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Globally regular solutions to the u^5 Klein-Gordon equation

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1. - Introduction

Consider the non-linear wave equation

$$u_{tt} - \Delta u + u^p = 0 \text{ in } \mathbb{R}^3 \times \mathbb{R}_+ \tag{1.1}$$

with initial data

$$u\,|_{t=0} = u_0, \;\; u_t\,|_{t=0} = u_1. \tag{1.2}$$

In 1961 K. Jörgens proved that, for $p < 5$, equation (1.1) admits a unique regular solution $u \in C^2$ for any Cauchy data $u_0 \in C^3$, $u_1 \in C^2$, see [1, Satz 2, p. 298].

The case $p = 5$ was later investigated by Rauch, who obtained global regularity for small initial energies

$$E_0 = \int_{\mathbb{R}^3} \left(\frac{1}{2} \left(|\nabla u_0|^2 + |u_1|^2 \right) + \frac{1}{6} \, |u_0|^6 \right) \, dx < \frac{\pi}{\sqrt{3}}, \tag{1.3}$$

see [3, Theorem, p. 347].

Rauch's approach, moreover, reveals that $p = 5$ arises as a limiting exponent for a Sobolev embedding relevant for problem (1.1), see [3, estimate (14), p. 346]. This and recent progress in elliptic equations involving limiting non-linearities has been our motivation for studying problem (1.1-2).

The supercritical case $p > 5$ seems to be open.

In this paper we show that Rauch's smallness assumption actually is unnecessary and that Jörgen's result continues to hold - at least for radially symmetric solutions - at the limiting exponent $p = 5$, which will be fixed from now on throughout this paper.

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THEOREM 1.1. *For any radially symmetric initial data* $u_0 \in C^3(\mathbb{R}^3)$, $u_1 \in C^2(\mathbb{R}^3)$, $u_0(x) = u_0(|x|)$, $u_1(x) = u_1(|x|)$, *there exists a unique, global, radially symmetric solution* $u \in C^2(\mathbb{R}^3 \times [0, \infty[)$, $u(x,t) = u(|x|, t)$ *to the Cauchy problem* (1.1-2), *with* $p = 5$.

The proof involves a blow-up analysis of possible singularities of equation (1.1). Thereby we heavily exploit "conformal invariance" of (1.1), i.e. invariance of (1.1) under scaling

$$u \to u_R(x,t) = R^{1/2} u(Rx, Rt). \tag{1.5}$$

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2. - Some fundamental estimates

In this section we recall Rauch's result for equation (1.1) and prove two basic integral estimates which result from testing (1.1) with suitable functions φ. Besides the standard choice $\varphi = u_t$ - which gives rise to the well-known "energy inequality", see Lemma 2.1 -, we will also use u and its radial derivative $x \cdot \nabla u$ as testing functions: the remaining components of the generator

$$\frac{d}{dR} \, u_R|_{R=1} = tu_t + x \cdot \nabla u + \frac{1}{2} \, u$$

of the family (1.5). This will give rise to the crucial "Pohožaev-type identity" Lemma 2.2 (see [2] for a related result in an elliptic setting).

2.1 *Notations*

Denote $z = (x,t)$ a generic point in space-time. The negative light-cone through $z_0 = (x_0, t_0)$ is given by

$$C(z_0) = \{(x,t) \mid t \le t_0, \ |x - x_0| \le t_0 - t\}.$$

Its mantle and space-like sections are denoted by

$$M(z_0) = \{(x,t) \in C(z_0) \mid |x - x_0| = t_0 - t\},$$

resp. by

$$D(z_0, t) = C(z_0) \cap (\mathbb{R}^3 \times \{t\}).$$

If $z_0 = (0,0)$ the point z_0 will be omitted from this notation.

Truncated cones will be denoted

$$C_s^t = C \cap (\mathbb{R}^3 \times [s,t]), \ C_s^0 = C_s, \ C_{-\infty}^t = C^t.$$

$B_R(x_0)$ denotes the Euclidean ball

$$B_R(x_0) = \{x \in \mathbb{R}^3 |\ |x - x_0| < R\}.$$

Again, if $x_0 = 0$ we simply write $B_R(0) = B_R$.

Finally, for a C^1-function u and a space-like region $\Omega(t) \subset (\mathbb{R}^3 \times \{t\})$,

$$E(u; \Omega(t)) = \int_{\Omega(t)} e(u) \mathrm{d}x$$

denotes the energy of u in $\Omega(t)$, with density

$$e(u) = \frac{1}{2} \left(|u_t|^2 + |\nabla u|^2\right) + \frac{1}{6} |u|^6.$$

The letters c, C will denote generic positive constants, occasionally numbered for clarity.

2.2 *The energy inequality*

Let $u \in C^2(\mathbb{R}^3 \times]-\infty, 0[)$ be a solution to (1.1). Actually, by finiteness of propagation speed, all estimates only require u to be C^2 near suitable sections of cones.

Multiply (1.1) by u_t. This gives the identity

$$\left(\frac{1}{2} \left(|u_t|^2 + |\nabla u|^2\right) + \frac{1}{6} |u|^6\right)_t - \operatorname{div}(\nabla u \cdot u_t) = 0. \tag{2.1}$$

If we integrate this expression over a section C_s^t of the negative light-cone, we obtain the following result:

$$\begin{aligned} &E(u; D(t)) - E(u; D(s)) \\ &\quad + \frac{1}{\sqrt{2}} \int_{M_s^t} \left(\frac{1}{2}(|u_t|^2 + |\nabla u|^2) + \frac{1}{6}|u|^6 - \frac{x}{|x|} \cdot \nabla u \ u_t\right) \mathrm{d}o = 0. \end{aligned} \tag{2.2}$$

Note that the outward normal to M_s^t is given by $n(x,t) = \frac{1}{\sqrt{2}} \left(\frac{x}{|x|}, 1\right)$; moreover, we recognize the energy density $e(u)$ inside the left bracket of (2.1).

Rauch [3, p. 345] interprets the boundary integrand as follows:

$$
\begin{aligned}
&\frac{1}{2}\left(|u_t|^2+|\nabla u|^2\right)+\frac{1}{6}|u|^6-\frac{x}{|x|}\cdot\nabla u\cdot u_t \\
&=\frac{1}{2}\left|\frac{x}{|x|}u_t-\nabla u\right|^2+\frac{1}{6}|u|^6=\frac{1}{2}|\nabla_y v|^2+\frac{1}{6}|v|^6,
\end{aligned}
\tag{2.3}
$$

where

$$
v(y)=u(y,\ -|y|). \tag{2.4}
$$

Thus we may state:

LEMMA 2.1. *For any* $s<t<0$ *there holds the energy estimate*

$$
E(u;D(t))+\int_{B_{|s|}\setminus B_{|t|}}\left\{\frac{1}{2}|\nabla v|^2+\frac{1}{6}|v|^6\right\}dy=E(u;D(s)),
$$

where v *is given by* (2.4).

2.3 *A Pohožaev-type identity*

The next result apparently is new. This and Lemma 3.3 are the crucial ingredients in the proof of Theorem 1.1.

LEMMA 2.2. *For* u *as above there holds*

$$
\begin{aligned}
&\frac{1}{3}\int_{C_{-1}}|u|^6dxdt+E(u;D(-1)) \\
&\leq\int_{D(-1)}u_t(x\cdot\nabla u+u)dx+\int_{B_1}\{|y|\ |\nabla v|^2+|\nabla v|\ |v|\}dy,
\end{aligned}
$$

where v *is given by* (2.4).

PROOF. Multiply (1.1) by $tu_t+x\cdot\nabla u+u$. By (2.1) the contribution from the first term is

$$
\begin{aligned}
0=&\left(t\left[\frac{1}{2}\left(|u_t|^2+|\nabla u|^2\right)+\frac{1}{6}|u|^6\right]\right)_t-\operatorname{div}(\nabla u\cdot tu_t) \\
&-\left(\frac{1}{2}(|u_t|^2+|\nabla u|^2)+\frac{1}{6}|u|^6\right).
\end{aligned}
$$

Similarly, we compute

$$\begin{aligned}0 &= (u_{tt} - \Delta u + u^5)(x \cdot \nabla u) \\ &= \operatorname{div} \left(-x \frac{|u_t|^2}{2} - \nabla u(x \cdot \nabla u) + x \frac{|\nabla u|^2}{2} + x \frac{|u|^6}{6}\right) \\ &+ \{x \cdot \nabla u \ u_t\}_t + \frac{3}{2} |u_t|^2 - \frac{1}{2} |\nabla u|^2 - \frac{1}{2} |u|^6.\end{aligned}$$

Finally,

$$\begin{aligned}0 &= (u_{tt} - \Delta u + u^5) \ u \\ &= \{u_t u\}_t - \operatorname{div}(\nabla u \cdot u) - |u_t|^2 + |\nabla u|^2 + |u|^6.\end{aligned}$$

Adding, we obtain that

$$\begin{aligned}0 &= \frac{1}{3} |u|^6 + \left(t \left[\frac{1}{2} \left(|u_t|^2 + |\nabla u|^2\right) + \frac{1}{6} |u|^6\right]\right)_t + \Big((x \cdot \nabla u + u)u_t\Big)_t \\ &- \operatorname{div} \left(x \frac{|u_t|^2}{2} - x \frac{|\nabla u|^2}{2} + \nabla u(x \cdot \nabla u + u) + \nabla u \cdot t u_t - x \frac{|u|^6}{6}\right).\end{aligned}$$

Thus, when we integrate this expression over the cone $C_{-1}^{-\varepsilon}$, we obtain

$$\begin{aligned}&\frac{1}{3} \int_{C_{-1}^{-\varepsilon}} |u|^6 \mathrm{d}x\mathrm{d}t + E(u; D(-1)) - \varepsilon \ E(u; D(-\varepsilon)) \\ &= \int_{D(-1)} u_t(x \cdot \nabla u + u)\mathrm{d}x - \int_{D(-\varepsilon)} u_t(x \cdot \nabla u + u)\mathrm{d}x + BI,\end{aligned} \tag{2.5}$$

with BI denoting the following boundary integral

$$\begin{aligned}BI &= \frac{1}{\sqrt{2}} \int_{M_{-1}^{-\varepsilon}} \left[|t| \left(|u_t|^2 + \frac{|x \cdot \nabla u|^2}{|x|^2}\right) - 2x \cdot \nabla u \ u_t - \left(u_t - \frac{x}{|x|} \cdot \nabla u\right) u\right] \mathrm{d}o \\ &= \frac{1}{\sqrt{2}} \int_{M_{-1}^{-\varepsilon}} \left[|t| \left|u_t - \frac{x}{|x|} \cdot \nabla u\right|^2 - \left(u_t - \frac{x}{|x|} \cdot \nabla u\right) \dot{u}\right] \mathrm{d}o \\ &\leq \int_{\{y:|y|\leq 1\}} \left[|y| \ |\nabla v|^2 + |\nabla v| \ |v|\right] \ \mathrm{d}y.\end{aligned}$$

By Lemma 2.1,

$$E(u; D(-\varepsilon)) \leq E(u; D(-1))$$

uniformly. Moreover, by Young's and Hölder's inequalities

$$\int_{D(-\varepsilon)} u_t(x\cdot\nabla u+u)\mathrm{d}x \le \varepsilon \int_{D(-\varepsilon)} \left[|u_t|^2+|\nabla u|^2+\varepsilon^{-2}|u|^2\right]\ \mathrm{d}x$$

$$\le \varepsilon \int_{D(-\varepsilon)} \left(|u_t|^2+|\nabla u|^2\right)\ \mathrm{d}x + \varepsilon\left(\int_{D(-\varepsilon)} |u|^6\mathrm{d}x\right)^{1/3} \le C\ \varepsilon.$$

Hence we may pass to the limit $\varepsilon\to 0$ in (2.5) and the proof is complete.

qed

2.4 *Small energy*

Finally, we recall the integral representation

$$u(0,0)=\underline{u}(0,0)-\frac{1}{4\pi}\int_{M_{t_0}} |t|^{-1}\ u^5(x,t)\mathrm{d}o \tag{2.6}$$

for the value $u(0,0)$ of a solution u of (1.1) in terms of the solution $\underline{u}$ of the homogeneous wave equation

$$\underline{u}_{tt}-\Delta\underline{u}=0, \tag{2.7}$$

sharing the Cauchy data u_0, u_1 of u at a time $t_0<0$:

$$\underline{u}=u_0=u,\ \frac{\partial}{\partial t}\ \underline{u}=u_1=\frac{\partial}{\partial t}\ u \text{ at } t=t_0, \tag{2.8}$$

see [3, (10), p. 342].

We will only apply this formula for functions u which are of class C^2 in a neighborhood of C_t, for some $t<0$.

Following Rauch [3], we turn (2.6) into a linear inequality for $\sup_{C_{t_0}}\ |u|$:

Suppose

$$\sup_{C_{t_0}}\ |u|=|u(0,0)|$$

is achieved at the origin. Then, if we let

$$\mu(s)=\frac{1}{4\pi}\int_{M_s} |t|^{-1}\ u^4(x,t)\mathrm{d}o=\frac{\sqrt{2}}{4\pi}\int_{B_{|s|}} |y|^{-1}\ v^4(y)\mathrm{d}y,$$

(2.6) implies the inequality, for any $s>t_0$,

$$(1-\mu(s))\ \sup_{C_{t_0}}|u|\le|\underline{u}(0,0)|-\frac{1}{4\pi}\int_{M^s_{t_0}} |t|^{-1}u^5(x,t)\mathrm{d}o. \tag{2.9}$$

By Hölder's inequality

$$\mu(s) = C \int_{B_{|s|}} |y|^{-1}\ v^4(y)\mathrm{d}y \le C \left(\int_{B_{|s|}} v^6 \mathrm{d}y \right)^{1/2} \left(\int_{B_{|s|}} \frac{v^2(y)}{|y|^2} \mathrm{d}y \right)^{1/2}. \tag{2.10}$$

Rauch now invokes Hardy's inequality

$$\int_{\mathbb{R}^3} \frac{\psi^2(y)}{|y|^2}\ \mathrm{d}y \le 4 \int_{\mathbb{R}^3} |\nabla\psi|^2 \mathrm{d}y,\ \forall \psi \in C_0^\infty(\mathbb{R}^3) \tag{2.11}$$

to estimate the last integral in (2.10).

If integration extends only over a bounded domain B_{2R}, (2.11) is not immediately applicable. However, if we truncate with a smooth localizing function $\eta \in C_0^\infty(\mathbb{R}^3)$ satisfying the conditions: $\eta \equiv 1$ on B_R, $\eta \equiv 0$ off B_{2R}, $0 \le \eta \le 1$, $|\nabla \eta| \le C/R$, then with absolute constants C there holds

$$\begin{aligned}
\int_{B_{2R}} \frac{\psi^2(y)}{|y|^2}\ \mathrm{d}y &\le \int_{B_{2R}} \frac{(\psi\eta)^2}{|y|^2}\ \mathrm{d}y + \int_{B_{2R}\setminus B_R} \frac{\psi^2}{|y|^2}\ \mathrm{d}y \\
&\le 4 \int_{B_{2R}} |\nabla(\psi\eta)|^2 \mathrm{d}y + R^{-2} \int_{B_{2R}} \psi^2\ \mathrm{d}y \\
&\le C \int_{B_{2R}} |\nabla\psi|^2 \mathrm{d}y + C\ R^{-2} \int_{B_{2R}} \psi^2\ \mathrm{d}y \\
&\le C \int_{B_{2R}} |\nabla\psi|^2 \mathrm{d}y + C \left(\int_{B_{2R}} \psi^6 \mathrm{d}y \right)^{1/3}, \\
&\forall \psi \in C^1(B_{2R}).
\end{aligned} \tag{2.12}$$

Using (2.12) and the energy inequality, Lemma 2.1, the number $\mu(s)$ from (2.10) may now be estimated

$$\begin{aligned}
\mu(s) &\le C \left(\int_{B_{|s|}} v^6 \mathrm{d}y \right)^{1/2} \left(\int_{B_{|s|}} |\nabla v|^2 \mathrm{d}y + \left(\int_{B_{|s|}} v^6 \mathrm{d}y \right)^{1/3} \right)^{1/2} \\
&\le C_0 \left[E(u; D(s)) + E(u; D(s))^{2/3} \right].
\end{aligned} \tag{2.13}$$

The remaining terms in (2.9) are easily bounded using Hölder's inequality

$$
\begin{aligned}
\int_{M_{t_0}^s} |t|^{-1}\, |u|^5 \mathrm{d}o &\le C \int_{B_{|t_0|}\setminus B_{|s|}} |y|^{-1}\, |v|^5 \mathrm{d}y \\
&\le C\, |s|^{-1/2} \left(\int_{B_{|t_0|}} v^6 \mathrm{d}y \right)^{5/6} \le C_1 |s|^{-1/2}\, E(u; D(t_0))^{5/6}.
\end{aligned}
\tag{2.14}
$$

From (2.9), (2.13) we immediately obtain Rauch's regularity result for small initial energies - while the more refined estimate (2.14) will be useful later on.

THEOREM 2.3. (Rauch [3]). *Suppose u is a C^2-solution of* (1.1) *in a neighborhood of C_{t_0} with initial data $u = u_0,\ u_t = u_1$ on $D(t_0)$. There exists an absolute constant $\varepsilon_0 > 0$ with the property*:

If $E(u; D(t_0)) \le \varepsilon_0$, then

$$|u(x,t)| \le 2 \sup_{C_{t_0}} |\underline{u}| \text{ for all } (x,t) \in C_{t_0},$$

where $\underline{u}$ denotes the solution to the Cauchy-problem (2.7-8) *for the homogeneous equation.*

PROOF. By passing to a smaller cone $\tilde{C} \subset C_{t_0}$, if necessary, we may assume that

$$\sup_{C_{t_0}} |u| = |u(0,0)|.$$

Determine $\varepsilon_0 > 0$ such that $C_0(\varepsilon_0 + \varepsilon_0^{2/3}) = \frac{1}{2}$. Applying (2.9), (2.13) with $s = t_0$ the Theorem follows.

qed

Note that there is a converse result to Theorem 2.3:

PROPOSITION 2.4. *Suppose u is a C^2-solution of* (1.1) *in a neighborhood of $C_{t_0}\setminus\{0\}$ with initial data $u = u_0 \in C^3,\ u_t = u_1 \in C^2$ on $D(t_0)$, and suppose that $|u(z)| \to \infty$ as $z \to 0,\ z \in C_{t_0}\setminus\{0\}$. Then for any $t \in [t_0, 0[$ there holds*

$$E(u; D(t)) \ge \varepsilon_0 > 0,$$

where ε_0 is the constant of Theorem 2.3.

PROOF. Suppose $E(u; D(t)) \le \varepsilon_0$ for some $t \in [t_0, 0[$. Note that, since by assumption $|u(z)| \to \infty$ as $z \to 0$, there exists a sequence $\delta_m \to 0,\ \delta_m > 0$, such that $\sup_{C_{t_0}^{-\delta_m}} |u|$ is attained in $D(-\delta_m)$. Hence (2.9), (2.13-14) are applicable in suitable cones $C_{t_0}(z_m) \subset C_{t_0}$, with $z_m \in D(-\delta_m)$, and we obtain

$$\frac{1}{2} \sup_{C_{t_0}^{-\delta_m}} |u| \leq \sup_{C_{t_0}} |\underline{u}| + C|t|^{-1/2}. \tag{2.15}$$

Since (2.15) holds for arbitrarily small $\delta_m > 0$, there results a contradiction, and the proof is complete.

qed

3. - Proof of Theorem 1.1

Let $u_0 \in C^3(\mathbb{R}^3)$, $u_1 \in C^2(\mathbb{R}^3)$ with $u_0(x) = u_0(|x|)$, $u_1(x) = u_1(|x|)$ be given radial functions. By [1; Satz 1, p. 297] the Cauchy problem (1.1-2) admits a unique (and hence radially symmetric) C^2-solution $u(x,t) = u(|x|,t)$ locally, i.e. in a neighborhood of $\mathbb{R}^3 \times \{0\}$.

Suppose (by contradiction) that u is not globally regular. Then there is a singularity $\overline{z} = (\overline{x}, \overline{t})$ such that $|u(z)| \to \infty$ as $z \to \overline{z}$, $z \in C_0(\overline{z})$, see [1, p. 301]. Replacing $\overline{z}_0$ by another singular point in $S = \{(x,t) : 0 \leq t < \overline{t},\ |x| \leq |\overline{x}| + \overline{t} - t\}$, if necessary, we may assume that $u \in C^2$ in S.

Radial symmetry implies

LEMMA 3.1. $\overline{x} = 0$, *in particular* $u \in C^2(\mathbb{R}^3 \times [0, \overline{t}[)$.

PROOF. By Proposition 2.4 and since $u(x,t) = u(|x|,t)$:

$$E(u; D(z,t)) \geq \varepsilon_0$$

for any $z = (x, \overline{t})$, $|x| = |\overline{x}|$, and any $t < \overline{t}$.

Now, if $|\overline{x}| > 0$, for any given $K \in \mathbb{N}$ we can choose points $x_1, \cdots, x_K$ satisfying $|x_k| = |\overline{x}|$, $1 \leq k \leq K$, and $t < \overline{t}$ such that with $z_k = (x_k, \overline{t})$ we have:

$$D(z_j, t) \cap D(z_k, t) = \emptyset, \ j \neq k.$$

But then, letting $T = |\overline{x}| + \overline{t}$, $Z = (0, T)$, by Lemma 2.1:

$$\begin{aligned} K\ \varepsilon_0 &\leq \sum_{k=1}^{K} E(u; D(z_k, t)) = E\left(u; \bigcup_{k=1}^{K} D(z_k, t)\right) \\ &\leq E(u; D(Z,t)) \leq E(u; D(Z,0)) < \infty, \end{aligned}$$

uniformly in K, and for large K we obtain a contradiction.

Hence a singularity is first encountered on the line $\{x = 0\}$.

qed

For convenience we shift coordinates such that $\bar{z}=0$ in our new coordinate frame, and denote $-\bar{t}=t_0$. Thus our solution u is transformed into a solution (indiscriminately denoted by u) of (1.1), of class C^2 in a neighborhood of $C_{t_0}\backslash\{0\}$, which becomes unbounded as $z\to 0$, $z\in C_{t_0}\backslash\{0\}$.

Denote by $\underline{u}$ the solution of the homogeneous wave equation (2.7) sharing the Cauchy data of u at t_0. $\underline{u}$ is uniformly bounded in a closed neighborhood of C_{t_0}.

For $R_m=2^{-m}$, $m\in\mathbb{N}$, define the blown-up functions

$$u_m(x,t)=R_m^{1/2}u(R_m x,\ R_m t).$$

Each u_m is of class C^2 in a neighborhood of a deleted cone $C_{t_m}\backslash\{0\}$, $t_m=t_0/R_m$.

As in (2.4) we denote the trace of u_m on M_{t_m} by

$$v_m(y)=u_m(y,-|y|)=R_m^{1/2}v(R_m y).$$

Relabelling $\{u_m\}$, if necessary, we may assume that $t_0\le -1$.

Note that for any m, any $t\in[t_m,0[$, by Lemma 2.1:

$$E(u_m;D(t))\le E(u_m;D(t_m))=E(u;D(t_0))=:E_0<\infty. \tag{3.1}$$

On the other hand, since u_m becomes unbounded at 0, by Proposition 2.4:

$$E(u_m;D(t))\ge\varepsilon_0,\qquad \text{for all } t\in[t_m,0[, \tag{3.2}$$

for any $m\in\mathbb{N}$.

By Lemma 2.1 the energy $E(u,D(t))$ is non-increasing in t, hence tends to a positive (by (3.2)) limit as $t\to 0$. But then, by Lemma 2.1 again,

$$\begin{aligned}(3.3)\qquad &\int_{B_1\backslash B_{|t|}}\left(\frac{1}{2}\,|\nabla v_m|^2+\frac{1}{6}\,|v_m|^6\right)\mathrm{d}y\\ &=\int_{B_{R_m}\backslash B_{|t|R_m}}\left(\frac{1}{2}\,|\nabla v|^2+\frac{1}{6}\,|v|^6\right)\mathrm{d}y\\ &\le E(u;D(-R_m))-E(u;D(tR_m))\to 0,\end{aligned}$$

as $m\to\infty$, uniformly in $t<0$.

LEMMA. 3.2. *There exists $t<0$ such that, for some $s_m\in[-1,t]$, there holds*

$$\int_{D(s_m)}(u_m)_t\ u_m\ \mathrm{d}x\le o(1), \tag{3.4}$$

where $o(1)\to 0$ as $m\to\infty$.

PROOF. We may assume that

$$\int_{D(-1)} u_m^2 \mathrm{d}x \geq C_2 > 0.$$

(Otherwise we can choose $s_m = -1$ to achieve our claim). Choose $t \in]-1, 0[$ such that

$$\int_{D(t)} u_m^2 \mathrm{d}x \leq ct^2 \left(\int_{D(t)} u_m^6 \mathrm{d}x \right)^{1/3}$$
$$\leq c\ t^2\ E(u_m; D(t))^{1/3} \leq c\ t^2 < C_2.$$

Suppose by contradiction that

$$\int_{D(s)} (u_m)_t\ u_m \mathrm{d}x \geq C_3 > 0$$

uniformly in $m \in \mathbb{N}$, for all $s \in [-1, t]$.

Then by (3.3) we obtain

$$\int_{D(t)} u_m^2 \mathrm{d}x - \int_{D(-1)} u_m^2 \mathrm{d}x$$
$$= 2 \int_{C_{-1}^t} (u_m)_t\ u_m \mathrm{d}x\mathrm{d}t - \frac{1}{\sqrt{2}} \int_{M_{-1}^t} u_m^2 \mathrm{d}o \geq 2(1+t)C_3 - o(1),$$

which, for large m, is in conflict with our choice of t.

qed

Since $\{s_m\}$ is bounded away from 0, we may scale with s_m to achieve (3.4) with $s_m = -1$ for all m. Note that with this change of scale the ratio R_m/R_{m+1} remains uniformly bounded, i.e. there exists $R > 0$ such that

$$0 < R^{-1} \leq |R_m/R_{m+1}| \leq R < \infty, \text{ for all } m. \tag{3.5}$$

Now apply Lemma 2.2:

$$\frac{1}{3} \int_{C_{-1}} u_m^6 \mathrm{d}x\mathrm{d}t + E(u_m; D(-1))$$
$$\leq \int_{D(-1)} (u_m)_t\ x \cdot \nabla u_m\ \mathrm{d}x + o(1).$$

It follows that

$$\int_{C_{-1}} |u_m|^6 dx dt$$

(3.6)
$$+ \int_{D(-1)} \Big\{ (1-|x|) \left[|(u_m)_t|^2 + |\nabla u_m|^2 \right]$$

$$+ |x| \left| \frac{x}{|x|}(u_m)_t - \nabla u_m \right|^2 + |u_m|^6 \Big\} \, dx \to 0,$$

as $m \to \infty$.

LEMMA 3.3. *There exists a sequence* $\Lambda \subset \mathbb{N}$ *such that*

$$\liminf_{m\to\infty, m\in\Lambda} \sup_{C_{t_m}^{-1}} |u_m| > 0.$$

PROOF. Suppose by contradiction that

$$\sup_{C_{t_m}^{-1}} |u_m| = R_m^{1/2} \sup_{C_{t_0}^{-R_m}} |u|$$

$$\geq R^{-1/2} \sup_{s\in[-R_{m-1}, -R_m]} \left(|s|^{1/2} \sup_{C_{t_0}^{s}} |u| \right) \to 0, \text{ as } m \to \infty.$$

I.e. if we let

$$g(t) = |t|^{1/2} \sup_{C_{t_0}^{t}} |u|,$$

g is continuous and satisfies

$$g(t) \to 0 \quad (t \to 0).$$

Also denote

$$h(t) = \sup_{s\geq \max\{t, -1\}} g(s).$$

Then h is continuous, non-increasing, and satisfies $h(t) \equiv h(-1)$, for $t \leq -1$, and $h(t) \to 0$ $(t \to 0)$.

Now the proof proceeds as follows: first we establish that $h(t)$ decays with a certain power of $|t| : h(t) \leq c|t|^{\varepsilon}$, $(t \to 0)$.

In a second step we use this decay estimate to prove that u is uniformly bounded near 0 - which will yield the desired contradiction.

i) Suppose $h(t) = g(s)$ for some $s \geq t \geq -1$ and that $g(s)$ is attained at $\tilde{z} = (\tilde{x}, \tilde{t})$, where $s \geq \tilde{t} = \lambda t$, $|u(\tilde{z})| = \sup_{C_{t_0}^{s}} |u| = \sup_{C_{t_0}^{\tilde{t}}} |u|$. Note that if $\lambda = \lambda(t) > 1$:

$$h(t) = g(s) = |s|^{1/2}|u(\tilde{z})| \leq |t|^{1/2}|u(\tilde{z})| \leq g(t) \leq h(t)$$

i.e.

$$s = t; \; h(t) = |t|^{1/2} \; |u(\tilde{z})|.$$

Similarly, for $\bar{t} \in]\tilde{t}, t]$,

$$h(\bar{t}) = \sup_{\bar{t} \leq s} g(s) = |\bar{t}|^{1/2}|u(\tilde{z})| = \left|\frac{\bar{t}}{\tilde{t}}\right|^{1/2} h(\tilde{t}).$$

In particular, $\lambda(\bar{t}) = \frac{\tilde{t}}{\bar{t}} > 1$.

Denote

$$J = \{t \in]-1, 0[: \lambda(t) > 1\}.$$

Remark that J consists of a union of left open intervals I and, for any pair $s \leq t$ belonging to such an interval I, there holds

$$h(t) = \left|\frac{t}{s}\right|^{1/2} h(s).$$

In particular, for any $\varepsilon \in]0, \mu]$, $0 < \mu \leq \frac{1}{2}$, there holds

$$(h(t) + |t|^{\mu}) \leq \left|\frac{t}{s}\right|^{\varepsilon} (h(s) + |s|^{\mu}), \quad \text{for all } s \leq t \in I. \tag{3.7}$$

On the other hand, if $\lambda \leq 1$, by (2.6)

$$\begin{aligned} h(t) &= g(s) = |s|^{1/2} \; |u(\tilde{z})| \\ &\leq |s|^{1/2}|\underline{u}(\tilde{z})| + c|s|^{1/2} \int_{M_{t_0}(\tilde{z})} |\tilde{t} - \tau|^{-1} \; |u(y, \tau)|^5 \, do \\ &\leq c|t|^{1/2} + c|\tilde{t}|^{1/2} \int_{-1}^{\tilde{t}} |\tilde{t} - \tau| \sup_{C_{t_0}^{\tau}} |u|^5 d\tau \\ &= c|t|^{1/2} + c|\tilde{t}|^{1/2} \int_{-1}^{\tilde{t}} |\tilde{t} - \tau| g^5(\tau) \; |\tau|^{-5/2} d\tau \\ &\leq c|t|^{1/2} + c|t|^{1/2} \int_{-1}^{\tilde{t}} \lambda^{1/2} g^5(\tau) \; |\tau|^{-3/2} d\tau \end{aligned} \tag{3.8}$$

$$\leq c|t|^{1/2} + c|t|^{1/2} \int_{-\lambda^{-1}}^{t} g^5(\lambda\tau)\ |\tau|^{-3/2} d\tau$$

$$\leq c|t|^{1/2} + c|t|^{1/2} \int_{-1}^{t} h^5(\tau)\ |\tau|^{-3/2} d\tau.$$

Now choose $\mu = \frac{1}{5}$, and denote $t_1 = t^\mu := t|t|^{\mu-1} < t$. Since h is non-increasing and bounded we obtain from (3.8):

$$\begin{aligned}
&(h(t) + |t|^\mu) \\
&\leq C|t|^{1/2} + |t|^\mu + C|t|^{1/2} \int_{t^\mu}^{t} h^5(\tau)\ |\tau|^{-3/2} d\tau + C|t|^{1/2} \int_{-1}^{t^\mu} h^5(\tau)\ |\tau|^{-3/2} d\tau \\
&\leq C|t|^{1/2} + |t|^\mu + C\ h^5(t^\mu) + C|t|^{1/2} \sup_{\tau<0}\ h^5(\tau) \left(|t|^{-\mu/2} - 1\right) \\
&\leq C|t|^{1/2} + |t|^\mu + C|t|^{(1-\mu)/2} + C\ h^5(t^\mu) \\
&\leq C\left(|t|^\mu + h^5(t^\mu)\right) \leq C_4 \left(h(t_1) + |t_1|^\mu\right)^5.
\end{aligned}$$

Iteratively define $t_k = t_{k-1}^\mu < t_{k-1},\ k = 1, \cdots, K$. Suppose $\lambda(t_k) \leq 1$ for all $k = 1, \cdots, K-1$. Then

$$\begin{aligned}
h(t) &\leq (h(t) + |t|^\mu) \leq C_4 \left(h(t_1) + |t_1|^\mu\right)^5 \\
&\leq C_4^6 \left(h(t_2) + |t_2|^\mu\right)^{25} \leq \cdots \leq C_4^{\sum_{k=0}^{K-1} 5^k} \left(h(t_K) + |t_K|^\mu\right)^{5^K} \\
&= \left[C_4^{1/4} \left(h(t_K) + |t_K|^\mu\right)\right]^{5^K - 1} \cdot \left(h(t_k) + |t_K|^\mu\right).
\end{aligned}$$

I.e., if for some $\varepsilon > 0$:

$$C_4^{1/4} \left(h(t_K) + |t_K|^\mu\right) \leq |t_K|^\varepsilon, \tag{3.9}$$

it follows that

$$\begin{aligned}
h(t) + |t|^\mu &\leq \left(\left|\frac{t_K^{5^K}}{t_K}\right|\right)^\varepsilon \left(h(t_K) + |t_K|^\mu\right) \\
&= \left|\frac{t}{t_K}\right|^\varepsilon \left(h(t_K) + |t_K|^\mu\right).
\end{aligned} \tag{3.10}$$

Note that, since $h(t) \to 0\ (t \to 0)$, there exist $T \in]-1, 0[, \varepsilon \in]0, \mu[$ such that (3.9) holds whenever $t_K \in [T^\mu, T]$. But then also (3.10) holds for all such t, t_K, provided $\lambda(t),\ \lambda(t_k) \leq 1, k = 1, \ldots, K-1$.

Now choose any $\tau = \tau_0 > T$ and define a sequence $\tau_1, \tau_2, \cdots \tau_K$ as follows:

$$\tau_{k+1} = \begin{cases} \tau_k^\mu, & \text{if } \lambda(\tau_k) \leq 1 \\ \tilde{\tau}_k, & \text{if } \lambda(\tau_k) > 1 \end{cases}, \quad k \in \mathbb{N}_0,$$

where $\tilde{\tau}_k$ denotes the left end-point of the interval $I \subset J$ containing τ_k, if $\lambda(\tau_k) > 1$, and where

$$K = \sup \{k \in \mathbb{N} \mid \tau_{k-1} > T\}.$$

Note that

$$|\tau_{k+2}/\tau_k| \geq |\tau_k|^{\mu-1} \geq |T|^{\mu-1} > 1,$$

if $\tau_k \geq \tau_{k+2} > T$. Hence K exists and is finite, for every $\tau < 0$.

Combining (3.7) and (3.10) we see that

$$h(\tau) \leq (h(\tau) + |\tau|^\mu) \leq \left|\frac{\tau}{\tau_K}\right|^\varepsilon (h(\tau_K) + |\tau_K|^\mu) \leq C|\tau|^\varepsilon,$$

i.e.

$$\sup_{C_{t_0}^t} |u| = g(t)\ |t|^{-1/2} \leq h(t)\ |t|^{-1/2} \leq C|t|^{\varepsilon - \frac{1}{2}}.$$

ii) Denote

$$\overline{\gamma} = \inf \left\{ \gamma > 0 : |t|^\gamma \sup_{C_{t_0}^t} |u| \leq C < \infty \text{ uniformly in } t \right\}.$$

By part i) $\overline{\gamma} < \frac{1}{2}$ and we may choose $\gamma > \overline{\gamma}$ such that $\mu := 5\gamma - 2 < \overline{\gamma},\ \mu \neq 0$.

Define

$$f(t) = |t|^\gamma \sup_{C_{t_0}^t} |u|.$$

Note that $f(t)$ is uniformly bounded, continuous and satisfies $f(t) \to 0$ as $t \to 0$.

By (2.6), for all $z = (x, t) \in C_{t_0} \backslash \{0\}$:

$$\begin{aligned} |u(z)| &\leq |\underline{u}(z)| + \frac{1}{4\pi} \int_{M_{t_0}(z)} |t-\tau|^{-1}\ |u(\xi,\tau)|^5\ do(\xi,\tau) \\ &\leq C + C \int_{t_0}^{t} |t-\tau|\ f^5(\tau)\ |\tau|^{-5\gamma} d\tau \qquad (3.11) \\ &\leq C + C \sup_{\tau<0} f^5(\tau) \int_{t_0}^{t} |\tau|^{1-5\gamma} d\tau \leq C + C|t|^{2-5\gamma}. \end{aligned}$$

First suppose $\mu > 0$. Then from (3.11) we obtain

$$|t|^{\mu}|u(x,t)| \leq C|t|^{\mu} + C \leq C,$$

uniformly for all $z = (x,t) \in C_{t_0} \setminus \{0\}$, which contradicts the definition of $\overline{\gamma}$.

Thus $\mu \leq 0$. But then by (3.11) u is uniformly bounded in $C_{t_0} \setminus \{0\}$, contrary to hypothesis.

qed

To proceed with the proof of Theorem 1.1, let $z_m = (x_m, s_m) \in C_{t_m}^{-1}$, $m \in \Lambda$, satisfy

$$|u_m(z_m)| = \sup_{C_{t_m}^{s_m}} |u_m| = \min\{1, \sup_{C_{t_m}^{-1}} |u_m|\} = R_m^{1/2} u(R_m x_m, R_m s_m). \tag{3.12}$$

Note that by Lemma 3.3

$$\liminf_{m \to \infty. m \in \Lambda} |u_m(z_m)| = 2\ c_5 > 0; \tag{3.13}$$

in particular, by (3.12)

$$R_m s_m \to 0,\ \ s_m \leq -1. \tag{3.14}$$

Now by (2.9), (2.13), (2.14), if $E\left(u_m; D(z_m, s)\right) < \varepsilon_0$ for some $s < s_m$:

$$\begin{aligned} c_5 \leq |u_m(z_m)| &\leq 2\ \left(R_m^{1/2} \underline{u}(R_m z_m) + c|s - s_m|^{-1/2}\right) \\ &\leq o(1) + 2\ c|s - s_m|^{-1/2} \leq o(1) + \frac{1}{2}\ c_5, \end{aligned}$$

provided $s \leq s_m - c_6$ for some $c_6 > 0$.

Since $c_5 > 0$, this is impossible for large m, and it follows that

$$E\left(u_m; D(z_m, s)\right) \geq \varepsilon_0,$$

for $s \in [t_m, s_m - c_6]$, $m \geq m_0$.

By radial symmetry

$$E(u_m; D(z, s)) \geq \varepsilon_0 \tag{3.15}$$

for such $s, m \geq m_0$, for all $z = (x, s_m)$ with $|x| = |x_m|$.

LEMMA 3.4. *For any $c > 0$, any family $\{x^k\}_{1 \leq k \leq K}$ in $\mathbb{R}^3$, with $|x^k| = r \geq 0$, $|x^j - x^k| \geq c^{-1} r$, $j \neq k$, there exists $\sigma_m \in [t_m, s_m - c_6]$ such that*

$$E\left(u_m;\ \bigcup_{j \neq k} D(z^j, \sigma_m) \cap D(z^k, \sigma_m)\right) \to 0 \tag{3.16}$$

as $m \to \infty$, $m \in \Lambda$, *where* $z^i = (x^i, s_m)$.

PROOF. Since $|s_m| \geq 1$, by uniform convexity of balls in $\mathbb{R}^3$, there exists $\varepsilon > 0$ such that

$$D(z^j, s) \cap D(z^k, s) \subset \{x \in D(s) \ : \ |x| < (1-\varepsilon)|s|\} =: D^\varepsilon(s)$$

for all $j \neq k$, $s \in [R(s_m - c_6), s_m - c_6] =: I_m$.

(Note that

$$|Rs_m| \leq \frac{R}{R_m} \ |R_m s_m| \leq c|t_m| \ |R_m s_m| = o(|t_m|),$$

by (3.14). Hence $R(s_m - c_6) \geq t_m$, for $m \geq m_0$).

Now by (3.5), there exists $k(m)$ such that

$$\sigma_m := -\frac{R_{k(m)}}{R_m} \in I_m.$$

Observe that by (3.14) again:

$$R_{k(m)} = -\sigma_m R_m \leq c|s_m| R_m \to 0,$$

hence $k(m) \to \infty$, $(m \to \infty, m \in \Lambda)$.

But then by (3.6)

$$\begin{aligned} E\left(u_m; D^\varepsilon(\sigma_m)\right) &= E\left(u_{k(m)}; D^\varepsilon(-1)\right) \\ &\leq c \cdot \varepsilon^{-1} \int\limits_{D(-1)} \Big\{(1-|x|) \ \left(|(u_{k(m)})_t|^2 + |\nabla u_{k(m)}|^2\right) \\ &+ |x| \ \left|\frac{x}{|x|}(u_{k(m)})_t - \nabla u_{k(m)}\right|^2 + |u_{k(m)}|^6\Big\} \ dx \to 0 \end{aligned}$$

as $m \to \infty$, $m \in \Lambda$.

This proves the claim.

qed

We can now complete the *proof of Theorem* 1.1.

Given $K \in \mathbb{N}$, we can find $c > 0$ such that, for any $m \in \mathbb{N}$, there are K points x_m^j, $1 \leq j \leq K$ such that $|x_m^j| = |x_m|$, $|x_m^j - x_m^k| \geq c^{-1} \ |x_m|$ for all $1 \leq j \neq k \leq K$. Denote $z_m^j = (x_m^j, s_m)$.

Let $\sigma_m \in [t_m, s_m]$ denote the number determined in Lemma 3.4 for the family $\{x_m^j\}$.

By (3.15-16) and (3.1)

$$
\begin{aligned}
K\varepsilon_0 &\le \sum_{j=1}^{K} E(u_m; D(z_m^j, \sigma_m)) \\
&\le E(u_m; \bigcup_{j=1}^{K} D(z_m^j, \sigma_m)) \\
&+ \sum_{j\neq k} E(u_m; D(z_m^j, \sigma_m) \cap D(z_m^k, \sigma_m)) \\
&\le E(u_m; D(\sigma_m)) + o(1) \le E_0 + o(1),
\end{aligned}
$$

where $o(1) \to 0$ as $m \to \infty$, $m \in \Lambda$.

For K sufficiently large we obtain a contradiction, and the proof is complete.

qed

4. - A Remark on the non-symmetric case

Estimate (3.6) suggests that, also in the non-symmetric case, singularities tend to build up in a rotationally symmetric pattern. Using this observation, it is possible to extend our results to arbitrary initial data $u \in C^3$, $u_1 \in C^2$, provided the modulus of continuity of the blow-up functions u_m, restricted to $C_{t_m}^{s_m}$ (where u_m is uniformly bounded by 1), can be uniformly bounded.

Added in proof

Generalizations of (1.1-2) to higher dimensions were studied for intance by Brenner and von Wahl [4] or Pecher [5], where results analogous to those found by Jörgens in dimension 3 were obtained. See [4] for further references.

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