpi\}$. *Assume* $\sum_{\substack{|j|\leq q-1\\ \partial=\partial_\alpha,\partial_\beta}} \|\Gamma^j \partial z\|_{H^{1/2}} = L < \infty$. *Then*

$$\sum_{\substack{|j|\leq q-1\\ \partial=\partial_\alpha,\partial_\beta}} \|\Gamma^j \partial(k-P)\|_{H^{1/2}} \leq c(L)L \tag{4.2}$$

for some constant $c(L)$ *depending on* L.

Proof Notice that for $P = (\alpha, \beta)$, $k - P = -\mathfrak{H}ze_3 + \mathfrak{K}ze_3$, and for $\partial = \partial_\alpha$, ∂_β,

$$\partial(k-P) = -[\partial, \mathfrak{H}]ze_3 - \mathfrak{H}\partial ze_3 + [\partial, \mathfrak{K}]ze_3 + \mathfrak{K}\partial ze_3.$$

(4.1) follows directly from applying Lemma 1.2, (2.6), (2.7), Propositions 2.6, 2.7, 2.9; the inequality (4.2) follows with a further application of Lemma 6.2 of [31] and interpolation. The rest of the statements in Lemma 4.1 part 1 follows straightforwardly from (4.1). □

Lemma 4.2 *Let* $q \geq 5$ *be an integer,* $0 < T < \infty$. *Assume that for each* $t \in [0, T]$, $k(\cdot, t) : \mathbb{R}^2 \to \mathbb{R}^2$ *is a diffeomorphism, and there are constants* $0 < c_1, c_2, \mu_1, \mu_2 < \infty$, *such that* $\mu_1 \leq J(k(t)) \leq \mu_2$ *and* $c_1|(\alpha, \beta) - (\alpha', \beta')| \leq |k(\alpha, \beta, t) - k(\alpha', \beta', t)| \leq c_2|(\alpha, \beta) - (\alpha', \beta')|$ *for* $(\alpha, \beta), (\alpha', \beta') \in \mathbb{R}^2$, $t \in [0, T]$. *Let* $s = 0$ *or* $\frac{1}{2}$.

1. *Let* $\Gamma = \{\partial_\alpha, \partial_\beta, L_0, \varpi\}$, *and assume* $\sum_{\substack{|j|\leq q-1\\ \partial=\partial_\alpha,\partial_\beta}} \|\Gamma^j \partial(k-P)(0)\|_{H^s} \leq L < \infty$. *Then*

$$\sum_{|j|\leq q} \|\Gamma^j (f \circ k)(0)\|_{H^s} \leq c(L) \sum_{|j|\leq q} \|\Gamma^j (f)(0)\|_{H^s},$$

$$\sum_{|j|\leq q} \|\Gamma^j (f \circ k^{-1})(0)\|_{H^s} \leq c(L) \sum_{|j|\leq q} \|\Gamma^j (f)(0)\|_{H^s}.$$

2. *Let* $\Gamma = \{\partial_t, \partial_\alpha, \partial_\beta, L_0, \varpi\}$. *Assume for* $t \in [0, T]$, $\sum_{\substack{|j|\leq q-1\\ \partial=\partial_\alpha,\partial_\beta}} \|\Gamma^j \partial(k - P)(t)\|_{H^s} \leq L$, $\sum_{|j|\leq q-1} \|\Gamma^j k_t(t)\|_{H^s} \leq L$, $L < \infty$. *Then for* $t \in [0, T]$,

$$\sum_{|j|\leq q} \|\Gamma^j (f \circ k)(t)\|_{H^s} \leq c(L) \sum_{|j|\leq q} \|\Gamma^j (f)(t)\|_{H^s},$$

$$\sum_{|j|\leq q} \|\Gamma^j (f \circ k^{-1})(t)\|_{H^s} \leq c(L) \sum_{|j|\leq q} \|\Gamma^j (f)(t)\|_{H^s}.$$

Here $c(L)$ *is a constant depending on* L *and* c_1, c_2, μ_1, μ_2, *and need not be the same in different contexts.*

Proof The proof of Lemma 4.2 is similar to that of Lemma 5.4 in [32]. The main difference is here we use the relations

$$\alpha\partial_\beta f-\beta\partial_\alpha f=\Upsilon f,\qquad \alpha\partial_\alpha f+\beta\partial_\beta f=L_0 f-\frac{1}{2}t\partial_t f$$

to derive that

$$\nabla_\perp f=\frac{(-\beta,\alpha)}{\alpha^2+\beta^2}\Upsilon f+\frac{(\alpha,\beta)}{\alpha^2+\beta^2}\left(L_0 f-\frac{1}{2}t\partial_t f\right)$$

and for $\Gamma=\varpi,\ L_0$, we use the identities

$$\begin{aligned}\Upsilon(f\circ k^{-1})&=\partial_\beta k^{-1}\cdot(\alpha\nabla f\circ k^{-1})-\partial_\alpha k^{-1}\cdot(\beta\nabla f\circ k^{-1}),\\ L_0(f\circ k^{-1})&=\partial_\alpha k^{-1}\cdot(\alpha\nabla f\circ k^{-1})+\partial_\beta k^{-1}\cdot(\beta\nabla f\circ k^{-1})\\&\quad+\frac{1}{2}t\partial_t(f\circ k^{-1}).\end{aligned}$$

The proof follows an inductive argument similar to that of Lemma 5.4 of [32], and in the case $s=\frac{1}{2}$, the proof further uses Lemma 6.2 of [31] and interpolation. We omit the details. □

We now present a local well-posedness result. Similar to (5.21)–(5.22) in [31], we first rewrite the quasilinear system (2.38)–(2.39)–(1.38) in a format for which local wellposedness can be proved by using energy estimates and iterative scheme. Let $\mathbf{n}=\frac{\mathcal{N}}{|\mathcal{N}|}$. From (1.23) we know $\mathbf{n}=\tilde{\mathbf{n}}=\frac{w+e_3}{|w+e_3|}$. From (1.23)–(1.24), and the fact that $A>0$ for nonself-intersecting interface (i.e. the Taylor sign condition holds, see [31]), we know $A|\mathcal{N}|=|w+e_3|$ and $(A\mathcal{N}\times\nabla)u=|w+e_3|\mathbf{n}\cdot\nabla_\xi^+ u$. Let $f=(I-\mathcal{H})(U_k^{-1}(\mathfrak{a}_t N))$. From the fact that $\mathfrak{a}_t$ is real valued, we know $U_k^{-1}(\mathfrak{a}_t N)=-\tilde{\mathbf{n}}(I+\tilde{\mathcal{K}^*})^{-1}(\mathrm{Re}\{\tilde{\mathbf{n}}f\})$, where $\tilde{\mathcal{K}^*}=\mathrm{Re}\,\tilde{\mathbf{n}}\mathcal{H}\tilde{\mathbf{n}}$. Therefore (2.38)–(2.39)–(1.38) can be rewritten as the following:

$$(\partial_t+b\cdot\nabla_\perp)^2u+a\mathbf{n}\cdot\nabla_\xi^+u=-\tilde{\mathbf{n}}(I+\tilde{\mathcal{K}^*})^{-1}(\mathrm{Re}\{\tilde{\mathbf{n}}f\}),\tag{4.3}$$

where

$$\begin{aligned}&a=|w+e_3|,\qquad \tilde{\mathbf{n}}=\frac{w+e_3}{|w+e_3|},\qquad w=(\partial_t+b\cdot\nabla_\perp)u,\\ &b=\frac{1}{2}(\mathcal{H}-\overline{\mathcal{H}})\overline{u}-[\partial_t+b\cdot\nabla_\perp,\mathcal{H}]\mathfrak{z}e_3+[\partial_t+b\cdot\nabla_\perp,\mathcal{K}]\mathfrak{z}e_3+\mathcal{K}u_3e_3,\\ &(\partial_t+b\cdot\nabla_\perp)\zeta=u,\qquad \mathfrak{U}=\frac{1}{2}(u+\mathcal{H}u),\end{aligned}$$

$$f = 2 \iint K(\zeta' - \zeta)(w - w') \times (\zeta'_{\beta'} \partial_{\alpha'} - \zeta'_{\alpha'} \partial_{\beta'}) \mathfrak{U}' \, d\alpha' d\beta' \tag{4.4}$$
$$+ \iint K(\zeta' - \zeta) \{((u - u') \times u'_{\beta'}) \mathfrak{U}'_{\alpha'} - ((u - u') \times u'_{\alpha'}) \mathfrak{U}'_{\beta'}\} \, d\alpha' d\beta'$$
$$+ 2 \iint K(\zeta' - \zeta)(u - u') \times (\zeta'_{\beta'} \partial_{\alpha'} - \zeta'_{\alpha'} \partial_{\beta'})(\partial'_t + b' \cdot \nabla'_\perp) \mathfrak{U}' \, d\alpha' d\beta'$$
$$+ \iint ((u' - u) \cdot \nabla) K(\zeta' - \zeta)(u - u') \times (\zeta'_{\beta'} \partial'_\alpha - \zeta'_{\alpha'} \partial'_\beta) \mathfrak{U}' \, d\alpha' d\beta'.$$

(4.3)–(4.4) is a well-defined quasilinear system. We give in the following the initial data for (4.3)–(4.4). As we know the initial data describing the water wave motion should satisfy the compatibility conditions given on pp. 464–465 of [31].

Assume that the initial interface $\Sigma(0)$ separates $\mathbb{R}^3$ into two simply connected, unbounded C^2 domains, $\Sigma(0)$ approaches the xy-plane at infinity, and assume that the water occupies the lower region $\Omega(0)$. Take a parameterization for $\Sigma(0) : \xi^0 = \xi^0(\alpha, \beta), (\alpha, \beta) \in \mathbb{R}^2$, such that $N_0 = \xi^0_\alpha \times \xi^0_\beta$ is an outward normal of $\Omega(0)$, $|\xi^0_\alpha \times \xi^0_\beta| \geq \mu$, and $|\xi^0(\alpha, \beta) - \xi^0(\alpha', \beta')| \geq C_0 |(\alpha, \beta) - (\alpha', \beta')|$ for $(\alpha, \beta), (\alpha', \beta') \in \mathbb{R}^2$ and some constants $\mu,\ C_0 > 0$. Let

$$\begin{aligned} &\xi(\alpha, \beta, 0) = (x^0, y^0, z^0) = \xi^0(\alpha, \beta), \qquad \xi_t(\alpha, \beta, 0) = \mathfrak{u}^0(\alpha, \beta), \\ &\xi_{tt}(\alpha, \beta, 0) = \mathfrak{w}^0(\alpha, \beta). \end{aligned} \tag{4.5}$$

Assume that the data in (4.5) satisfy the compatibility conditions (5.29)–(5.30) of [31], that is $\mathfrak{u}^0 = \mathfrak{H}_{\Sigma(0)} \mathfrak{u}^0$, and

$$\mathfrak{w}^0 = -e_3 + (\mathbf{n}_0 \cdot e_3)\mathbf{n}_0 - \mathbf{n}_0 (I + \mathcal{K}_0^*)^{-1} (\operatorname{Re} \mathbf{n}_0 [\partial_t, \mathfrak{H}_{\Sigma(0)}] \mathfrak{u}_0$$
$$+ \mathfrak{H}^*_{\Sigma(0)} (\mathbf{n}_0 \times e_3)\}), \tag{4.6}$$

where $\mathbf{n}_0 = \frac{N_0}{|N_0|}$, $\mathfrak{H}^*_{\Sigma(0)} = \mathbf{n}_0 \mathfrak{H}_{\Sigma(0)} \mathbf{n}_0$ and $\mathcal{K}_0^* = \operatorname{Re} \mathfrak{H}^*_{\Sigma(0)}$. Assume that $k(0) = k_0 = \xi^0 - (I + \mathfrak{H}_{\Sigma(0)}) z^0 e_3 + \mathfrak{K}^0 z^0 e_3$, where $\mathfrak{K}^0 = \operatorname{Re} \mathfrak{H}_{\Sigma(0)}$, as defined in (1.28) is a diffeomorphism with its Jacobian $\nu_1 \leq J(k_0) \leq \nu_2$, and $c_1 |(\alpha, \beta) - (\alpha', \beta')| \leq |k_0(\alpha, \beta) - k_0(\alpha', \beta')| \leq c_2 |(\alpha, \beta) - (\alpha', \beta')|$ for $(\alpha, \beta), (\alpha', \beta') \in \mathbb{R}^2$ and some constants $0 < \nu_1,\ \nu_2,\ c_1,\ c_2 < \infty$. Let

$$\begin{aligned} &\zeta(\cdot, 0) = \zeta^0(\cdot) = P + \lambda^0(\cdot), \qquad u(\cdot, 0) = u^0(\cdot), \\ &(\partial_t + b \cdot \nabla_\perp) u(\cdot, 0) = w^0(\cdot), \end{aligned} \tag{4.7}$$

where

$$\lambda^0(\cdot) = \xi^0 \circ k_0^{-1}(\cdot) - P, \qquad u^0(\cdot) = \mathfrak{u}^0 \circ k_0^{-1}(\cdot),$$
$$w^0(\cdot) = \mathfrak{w}^0 \circ k_0^{-1}(\cdot). \tag{4.8}$$

Let $s \geq 5$ be an integer. Assume that for $\Gamma = \partial_\alpha,\ \partial_\beta,\ L_0,\ \varpi$,

$$\sum_{\substack{|j|\leq s-1 \\ \partial=\partial_\alpha,\partial_\beta}} \|\Gamma^j \partial \lambda^0\|_{H^{1/2}} + \|\Gamma^j u^0\|_{H^{1/2}} + \|\Gamma^j \partial u^0\|_{H^{1/2}}$$
$$+ \|\Gamma^j w^0\|_{L^2} + \|\Gamma^j \partial w^0\|_{L^2} < \infty. \tag{4.9}$$

We have the following local well-posedness result for the initial value problem (4.3)–(4.4)–(4.7) with a non-blow-up criteria.

Theorem 4.3 (Local existence) 1. *There exists $T > 0$, depending on the norm of the initial data, so that the initial value problem (4.3)–(4.4)–(4.7) has a unique solution $(u, \zeta) = (u(\alpha, \beta, t), \zeta(\alpha, \beta, t))$ for $t \in [0, T]$, satisfying for $|j| \leq s - 1$, $\Gamma = \partial_\alpha, \partial_\beta, L_0, \varpi$, $\partial = \partial_\alpha, \partial_\beta$,*

$$\Gamma^j \partial(\zeta - P), \Gamma^j u, \Gamma^j \partial u \in C([0, T], H^{1/2}(\mathbb{R}^2)),$$
$$\Gamma^j w, \Gamma^j \partial w \in C([0, T], L^2(\mathbb{R}^2)), \tag{4.10}$$

and $|\zeta_\alpha \times \zeta_\beta| \geq \nu$, $|\zeta(\alpha, \beta, t) - \zeta(\alpha', \beta', t)| \geq C_1|(\alpha, \beta) - (\alpha', \beta')|$ for all $(\alpha, \beta), (\alpha', \beta') \in \mathbb{R}^2$ and $t \in [0, T]$, for some constants $C_1,\ \nu > 0$.

Moreover, if T^* is the supremum over all such times T, then either $T^* = \infty$ or

$$\sum_{|j|\leq[\frac{s}{2}]+3} \|\Gamma^j w(t)\|_{L^2} + \|\Gamma^j u(t)\|_{L^2}$$
$$+ \sup_{(\alpha,\beta)\neq(\alpha',\beta')} \frac{|(\alpha, \beta) - (\alpha', \beta')|}{|\zeta(\alpha, \beta, t) - \zeta(\alpha', \beta', t)|}$$
$$+ \left|\frac{1}{|(\zeta_\alpha \times \zeta_\beta)(t)|}\right|_{L^\infty} \notin L^\infty[0, T^*). \tag{4.11}$$

2. *Let $P : \mathbb{R}^2 \to \mathbb{R}^2$ be the identity map: $P(\alpha, \beta) = (\alpha, \beta)$ for $(\alpha, \beta) \in \mathbb{R}^2$,*

$$h_t(\cdot, t) = b(h(\cdot, t), t), \qquad h(\cdot, 0) = P(\cdot), \tag{4.12}$$

and $T < T^*$. Then for $t \in [0, T]$, $h(\cdot, t) : \mathbb{R}^2 \to \mathbb{R}^2$ exists and is a diffeomorphism, with its Jacobian $c_1(T) \leq J(h(t)) \leq c_2(T)$ for some constants $c_1(T), c_2(T) > 0$; and $\xi(\cdot, t) = \zeta(h(k_0(\cdot), t), t)$ is the solution of the water *wave system* (1.23)–(1.24), *satisfying the initial condition* (4.5). *Furthermore*, $h(k_0(\cdot), t) = k(\cdot, t)$ *for* $t \in [0, T^*)$, *where* $k(\cdot, t)$ *is as defined in* (1.28), *and* $\zeta \circ k = \xi$, $u \circ k = \xi_t$, $w \circ k = \xi_{tt}$.

Proof The proof of part 1 is very much the same as that in [31]. The main modification is to use the vector fields $\Gamma = \partial_\alpha, \partial_\beta, L_0, \varpi$ instead of using only $\partial_\alpha, \partial_\beta$ as in [31], and use $\partial_t + b \cdot \nabla_\perp$ instead of ∂_t. We omit the details.

Let $T < T^*$. Notice that for the solution obtained in part 1, $b = b(\cdot, t)$ is defined for $t \in [0, T]$. Furthermore by applying Lemma 1.2, Proposition 2.2, (2.6), (2.7), Propositions 2.6, 2.7, 2.9, Lemma 6.2 of [31] and interpolation, and (4.10), we know for $\partial = \partial_\alpha, \partial_\beta$, and $|j| \le s-1$, $\Gamma^j b$, $\Gamma^j \partial b \in C([0, T], H^{1/2}(\mathbb{R}^2))$. Therefore for $|j| \le 3$, $\partial^j b \in C([0, T], C(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2))$. Thus from the classical ODE theory we know (4.12) has a unique solution $h(\cdot, t)$ on $[0, T]$, $h(\cdot, t) : \mathbb{R}^2 \to \mathbb{R}^2$ is a diffeomorphism with $c_1 \le J(h(t)) \le c_2$, $c_3|(\alpha, \beta) - (\alpha', \beta')| \le |h(\alpha, \beta, t) - h(\alpha', \beta', t)| \le c_4|(\alpha, \beta) - (\alpha', \beta')|$ for $(\alpha, \beta), (\alpha', \beta') \in \mathbb{R}^2$, $t \in [0, T]$ and some constants $0 < c_i < \infty$, $i = 1, \dots, 4$; and $\partial^j(h - P) \in C([0, T], H^{1/2}(\mathbb{R}^2))$, for $|j| \le s$. Let $\mathfrak{u} = u \circ h \circ k_0$, $\xi = \zeta \circ h \circ k_0$. From the chain rule we know $\mathfrak{u} = \xi_t$, and for $t \in [0, T^*)$, $(\mathfrak{u}, \xi)$ satisfies the quasilinear system (5.21)–(5.22) in [31]. Therefore as was proved in [31], ξ solves the water wave system (1.23)–(1.24) with initial data satisfying (4.5). Furthermore, for k as defined in (1.28), we know $k_t = (h \circ k_0)_t$ (see (1.42)), and $k(0) = (h \circ k_0)(0)$. Therefore $k(\cdot, t) = h(k_0(\cdot), t)$ for $t \in [0, T^*)$, so $k(t) : \mathbb{R}^2 \to \mathbb{R}^2$ is a diffeomorphism and $J(k(t)) > 0$ for each $t \in [0, T^*)$. $w \circ k = \xi_{tt}$ follows straightforwardly from the chain rule. □

Remark 4.4 Let ξ be the solution obtained in Theorem 4.3. As a consequence of Theorem 4.3 part 2, we know for $t \in [0, T^*)$, the mapping $k = k(\cdot, t)$ defined in (1.28) is a diffeomorphism and the solution (u, ζ) for (4.3)–(4.4)–(4.7) coincides with those defined in (1.31). Recall $\lambda = \zeta - P$. Notice that $\partial_t \lambda = u - b - b \cdot \nabla_\perp \lambda$, $\partial_t u = w - b \cdot \nabla_\perp u$. By taking successive derivatives to t to (2.38) (or equivalently (4.3)), we know that in fact for $|j| \le s - 1$, and $\Gamma = \partial_t, \partial_\alpha, \partial_\beta, L_0, \varpi$, $\partial = \partial_\alpha, \partial_\beta$,

$$\begin{aligned} &\Gamma^j \partial_t \lambda, \Gamma^j \partial \lambda, \Gamma^j u, \Gamma^j \partial_t u, \Gamma^j \partial u \in C([0, T^*), H^{1/2}(\mathbb{R}^2)), \\ &\Gamma^j w, \Gamma^j \partial_t w, \Gamma^j \partial w \in C([0, T^*), L^2(\mathbb{R}^2)). \end{aligned} \tag{4.13}$$

Remark 4.5 Notice that $\eta = \xi(k_0^{-1}(\cdot), t) = \zeta \circ h(\cdot, t)$ is a solution of the water wave equation (1.23)–(1.24) with data $\eta(\cdot, 0) = \xi^0 \circ k_0^{-1}(\cdot)$, $\eta_t(\cdot, 0) = \mathfrak{u}^0 \circ k_0^{-1}(\cdot)$. Let $|j| \le s - 1$, and $\Gamma = \partial_t, \partial_\alpha, \partial_\beta, L_0, \varpi$, $\partial = \partial_\alpha, \partial_\beta$. Using (4.13), Lemma 1.2, Proposition 2.2, (2.6), (2.7), Propositions 2.6, 2.7, 2.9, and Lemma 6.2 of [31] and interpolation, we know that the function b defined in (4.4) satisfies $\Gamma^j b,\ \Gamma^j \partial b \in C([0,T^*), H^{1/2}(\mathbb{R}^2))$. Therefore we have for h the solution of (4.12), $\Gamma^j(h-P), \Gamma^j \partial_t(h-P), \Gamma^j \partial(h-P) \in C([0,T^*), H^{1/2}(\mathbb{R}^2))$. This implies the solution η satisfies

$$\begin{aligned}&\Gamma^j \partial_t \eta, \Gamma^j \partial(\eta - P), \Gamma^j \partial_t \eta_t, \Gamma^j \partial \eta_t \in C([0,T^*), H^{1/2}(\mathbb{R}^2)),\\ &\Gamma^j \partial \eta_{tt}, \Gamma^j \partial_t \eta_{tt} \in C([0,T^*), L^2(\mathbb{R}^2)).\end{aligned} \tag{4.14}$$

From Proposition 2.9, we know there is $N_1 > 0$ small enough, such that whenever $\sum_{\substack{|i|\le 2\\ \partial=\partial_\alpha,\partial_\beta}} \|\partial^i \partial \lambda(t)\|_2 \le N_1$, $|\partial_\alpha \lambda(t)|_\infty + |\partial_\beta \lambda(t)|_\infty \le \frac{1}{4}$; this in turn implies that

$$|\zeta(\alpha,\beta,t) - \zeta(\alpha',\beta',t)| \ge \frac{1}{4}(|\alpha-\alpha'| + |\beta-\beta'|), \qquad |\zeta_\alpha \times \zeta_\beta| \ge \frac{1}{4},$$

and $\Sigma(t): \zeta = \zeta(\alpha,\beta,t),\ (\alpha,\beta)\in\mathbb{R}^2$ is a graph.

We now present a global in time well-posedness result. Let $s \ge 27$, $\max\{[\frac{s}{2}]+1, 17\} \le l \le s-10$, and the initial interface $\Sigma(0)$ be a graph given by $\xi^0 = (\alpha,\beta,z^0(\alpha,\beta))$, satisfying $N = \sum_{\substack{|i|\le 2\\ \partial=\partial_\alpha,\partial_\beta}} \|\partial^i \partial z^0\|_2 \le N_0$, where N_0 is the constant in Lemma 4.1, part 1. Therefore the corresponding mapping $k(0) = k_0$ defined in (1.28) is a diffeomorphism with its Jacobian $1/4 \le J(k_0) \le 2$ and $\frac{1}{4}(|\alpha-\alpha'| + |\beta-\beta'|) \le |k_0(\alpha,\beta) - k_0(\alpha',\beta')| \le 2(|\alpha-\alpha'| + |\beta-\beta'|)$. Assume that the initial data satisfies (4.5)–(4.9), and for $\Gamma = \partial_\alpha, \partial_\beta, L_0, \varpi$,

$$\begin{aligned} L = &\sum_{\substack{|j|\le l+9\\ \partial=\partial_\alpha,\partial_\beta}} \|\Gamma^j |D|^{1/2} \mathfrak{z}^0\|_2 + \|\Gamma^j \partial \lambda^0\|_2 \\ &+ \|\Gamma^j u^0\|_{H^{1/2}} + \|\Gamma^j w^0\|_2 < \infty, \end{aligned} \tag{4.15}$$

here $\mathfrak{z}^0 = z^0 \circ k_0^{-1}$. Let

$$\epsilon = \sum_{\substack{|j|\le l+3\\ \partial=\partial_\alpha,\partial_\beta}} \|\Gamma^j |D|^{1/2} \mathfrak{z}^0\|_2 + \|\Gamma^j \partial \lambda^0\|_2 + \|\Gamma^j u^0\|_{H^{1/2}} + \|\Gamma^j w^0\|_2, \tag{4.16}$$

and assume $\epsilon \le N_1$. An argument as that in Remark 4.4 and an application of Lemma 1.2, Proposition 2.2, (2.7), (2.6), Propositions 2.6, 2.7, 2.9 gives that for $\Gamma = \partial_t,\ \partial_\alpha,\ \partial_\beta,\ L_0,\ \varpi$,

$$\mathcal{M}_0 = \sum_{\substack{|j|\le l+2\\ \partial=\partial_\alpha,\partial_\beta}} (\|\Gamma^j \partial \lambda^0\|_2 + \|\Gamma^j \partial \mathfrak{z}^0\|_2$$

$$+\|\Gamma^j\mathfrak{v}(0)\|_2+\|\Gamma^j(\partial_t+b\cdot\nabla_\perp)\mathfrak{v}(0)\|_2)\le c_1(\epsilon)\epsilon<\infty \tag{4.17}$$

and a further application of Lemma 6.2 of [31] and interpolation gives that (for $\epsilon>0$ small enough such that $c_1(\epsilon)\epsilon\le M_0$)

$$\begin{aligned}&\mathfrak{F}_{l+2}(0)\le c_2(\epsilon)\epsilon^2, \qquad \mathcal{F}_{l+3}(0)\le c_3(\epsilon)\epsilon^2,\\ &\mathcal{F}_{l+9}(0)=c_4(L)<\infty.\end{aligned} \tag{4.18}$$

Here $c_i(p)$, $i=1,2,3,4$ are constants depending on p.

Take M_0 such that $0<M_0\le N_1$ and all the estimates derived in Sect. 3 holds.

Theorem 4.6 (Global well-posedness) *There exists $\epsilon_0>0$, depending on M_0, L, where L is as in (4.15), such that for $0\le\epsilon\le\epsilon_0$, the initial value problem (1.23)–(1.24)–(4.5) has a unique classical solution globally in time. For $0\le t<\infty$, the solution satisfies (4.13), (4.14), the interface is a graph, and*

$$(1+t)\sum_{\substack{|j|\le l-3\\ \partial=\partial_\alpha,\partial_\beta}}(|\partial\Gamma^j\chi(t)|_\infty+|\partial\Gamma^j\mathfrak{v}(t)|_\infty)\lesssim\mathfrak{F}_{l+2}^{1/2}(t)\le C(M_0,L)\epsilon. \tag{4.19}$$

Here $C(M_0,L)$ is a constant depending on M_0,L.

Proof From Theorem 4.3, Remarks 4.4, 4.5, we know there exists a unique solution $\xi=\xi(\cdot,t)$ for $t\in[0,T^*)$ of (1.23)–(1.24)–(4.5), with $k(\cdot,t):\mathbb{R}^2\to\mathbb{R}^2$ as defined in (1.28) being a diffeomorphism, λ, u, w as defined in (1.31), (1.36) satisfying (4.13) for $t\in[0,T^*)$, and $\eta=\xi\circ k_0^{-1}$ satisfying (4.14). Applying Lemma 1.2, Proposition 2.2, (2.6), (2.7), Propositions 2.6, 2.7, 2.9, Lemma 6.2 of [31] and interpolation, and the fact that $\mathfrak{z}(\cdot,t)=\mathfrak{z}^0(\cdot)+\int_0^t(u_3-b\cdot\nabla_\perp\mathfrak{z})(\cdot,s)\,ds$, here u_3 is the e_3 component of u, we have $\mathcal{F}_n(t)$, $\mathfrak{F}_n(t)\in C^1[0,T^*)$ for $n\le l+9$. Let $0<\epsilon_1\le N_1$ be small enough such that for $\epsilon\le\epsilon_1$, $\mathcal{M}_0\le c_1(\epsilon)\epsilon\le\frac{M_0}{2}$. Let $T_1\le T_*$ be the largest such that for $t\in[0,T_1)$, (3.13) holds. From Theorem 3.101, Lemma 3.4, we know there is a $0<\epsilon_2\le\epsilon_1$, such that when $0<\epsilon\le\epsilon_2$, $\sup_{[0,T_1)}E_{l+2}(t)\lesssim\mathfrak{F}_{l+2}(t)\le c(M_0,L)^2\epsilon^2$ for some constant $c(M_0,L)$ depending on M_0,L. On the other hand from Proposition 2.16 we have that for $t\in[0,T_1)$,

$$\begin{aligned}&\sum_{\substack{|j|\le l+2\\ \partial=\partial_\alpha,\partial_\beta}}(\|\Gamma^j\partial\lambda(t)\|_2+\|\Gamma^j\partial\mathfrak{z}(t)\|_2+\|\Gamma^j\mathfrak{v}(t)\|_2+\|\Gamma^j(\partial_t+b\cdot\nabla_\perp)\mathfrak{v}(t)\|_2)\\ &\quad\le C(M_0)E_{l+2}(t)^{1/2},\end{aligned}$$

where $C(M_0)$ is a constant depending on M_0. Taking $\epsilon_0 \le \epsilon_2$, such that $C(M_0)c(M_0, L)\epsilon_0 \le \frac{3M_0}{4}$. Therefore when $\epsilon \le \epsilon_0$, we have for $t \in [0, T_1)$,

$$\sum_{\substack{|j|\le l+2\\ \partial=\partial_\alpha,\partial_\beta}} (\|\Gamma^j \partial\lambda(t)\|_2 + \|\Gamma^j \partial\mathfrak{z}(t)\|_2 + \|\Gamma^j \mathfrak{v}(t)\|_2 + \|\Gamma^j (\partial_t + b\cdot\nabla_\perp)\mathfrak{v}(t)\|_2) \le \frac{3M_0}{4}.$$

This implies that $T_1 = T^*$ or otherwise it contradicts with the assumption that T_1 is the largest. Applying Proposition 2.16 again we deduce that

$$\sum_{|j|\le l+2} \|\Gamma^j w(t)\|_{L^2} + \|\Gamma^j u(t)\|_{L^2} \in L^\infty[0, T^*). \tag{4.20}$$

Furthermore from $M_0 \le N_1$ we have

$$\sup_{(\alpha,\beta)\ne(\alpha',\beta')} \frac{|(\alpha,\beta)-(\alpha',\beta')|}{|\zeta(\alpha,\beta,t)-\zeta(\alpha',\beta',t)|} + \left|\frac{1}{|\zeta_\alpha \times \zeta_\beta(t)|}\right|_{L^\infty} \in L^\infty[0, T^*) \tag{4.21}$$

and $\Sigma(t) : \zeta = \zeta(\cdot, t)$ defines a graph for $t \in [0, T^*)$. Now from our assumption we know $[\frac{s}{2}] + 3 \le l + 2$. Applying (4.11), we obtain $T^* = \infty$. (4.19) is a consequence of Lemma 3.3. □

Remark 4.7 As a consequence of (4.19) and Proposition 2.17, the steepness, the acceleration of the interface and the derivative of the velocity on the interface decay at the rate $\frac{1}{t}$.

Remark 4.8 Instead of (4.9), (4.15), (4.16), we may assume for $|j| \le s-1$, and $\Gamma = \partial_\alpha, \partial_\beta, L_0, \varpi$,

$$\sum_{\substack{|j|\le s-1\\ \partial=\partial_\alpha,\partial_\beta}} \|\Gamma^j \partial z^0\|_{H^{1/2}} + \|\Gamma^j \mathfrak{u}^0\|_{H^{3/2}} + \|\Gamma^j \mathfrak{w}^0\|_{H^1} < \infty; \tag{4.22}$$

$$L = \sum_{\substack{|j|\le l+9\\ \partial=\partial_\alpha,\partial_\beta}} \|\Gamma^j |D|^{1/2} z^0\|_2 + \|\Gamma^j \partial z^0\|_2 + \|\Gamma^j \mathfrak{u}^0\|_{H^{1/2}} + \|\Gamma^j \mathfrak{w}^0\|_2 < \infty; \tag{4.23}$$

and let

$$\epsilon = \sum_{\substack{|j|\le l+3\\ \partial=\partial_\alpha,\partial_\beta}} \|\Gamma^j |D|^{1/2} z^0\|_2 + \|\Gamma^j \partial z^0\|_2 + \|\Gamma^j \mathfrak{u}^0\|_{H^{1/2}} + \|\Gamma^j \mathfrak{w}^0\|_2. \tag{4.24}$$

We know from Lemma 4.1 and Lemma 4.2 that (4.22), (4.23), (4.24) implies

$$\sum_{\substack{|j|\le s-1\\ \partial=\partial_\alpha,\partial_\beta}} \|\Gamma^j\partial\lambda^0\|_{H^{1/2}}+\|\Gamma^j u^0\|_{H^{3/2}}+\|\Gamma^j w^0\|_{H^1}<\infty \quad \text{and} \tag{4.25}$$

$$\sum_{\substack{|j|\le l+9\\ \partial=\partial_\alpha,\partial_\beta}} \|\Gamma^j|D|^{1/2}\mathfrak{z}^0\|_2+\|\Gamma^j\partial\lambda^0\|_2+\|\Gamma^j u^0\|_{H^{1/2}}+\|\Gamma^j w^0\|_2$$
$$\le c_5(L)L<\infty, \tag{4.26}$$

$$\sum_{\substack{|j|\le l+3\\ \partial=\partial_\alpha,\partial_\beta}} \|\Gamma^j|D|^{1/2}\mathfrak{z}^0\|_2+\|\Gamma^j\partial\lambda^0\|_2+\|\Gamma^j u^0\|_{H^{1/2}}+\|\Gamma^j w^0\|_2$$
$$\le c_5(\epsilon)\epsilon<\infty \tag{4.27}$$

for some constants $c_5(L)$, $c_6(\epsilon)$ depending on L, ϵ respectively. Therefore the same conclusions of Theorem 4.6 hold, and furthermore by using Lemmas 4.1, 4.2, we have for $\xi=\eta\circ k_0$ the solution of the initial value problem of the water wave equations (1.23)–(1.24)–(4.5), and $|j|\le s-1$, $\Gamma=\partial_t,\partial_\alpha,\partial_\beta,L_0,\varpi$, (notice that $k_0=k_0(\alpha,\beta)$ is independent of t).

$$\Gamma^j\partial_t\xi,\Gamma^j\partial(\xi-P),\Gamma^j\partial_t\xi_t,\Gamma^j\partial\xi_t\in C([0,T^*),H^{1/2}(\mathbb{R}^2)),$$
$$\Gamma^j\partial\xi_{tt},\Gamma^j\partial_t\xi_{tt}\in C([0,T^*),L^2(\mathbb{R}^2)).$$

Appendix A: Notations

Here we summarize a partial list of quantities introduced in this paper.

Let $\Sigma(t):\xi=\xi(\alpha,\beta,t)=x(\alpha,\beta,t)e_1+y(\alpha,\beta,t)e_2+z(\alpha,\beta,t)e_3$ be the interface at time t in Lagrangian coordinates (α,β), and $\mathfrak{H}=\mathfrak{H}_{\Sigma(t)}$ be the associated Hilbert transform (see (1.10)). We defined:

1. $\pi=(I-\mathfrak{H})ze_3$.
2. The change of coordinates k given in (1.28):

$$k=k(\alpha,\beta,t)=\xi(\alpha,\beta,t)-(I+\mathfrak{H})z(\alpha,\beta,t)e_3+\mathfrak{K}z(\alpha,\beta,t)e_3, \tag{A.1}$$

   where $\mathfrak{K}=\operatorname{Re}\mathfrak{H}$ is the double layered potential operator. Define $U_g f(\alpha,\beta,t)=f(g(\alpha,\beta,t),t)=f\circ g(\alpha,\beta,t)$.
3. We then defined (see (1.31), (1.32))

$$\begin{aligned}&\zeta=\xi\circ k^{-1}=\mathfrak{x}e_1+\mathfrak{y}e_2+\mathfrak{z}e_3,\\ &u=\xi_t\circ k^{-1},\quad \text{and}\quad w=\xi_{tt}\circ k^{-1};\end{aligned} \tag{A.2}$$

$$\begin{aligned} &b = k_t \circ k^{-1}, \\ &A \circ k e_3 = \mathfrak{a} J(k) e_3 = \mathfrak{a} k_\alpha \times k_\beta, \quad \text{and} \quad \mathcal{N} = \zeta_\alpha \times \zeta_\beta; \end{aligned} \tag{A.3}$$

4. $\chi = \pi \circ k^{-1} = (I - \mathcal{H})\mathfrak{z} e_3$, where $U_k^{-1} \mathfrak{H} U_k = \mathcal{H}$ is the Hilbert transform associated to ζ.
5. Let $\mathcal{K} = \operatorname{Re} \mathcal{H}$. We defined by $P = \alpha e_1 + \beta e_2 = (\alpha, \beta)$ the horizontal plane,

$$\begin{aligned} \Lambda^* &= (I + \mathfrak{H}) z e_3, \qquad \Lambda = (I + \mathfrak{H}) z e_3 - \mathfrak{K} z e_3, \\ \lambda^* &= (I + \mathcal{H}) \mathfrak{z} e_3, \qquad \lambda = \lambda^* - \mathcal{K} \mathfrak{z} \mathfrak{e}_3. \end{aligned} \tag{A.4}$$

   So $\zeta = P + \lambda$. See (1.36).
6. We defined $\mathfrak{v} = (\partial_t + b \cdot \nabla_\perp)\chi$.
7. We defined (see (2.37))

$$E_m(t) = \sum_{|j| \le m} (\|(\partial_t + b \cdot \nabla_\perp)\Gamma^j \chi(t)\|_2^2 + \|(\partial_t + b \cdot \nabla_\perp)\Gamma^j \mathfrak{v}(t)\|_2^2). \tag{A.5}$$

8. For a Clifford number $\sigma \in \mathcal{C}(V_2)$, we defined $\overline{\sigma} = e_3 \sigma e_3$.

Appendix B: Traveling waves

Here we consider the full water wave equation (1.1) with the interface $\Sigma(t)$ tending to the horizontal plane, and the velocity $\mathbf{v} \to c$ at the spatial infinity, where $c = (c_1, \dots, c_{n-1}, 0)$ is a constant vector. We consider only the case $n = 3$ and are interested in the global in time behavior of solutions with small initial data. The analysis in the main body of this paper carries over without difficulties to this case, so we only indicate what would be the transformed equation (1.25), what would be the change of coordinates k and relations between various quantities in this framework. We adopt the same notations as in the main body of this paper.

Let $\Sigma(t) : \xi = \xi(\alpha, \beta, t)$ be the interface in Lagrangian coordinate (α, β). Assume $\mathbf{v} \to c$ at the spatial infinity. Then (1.1) is equivalent to

$$\begin{cases} \xi_{tt} + e_3 = \mathfrak{a} N, \\ \xi_t - c = \mathfrak{H}(\xi_t - c). \end{cases} \tag{B.1}$$

First, Proposition 1.3 still holds. The same proof, with ξ_t replaced by $\xi_t - c$ in various places, works. Let $\pi = (I - \mathfrak{H}) z e_3$.

Proposition B.1 *We have*

$$(\partial_t^2 - \mathfrak{a} N \times \nabla)\pi$$

$$= \iint K(\xi' - \xi)(\xi_t - \xi_t') \times (\xi_{\beta'}' \partial_{\alpha'} - \xi_{\alpha'}' \partial_{\beta'}) \overline{\xi_t'} \, d\alpha' d\beta'$$
$$- \iint K(\xi' - \xi)(\xi_t - \xi_t') \times (\xi_{t\beta'}' \partial_{\alpha'} - \xi_{t\alpha'}' \partial_{\beta'}) z' \, d\alpha' d\beta' e_3$$
$$- \iint \partial_t K(\xi' - \xi)(\xi_t - \xi_t') \times (\xi_{\beta'}' \partial_{\alpha'} - \xi_{\alpha'}' \partial_{\beta'}) z' \, d\alpha' d\beta' e_3. \quad \text{(B.2)}$$

The change of coordinates is a small modification of that in (1.28):

$$k = k(\alpha, \beta, t) = \xi(\alpha, \beta, t) - ct - (I + \mathfrak{H}) z(\alpha, \beta, t) e_3 + \mathfrak{K} z(\alpha, \beta, t) e_3 \quad \text{(B.3)}$$

where $\mathfrak{K} = \operatorname{Re} \mathfrak{H}$ is the double layered potential operator.

Now we also have the following proposition, showing that b and $A - 1$ are quadratic.

Proposition B.2 *Let $b = k_t \circ k^{-1}$ and $A \circ k = \mathfrak{a} J(k)$. We have*

$$b = \frac{1}{2}(\mathcal{H} - \overline{\mathcal{H}})(\overline{u} - c) - [\partial_t + b \cdot \nabla_\perp, \mathcal{H}] \mathfrak{z} e_3$$
$$+ [\partial_t + b \cdot \nabla_\perp, \mathcal{K}] \mathfrak{z} e_3 + \mathcal{K} u_3 e_3, \quad \text{(B.4)}$$
$$(A - 1) e_3 = \frac{1}{2}(-\mathcal{H} + \overline{\mathcal{H}}) \overline{w} + \frac{1}{2}([\partial_t + b \cdot \nabla_\perp, \mathcal{H}](u - c)$$
$$- \overline{[\partial_t + b \cdot \nabla_\perp, \mathcal{H}](u - c)})$$
$$+ [A \mathcal{N} \times \nabla, \mathcal{H}] \mathfrak{z} e_3 - A \zeta_\beta \times (\partial_\alpha \mathcal{K} \mathfrak{z} e_3)$$
$$+ A \zeta_\alpha \times (\partial_\beta \mathcal{K} \mathfrak{z} e_3) + A \partial_\alpha \lambda \times \partial_\beta \lambda. \quad \text{(B.5)}$$

Here $\overline{\mathcal{H}} f = e_3 \mathcal{H}(e_3 f) = \iint e_3 K \mathcal{N}' e_3 f'$.

Proof The same proof as that of Proposition 1.4, with ξ_t replaced by $\xi_t - c$ works. □

The relations between quantities is a small modification of that in Lemma 2.14. The main change is in (B.10).

Lemma B.3 *We have*

$$\overline{\lambda} + \chi = (\overline{\mathcal{H}} - \mathcal{H}) \mathfrak{z} e_3 + \mathcal{K} \mathfrak{z} e_3, \qquad \overline{\lambda^*} + \chi = (\overline{\mathcal{H}} - \mathcal{H}) \mathfrak{z} e_3, \quad \text{(B.6)}$$
$$\partial_\alpha \mathfrak{z} = -\mathcal{N} \cdot e_1 + (\partial_\alpha \lambda \times \partial_\beta \lambda) \cdot e_1,$$
$$\partial_\beta \mathfrak{z} = -\mathcal{N} \cdot e_2 + (\partial_\alpha \lambda \times \partial_\beta \lambda) \cdot e_2, \quad \text{(B.7)}$$
$$\mathcal{N} = e_3 + \partial_\alpha \lambda \times e_2 - \partial_\beta \lambda \times e_1 + \partial_\alpha \lambda \times \partial_\beta \lambda, \quad \text{(B.8)}$$

$$A\mathcal{N} - e_3 = w, \tag{B.9}$$

$$\begin{aligned} 2(\overline{u} - c + (\partial_t + b\cdot\nabla_\perp)\chi) =& (\mathcal{H} - \overline{\mathcal{H}})(\overline{u} - c) \\ &- 2[\partial_t + b\cdot\nabla_\perp, \mathcal{H}]\mathfrak{z}e_3, \end{aligned} \tag{B.10}$$

$$\begin{aligned} 2(\overline{w} + (\partial_t + b\cdot\nabla_\perp)\mathfrak{v}) =& (\mathcal{H} - \overline{\mathcal{H}})\overline{w} + [\partial_t + b\cdot\nabla_\perp, \mathcal{H} - \overline{\mathcal{H}}](\overline{u} - c) \\ &- 2(\partial_t + b\cdot\nabla_\perp)[\partial_t + b\cdot\nabla_\perp, \mathcal{H}]\mathfrak{z}e_3, \end{aligned} \tag{B.11}$$

$$(\mathcal{H} - \overline{\mathcal{H}})f = -2\iint K\cdot\mathcal{N}'f' + 2\iint (K_1\mathcal{N}_2' - K_2\mathcal{N}_1')e_3 f', \tag{B.12}$$

where $K = K_1e_1 + K_2e_2 + K_3e_3$, $\mathcal{N} = \mathcal{N}_1e_1 + \mathcal{N}_2e_2 + \mathcal{N}_3e_3$, *and* f *is a function.*

For the 2D water wave one may do similar modifications. We omit.

Appendix C: Normal Forms

Since the publication of [32], there have been questions on whether one may just use a normal form transformation containing only linear and quadratic terms (we call such transformation a bilinear transformation) and obtain the same results as in [32] and the present paper. Indeed, if the projection $(I - \mathfrak{H})$ and the change of coordinate k as in (1.28) together produce a quantity $\pi \circ k^{-1} = (I - \mathcal{H})\mathfrak{z}e_3$ that satisfies (1.35) which doesn't contain quadratic terms, then the quantity consisting of the linear and quadratic parts of this transformation only should also satisfy an equation without quadratic terms, since the terms of cubic and higher orders in the transformation should be redundant in canceling the quadratic terms in the water wave equation.

We recall that the method of using only linear and quadratic terms for the transformation was carried out successfully by Shatah [23] in obtaining a global well-posedness result for the Klein-Gordon equations in $\mathbb{R}^3$. The advantage of this method is that it is algorithmic.

A key requirement for the method of normal forms to work is that the transformed quantity and the original unknown should be equivalent in the various norms involved in the course of analysis. We will see in this note that retaining only the linear and quadratic terms in our transformation fails for the water wave equation, as this partial transformation doesn't give a quantity that has equivalent norms as the original unknown in the functional space considered in this paper. The failure is due to the coordinate change part of the transformation.

In what follows, we first use the method of Shatah to find a bilinear transformation that cancels the quadratic terms in the water wave equation; we next analyze the partial transformation containing only the linear and quadratic parts of the transformation $(I-\mathcal{H})\mathfrak{z}e_3$. We then give some concluding remarks.

In this note, we sometimes do not observe consistency of notations with the main body of the paper. For example, ξ, η here denote variables in the Fourier space, h denotes the hight of the interface.

Let $\hat{f}$ be the Fourier transform of f, and

$$B(f,g)=\int e^{ix\cdot(\xi+\eta)}m(\xi,\eta)\hat{f}(\xi)\hat{g}(\eta)\,d\xi\,d\eta \tag{C.1}$$

be a bilinear operator. We call $m(\xi,\eta)$ the Fourier symbol of B. In particular, if $m(\xi,\eta)=1$, then $B(f,g)=fg$. We note that if a normal form transformation is given by

$$V=U+B(U,U)$$

and if B is bounded: $\|B(U,U)\|\lesssim\|U\|^2$, then $\|V\|\approx\|U\|$ provided $\|U\|$ is sufficiently small.

C.1 The bilinear normal form transformation

Assume that the interface at time t is a graph given by $\Sigma(t):(X,z)=(X,h(X,t))$, $X\in\mathbb{R}^{n-1}$, and let ϕ be the velocity potential, $\psi(X,t)$ be the trace of the velocity potential: $\psi(X,t)=\phi(X,h(X,t),t)$. The system (1.1) is equivalent to (see [27])

$$\begin{cases}\partial_t h=G(h)\psi,\\ \partial_t\psi=-h-\frac{1}{2}|\nabla\psi|^2+\frac{1}{2(1+|\nabla h|^2)}(G(h)\psi+\nabla h\cdot\nabla\psi)^2,\end{cases} \tag{C.2}$$

where $G(h)=\sqrt{1+|\nabla h|^2}\nabla_{\mathbf{n}}$, $\nabla_{\mathbf{n}}$ is the Dirichlet-Neumann operator associated to $\Omega(t)$, $\nabla=(\partial_{x_1},\dots,\partial_{x_{n-1}})$ for $X=(x_1,\dots,x_{n-1})$. Notice that the right hand sides of the equations in (C.2) are dependent of ∇h, $\nabla\psi$ only. Furthermore, we know (C.2) can be expanded as the following (see [27])

$$\begin{cases}\partial_t h=|D|\psi-\nabla h\cdot\nabla\psi+[h,|D|]|D|\psi+C_1(\nabla h,\nabla\psi),\\ \partial_t\psi=-h-\frac{1}{2}|\nabla\psi|^2+\frac{1}{2}(|D|\psi)^2+C_2(\nabla h,\nabla\psi),\end{cases} \tag{C.3}$$

where $C_i(\nabla h,\nabla\psi)$, $i=1,2$ are terms of cubic and higher orders in ∇h and $\nabla\psi$, $|D|=\sqrt{-\nabla\cdot\nabla}=\sqrt{-\Delta}$. (C.2) is equivalent to (1.23)–(1.24). We choose to work on (C.2) here for the sake of convenience.

We first find the bilinear norm form transformation that cancels out the quadratic terms in (C.3). The ansatz (C.6)–(C.7) we use is similar to that of Shatah [23].

Let M^T denote the transpose of a matrix M, $\tilde{m}(\xi,\eta)=m(\eta,\xi)$; C denotes a term that is at least of cubic order in h,ψ; the Cs appearing in different equations and contexts need not be the same.

Let $U=\binom{h}{\psi}$, $\mathcal{A}=\begin{pmatrix}0 & |D|\\ -1 & 0\end{pmatrix}$, $Q=\binom{Q_1}{Q_2}$, $C=\binom{C_1}{C_2}$, where

$$\begin{aligned}\hat{Q}_1(\eta) &= \int (\eta\cdot\xi-|\eta||\xi|)\hat{h}(\eta-\xi)\hat{\psi}(\xi)\,d\xi\\ &= \int (\eta\cdot(\eta-\xi)-|\eta||\eta-\xi|)\hat{h}(\xi)\hat{\psi}(\eta-\xi)\,d\xi,\\ \hat{Q}_2(\eta) &= \int \frac{1}{2}((\eta-\xi)\cdot\xi+|\eta-\xi||\xi|)\hat{\psi}(\xi)\hat{\psi}(\eta-\xi)\,d\xi.\end{aligned}\tag{C.4}$$

We can rewrite (C.3) as

$$\partial_t U=\mathcal{A}U+Q+C.\tag{C.5}$$

Let

$$V=U+B(U,U),\tag{C.6}$$

where $B(U,U)=\binom{B_1(U,U)}{B_2(U,U)}$ is bilinear, with

$$\begin{aligned}&\hat{B}_i(F,G)(\eta,t)\\ &\quad=\int \hat{F}^T(\xi,t)\begin{pmatrix}K_i(\xi,\eta-\xi) & L_i(\xi,\eta-\xi)\\ M_i(\xi,\eta-\xi) & N_i(\xi,\eta-\xi)\end{pmatrix}\hat{G}(\eta-\xi,t)\,d\xi.\end{aligned}\tag{C.7}$$

We know

$$\begin{aligned}\hat{B}_i(U,U)(\eta,t) &= \int (\hat{h}(\xi,t)K_i(\xi,\eta-\xi)\hat{h}(\eta-\xi,t)\\ &\quad+\hat{\psi}(\xi,t)N_i(\xi,\eta-\xi)\hat{\psi}(\eta-\xi,t))\,d\xi\\ &\quad+\int \hat{h}(\xi,t)(L_i+\tilde{M}_i)(\xi,\eta-\xi)\hat{\psi}(\eta-\xi,t)\,d\xi.\end{aligned}\tag{C.8}$$

We want to find $K_i, L_i+\tilde{M}_i, N_i$, $i=1,2$ so that V satisfies an equation of the form

$$\partial_t V=\mathcal{A}V+C.\tag{C.9}$$

We calculate

$$\begin{aligned}\partial_t V &= \partial_t U+B(\partial_t U,U)+B(U,\partial_t U)\\ &= \mathcal{A}U+Q+B(\mathcal{A}U,U)+B(U,\mathcal{A}U)+C,\end{aligned}$$

$$\mathcal{A}V = \mathcal{A}U + \mathcal{A}B(U, U).$$

In order for V to satisfy $\partial_t V = \mathcal{A}V + C$, we need

$$Q + B(\mathcal{A}U, U) + B(U, \mathcal{A}U) = \mathcal{A}B(U, U). \tag{C.10}$$

Let $m_1(\xi, \eta - \xi) = \eta \cdot (\eta - \xi) - |\eta||\eta - \xi|$, $\tilde{m}_1(\xi, \eta - \xi) = \eta \cdot \xi - |\eta||\xi|$, $m_2(\xi, \eta - \xi) = (\eta - \xi) \cdot \xi + |\eta - \xi||\xi|$. Solving (C.10), we find

$$\begin{aligned}
&K_2 = N_2 = M_1 = L_1 = 0,\\
&(L_2 + \tilde{M}_2)(\xi, \eta - \xi)\\
&\quad = -\frac{(|\eta - \xi| + |\xi| - |\eta|)(m_1 + m_2) - 2|\eta - \xi|(\tilde{m}_1 + m_2)}{(|\eta - \xi| + |\xi| - |\eta|)^2 - 4|\xi||\eta - \xi|},\\
&K_1(\xi, \eta - \xi) = \frac{1}{2}(L_2 + \tilde{M}_2)(\xi, \eta - \xi) + \frac{1}{2}(L_2 + \tilde{M}_2)(\eta - \xi, \xi),\\
&N_1(\xi, \eta - \xi) = -\frac{1}{2}(m_2 + |\xi|(L_2 + \tilde{M}_2)(\xi, \eta - \xi)\\
&\qquad\qquad + |\eta - \xi|(L_2 + \tilde{M}_2)(\eta - \xi, \xi)).
\end{aligned} \tag{C.11}$$

Notice that the denominator $(|\eta - \xi| + |\xi| - |\eta|)^2 - 4|\xi||\eta - \xi| = 0$ if and only if $\xi = 0$ or $\eta = 0$ or $\eta - \xi = 0$. To understand better the nature of the zeros of $(|\eta - \xi| + |\xi| - |\eta|)^2 - 4|\xi||\eta - \xi| = 0$, we present the following identity.

Lemma C.1 *We have*[14]

$$\begin{aligned}
&((|\eta - \xi| + |\xi| - |\eta|)^2 - 4|\xi||\eta - \xi|)((|\eta - \xi| + |\xi| + |\eta|)^2 - 4|\xi||\eta - \xi|)\\
&\quad = -16|\xi||\eta - \xi||\eta|^2 + (|\eta|^2 - (|\eta - \xi| - |\xi|)^2)^2.
\end{aligned} \tag{C.12}$$

The checking of the identity (C.12) is straightforward, we omit the details.

Remark C.2 The following identity holds:

$$\begin{aligned}
&\prod_{i,j=0,1} (|\eta|^{1/2} + (-1)^i |\eta - \xi|^{1/2} + (-1)^j |\xi|^{1/2})\\
&\quad = (|\eta - \xi| + |\xi| - |\eta|)^2 - 4|\xi||\eta - \xi|.
\end{aligned} \tag{C.13}$$

[14]The identity (C.12) was found during our effort in finding a nonlinear normal form transformation for the water wave equation in 3D. Multiplying $(|\eta - \xi| + |\xi| - |\eta|)^2 - 4|\xi||\eta - \xi|$ by the factor $(|\eta - \xi| + |\xi| + |\eta|)^2 - 4|\xi||\eta - \xi|$ is natural in the Clifford analysis framework.

Now

$$|\eta|^2 - (|\eta-\xi| - |\xi|)^2 = 2(\eta-\xi)\cdot\xi + 2|\eta-\xi||\xi|;$$

therefore

$$(|\eta|^2 - (|\eta-\xi| - |\xi|)^2)^2 \leq 4|\xi||\eta-\xi||\eta|^2. \tag{C.14}$$

On the other hand,

$$(|\eta-\xi|+|\xi|+|\eta|)^2 - 4|\xi||\eta-\xi| = (|\eta-\xi|-|\xi|)^2 + |\eta|^2 + 2(|\eta-\xi|+|\xi|)|\eta|.$$

So

$$\begin{aligned}(|\eta-\xi|+|\xi|+|\eta|)|\eta| &\leq (|\eta-\xi|+|\xi|+|\eta|)^2 - 4|\xi||\eta-\xi| \\ &\leq 2(|\eta-\xi|+|\xi|+|\eta|)\,|\eta|.\end{aligned} \tag{C.15}$$

Combining (C.12), (C.14), (C.15), we obtain

$$\begin{aligned}\frac{|\eta-\xi|+|\xi|+|\eta|}{16|\eta-\xi||\xi||\eta|} &\leq -\frac{1}{(|\eta-\xi|+|\xi|-|\eta|)^2 - 4|\xi||\eta-\xi|} \\ &\leq \frac{|\eta-\xi|+|\xi|+|\eta|}{6|\eta-\xi||\xi||\eta|}.\end{aligned} \tag{C.16}$$

Notice that $|m_1|, |\tilde{m}_1|, |m_2| \leq 2\min\{|\xi||\eta-\xi|, |\xi||\eta|, |\eta||\eta-\xi|\}$. Therefore

$$\begin{aligned}&|(L_2+\tilde{M}_2)(\xi,\eta-\xi)| \lesssim |\eta-\xi|, \\ &|K_1(\xi,\eta-\xi)| \lesssim |\eta-\xi|+|\xi|, \\ &|N_1(\xi,\eta-\xi)| \lesssim |\xi||\eta-\xi|.\end{aligned} \tag{C.17}$$

Moreover we have

$$\begin{aligned}&\overline{\lim_{\xi\to 0}}(L_2+\tilde{M}_2)(\xi,\eta-\xi) = |\eta|, \\ &\overline{\lim_{\xi\to 0}}\, K_1(\xi,\eta-\xi) = \frac{1}{2}|\eta|, \\ &\overline{\lim_{\eta-\xi\to 0}}\, K_1(\xi,\eta-\xi) = \frac{1}{2}|\xi|.\end{aligned} \tag{C.18}$$

C.2 Analysis of the bilinear normal form transformation

Notice that in this paper, the unknown of the water wave equation is $(|D|^{1/2}h, |D|\psi)$ and only $(|D|^{1/2}h, |D|\psi)$ and its derivatives are assumed small and localized.[15,16] We remark that it makes sense to take $(|D|^{1/2}h, |D|\psi)$ as the unknown for the water wave equation (C.2) since the right hand sides of the equations in (C.2) are dependent on ∇h, $\nabla\psi$ only.

We are also interested in understanding the normal form transformation for the case when $(h, |D|^{1/2}\psi)$ is assumed small and localized. In what follows we find the bilinear normal form transformations that cancel out quadratic terms in the water wave equation for the unknowns $U_s = E_s U = |D|^s E_0 U = \begin{pmatrix} |D|^s h \\ |D|^{s+1/2}\psi \end{pmatrix}$, where $E_s = \begin{pmatrix} |D|^s & 0 \\ 0 & |D|^{s+1/2} \end{pmatrix}$, $s = 0$ or $s = 1/2$. It is easy to see that the bilinear normal form transformation for U_s is given by $V_s = E_s V$:

$$V_s = U_s + B_s(U_s, U_s),$$

where $B_s(U_s, U_s) = E_s B(E_{-s}U_s, E_{-s}U_s)$, and B is as given by (C.7)–(C.11). We have $B_s(F, G) = \begin{pmatrix} B_{1,s}(F,G) \\ B_{2,s}(F,G) \end{pmatrix}$, with

$$\begin{aligned} &\hat{B}_{i,s}(F, G)(\eta, t) \\ &\quad = \int \hat{F}^T(\xi, t) \begin{pmatrix} K_{i,s}(\xi, \eta-\xi) & L_{i,s}(\xi, \eta-\xi) \\ M_{i,s}(\xi, \eta-\xi) & N_{i,s}(\xi, \eta-\xi) \end{pmatrix} \hat{G}(\eta-\xi, t)\, d\xi, \end{aligned} \tag{C.19}$$

where

$$\begin{aligned} &K_{1,s}(\xi, \eta-\xi) = |\eta|^s |\xi|^{-s} |\eta-\xi|^{-s} K_1(\xi, \eta-\xi), \\ &N_{1,s}(\xi, \eta-\xi) = |\eta|^s |\xi|^{-s-1/2} |\eta-\xi|^{-s-1/2} N_1(\xi, \eta-\xi), \\ &(L_{2,s} + \tilde{M}_{2,s})(\xi, \eta-\xi) \\ &\quad = |\eta|^{s+1/2} |\xi|^{-s} |\eta-\xi|^{-s-1/2} (L_2 + \tilde{M}_2)(\xi, \eta-\xi), \\ &K_{2,s} = N_{2,s} = M_{1,s} = L_{1,s} = 0. \end{aligned} \tag{C.20}$$

If $s = 0$, i.e. if $(h, |D|^{1/2}\psi)$ is the unknown and assumed small and localized, we see from (C.17) that the Fourier symbols $K_{i,0}$, $L_{i,0} + \tilde{M}_{i,0}$ and $N_{i,0}$ of the bilinear operator B_0 are locally bounded. We know from the Coifman-Meyer theory [7] that the Fourier symbol being bounded is the most basic assumption for obtaining boundedness of the bilinear operator.[17] For B_0 in particular, its boundedness properties can be derived from results in harmonic analysis [7, 25] by using the identity (C.12) and inequality (C.14).

[15] We say a function f is localized if f tends to 0 at the spatial infinity.

[16] Let the trace of the velocity $\mathbf{v}(X, h(X,t), t) = (\mathbf{v}_1, \ldots, \mathbf{v}_n)$, $X = (x_1, \ldots, x_{n-1})$. From the chain rule, we have $\partial_{x_i}\psi = \mathbf{v}_i + \mathbf{v}_n \partial_{x_i} h$, $i = 1, \ldots, n-1$.

[17] Here we restrict ourself to discussing the type of bilinear operators given by (C.1) only.

On the other hand when $(|D|^{1/2}h, |D|\psi)$ is taken as the unknown and assumed small and localized as in this paper, i.e. when $s = 1/2$, we know from (C.18) that of $B_{1/2}$ the Fourier symbols: $(L_{2,1/2} + \tilde{M}_{2,1/2})(\xi, \eta - \xi)$ has a small divisor $1/|\xi|^{1/2}$, $K_{1,1/2}(\xi, \eta - \xi)$ has small divisors $1/|\xi|^{1/2}$ and $1/|\eta - \xi|^{1/2}$. We will see next that the difficulty of these small divisors amounts to proving such inequality $\|f\|_{L^2(\mathbb{R}^{n-1})} \lesssim \||D|^{1/2} f\|_{L^2(\mathbb{R}^{n-1})}$. This is impossible even if f satisfies moment conditions.[18]

C.3 The bilinear partial transformation of $(I - \mathcal{H})\mathfrak{z}e_3$

We now analyze the partial transformation consisting only of the linear and bilinear terms of $(I - \mathcal{H})\mathfrak{z}e_3 = U_k^{-1}(I - \mathfrak{H})ze_3$.

First it is not difficult to see that the bilinear term in the projection $(I - \mathfrak{H})ze_3$ is a bounded operator belonging to the class considered in Propositions 2.6, 2.7 and has locally bounded Fourier symbols. On the other hand, if a normal form transformation G is exactly obtained from the original unknown g by a change of coordinates k^{-1}, i.e. $G(X) = g(k^{-1}(X))$, with $k(X) - X$ a small quantity, then from the Taylor expansion

$$G(X) = g(k^{-1}(X)) = g(X) + \nabla g(X) \cdot (k^{-1}(X) - X) + \cdots, \tag{C.21}$$

the partial transformation consisting only of the linear and quadratic terms in $G(X)$ is

$$G_1(X) = g(X) + \nabla g(X) \cdot (k^{-1}(X) - X).^{19} \tag{C.22}$$

To obtain $\|G_1\| \approx \|g\|$ in various norms $\|\cdot\|$ for small $\|g\|$, one cannot avoid engaging the smallness of the norms of $k^{-1}(X) - X$, and the dependence on ∇g in the quadratic term can also be problematic. On the other hand, Lemma 4.2 shows that for the full coordinate change $G = g \circ k^{-1}$, $\|G\| \approx \|g\|$ in various norms $\|\cdot\|$ provided the norms of $\nabla k^{-1} - I$ are finite, here I is the identity map; we do not need to know the norm of the primitive $k^{-1}(X) - X$ itself if we consider the full coordinate change. As we know, the composition of the partial transformation consisting of the linear and bilinear parts of the projection $(I - \mathfrak{H})$ and the coordinate change U_k^{-1} gives in part the bilinear normal form transformation obtained in (C.20) (for $s = 1/2$). We conclude that the coordinate change U_k^{-1} has in part taken care of the difficulty of the small divisor associated to the case $s = 1/2$ considered in this paper.[20]

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[18] We remark that for periodic domains $\mathbb{T}^d$, such inequality $\|f\|_{L^2(\mathbb{T}^d)} \lesssim \||D|^{1/2} f\|_{L^2(\mathbb{T}^d)}$ holds for f satisfying the moment condition $\int_{\mathbb{T}^d} f = 0$.

[19] This suggests an algorithm to handle problems where the bilinear normal form transformation contain small divisors attributable to a change of coordinates.

[20] Note that the differences between $\partial_t^2 - i\mathfrak{a}\partial_\alpha$ and $\partial_t^2 + |D|$ in 2D; $\partial_t^2 - e_2\partial_\alpha + e_1\partial_\beta$ and $\partial_t^2 + |D|$, and the two quadratic terms in (1.35) in 3D are not accounted for here.

Indeed, after some further analysis and reformulation of the water wave equations, we realized that the small divisor could be taken care of by a properly constructed coordinate change, we therefore looked further for a better understanding of the bounded part of the bilinear normal form transformation. While for 3D water wave it was quite difficult, it was possible for us to find that the projection $(I - \mathfrak{H})$ is responsible for the bounded part in the bilinear normal form transformation for the 2D water waves. We then used the knowledge of the 2D water wave to derive corresponding results for 3D.

And indeed when applying an ODE method such as the method of normal forms to a PDE, the least one should take into consideration is the possibility of a better coordinate system.

Finally, we mention the recent work of Germain, Masmoudi, Shatah [13], in which they assumed smallness of $(h, |D|^{1/2}\psi)$ in $L^2(|x|^2\,dx) \cap W^{6,1}(dx) \cap H^N(dx)$ for some large N initially and studied the global existence and scattering of the 3D water wave through analyzing the space-time resonance. Here H^N is the L^2 Sobolev space with N derivatives, $W^{6,1}$ is the L^1 Sobolev space of 6 derivatives. As discussed earlier, this is the case when the bilinear normal form transformation have locally bounded Fourier symbols. While their method is algorithmic, and applicable to some other problems [12], we note that their assumptions on the initial data implies some quite strong decay properties of the velocity potential ψ. As we know ([25], p. 117)

$$c_0\psi(x) = \int \frac{1}{|x-y|^{2-1/2}} |D|^{1/2}\psi(y)\,dy$$
$$= \left(\int_{|y|\le\frac{1}{2}|x|} + \int_{|y|\ge 2|x|} + \int_{\frac{1}{2}|x|\le|y|\le 2|x|}\right) \frac{1}{|x-y|^{2-1/2}} |D|^{1/2}\psi(y)\,dy,$$

where

$$\left|\left(\int_{|y|\le\frac{1}{2}|x|} + \int_{|y|\ge 2|x|}\right) \frac{1}{|x-y|^{2-1/2}} |D|^{1/2}\psi(y)\,dy\right|$$
$$\lesssim \frac{1}{|x|^{3/2}} (\||D|^{1/2}\psi\|_{L^1(dx)} + \||D|^{1/2}\psi\|_{L^2(|x|^2dx)})$$

and

$$\int \left|\int_{\frac{1}{2}|x|\le|y|\le 2|x|} \frac{1}{|x-y|^{2-1/2}} |D|^{1/2}\psi(y)\,dy\right|^2 |x|\,dx$$
$$\le \int \left(\int_{\frac{1}{2}|x|\le|y|\le 2|x|} \frac{1}{|x-y|^{3/2}}\,dy\right)$$

$$\times \left(\int_{\frac{1}{2}|x| \le |y| \le 2|x|} \frac{1}{|x-y|^{3/2}} ||D|^{1/2}\psi(y)|^2\, dy \right) |x|\, dx$$

$$\lesssim \int \left(\int_{\frac{1}{2}|y| \le |x| \le 2|y|} \frac{|x|^{3/2}}{|x-y|^{3/2}}\, dx \right) ||D|^{1/2}\psi(y)|^2\, dy$$

$$\lesssim \int |y|^2 ||D|^{1/2}\psi(y)|^2\, dy.$$

So if ψ satisfies $|D|^{1/2}\psi \in L^1(dx) \cap L^2(|x|^2 dx)$ as assumed in [13], it is necessary then that $\psi(x)$ decays at a rate no slower than $1/|x|^{3/2}$ as $|x| \to \infty$. This is a rather restrictive assumption on the initial data. As while it is physically reasonable to assume the velocity field $\mathbf{v} \to 0$ at the spatial infinity, the velocity potential itself does not decay, in particular does not decay at such rates at the spatial infinity in general.

References

1. Ambrose, D., Masmoudi, N.: The zero surface tension limit of two-dimensional water waves. Commun. Pure Appl. Math. **58**(10), 1287–1315 (2005)
2. Beale, T., Hou, T., Lowengrub, J.: Growth rates for the linearized motion of fluid interfaces away from equilibrium. Commun. Pure Appl. Math. **46**(9), 1269–1301 (1993)
3. Birkhoff, G.: Helmholtz and Taylor instability. In: Proc. Symp. in Appl. Math. Vol. XIII, pp. 55–76
4. Chen, S., Zhou, Y.: Decay rate of solutions to hyperbolic system of first order. Acta Math. Sin. Engl. Ser. **15**(4), 471–484 (1999)
5. Christodoulou, D., Lindblad, H.: On the motion of the free surface of a liquid. Commun. Pure Appl. Math. **53**(12), 1536–1602 (2000)
6. Coutand, D., Shkoller, S.: Wellposedness of the free-surface incompressible Euler equations with or without surface tension. J. Am. Math. Soc. **20**(3), 829–930 (2007)
7. Coifman, R.R., Meyer, Y.: Au delá des opérateurs pseudodifférentials. Asterisque, vol. 57. Societé Math. de France, Paris (1978)
8. Coifman, R., McIntosh, A., Meyer, Y.: Líntegrale de Cauchy definit un operateur borne sur L^2 pour les courbes lipschitziennes. Ann. Math. **116**, 361–387 (1982)
9. Coifman, R., David, G., Meyer, Y.: La solution des conjectures de Calderón. Adv. Math. **48**, 144–148 (1983)
10. Craig, W.: An existence theory for water waves and the Boussinesq and Korteweg-deVries scaling limits. Commun. Partial Differ. Equ. **10**(8), 787–1003 (1985)
11. Craig, W.: Non-existence of solitary water waves in three dimensions. Philos. Trans. R. Soc., Ser. A, Math. Phys. Eng. Sci. **360**(1799), 2127–2135 (2002)
12. Germain, P., Masmoudi, N., Shatah, J.: Global solutions for 3D quadratic Schrödinger equations. IMRN **3**, 414–432 (2009)
13. Germain, P., Masmoudi, N., Shatah, J.: Global solutions of the gravity water wave equation in dimension 3. Preprint July 2009
14. Gilbert, J., Murray, M.: Clifford Algebras and Dirac Operators in Harmonic Analysis. Cambridge University Press, Cambridge (1991)
15. Iguchi, T.: Well-posedness of the initial value problem for capillary-gravity waves. Funkc. Ekvacioj **44**(2), 219–241 (2001)

16. Kenig, C.: Elliptic boundary value problems on Lipschitz domains. In: Stein, E.M. (ed.) Beijing Lectures in Harmonic Analysis, pp. 131–183. Princeton Univ. Press, Princeton (1986)
17. Klainerman, S.: The null condition and global existence to nonlinear wave equations. Lect. Appl. Math. **23**, 293–325 (1986)
18. Lannes, D.: Well-posedness of the water-wave equations. J. Am. Math. Soc. **18**, 605–654 (2005)
19. Lindblad, H.: Well-posedness for the motion of an incompressible liquid with free surface boundary. Ann. Math. **162**(1), 109–194 (2005)
20. Nalimov, V.I.: The Cauchy-Poisson problem. Dynamika Splosh. Sredy **18**, 104–210 (1974) (in Russian)
21. Ogawa, M., Tani, A.: Free boundary problem for an incompressible ideal fluid with surface tension. Math. Models Methods Appl. Sci. **12**(12), 1725–1740 (2002)
22. Sogge, C.: Lectures on Nonlinear Wave Equations. International Press, Boston (1995)
23. Shatah, J.: Normal forms and quadratic nonlinear Klein-Gordon equations. Commun. Pure Appl. Math. **38**, 685–696 (1985)
24. Shatah, J., Zeng, C.: Geometry and a priori estimates for free boundary problems of the Euler's equation. Commun. Pure Appl. Math. **61**(5), 698–744 (2008)
25. Stein, E.M.: Singular Integrals and Differentiability Properties of Functions. Princeton University Press, Princeton (1970)
26. Strauss, W.: Nonlinear wave equations CBMS No. 73. AMS (1989)
27. Sulem, C., Sulem, P.-L.: The Nonlinear Schrodinger Equation. Self-focusing and Wave Collapse. Applied Mathematical Sciences, vol. 139. Springer, New York (1999)
28. Sun, S.M.: Some analytical properties of capillary-gravity waves in two-fluid flows of infinite depth. Proc. R. Soc. Lond., Ser. A **453**(1961), 1153–1175 (1997)
29. Taylor, G.I.: The instability of liquid surfaces when accelerated in a direction perpendicular to their planes I. Proc. R. Soc. Lond., Ser. A **201**, 192–196 (1950)
30. Wu, S.: Well-posedness in Sobolev spaces of the full water wave problem in 2-D. Invent. Math. **130**, 39–72 (1997)
31. Wu, S.: Well-posedness in Sobolev spaces of the full water wave problem in 3-D. J. Am. Math. Soc. **12**(2), 445–495 (1999)
32. Wu, S.: Almost global wellposedness of the 2-D full water wave problem. Invent. Math. **177**(1), 45–135 (2009)
33. Yosihara, H.: Gravity waves on the free surface of an incompressible perfect fluid of finite depth. RIMS Kyoto **18**, 49–96 (1982)
34. Zhang, P., Zhang, Z.: On the free boundary problem of 3-D incompressible Euler equations. Commun. Pure Appl. Math. **61**(7), 877–940 (2008)