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GLOBALLY HYPOELLIPTIC AND GLOBALLY SOLVABLE FIRST ORDER EVOLUTION EQUATIONS

BY

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ABSTRACT. We consider global hypoellipticity and global solvability of abstract first order evolution equations defined either on an interval or in the unit circle, and prove that it is equivalent to certain conditions bearing on the total symbol. We relate this to known results about hypoelliptic vector fields on the 2-torus.

0. Introduction. Let A denote a linear selfadjoint operator, densely defined in a complex Hilbert space H, which is unbounded, positive, and has a bounded inverse A^{-1}. Such an operator defines a scale of Sobolev spaces H^s ($s \in R$). Their intersection $\bigcap_s H^s$ is denoted by H^∞ and their union $\bigcup_s H^s$ by $H^{-\infty}$.

Let Ω be either an open interval of $\mathbf{R}$ or the unit circle S^1. We will consider first-order evolution operators of the form

$$\mathbf{L} = \partial_t + b(t, A)A, \qquad t \in \Omega, \tag{0.1}$$

where ∂_t means $\partial/\partial t$ and the coefficient $b(t, A)$ belongs to the ring $\mathfrak{Q}_A$ of series in the nonnegative powers of A^{-1} with complex coefficients in $C^\infty(\Omega)$, which converge in $\mathcal{L}(H, H)$ as well as each one of their t-derivatives, uniformly with respect to t in compact subsets of Ω. (For more details on these definitions see [1], [3].)

We denote by $C^\infty(\Omega; H^\infty)$ ($C^\infty(\Omega; H^{-\infty})$) the space of smooth functions defined in Ω and valued in H^∞ ($H^{-\infty}$), and by $C_c^\infty(\Omega; H^{\pm\infty})$ the compactly supported functions of $C^\infty(\Omega; H^{\pm\infty})$.

REMARK 0.1. The requirement that A be strictly positive is inessential, since one can always work with the scale of spaces defined by $(I + A^2)^{1/2}$. That is, for instance, the case when $A = (1/i)(\partial/\partial x)$ in $\mathbf{R}$ or S^1.

DEFINITION 0.1. *Let t_0 be any point of Ω. We say that $\mathbf{L}$ is locally solvable at t_0 if there is an open neighborhood $V \subset \Omega$ of t_0 such that, to every $f \in C_c^\infty(V; H^\infty)$ there is $u \in C^\infty(\Omega; H^{-\infty})$ satisfying*

$$\mathbf{L}u = f \quad \textit{in } V. \tag{0.2}$$

We say that $\mathbf{L}$ is locally solvable in Ω if $\mathbf{L}$ is locally solvable for all $t_0 \in \Omega$.

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Definition 0.2. *We say that* $\mathbf{L}$ *is hypoelliptic in* Ω *if, given any open subset* V *of* Ω *and* $u \in C^\infty(\Omega; H^{-\infty})$,

$$\mathbf{L}u \in C^\infty(V; H^\infty) \Rightarrow u \in C^\infty(V; H^\infty). \tag{0.3}$$

Definition 0.3. *We say that* $\mathbf{L}$ *is globally hypoelliptic in* Ω *if, given* $u \in C^\infty(\Omega; H^{-\infty})$,

$$\mathbf{L}u \in C^\infty(\Omega; H^\infty) \Rightarrow u \in C^\infty(\Omega; H^\infty). \tag{0.4}$$

Remark 0.2. There is no gain in generality in Definitions 0.1, 0.2 and 0.3, if we replace our "space of solutions" $C^\infty(\Omega; H^{-\infty})$ by the bigger space $\mathcal{D}'(\Omega; H^{-\infty})$ (defined in [3]), since $\mathbf{L}u \in C^\infty(V; H^{-\infty})$ and $u \in \mathcal{D}'(V; H^{-\infty})$ imply that $u \in C^\infty(V; H^{-\infty})$, whatever $V \subset \Omega$. This is a particular case of Proposition III.2.2 in [4], and may be regarded as a partial hypoellipticity result in the t-direction.

We will need the following results.

Proposition 0.1. *Suppose that* $\mathbf{L}$ *is globally hypoelliptic in* Ω. *Then for every integer* $N \geqslant 0$, *there is another integer* $M \geqslant 0$ *and a constant* $C > 0$ *such that, for all u in* $C^\infty(\Omega, H^\infty)$ *bounded* (*with bounded derivatives*)

$$\sup_{t\in\Omega} \sum_{j+k\leqslant N} \|A^j\partial_t^k u\|_0 \leqslant C\Big(\sup_{t\in\Omega}\|u\|_0 + \sum_{j+k\leqslant M}\|A^j\partial_t^k \mathbf{L}u\|_0\Big).$$

The proof is a variation of [5, Theorem 52.2].

We denote by $\mathbf{L}^*$ the formal adjoint of $\mathbf{L}$, i.e. the operator defined by

$$\int(\mathbf{L}^*u, v)_0\, dt = \int (u, \mathbf{L}v)_0\, dt, \qquad u, v \in C_c^\infty(\Omega; H^\infty), \tag{0.5}$$

where $(\ ,\ )_0$ indicates the inner product in $H = H^0$. If $\mathbf{L}$ is given by (0.1) then its adjoint will be

$$\mathbf{L}^* = -\partial_t + b(t, A)^*A = -\partial_t + \bar{b}(t, A)A \tag{0.6}$$

where the coefficients in the development of $\bar{b}(t, A)$ are the complex conjugates of those of $b(t, A)$.

Proposition 0.1 has the following standard

Corollary 0.1. *If* $\mathbf{L}$ *is globally hypoelliptic in* Ω, *its formal adjoint* $\mathbf{L}^*$ *is locally solvable.*

I am greatly indebted to Professor F. Tréves who has permitted the inclusion of interesting unpublished results of his in §3. They add interest to the hypoellipticity sections, which were the original subject of this paper.

1. The global hypoellipticity when Ω is an interval. The local solvability and hypoellipticity of the first-order evolution operator (0.1) has been thoroughly studied (see [3]). Here the geometry of Ω does not enter the picture, due to the local nature of these properties. On the other hand, the study of the global hypoellipticity will reveal quite different answers in the cases $\Omega = (a, b) \subset \mathbf{R}$ and $\Omega = S^1$.

In this section Ω will be an open interval of $\mathbf{R}$, which we will take to be $(0, 1)$. We recall the following theorems:

THEOREM 1.1. *The differential operator* $\mathbf{L}$, *given by* (0.1), *is locally solvable in* Ω *if and only if the following is true*:

(ψ) *if* $\operatorname{Re} b_0(t_0) > 0$ *for some* $t_0 \in \Omega$, *then* $\operatorname{Re} b_0(t) \geqslant 0$ *for all* $t \in \Omega$, $t > t_0$, *where* $b_0(t)$ *is the leading coefficient in* $b(t, A)$.

THEOREM 1.2. *The differential operator* $\mathbf{L}$ *is hypoelliptic in* Ω *if and only if the following two conditions are satisfied*:

(ψ^*) *if* $\operatorname{Re} b_0(t_0) < 0$ *for some* $t_0 \in \Omega$, *then* $\operatorname{Re} b_0(t) \leqslant 0$ *for all* $t \in \Omega$, $t > t_0$;

($\mathcal{Q}$) $\operatorname{Re} b_0$ *does not vanish identically in any* (*nonempty*) *open subinterval of* Ω.

REMARK 1.1. Condition (ψ^*) implies that there exists a number $c \in \bar{\Omega} = [0, 1]$ such that $\operatorname{Re} b_0$ is nonnegative in $(0, c]$ and nonpositive in $[c, 1)$.

We introduce the following open set

$$\mathfrak{T} = \operatorname{int}\{t \in (0, 1) | \operatorname{Re} b_0(t) = 0\}. \tag{1.1}$$

REMARK 1.2. With notation (1.1), condition ($\mathcal{Q}$) reads: $\mathfrak{T}$ is empty.

We are going to need also two conditions defined for a component (α, β) of $\mathfrak{T}$:

(τ_1) $\alpha > 0$ and $\operatorname{Re} b_0(t) \geqslant 0$ for all $t \in (0, \alpha)$;

(τ_2) $\beta < 1$ and $\operatorname{Re} b_0(t) \leqslant 0$ for all $t \in (\beta, 1)$.

DEFINITION 1.1. *Condition* (τ) *holds if every component* (α, β) *of* $\mathfrak{T}$ *verifies either* (τ_1) *or* (τ_2).

We observe that if $\mathfrak{T} = \varnothing$, ($\tau$) is satisfied trivially so $(\mathcal{Q}) \Rightarrow (\tau)$. On the other hand it is clear that $(\tau) \nRightarrow (\mathcal{Q})$. The main reason for the introduction of condition (τ) is the following.

THEOREM 1.3. *The differential operator* $\mathbf{L}$, *given by* (0.1), *is globally hypoelliptic in* Ω *if and only if conditions* (ψ^*) *and* (τ) *hold.*

PROOF. If $\mathbf{L}$ is globally hypoelliptic, $\mathbf{L}^*$, its formal adjoint, is locally solvable (Corollary 0.1), so condition (ψ) holds for $\mathbf{L}^*$ (Theorem 1.1) which naturally implies that (ψ^*) holds for L. Let us suppose that (τ) does not hold and let (α, β) be a component of $\mathfrak{T}$. If $\alpha = 0$ and $\beta = 1$, $\mathbf{L} \sim \partial_t$ is obviously not globally hypoelliptic. Say $\alpha > 0$; we may find t_0, $0 < t_0 < \alpha$, such that $\operatorname{Re} b_0(t_0) < 0$ and (ψ^*) implies readily that $\beta = 1$. Consider an element $h_0 \in H^0 \setminus H^\infty$, and a C^∞ function $\phi(t)$ in Ω which is zero for $t \leqslant t_0$ and identically one if $t \geqslant t_0 + \varepsilon$. Here $\varepsilon > 0$ is chosen so that

$$\int_t^1 \operatorname{Re} b_0(s)\, ds \leqslant -\rho < 0 \quad \text{for all } t,\ t_0 \leqslant t \leqslant t_0 + \varepsilon. \tag{1.2}$$

Let us define

$$u(t) = \phi(t) \cdot \exp\left\{ \int_t^1 b(s, A)A \, ds \right\} h_0. \tag{1.3}$$

Since $u(\alpha) \notin H^\infty$ it is clear that $u \notin C^\infty(\Omega; H^\infty)$. On the other hand,

$$\mathbf{L}u(t) = \phi'(t) \exp\left\{ \int_t^1 b(s, A)A \, ds \right\} h_0 \tag{1.4}$$

so $\mathbf{L}u \in C^\infty(\Omega; H^\infty)$. Indeed, in a neighborhood V of the set where $\phi'(t) = 0$, the operator $\exp(\int_t^1 b(s, A)A \, ds)$ maps $C^\infty(V; H^{-\infty})$ into $C^\infty(V; H^\infty)$, as follows from (1.2). This contradicts the global hypoellipticity of $\mathbf{L}$.

To prove the "if" part we will need

LEMMA 1.1. *Suppose that* (ψ^*) *and* (τ) *hold. Then every* $x \in (0, 1)$ *either verifies*

(i) $\operatorname{Re} b_0(t) \geq 0$ *for all* $t \in (0, x)$ *and* $\int_0^x \operatorname{Re} b_0(t) \, dt > 0$, *or*

(ii) $\operatorname{Re} b_0(t) \leq 0$ *for all* $t \in (x, 1)$ *and* $\int_1^x \operatorname{Re} b_0(t) \, dt > 0$.

PROOF. Since $\operatorname{Re} b_0$ changes sign at most once, and in that case from positive to negative, it is clear that, if $\operatorname{Re} b_0(x) > 0$ (i) holds, and if $\operatorname{Re} b_0(x) < 0$ (ii) holds. Suppose then that $\operatorname{Re} b_0(x) = 0$ and that neither (i) nor (ii) holds.

If there is a $t_0 < x$ such that $\operatorname{Re} b_0(t_0) < 0$ then $\operatorname{Re} b_0(t) \leq 0$ for all $t \geq t_0$. Since $\int_1^x \operatorname{Re} b_0 \leq 0$ we conclude that $\operatorname{Re} b_0(t) = 0$ for $t \geq x$. Hence, there is a connected component $(\alpha, 1)$ in $\mathfrak{T}$ with $\alpha \leq x$. This contradicts (τ_2) and also (τ_1). So we may assume that $\operatorname{Re} b_0(t) \geq 0$ for $t \leq x$. Using $\int_0^x \operatorname{Re} b_0 \leq 0$ we find as before a component $(0, \beta)$ with $x \leq \beta \leq 1$. So (τ_1) does not hold and (τ_2) must. Then $\int_1^x \operatorname{Re} b_0 = \int_1^\beta \operatorname{Re} b_0 > 0$ and the negation of (ii) implies that there is a $t_1 > \beta$ with $\operatorname{Re} b_0(t_1) > 0$. This is incompatible with (τ_2).

END OF THE PROOF OF THEOREM 1.3. Let $u \in C^\infty(\Omega; H^{-\infty})$ such that $\mathbf{L}u = f \in C^\infty(\Omega; H^\infty)$ and fix $x \in (0, 1) = \Omega$. According to Lemma 1.1 we may suppose that, say, $\operatorname{Re} b_0(t) \geq 0$ for all $0 < t < x$ and $\int_0^x \operatorname{Re} b_0(t) \, dt > 0$. We choose $0 < \eta < x$ so that

$$\int_s^t \operatorname{Re} b_0(t') \, dt' \geq \eta \quad \text{if } 0 \leq s \leq \eta, \;\; x - \eta \leq t \leq x + \eta. \tag{1.5}$$

Next we pick a function $\phi \in C^\infty(\Omega)$ such that $\phi(0) = 0$ and $\phi(t) = 1$ if $t \geq \eta$. Finally consider the operator

$$(K_1 g)(t) = \int_0^t \exp\left(-\int_s^t b(t', A) \, dt' A \right) g(s) \, ds. \tag{1.6}$$

If $c = \sup\{t | \operatorname{Re} b_0(t) \geq 0\}$, it is clear that K_1 maps $C^\infty([0, c]; H^\infty)$ (resp. $C^\infty([0, c]; H^{-\infty})$) into itself. Furthermore, if $g(0) = 0$, we see by integration by parts that

$$(K_1\mathbf{L}g)(t) = g(t) - \exp\left(-\int_0^t b(t', A)\, dt' A\right)g(0) = g(t). \tag{1.7}$$

In particular, when $g = \phi u$, $\mathbf{L}g = \phi f + \phi' u$, so

$$\phi u = K_1(\phi f) + K_1(\phi' u). \tag{1.8}$$

Now $f \in C^\infty(\Omega; H^\infty)$ so $\phi f \in C^\infty([0, c]; H^\infty)$. On the other hand, if $|x - t| < \eta$ and $s \in \operatorname{supp} \phi'$, we have, using (1.5) that

$$-\int_s^t \operatorname{Re} b_0(t')\, dt' \leq -\eta \tag{1.9}$$

so it follows easily that $K_1(\phi' u)$ belongs to $C^\infty((x - \eta, x + \eta); H^\infty)$. Since $x \leq c$, we conclude that ϕu, hence u, is in $C^\infty((x - \eta, c]; H^\infty)$. If $x < c$ this shows that $u(t)$ is smooth with values in H^∞ in a neighborhood of x. If $x = c$, $\operatorname{Re} b_0(t) \leq 0$ for $t > c$ and similar reasoning with

$$(K_2 g)(t) = \int_1^t \exp\left(-\int_s^t b(t', A)\, dt' A\right)g(s)\, ds \tag{1.10}$$

substituted for K_1 shows that $u \in C^\infty([x, x + \varepsilon); H^\infty)$ for a certain $\varepsilon > 0$. Since x is arbitrary, $u \in C^\infty(\Omega; H^\infty)$. Q.E.D.

We may now obtain Theorem 1.2 as a corollary of the global result.

COROLLARY 1.1. *The differential operator* $\mathbf{L}$ *is hypoelliptic in* Ω *iff* (ψ^*) *and* $(\mathcal{Q})$ *hold in* Ω.

PROOF. $\mathbf{L}$ is hypoelliptic in Ω if and only if it is globally hypoelliptic in every subinterval $\Omega' \subset \Omega$. Furthermore it is clear that (ψ^*) holds in Ω if and only if it does in every $\Omega' \subset \Omega$ and that (τ) holds in every $\Omega' \subset \Omega$ if and only if $\mathcal{T} = \varnothing$, i.e., $(\mathcal{Q})$ holds in Ω. Q.E.D.

2. The periodic case ($\Omega = S^1$). We now consider the evolution operator given by (0.1) defined in the unit circle S^1, which we will identify with $T^1 = \mathbf{R}/2\pi\mathbf{Z}$. Thus, $\mathbf{L}$ may be thought of as an operator on $\mathbf{R}$ with periodic coefficients, and we look at periodic solutions, when periodic data are prescribed.

Theorem 1.3 characterizes globally hypoelliptic evolution operators of order one exclusively in terms of properties of the real part of the leading coefficient, $\operatorname{Re} b_0(t)$. In particular, the structure of the spectrum of A, $\sigma(A)$, does not enter into the picture. On the other hand, when $\Omega = T^1$, the behavior of $\sigma(A)$ in a neighborhood of infinity will play an important role.

According to the hypotheses made on A, $\sigma(A) \subseteq [\rho, \infty) \subseteq \mathbf{R}$, for a certain positive ρ. We introduce a function

$$r(z) = \frac{1}{2\pi i}\int_0^{2\pi} z b(t, z)\, dt = r_0 z + r_1 + r_2 z^{-1} + \dots, \qquad z \in \mathbf{C}, \tag{2.1}$$

which is analytic on $|z| > \rho$, and either has a pole of order one or a removable singularity at $z = \infty$. We will also make use of the subset of $\mathbf{C}$ given by

$$\Lambda = \{z \mid |z| > \rho, r(z) \in \mathbf{Z}\} \tag{2.2}$$

where $r(z)$ takes integral values. It is clear that Λ is a discrete set when $r(z)$ is not a constant. We denote by

$$d(\xi) = d(\xi, \Lambda) = \inf_{z \in \Lambda} |\xi - z|, \qquad \xi \in \mathbf{R}, \tag{2.3}$$

the distance from $\xi \in \mathbf{R}$ to the set Λ.

THEOREM 2.1. *The differential operator* $\mathbf{L}$, *given by* (0.1) *and defined on* T^1, *is globally hypoelliptic in* T^1 *if and only if the following conditions hold*:

($\mathcal{P}$) Re b_0 *does not change sign in* T^1;

($\mathcal{G}$) *if* Re $b_0 \equiv 0$, *there exist a positive constant c and a positive integer N such that*

$$d(\xi) > c\xi^{-N} \tag{2.4}$$

for all $\xi \in \sigma(A)$ *sufficiently large.*

We will refer to an estimate like (2.4) saying that $d(\xi)$ decreases slowly at infinity on $\sigma(A)$.

We recall that if $\mathbf{L}$ is a differential operator on a smooth orientable manifold M, we say that $\mathbf{L}$ is globally hypoelliptic if $u \in \mathcal{D}'(M)$ (the distributions on M) and $\mathbf{L}u \in C^\infty(M)$ imply that $u \in C^\infty(M)$.

THEOREM 2.2. *Let $b(t)$ be a smooth complex function on T^1 and consider the vector field*

$$\mathbf{L} = \frac{\partial}{\partial t} - b(t)\frac{\partial}{\partial x} \tag{2.5}$$

defined on the 2-torus $T^2 = T^1 \times T^1 = \{e^{it}\} \times \{e^{ix}\}$. *Then* $\mathbf{L}$ *is globally hypoelliptic if and only if the following conditions hold*:

($\mathcal{P}$) Im $b(t)$ *does not change sign*;

($\mathcal{W}$) *if* Im $b(t) \equiv 0$, $\gamma = (2\pi)^{-1}\int_0^{2\pi}$Re $b(t)\,dt$ *is an irrational non-Liouville number.*

PROOF. Take A to be (a selfadjoint extension of) $(1/i)(\partial/\partial x)$ in $H_x = L^2(\{e^{ix}\})$. Then $(I + A^2)^{1/2}$ defines the usual scale of Sobolev spaces H_x^s in T^1, in particular $C^\infty(T_t^1; H_x^\infty) = C^\infty(T^2)$, $C^\infty(T_t^1; H_x^{-\infty}) = C^\infty(T_t^1; \mathcal{D}'(T_x^1))$ and $\mathcal{D}'(T_t^1; H_x^{-\infty}) = \mathcal{D}'(T^2)$. In view of Remarks 0.1 and 0.2, we see that L is globally hypoelliptic in the usual sense iff it is globally hypoelliptic in the sense of Definition 0.3, so Theorem 2.1 applies. Here $\sigma(A) = \mathbf{Z}$ and $\Lambda = \{z \mid \gamma z \in \mathbf{Z}\}$ so it is not difficult to see ($\mathcal{G}$) specializes to ($\mathcal{W}$). Q.E.D.

When $b(t)$ is constant in (2.5), Theorem 2.2 yields a result due to Greenfield and Wallach (see [2]) namely

COROLLARY 2.1. *The complex vector field* $L = \partial_t - b\partial_x$, $b \in \mathbf{C}$, *is globally hypoelliptic in* T^2 *if and only if* $\operatorname{Im} b \neq 0$ *or* $\operatorname{Re} b$ *is an irrational non-Liouville number.*

When $\sigma(A) = [\rho, \infty)$, condition $(\mathcal{G})$ can be simply expressed in terms of the coefficients of $r(z)$. We have

THEOREM 2.3. *Let* $\mathbf{L}$ *be as in Theorem* 2.1 *and assume that* $\sigma(A) = [\rho, \infty)$. *Then* $\mathbf{L}$ *is globally hypoelliptic unless*

(i) $\operatorname{Re} b_0(t)$ *changes sign, or*

(ii) $\operatorname{Re} b_0 \equiv 0$ *and all coefficients* r_k, $k = 0, 1, 2, \ldots,$ *are real. Moreover, if* $r_0 = 0$ *and* r_1 *is an integer all coefficients* r_k *must vanish for* $k \geq 2$.

In the proof of Theorem 2.1, we will need a few lemmas.

LEMMA 2.1. *Let* $r(z)$, $d(\xi)$ *be the functions defined in* (2.1) *and* (2.3), *respectively, and suppose that there is a coefficient* r_i *in the development* (2.1) *of* $r(z)$ *which is not real. Then, there are positive constants* c, N, *such that*

$$d(\xi) \geq c\xi^{-N} \tag{2.6}$$

for all $\xi \in \mathbf{R}$ *sufficiently large.*

PROOF. Set $\tilde{\Lambda} = \{z \in \mathbf{C} \mid |z| \geq \rho,\ r(z) \in \mathbf{R}\}$, $\tilde{d}(\xi) = d(\xi, \tilde{\Lambda})$. It will be enough to prove an estimate like

$$\tilde{d}(\xi) \geq c\xi^{-N}, \qquad \xi \text{ real and large}, \tag{2.7}$$

since $\tilde{d}(\xi) \leq d(\xi)$.

Assume first that $r_0 = 0$, so $r(z) = r_1 + [g(1/z)]^k$ for a certain $k \in \mathbf{N}$ and g analytic in a neighborhood of the origin, $g(0) = 0$, $g'(0) \neq 0$. If $\operatorname{Im} r_1 \neq 0$, $\tilde{d}(\xi)$ is bounded away from zero as $\xi \to \infty$, $\xi \in \mathbf{R}$. If $\operatorname{Im} r_1 = 0$,

$$\tilde{\Lambda} = \left\{ z \,\middle|\, g\left(\frac{1}{z}\right)^k \in \mathbf{R} \right\} = \bigcup_{j=0}^{k-1} \left\{ z \,\middle|\, \arg g\left(\frac{1}{z}\right) = \frac{j2\pi}{k} \right\} = \bigcup_{j=0}^{k-1} \left\{ z \,\middle|\, \frac{1}{z} \in \gamma_j \right\}.$$

The γ_j are analytic curves through the origin distinct from the real line, so they have finite order contact with the real axis. If (2.7) were false we could pick j, $0 \leq j \leq k-1$, and sequences $(\xi_n) \subseteq \mathbf{R}$, $(z_n) \subseteq \mathbf{C}$, $1/z_n \in \gamma_j$, so that $\xi_n \to \infty$ and $|\xi_n - z_n| \leq \xi_n^{-n}$. But then

$$\left| \frac{1}{\xi_n} - \frac{1}{z_n} \right| \leq \frac{1}{\xi_n |z_n|} \xi_n^{-n} \leq \xi_n^{-n-1} \qquad \text{as } \xi_n \to \infty,$$

which contradicts the fact that γ_j has at the origin finite order contact with the real axis.

If $r_0 \neq 0$, $r(z) = z(r_0 + r_1 z^{-1} + \dots) = zh(1/z)$, $h(0) \neq 0$ so we may write $r(z) = 1/f(1/z)$ with f analytic in a neighborhood of the origin, $f(0) = 0$, $f'(0) \neq 0$. Now $r(z)$ is real iff $f(1/z)$ is real and we proceed as before.

LEMMA 2.2. *Suppose that* Re $b_0 \equiv 0$ *and* $(\mathcal{G})$ *holds. Then there are positive constants c and N such that*

$$|1 - e^{-2\pi i r(\xi)}| \geqslant c\xi^{-N} \tag{2.8}$$

for all $\xi \in \sigma(A)$ *sufficiently large.*

PROOF. Suppose that there is a first coefficient r_k in the expansion of $r(z)$, such that $\operatorname{Im} r_k \neq 0$. If $k = 0$ or 1 it is clear that $|1 - e^{-2\pi i r(\xi)}|$ is bounded away from zero as $\xi \to \infty$. For $k \geqslant 2$, write $r_k = \alpha + i\beta$ ($\beta \neq 0$) and observe that

$$|1 - e^{-2\pi i r(\xi)}| \geqslant |1 - |e^{-2\pi i r(\xi)}|| = |1 - e^{2\pi \operatorname{Im} r(\xi)}| \geqslant |\beta|\xi^{1-k} + O(\xi^{-k})$$

as $\xi \to \infty$, so (2.8) holds.

Now suppose that $r(\xi)$ is real for ξ real, so $|1 - e^{-2\pi i r(\xi)}|^2 = 2(1 - \cos[2\pi r(\xi)])$. If $r_0 = 0$ $\cos(2\pi r(\xi)) \to \cos(2\pi r_1)$ so $|1 - e^{-2\pi i r(\xi)}|$ is bounded away from zero if $r_1 \notin \mathbf{Z}$. If $r_1 \in \mathbf{Z}$, there is a first $r_k \neq 0$, $k \geqslant 2$ (otherwise $\Lambda \supseteq (\rho, \infty)$ and $d(\xi) \equiv 0$ contradicting $(\mathcal{G})$) so for ξ large

$$1 - \cos(2\pi r(\xi)) = 1 - \cos(2\pi(r(\xi) - r_1)) = 2\pi^2 (r_k \xi^{1-k})^2 + O(\xi^{-2k})$$

and (2.8) holds.

Now assume $r_0 \neq 0$, and suppose that there is a sequence $\xi_n \in \sigma(A)$, such that $\xi_n \to \infty$ and $d(\xi_n) \to 0$, and take $\lambda_n \in \Lambda$ such that $d(\xi_n) = |\xi_n - \lambda_n|$. Now set $r(z) = r_0 z + r_1 + R_2(z)$. For n sufficiently large

$$|r(\xi_n) - r(\lambda_n)| \geqslant |r_0(\xi_n - \lambda_n)| - |R_2(\lambda_n) - R_2(\xi_n)| \geqslant \left|\frac{r_0}{2}\right| d(\xi_n),$$

so

$$\begin{aligned}(1 - \cos 2\pi r(\xi_n))^{1/2} &\geqslant \left[1 - \cos(\pi |r_0|\, d(\xi_n))\right]^{1/2} \\ &\geqslant \frac{\pi |r_0|}{2} d(\xi_n) + o(d(\xi_n))\end{aligned}$$

and (2.4) implies (2.8). Q.E.D.

LEMMA 2.3. *The function* $d(\xi)$ *defined in* (2.3) *decreases slowly at infinity on* $\sigma(A)$ *if and only if* $|1 - e^{-2\pi i r(\xi)}|$ *decreases slowly at infinity on* $\sigma(A)$.

PROOF. In view of the proofs of Lemmas 2.1 and 2.2, we only have to prove that $d(\xi)$ decreases slowly when $r(\xi)$ is real and $|1 - e^{-2\pi i r(\xi)}|$ decreases slowly. This can be easily checked along the lines of those lemmas.

PROOF OF THEOREM 2.1. *Sufficiency of* $(\mathcal{P})$ *and* $(\mathcal{G})$. Suppose that there is a point $t_0 \in T^1$ such that Re $b_0(t_0) > 0$ so Re $b_0(t) \geqslant 0$ in T^1 and look at $\mathbf{L}$ on the interval $\Omega' = (t_0, t_0 + 2\pi) \subseteq \mathbf{R}$. It is clear that condition (τ_1) of Definition 1.1 holds for $\mathbf{L}$ on Ω', so according to Theorem 1.3 $\mathbf{L}$ is globally hypoelliptic in any set of the form $T^1 - \{t_0\}$ with $\operatorname{Re} b_0(t_0) > 0$. Two such sets cover T^1 and we conclude that $\mathbf{L}$ is globally hypoelliptic in T^1. A similar reasoning takes care of the case $\operatorname{Re} b_0(t) \leq 0$, $\operatorname{Re} b_0(t) \not\equiv 0$.

Suppose now that $\operatorname{Re} b_0(t) \equiv 0$. Condition $(\mathcal{G})$ implies that there exist $M > 0$ so that whenever $\xi \in \sigma(A)$ is larger than M, the O.D.E.

$$\left(\frac{d}{dt} + b(t, \xi)\xi\right)u = f, \qquad 0 \leq t \leq 2\pi,$$
$$u(0) = u(2\pi) \tag{2.9}$$

has a unique solution, since all nontrivial solutions of the homogeneous problem are not 2π-periodic (indeed, all solutions of the homogeneous O.D.E. are multiples of $v(t) = \exp(-\int_0^t b(s, \xi)\xi\, ds)$ and $v(2\pi) = v(0) = 1$ implies that $r(\xi) \in \mathbf{Z}$, so $d(\xi) = 0$).

Now let $u \in C^\infty(T^1; H^{-\infty})$ be such that $\mathbf{L}u = f \in C^\infty(T^1; H^\infty)$ and set

$$I - \Pi = \int_M^\infty dE(\lambda) \tag{2.10}$$

where $\{E(\lambda)\}$ is the spectral resolution of A.

We may write $f = f_1 + f_2 = \Pi f + (I - \Pi)f$ so $f_1 \in C^\infty(T^1; \Pi(H^\infty))$. Since the restriction of A to $\Pi(H)$ is bounded and A commutes with Π, we see that $u_1 = \Pi u$ satisfies $\mathbf{L}u_1 = f_1$ and $u_1 \in C^\infty(H^1; \Pi(H^{-\infty})) \subseteq C^\infty(T^1; H^\infty)$. Hence we only need to show that $u_2 = u - u_1 \in C^\infty(T^1; H^\infty)$ and there is no restriction if we assume from the start that f is valued in $H^\infty \cap (I - \Pi)H$. In particular, $u(t)$ is uniquely determined by the formulas

$$u(t) = \int_0^t k_1(s, t, A)f(s)\, ds + \int_t^{2\pi} k_2(s, t, A)f(s)\, ds, \tag{2.11}$$

$$k_1(s, t, A) = \left[I - e^{-2\pi i r(A)}\right]^{-1} \cdot \exp\left[B(s, A) - B(t, A)\right],$$
$$k_2(s, t, A) = \left[I - e^{-2\pi i r(A)}\right]^{-1} \cdot \exp\left[B(s, A) - B(t, A) - B(2\pi, A)\right], \tag{2.12}$$

where we have used the following notation

$$B(t, A) = \int_0^t b(s, A)A\, ds = \int_0^t \int_{\sigma(A)} b(s, \lambda)\lambda\, dE(\lambda)\, ds,$$

$$2\pi i r(A) = 2\pi i \int_{\sigma(A)} r(\lambda)\, dE(\lambda) = \int_0^{2\pi} Ab(s, A)\, ds = B(2\pi, A). \tag{2.13}$$

In view of (2.11), (2.12) and (2.13) u will belong to $C^\infty(T^1; H^\infty)$ if we prove that $k_i(s, t, \lambda)$ and its t-derivatives ($i = 1, 2$) grow slower than powers of λ on $\sigma(A)$. Since Re $b_0 \equiv 0$, $\exp(B(t, \lambda))$ is bounded and its derivatives of order k grow slower than λ^k, it is a matter of checking that $[1 - \exp(-2\pi i r(\lambda))]^{-1}$ grows at infinity slower than a power of λ. This is precisely the content of Lemma 2.2.

Necessity of $(\mathcal{P})$ *and* $(\mathcal{G})$. If $\mathbf{L}$ is globally hypoelliptic, $\mathbf{L}^*$ is locally solvable, so Re b_0 cannot change sign from minus to plus, say, in the counter-clockwise direction, and thus cannot change sign at all on T^1 and $(\mathcal{P})$ holds. Suppose that $(\mathcal{G})$ does not hold. Hence we may assume Re $b_0 \equiv 0$ and (by Lemma 2.3) that there is an increasing sequence $\xi_n \in \sigma(A)$, $\xi_n \to \infty$ such that

$$|1 - \exp[-2\pi i r(\xi_n)]| < \xi_n^{-n}, \qquad n = 1, 2, \ldots . \tag{2.14}$$

Let $\beta(\xi)$ the function defined by $\beta(\xi) = n - 1$ if $\xi_{n-1} < \xi \leq \xi_n$, $n = 2, 3, \ldots$, and pick a continuous function $\alpha\colon R^+ \to [0, 1]$ such that $\alpha(\xi_n) = 1$ $\forall n \in N$ and

$$|1 - \exp[-2\pi i r(\xi)]|\alpha(\xi) \leq \xi^{-\beta(\xi)} \qquad \forall \xi \in [\rho, \infty). \tag{2.15}$$

Now we choose positive numbers η_k so that $\xi_{k-1} < \eta_k < \xi_k$, $k = 2, 3, \ldots$, and such that

$$\alpha(t) \geqslant \tfrac{1}{2} \quad \text{if } \eta_k \leqslant t \leqslant \xi_k. \tag{2.16}$$

We also choose unit vectors $(h_n) \in H$ and positive numbers (a_n) for every $n \in \mathbf{N}$ so that the following properties are satisfied

(i) $h_n \in \mathcal{R}(E(\xi_n) - E(\eta_n))$, so in particular $h_n \perp h_m$ if $n \neq m$ (this is possible for $\xi_n \in \sigma(A)$ so $E(\xi_n - 0) \neq E(\xi_n)$),

(ii) $\Sigma a_n^2 < \infty$,

(iii) $\Sigma \xi_{n-1}^2 a_n^2 = \infty$.

We define $h_0 = \Sigma a_n h_n \in H$. Since

$$\|\alpha(A) A h_n\|_0^2 = \int_{\eta_n}^{\xi_n} \alpha(\lambda)^2 \lambda^2 d\|E(\lambda) h_n\|_0^2 \geq \tfrac{1}{4}\eta_n^2 \int_{\eta_n}^{\xi_n} d\|E(\lambda) h_n\|_0^2 \geq \tfrac{1}{4}\xi_{n-1}^2$$

we see that

$$\int_{\sigma(A)} \alpha^2(\lambda)\lambda^2 d\|E(\lambda) h_0\|_0^2 = \infty,$$

so $\sigma(A) h_0 \notin H^1$. Finally, take a function $\gamma \in C^\infty(T^1; \mathbf{R})$ which vanishes in a neighborhood of $t = 0$ and verifies $\int_0^{2\pi} \gamma(t)\, dt = 1$, and set

$$u(t) = \alpha(A)\left[\int_0^t \gamma(s)\, ds I + \int_t^{2\pi} \gamma(s)\, ds e^{-B(2\pi, A)}\right] e^{-B(t, A)} h_0. \tag{2.17}$$

Since Re $b_0 \equiv 0$ and u is periodic, it is easy to verify that $u \in C^\infty(T^1; H^{-\infty})$ but $u \notin C^\infty(T^1; H^\infty)$. Indeed $u(0) = e^{-B(2\pi, A)}\alpha(A) h_0$ and $\alpha(A) h_0 \notin H^1$, so $u(0) \notin H^1$.

On the other hand

$$\mathbf{L}u(t) = \gamma(t)(I - e^{-2\pi i r(A)})e^{-B(t,A)}\alpha(A)h_0$$

and (2.15) implies right away that $\mathbf{L}u \in C^\infty(T^1, H^\infty)$. Q.E.D.

3. Global solvability in T^1. We study the solvability of the equation

$$\mathbf{L}u = (\partial_t + b(t, A)A)u = f, \qquad t \in T^1, \tag{3.1}$$

where u and f are smooth functions of t valued in $H^{\pm\infty}$. Bearing in mind the notations of §2, suppose that there is an eigenvalue λ of A such that $\lambda \in \Lambda$. If (3.1) has a solution, and we call $P = E(\lambda) - E(\lambda -)$ the orthogonal projection associated to λ, we have

$$\begin{aligned}(\partial_t + b(t, A)A)Pu &= (\partial_t + b(t, \lambda)\lambda)Pu \\ &= e^{-B(t,\lambda)}\partial_t(e^{B(t,\lambda)}Pu) = Pf\end{aligned}$$

since $e^{B(0,\lambda)} = e^{B(2\pi,\lambda)} = 1$. In particular

$$\int_0^{2\pi} e^{B(t,\lambda)}(E(\lambda) - E(\lambda -))f(t)\,dt = 0 \tag{3.2}$$

will be a compatibility condition for the existence of solutions to (3.1).

On the other hand, if we assume that $r(z)$ reduces to an integer, we obtain as in [1], another compatibility condition:

$$\int_0^{2\pi} e^{B(t,A)}E(\lambda)f(t)\,dt = 0 \quad \text{a.e. } \lambda \in \sigma(A). \tag{3.3}$$

Definition 3.1. *We say that $f \in C^\infty(T^1; H^{\pm\infty})$ is in $B_{\mathbf{L}}C^\infty(T^1; H^{\pm\infty})$ if* (3.2) *holds for every $\lambda \in \sigma(A) \cap \Lambda$ and* (3.3) *holds when $r(z)$ is an integral constant.*

We observe that condition (3.2) is empty if Λ does not meet the discrete spectrum of A, and also that $B_{\mathbf{L}}C^\infty(T^1; H^\infty)$ is a Fréchet space in any case.

We introduce the function

$$\begin{aligned}\tilde{B}(t, \lambda) &= \sum_{i=0}^{\infty} \tilde{B}_i(t)\lambda^{i-i} = B(t, \lambda) - ir(\lambda)t \\ &= \int_0^t b(s, \lambda)\,dt - \frac{t}{2\pi}\int_0^{2\pi} b(s, \lambda)\,ds.\end{aligned}$$

It is clear that the coefficients $B_i(t) \in C^\infty(T^1)$.

For any real number r, we write

$$\Omega_r = \left\{t \in T^1 | \operatorname{Re} \tilde{B}_0(t) < r\right\}. \tag{3.4}$$

Theorem 3.1. *The following conditions are equivalent*:

$$\forall f \in B_{\mathbf{L}}C^\infty(T^1; H^\infty), \quad \exists u \in C^\infty(T^1; H^{-\infty}) \quad \textit{such that } (3.1) \textit{ holds}; \tag{3.5}$$

$$\forall f \in B_{\mathbf{L}}C^\infty(T^1; H^\infty), \quad \exists u \in C^\infty(T^1; H^\infty) \quad \textit{such that } (3.1) \textit{ holds}; \tag{3.6}$$

$$\forall f \in B_{\mathbf{L}}C^\infty(T^1; H^{-\infty}), \quad \exists u \in C^\infty(T^1; H^{-\infty}) \quad \textit{such that } (3.1) \textit{ holds}; \tag{3.7}$$

(3.8)

(i) *if* $r(z)$ *does not reduce to an integer* $\sigma(A) \cap \Lambda$ *only contains isolated points of* $\sigma(A)$;

(ii) *if* $\sigma(A) \cap \Lambda$ *is unbounded, the sets* (3.4) *are connected*;

(iii) *if* $\sigma(A) \cap \mathbf{C}\Lambda$ *is unbounded,* $\operatorname{Re} b_0(t)$ *does not change sign; furthermore, if* $\operatorname{Re} b_0 \equiv 0$, $d(\xi)$ *decreases slowly at infinity on* $\sigma(A) \cap \mathbf{C}\Lambda$.

The function $d(\xi)$ appearing in (3.8)(iii), was defined in §2. A first order evolution operator verifying any of equivalent conditions (3.5), (3.6), (3.7), (3.8) will be called globally solvable.

REMARK 3.1. When $r(z) \equiv 0$, $b(t, \lambda)\,dt$ is exact for a.e. $\lambda \in \sigma(A)$ and Theorem 3.1 reduces to Theorem 1.1 of **[1]** on the circle. Thus it is a generalization of this result when the one-form $b(t, \lambda)\,dt$ generating the complex is closed, for the special case of S^1. The global solvability of complexes when the generators are closed remains an open problem for manifolds of higher dimensions.

REMARK 3.2. If the spectrum of A does not contain isolated points, and $r(z) \not\equiv k \in \mathbf{Z}$ Theorems 2.1 and 3.1 imply that $\mathbf{L}$ is globally solvable if and only if it is globally hypoelliptic and $\sigma(A) \cap \Lambda = \varnothing$, but in general there is no relationship between global solvability and global hypoellipticity.

Let $\mathbf{L}$ be the vector field (2.5) of Theorem 2.2 and set

$$r_0 = \frac{1}{2\pi}\int_0^{2\pi} b(t)\,dt, \qquad \tilde{B}(t) = \int_0^t b(s)\,ds - ir_0 t.$$

We say that a smooth function $f \in C^\infty(T^2)$ belongs to $B_{\mathbf{L}}C^\infty(T^2)$ if given any $n \in \mathbf{Z}$ such that

$$nr_0 = \frac{n}{2\pi}\int_0^{2\pi} b(t)\,dt \in \mathbf{Z} \tag{3.9}$$

we have

$$\int_0^{2\pi} \exp\left(\int_s^t b(t')\,dt'\right)\hat{f}(s, n)\,ds = 0, \tag{3.10}$$

where $\hat{f}(s, n)$ is the Fourier coefficient of f in the second variable. A straightforward application of Theorem (3.1) yields.

THEOREM 3.2. *Let* $\mathbf{L}$ *be the vector field* (2.5). *The following conditions are equivalent*:

$$\forall f \in B_{\mathbf{L}}C^{\infty}(T^2), \quad \exists u \in \mathfrak{D}'(T^2) \text{ such that } \mathbf{L}u = f; \tag{3.11}$$

$$\forall f \in B_{\mathbf{L}}C^{\infty}(T^2), \quad \exists u \in C^{\infty}(T^2) \text{ such that } \mathbf{L}u = f; \tag{3.12}$$

(i) *if* r_0 *is an integer, the sets* $\Omega_r = \{t \in T^1;$ $\operatorname{Im} \tilde{B}(t) < r\}$ *are connected* $\forall r \in \mathbf{R}$;

(ii) *if* r_0 *is rational but not an integer, the sets* Ω_r *are connected and* $\operatorname{Im} b(t)$ *does not change sign*;

(iii) *if* r_0 *is real but not rational,* $\operatorname{Im} b(t)$ *does not change sign and whenever* $\operatorname{Im} b(t) \equiv 0$, r_0 *is non-Liouville*;

(iv) *if* r_0 *is nonreal,* $\operatorname{Im} b(t)$ *does not change sign.* (3.13)

This theorem in conjunction with the results of §2 gives

COROLLARY 3.1. *If the vector field* (2.5) *is globally hypoelliptic, it is globally solvable. The converse is false.*

PROOF OF (3.8) $\Rightarrow$ (3.7) AND (3.6). Assume first that $r(z)$ does not reduce to an integer. Choose $t_0 \in T^1$ such that $\operatorname{Re} \tilde{B}_0(t_0) = \inf \operatorname{Re} \tilde{B}_0(t)$, and let us enumerate the set $\sigma(A) \cap \Lambda = \{\lambda_1, \lambda_2, \dots\}$. We shall write $P_k = E(\lambda_k) - E(\lambda_k -)$ and shall call χ the characteristic function of the set $\sigma(A) \cap \complement\Lambda$. Let us set

$$u_1(t) = \chi(A)\left(\int_0^t k_1(s, t, A)f(s) + \int_t^{2\pi} k_2(s, t, A)f(s)\, ds\right), \tag{3.14}$$

$$u_2(t) = \sum_{k=1}^{\infty} \int_{t_0}^{t} e^{B(s,\lambda_k) - B(t,\lambda_k)} Puf(s)\, ds, \tag{3.15}$$

with k_1, k_2 given by (2.12), (2.13).

It is easy to check that $u_1(0) = u_1(2\pi)$, $u_2(0) = u_2(2\pi)$, $\mathbf{L}u_1 = \chi(A)f$, $\mathbf{L}u_2 = (I - \chi(A))f$, so we only need to prove that they are smooth functions valued in $H^{\pm\infty}$, when this is true of f, to conclude that $u = u_1 + u_2$ will be a solution of (3.1) in the required space.

If $\sigma(A) \cap \Lambda$ is bounded the sum that defines u_2 is finite. If $\sigma(A) \cap \Lambda$ is not bounded, the sets (3.4) are connected and $t_0 \in \Omega_r$ whenever $\Omega_r \neq \varnothing$. In particular, for all t there will be a path γ_t, which we may take to be an arc of circumference, joining t_0 to t, such that

$$\operatorname{Re} \tilde{B}_0(s) < \operatorname{Re} \tilde{B}_0(t) \qquad \forall s \in \gamma_t. \tag{3.16}$$

According to (3.2) we may write

$$u_2(t) = \sum_k \int_{\gamma_t} e^{B(s,\lambda_k)} P_k f(s)\, ds \tag{3.17}$$

for the value of the integral will be independent of the path. In view of (3.16), the definition of $\tilde{B}$, and the fact that $ir(\lambda_k)$ is purely imaginary, we see that

$$\operatorname{Re}(B(s,\lambda_k) - B(t,\lambda_k)) = \operatorname{Re}(\tilde{B}(s,\lambda_k) - \tilde{B}(t,\lambda_k)) < M \tag{3.18}$$

for every $s \in \gamma_t$, $\lambda_k \in \sigma(A) \cap \Lambda$, where M is a constant independent of s, t, k. It is now clear that $u_2 \in C^\infty(T^1; H^{\pm\infty})$.

Let us now look at $u_1(t)$. Since the points of $\sigma(A) \cap \Lambda$ are isolated, $1 - e^{-2\pi i r(\lambda)}$ is bounded away from zero on bounded subsets of $\sigma(A) \cap \mathbf{C}\Lambda$. Suppose $\operatorname{Re} b_0(t) \geq 0$. Then $\operatorname{Re}(B_0(s) - B_0(t)) < 0$ when $0 < s < t$ and $\operatorname{Re}(B_0(s) - B_0(t) - B_0(2\pi)) < 0$ when $t < s < 2\pi$, so the exponentials appearing in (3.14) are bounded. When $\operatorname{Re} b_0(t) < 0$ we reach a similar conclusion by rewriting (3.14) in the following fashion

$$u_1(t) = \chi(A)\int_0^t k_3(s,t,A) f(s) + \int_t^{2\pi} k_4(s,t,A) f(s)\, ds,$$

$$k_3(s,t,A) = \left[e^{2\pi i r(A)} - I\right]^{-1} \exp\left[B(s,A) - B(t,A) + B(2\pi,A)\right], \tag{3.19}$$

$$k_4(s,t,A) = \left[e^{2\pi i r(A)} - I\right]^{-1} \exp\left[B(s,A) - B(t,A)\right]. \tag{3.20}$$

Therefore we only need to verify that $(1 - e^{\pm 2\pi i r(\lambda)})$ grows slowly at infinity on $\sigma(A) \cap \mathbf{C}\Lambda$ to conclude that $u_1(t) \in C^\infty(T^1; H^{\pm\infty})$. This is a consequence of Lemma 2.3 when $\operatorname{Re} b_0 \equiv 0$ and a consequence of $\operatorname{Im} r_0 \neq 0$ when $\operatorname{Re} b_0 \not\equiv 0$.

When $r(z) \equiv k \in \mathbf{Z}$, one defines

$$u(t) = \int_{t_0}^t e^{B(s,A) - B(t,A)} f(s)\, ds, \tag{3.21}$$

and uses (3.3) and (3.8)(ii) to check that $u \in C^\infty(T^1; H^{\pm\infty})$ and $\mathbf{L}u = f$. Q.E.D.

It is plain that any of (3.6), (3.7) implies trivially (3.5) so we only need to prove that (3.5) implies (3.8). For that we need the following version of a well-known lemma (Lemma II.2.1, in [4]).

LEMMA 3.1. *Suppose that* (3.5) *holds. Then, there are positive constants m and C such that*

$$\left|\int_0^{2\pi} (f(t), v(t))_0\, dt\right| \leq C \cdot \sup_{t\in T^1} \sum_{\alpha=0}^m \|\partial_t^\alpha f\|_m \cdot \sup_{t \in T^1} \sum_{\alpha=0}^m \|\partial_t^\alpha (L^* v)\|_m \tag{3.22}$$

for every $f \in B_L C^\infty(T^1; H^\infty)$ and every $v \in C^\infty(T^1; H^\infty)$, with L^* given by (0.6).

(3.5) $\Rightarrow$ (3.8)(i). Suppose that there exists $\lambda_0 \in \sigma(A) \cap \Lambda$ which is a limit point of $\sigma(A)$, and $r(z)$ does not reduce to an integer. Since Λ is discrete we may find a sequence of intervals $I_n = (\alpha_n, \beta_n)$ and $\lambda_n \in \sigma(A)$ such that

(a) $\alpha_n \to \lambda_0, \quad \beta_n \to \lambda_0,$
(b) $I_n \cap \Lambda = \varnothing \quad \forall n,$
(c) $\lambda_n \in I_n \cap \sigma(A) \quad \forall n,$
(d) $I_n \cap I_m = \varnothing, \quad n \neq m.$

Choose $h_n \in H$ such that $\|h_n\| = 1$, $h_n \in \mathfrak{R}(E(\beta_n) - E(\alpha_n))$, and a nonnegative function $\sigma \in C^\infty(T^1; R)$ which vanishes in a neighborhood of $t = 0$ and verifies $\int_0^{2\pi}\sigma(s)\,ds = 1$. Write $\Gamma(t) = \int_0^t \sigma(s)\,ds$ and set

$$v_n(t) = \left[\Gamma(t) + (1 - \Gamma(t))e^{\bar{B}(2\pi,A)}\right]e^{\bar{B}(t,A)}h_n,$$
$$f_n(t) = \sigma(t)e^{-B(t,A)}h_n, \qquad n = 1, 2, \ldots. \tag{3.23}$$

It is easy to check that $f_n \in B_L C^\infty(H^1; H^\infty)$, $v_n \in C^\infty(T^1; H^\infty)$ and

$$L^* v_n = e^{\bar{B}(t,A)}(I - e^{\bar{B}(2\pi,A)})h_n, \qquad n = 1, 2, \ldots. \tag{3.24}$$

The reader may verify that for a certain positive constant M, depending on m but independent of n

$$\sum_{\alpha=0}^{m} \|\partial_t^\alpha(L^* v_n)\|_m < M \cdot \sup_{\lambda \in I_n} |1 - e^{\bar{B}(2\pi,\lambda)}| \to 0 \quad \text{as } n \to \infty, \tag{3.25}$$

$$\sum_{\alpha=0}^{m} \|\partial_t^\alpha(f_n)\|_m < M, \tag{3.26}$$

$$(f_n(t), v_n(t))_0 = \sigma(t)\left[\Gamma(t) + (1 - \Gamma(t))\left(h_n, e^{\bar{B}(i\pi,A)}h_n\right)_0\right]$$
$$\to \Gamma(t)\sigma(t) \quad \text{as } n \to \infty. \tag{3.27}$$

Therefore

$$\int_0^{2\pi}(f_n(t), v_n(t))\,dt \to \int_0^{2\pi}\Gamma(t)\sigma(t)\,dt = k_0 > 0. \tag{3.28}$$

so (3.25), (3.26) and (3.28) contradict (3.22). Q.E.D.

(3.5) $\Rightarrow$ (3.8)(ii). Here we follow the lines of Theorem 1.1 of [**1**], and omit details. If Ω_r is not connected we may find a real number $r_0 < r$, and smooth real functions f_0, v_0 on T^1 such that

(a) $\int_0^{2\pi} f_0(t)\,dt = 0,$
(b) $\operatorname{supp} \partial_t v_0 \subseteq \Omega_{r_0},$
(c) $\operatorname{supp} f_0 \cap \Omega_{r_0} = \varnothing,$
(d) $\int_0^{2\pi} f_0(t)v_0(t)\,dt > 0.$

If $r(z)$ reduces to a constant and $\sigma(A) \cap \Lambda$ is unbounded, the constant must be an integer. In this case, we take a sequence $\lambda_n \to \infty$ in $\sigma(A) \cap \Lambda = \sigma(A)$ and unit vectors $h_n \in \mathfrak{R}(E(\lambda_n + 1) - E(\lambda_n - 1))$. Then

$$\begin{aligned} f_n(t) &= e^{-\bar{B}(t,A)} f_0(t) h_n \in B_{\mathrm{L}} C^\infty(T^1; H^\infty), \\ v_n(t) &= e^{B(t,A)} v_0(t) h_n \in C^\infty(T^1; H^\infty), \end{aligned} \tag{3.29}$$

will violate (3.22) as $n \to \infty$.

If $r(z)$ is nonconstant and (3.5) holds, we already know that Λ can only meet $\sigma(A)$ at the point spectrum. If there is a sequence $\lambda_n \in \sigma(A) \cap \Lambda$ that goes off to infinity, we take unit eigenvectors $h_n \in \mathfrak{R}(E(\lambda_n) - E(\lambda_n -))$ and set

$$\begin{aligned} f_n(t) &= e^{-\bar{B}(t,\lambda_n)} f_0(t) h_n = f_0(t) e^{-\bar{B}(t,A)} h_n \in B_{\mathrm{L}} C^\infty(T^1; H^\infty), \\ v_n(t) &= e^{B(t,\lambda_n)} v_0(t) h_n = v_0(t) e^{B(t,A)} h_n \in C^\infty(T^1; H^\infty). \end{aligned} \tag{3.30}$$

Again (3.22) cannot hold for the pair f_n, v_n as $n \to \infty$. Q.E.D.

(3.5) $\Rightarrow$ (3.8)(iii). Let $\lambda_n \to \infty$ be a sequence in $\sigma(A) \cap \mathbf{C}\Lambda$, take closed disjoint intervals I_n, such that $\lambda_n \in I_n \subseteq \mathbf{C}\Lambda$, and set

$$P = \int_{\bigcup I_n} dE(\lambda); \quad H = \mathfrak{R}P, \qquad A' = A|_{H'}. \tag{3.31}$$

Then A' is a selfadjoint unbounded operator on the Hilbert space H', with bounded inverse, and defines Sobolev scales $H'^s = PH^s$, $s \in \mathbf{R}$. Multiplying (3.1) through by P, we see that

$$\forall f \in B_{\mathrm{L}} C^\infty(T^1; H^\infty), \quad \exists u \in C^\infty(T^1, H'^{-\infty}) \text{ such that } \mathrm{L}'_n = Pf, \tag{3.32}$$

where $L' = \partial_t + b(t, A')A'$. Since Λ does not meet $\sigma(A')$, $PB_{\mathrm{L}} C^\infty(T^1; H^\infty) = C^\infty(T^1; H'^\infty)$, so in particular L' is locally solvable at any point of T^1 and this implies that $\operatorname{Re} b_0$ cannot change sign. If $\operatorname{Re} b_0 \equiv 0$, a straightforward combination of the techniques used in the proof of (3.5) $\Rightarrow$ (3.8)(i) and in the necessity of ($\mathcal{G}$) in Theorem 2.1 allows us to conclude that $d(\xi)$ must decrease slowly at infinity on $\sigma(A) \cap \mathbf{C}\Lambda$. We leave details to the reader. Q.E.D.

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