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GMM versus GQL inferences in semiparametric linear dynamic mixed models

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Abstract. Linear dynamic mixed models are commonly used for continuous panel data analysis in economic statistics. There exists generalized method of moments (GMM) and generalized quasi-likelihood (GQL) inferences for binary and count panel data models, the GQL estimation approach being more efficient than the GMM approach. The GMM and GQL estimating equations for the linear dynamic mixed model can not, however, be obtained from the respective estimating equations under the nonlinear models for binary and count data. In this paper, we develop the GMM and GQL estimation approaches for the linear dynamic mixed models and demonstrate that the GQL approach is more efficient than the GMM approach, also under such linear models. This makes the GQL approach uniformly more efficient than the GMM approach in estimating the parameters of both linear and nonlinear dynamic mixed models.

1 Introduction

Let y_{it} denote a continuous response for the ith ($i = 1, \dots, I$) individual recorded at time t ($t = 1, \dots, T$). Let x_{it} be the $p \times 1$ vector of fixed covariates corresponding to y_{it}, and β be the $p \times 1$ vector of fixed effects of x_{it} on y_{it}. Further suppose that the response y_{it} is influenced by $y_{i,t-1}$ for $t = 2, \dots, T$, as well as it is also influenced by an unobservable random effect γ_i^* which is shared by all responses of the ith individual recorded over T periods of time. This type of data can be explained by using a linear dynamic panel data model (LDPDM) given by

$$\begin{aligned} y_{i1} &= x_{i1}'\beta + z_i\gamma_i^* + \epsilon_{i1}, \\ y_{it} &= x_{it}'\beta + \theta(y_{i,t-1} - x_{i,t-1}'\beta) + z_i\gamma_i^* + \epsilon_{it} \qquad \text{for } t = 2, \dots, T, \end{aligned} \tag{1.1}$$

where z_i is an additional covariate for the ith individual on top of the fixed covariates x_{it}, $\epsilon_{it} \overset{\text{iid}}{\sim} (0, \sigma_\epsilon^2)$ and $\gamma_i^* \overset{\text{iid}}{\sim} (0, \sigma_\gamma^2)$. Also, ϵ_{it} and γ_i^* are independent. In (1.1), θ is referred to as the lag 1 dynamic dependence parameter. Note that the linear dynamic mixed model in (1.1) is semiparametric by nature. This is because, the random effects and the errors of the model are assumed to have their means and

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variances, but, their distributional forms are unknown. We refer to Bun and Carree (2005), for example, and the references therein for this type of linear dynamic mixed model based inferences.

Note that the LDPDM (1.1) produces the marginal mean as

$$E[Y_{it}] = x'_{it}\beta = \mu_{it} \qquad \text{(say)}, \tag{1.2}$$

which depends on the time dependent covariate at time t, and is a function of the regression parameter vector β. Further, the marginal variance and the bivariate auto-covariances produced by the model (1.1) have the formulas given by

$$\mathrm{var}[Y_{it}] = \sigma_{itt} = z_i^2\sigma_\gamma^2\left\{\sum_{j=0}^{t-1}\theta^j\right\}^2 + \sigma_\epsilon^2\sum_{j=0}^{t-1}\theta^{2j} \qquad \text{for } t = 1,\ldots,T \tag{1.3}$$

and

$$\mathrm{cov}[Y_{iu}, Y_{it}] = \sigma_{iut} = z_i^2\sigma_\gamma^2\sum_{j=0}^{t-1}\theta^j\sum_{k=0}^{u-1}\theta^k + \sigma_\epsilon^2\sum_{j=0}^{u-1}\theta^{t-u+2j} \qquad \text{for } u < t, \tag{1.4}$$

respectively, and they are the functions of the dynamic dependence parameter θ, random effects variance σ_γ^2, and the error variance σ_ϵ^2. It is of scientific interest to estimate these parameters, namely $\beta, \theta, \sigma_\gamma^2$ and σ_ϵ^2, consistently and as efficiently as possible.

Note that for the above linear dynamic mixed model (1.1), many econometricians such as Arellano and Bond (1991), Ahn and Schmidt (1995), Blundell and Bond (1998), and Imbens (2002) [see also Chamberlain (1992), Keane and Runkle (1992) and Bond, Bowsher and Windmeijer (2001)] have bypassed the estimation of the variance parameters and estimated the regression parameter β and the dynamic dependence parameter θ by using the well-known generalized method of moments (GMM) due to Hansen (1982). To be specific, to bypass the variance parameters, these authors utilize the differences of the responses such as $z_{it} = y_{it} - y_{i,t-1}$ and construct suitable moment functions for the ith individual as $\psi_i(z_{i1},\ldots,z_{iT};\beta,\theta)$ so that $E[\psi_i(z_{i1},\ldots,z_{iT};\beta,\theta)] = 0$. This leads to the $p+1$ dimensional moment estimating equations for β and θ given by

$$I^{-1}\sum_{i=1}^{I}\psi_i(z_i,\beta,\theta) = 0, \tag{1.5}$$

where $z_i \equiv [z_{i1},\ldots,z_{iT}]'$. The moment estimators for β and θ obtained from (1.5) are consistent but they may be inefficient. As an improvement over (1.5), some of the above-mentioned authors estimate $\eta = (\beta',\theta)'$ by minimizing the quadratic form

$$I^{-1}\left[\sum_{i=1}^{I}\psi_i(z_i,\eta)\right]' C\left[\sum_{i=1}^{I}\psi_i(z_i,\eta)\right] \tag{1.6}$$

for some positive definite $m \times m$ symmetric matrix C. The resulting estimators are referred to as the GMM estimators due to Hansen (1982).

In Section 2, following Hansen (1982), we develop the GMM approach to estimate all parameters of the model (1.1), namely $\eta = (\beta', \theta)'$, σ_γ^2 and σ_ϵ^2. Note that, in our approach, unlike the aforementioned authors, we do not need to bypass the estimation of the random effects variance σ_γ^2. Next, following Sutradhar (2004) [see also Jiang and Zhang (2001)], in Section 3, we develop a generalized quasi-likelihood (GQL) approach for the estimation of all parameters of the model (1.1). Note that both GQL and GMM are moments based estimation approaches. But, unlike the GMM approach, the GQL estimating equations are constructed by pooling the covariance matrix based standard distance functions of all independent individuals. This makes the GQL approach more efficient than the GMM approach. Further note that even though these GMM and GQL approaches were developed recently by Sutradhar, Rao and Pandit (2008) for binary panel data, and by Jowaheer and Sutradhar (2009) for panel count data, these approaches, however, do not have the so-called pedagogical virtue of reducing from the discrete data case to the continuous data case following the model (1.1).

We remark that the construction of the GMM and GQL estimating equations requires the computation of the fourth-order moments based certain weight matrices similar to the C matrix in (1.6). These weight matrices are approximated by pretending that the data follow the normal distribution but with correct means and variances under the semiparametric model (1.1). This approximation for the weight matrices helps to increase the efficiency of the estimators as compared to using certain independence assumption based identity weight matrices. Next, for efficiency comparison between the two proposed GMM and GQL approaches, in Section 4, we provide the formulas for the asymptotic variances of the estimators [see equations (4.1) and (4.5)] without making any distributional assumptions for the errors and random effects in model (1.1). Note that these variances were computed based on the GMM estimating equation (2.7) and the GQL estimating equations (3.2) and (3.4), which were also developed under the semiparametric model (1.1). However, for an empirical efficiency comparison between the GMM and GQL approaches, we may consider normal errors and random effects, under the model (1.1), without any loss of generality. This normality consideration does not put one approach in any disadvantageous situation as compared to the other. The asymptotic variances under such a special normal case are also given in the same section [see equations (4.2) and (4.6)]. The efficiency results for selected values of the parameters are displayed in Table 1 in Section 4. As expected, the GQL approach appears to be uniformly more efficient than the GMM approach in estimating all parameters of the dynamic mixed model (1.1).

2 GMM estimation approach

In this approach, one first writes appropriate standard unbiased moment functions for the parameters of interest. Suppose that ψ_1, ψ_2, ψ_3 and ψ_4 are four such functions for the parameters of interest, namely for $\beta, \theta, \sigma_\gamma^2$ and σ_ϵ^2 in the present linear dynamic mixed model setup. Here, $E[\psi_m] = 0$, for all $m = 1, \ldots, 4$. Next, by constructing a vector of all moments functions, such as

$$\psi = [\psi_1', \psi_2, \psi_3, \psi_4]',$$

a quadratic form, namely

$$Q = \psi' C \psi \tag{2.1}$$

[Jiang and Zhang (2001, Section 1, equation (4))] is minimized in order to obtain the GMM (or optimal moment) estimates for the desired parameters $\alpha = [\beta', \theta, \sigma_\gamma^2, \sigma_\epsilon^2]'$. In (2.1), C is a suitable $(p+3) \times (p+3)$ positive definite weight matrix C, with $C = [\text{cov}(\psi)]^{-1}$ as an optimal choice [Jiang and Zhang (2001), Hansen (1982)].

Now to construct the vector of moment functions, we consider

$$\psi_1 = \sum_{i=1}^{I} \sum_{t=1}^{T} x_{it}[y_{it} - x_{it}'\beta], \tag{2.2}$$

$$\psi_2 = \sum_{i=1}^{I} \sum_{t=1}^{T-1} [\{(y_{it} - x_{it}'\beta)(y_{i,t+1} - x_{i,t+1}'\beta)\} - \sigma_{it,t+1}], \tag{2.3}$$

$$\psi_3 = \sum_{i=1}^{I} \sum_{u<t} [\{(y_{iu} - x_{iu}'\beta)(y_{it} - x_{it}'\beta)\} - \sigma_{iut}] \tag{2.4}$$

and

$$\begin{aligned} \psi_4 = & \sum_{i=1}^{I} \sum_{t=1}^{T} [\{y_{it} - x_{it}'\beta\}^2 - \sigma_{itt}]/IT \\ & - 2\sum_{i=1}^{I} \sum_{u<t} [\{(y_{iu} - x_{iu}'\beta)(y_{it} - x_{it}'\beta)\} - \sigma_{iut}]/IT(T-1) \end{aligned} \tag{2.5}$$

for the estimation of $\beta', \theta, \sigma_\gamma^2$ and σ_ϵ^2, respectively. Note that these functions are unbiased for zero, that is, $E[\psi_m] = 0$ for $m = 1, \ldots, 4$. Further note that since the parameters $\beta', \theta, \sigma_\gamma^2$ and σ_ϵ^2, have different interpretation in the present linear dynamic mixed model setup as opposed to those of the panel count model considered by Jowaheer and Sutradhar (2009) and binary panel model by Sutradhar, Rao and Pandit (2008), the moment functions in (2.2)–(2.5) are different than those of the count and binary data models.

Note that the estimate of $\alpha = [\beta', \theta, \sigma_\gamma^2, \sigma_\epsilon^2]'$ that minimizes the quadratic form (2.1) is obtained by solving

$$\frac{\partial \psi'}{\partial \alpha} C \psi = \frac{\partial \psi'}{\partial \alpha} [\mathrm{cov}(\psi)]^{-1} \psi = 0. \tag{2.6}$$

But, since the computation of the $\mathrm{cov}(\psi)$ matrix in (2.6) requires the formulas for the third and fourth-order moments as well, one cannot compute such a covariance matrix provided the error distributions for the model (1.1) are known. However, as the consistency of the estimator of α will not be effected by the choice of the weight matrix, a possible resolution is to use a normality (N) based "working" C_N, say, matrix, and solve the estimating equation

$$\frac{\partial \psi'}{\partial \alpha} C_N \psi = \frac{\partial \psi'}{\partial \alpha} [\mathrm{cov}_N(\psi)]^{-1} \psi = 0. \tag{2.7}$$

We remark that even though it is suggested that the weight matrix C be computed under "working" normality, the distance function ψ is computed based on σ_γ^2 and σ_ϵ^2, the variances of the true distributions of γ_i^* and ϵ_{it}, respectively. The estimating equation (2.7) may be solved by using the Gauss–Newton iterative equation

$$\hat{\alpha}_{\mathrm{GMM}}(r+1) = \hat{\alpha}_{\mathrm{GMM}}(r) + \left[\frac{\partial \psi'}{\partial \alpha} C_N \frac{\partial \psi}{\partial \alpha'}\right]_r^{-1} \left[\frac{\partial \psi'}{\partial \alpha} C_N \psi\right]_r, \tag{2.8}$$

where $[\cdot]_r$ denotes that the expression within the square bracket is evaluated at $\alpha = \hat{\alpha}_{\mathrm{GMM}}(r)$, the estimate obtained for the rth iteration. Let the final solution obtained from (2.8) is denoted by $\hat{\alpha}_{\mathrm{GMM}}$.

2.1 Construction of the "working" weight matrix C_N

Recall that $C = [\mathrm{cov}(\psi)]^{-1}$, where for $m, v = 1, \ldots, 4$,

$$\mathrm{cov}(\psi) = (\mathrm{cov}(\psi_m, \psi_v)) \tag{2.9}$$

with ψ_1, ψ_2, ψ_3 and ψ_4 are as in (2.2), (2.3), (2.4) and (2.5), respectively. Note that the computation of this covariance matrix requires the formulas for all possible second, third and fourth-order moments. Computing C_N means that the elements of the C matrix be computed by pretending that the data, that is, the response vector $y_i = (y_{i1}, \ldots, y_{it}, \ldots, y_{iT})'$ follows a T-dimensional multinormal distribution with true mean vector $\mu_i = (\mu_{i1}, \ldots, \mu_{it}, \ldots, \mu_{iT})'$, and the $T \times T$ true covariance matrix $\Sigma_i = (\sigma_{iut})$, where $\mu_{it} = x_{it}'\beta$ by (1.2) and the formulas for σ_{itt} and σ_{iut} are as in (1.3) and (1.4), respectively.

For convenience we provide all third and fourth-order moments as in the following two lemmas.

Lamma 2.1. *Let $\delta_{iu\ell t} = E[(Y_{iu} - \mu_{iu})(Y_{i\ell} - \mu_{i\ell})(Y_{it} - \mu_{it})]$. Under normality*

$$\delta_{iu\ell t} = 0. \tag{2.10}$$

Lamma 2.2. *Let* $\phi_{iu\ell mt} = E[(Y_{iu} - \mu_{iu})(Y_{i\ell} - \mu_{i\ell})(Y_{im} - \mu_{im})(Y_{it} - \mu_{it})]$. *Under normality*

$$\phi_{iu\ell mt} = \sigma_{iu\ell}\sigma_{imt} + \sigma_{ium}\sigma_{i\ell t} + \sigma_{iut}\sigma_{i\ell m}. \tag{2.11}$$

We now use the above two lemmas and derive the elements of $\mathrm{cov}(\psi)$ in (2.9) as follows. The resulting matrix is denoted by $\mathrm{cov}_N(\psi)$ so that $C_N = [\mathrm{cov}_N(\psi)]^{-1}$, as shown in (2.7) and (2.8).

The variances are given by

$$\mathrm{var}_N(\psi_1) = \sum_{i=1}^{I}\sum_{u=1}^{T}\sum_{t=1}^{T}\sigma_{iut}x_{iu}x_{it}', \tag{2.12}$$

$$\mathrm{var}_N(\psi_2) = \sum_{i=1}^{I}\sum_{u=1}^{T-1}\sum_{t=1}^{T-1}\bigl[\phi_{iu(u+1)t(t+1)} - \sigma_{iu(u+1)}\sigma_{it(t+1)}\bigr], \tag{2.13}$$

$$\mathrm{var}_N(\psi_3) = \sum_{i=1}^{I}\sum_{u<\ell}^{T}\sum_{m<t}^{T}[\phi_{iu\ell mt} - \sigma_{iu\ell}\sigma_{imt}], \tag{2.14}$$

$$\begin{aligned}\mathrm{var}_N(\psi_4) = {} & (IT)^{-2}\sum_{i=1}^{I}\sum_{u=1}^{T}\sum_{t=1}^{T}[\phi_{iuutt} - \sigma_{iuu}\sigma_{itt}] \\ & - 2(IT)^{-1}\bigl(IT(T-1)\bigr)^{-1}\sum_{i=1}^{I}\sum_{u=1}^{T}\sum_{m<t}^{T}[\phi_{iuumt} - \sigma_{iuu}\sigma_{imt}] \\ & + 4\bigl(IT(T-1)\bigr)^{-2}\sum_{i=1}^{I}\sum_{u<\ell}^{T}\sum_{m<t}^{T}[\phi_{iu\ell mt} - \sigma_{iu\ell}\sigma_{imt}].\end{aligned} \tag{2.15}$$

All covariances for ψ_1 with other functions are zero. That is,

$$\mathrm{cov}_N(\psi_1, \psi_2) = \mathrm{cov}_N(\psi_1, \psi_3) = \mathrm{cov}_N(\psi_1, \psi_4) = 0. \tag{2.16}$$

The remaining covariances have the formulas as

$$\mathrm{cov}_N(\psi_2, \psi_3) = \sum_{i=1}^{I}\sum_{u=1}^{T-1}\sum_{m<t}^{T}[\phi_{iu,u+1,mt} - \sigma_{iu,u+1}\sigma_{imt}], \tag{2.17}$$

$$\begin{aligned}\mathrm{cov}_N(\psi_2, \psi_4) = {} & (IT)^{-1}\sum_{i=1}^{I}\sum_{u=1}^{T-1}\sum_{t=1}^{T}\bigl[\phi_{iu(u+1)tt} - \sigma_{iu(u+1)}\sigma_{itt}\bigr] \\ & - 2\{IT(T-1)\}^{-1} \\ & \times \sum_{i=1}^{I}\sum_{u=1}^{T-1}\sum_{m<t}^{T}\bigl[\phi_{iu(u+1)mt} - \sigma_{iu(u+1)}\sigma_{imt}\bigr],\end{aligned} \tag{2.18}$$

$$\begin{aligned}\mathrm{cov}_N(\psi_3, \psi_4) = (IT)^{-1} \sum_{i=1}^{I} \sum_{u=1}^{T} \sum_{m<t}^{T} [\phi_{iuumt} - \sigma_{iuu}\sigma_{imt}] \\ - 2\{IT(T-1)\}^{-1} \sum_{i=1}^{I} \sum_{u<\ell}^{T} \sum_{m<t}^{T} [\phi_{iu\ell mt} - \sigma_{iu\ell}\sigma_{imt}].\end{aligned} \tag{2.19}$$

3 GQL estimation approach

3.1 GQL estimating equation for β

Let $y_i = [y_{i1}, \ldots, y_{it}, \ldots, y_{iT}]'$ be the $T \times 1$ vector of first-order responses. Since $E[Y_{it} = x'_{it}\beta = \mu_{it}$ by (1.2), one obtains

$$\begin{aligned}E[Y_i] &= [\mu_{i1}, \ldots, \mu_{it}, \ldots, \mu_{iT}]' = [x'_{i1}, \ldots, x'_{it}, \ldots, x'_{iT}]'\beta \\ &= X_i\beta = \mu_i \qquad \text{(say)}.\end{aligned}$$

One may then write the so-called GQL estimating equation for β as

$$\sum_{i=1}^{I} \frac{\partial \mu'_i}{\partial \beta} \Sigma_i^{-1}(y_i - \mu_i) = \sum_{i=1}^{I} X'_i \Sigma_i^{-1}(y_i - X_i\beta) = 0, \tag{3.1}$$

yielding the GQL estimator of β given by

$$\hat{\beta}_{\mathrm{GQL}} = \left[\sum_{i=1}^{I} X'_i \Sigma_i^{-1} X_i\right]^{-1} \sum_{i=1}^{I} X'_i \Sigma_i^{-1} y_i. \tag{3.2}$$

Note that this GQL estimator is in fact the well-known generalized least squared (GLS) estimator, with Σ_i as the covariance matrix of y_i, where its diagonal elements σ_{itt} are defined in (1.3), and its off-diagonal elements σ_{iut} have the formulas given by (1.4).

3.2 GQL estimating equation for $\xi = (\theta, \sigma^2_\gamma, \sigma^2_\epsilon)'$

To estimate all three scale parameters, we follow Jowaheer and Sutradhar (2009, Section 2.2.2) but utilize a vector of basic statistic consisting of the corrected squares and the pair-wise products of the responses, given by

$$\begin{aligned}s_i = [&(y_{i1} - \mu_{i1})^2, \ldots, (y_{iT} - \mu_{iT})^2, (y_{i1} - \mu_{i1})(y_{i2} - \mu_{i2}), \\ &\ldots, (y_{iu} - \mu_{iu})(y_{it} - \mu_{it}), \ldots, (y_{i,T-1} - \mu_{i,T-1})(y_{iT} - \mu_{iT})]'.\end{aligned} \tag{3.3}$$

The GQL estimating equation for ξ has the form

$$\sum_{i=1}^{I} \frac{\partial \sigma'_i}{\partial \xi} \Omega_i^{-1}(s_i - \sigma_i) = 0, \tag{3.4}$$

where $\sigma_i = (\sigma_{i11}, \ldots, \sigma_{itt}, \ldots, \sigma_{iTT}, \sigma_{i12}, \ldots, \sigma_{iut}, \ldots, \sigma_{i,T-1,T})' = E(s_i)$ and $\Omega_i = \text{cov}(s_i)$. Note that similar to the computational difficulty for the weight matrix C under the GMM approach, we encounter difficulties to compute the fourth-order moment matrix Ω_i. This is because the true distributions of the random effects and errors of the model (1.1) are not known. Let $\Omega_{i,N}$ denote the "working" matrix to be computed by pretending that the responses follow the multivariate normal distribution as pointed out in the last section. Thus we solve the "working" GQL estimating equation

$$\sum_{i=1}^{I} \frac{\partial \sigma_i'}{\partial \xi} \Omega_{iN}^{-1} (s_i - \sigma_i) = 0 \tag{3.5}$$

instead of (3.4), to obtain the GQL estimate for ξ.

Note that as s_i contains corrected squares and pairwise products of the responses, the normality based fourth-order moments matrix Ω_{iN} may be computed by using the general fourth order moments from Lemma 2.2.

4 Asymptotic efficiency comparison

4.1 Asymptotic covariance matrix of the GMM estimator

Recall that the GMM estimate for $\alpha = (\beta', \theta, \sigma_\gamma^2, \sigma_\epsilon^2)'$ is obtained from (2.8). Under some mild regularity condition it may be shown that as $I \to \infty$,

$$\begin{aligned} &I^{1/2}(\hat{\alpha}_{\text{GMM}} - \alpha) \\ &\quad \sim N\left[0, I\left\{\frac{\partial \psi'}{\partial \alpha} C_N \frac{\partial \psi}{\partial \alpha}\right\}^{-1} \left(\frac{\partial \psi'}{\partial \alpha} C_N C^{-1} C_N \frac{\partial \psi}{\partial \alpha'}\right) \left\{\frac{\partial \psi'}{\partial \alpha} C_N \frac{\partial \psi}{\partial \alpha'}\right\}^{-1}\right], \end{aligned} \tag{4.1}$$

where $C^{-1} = \text{cov}(\psi)$ is the true covariance matrix for ψ based on the true data such as Gaussian or elliptic or any other symmetric continuous data. Note that if the true distributions of the errors are normal, then $C = C_N$. This leads to the covariance matrix of $\hat{\alpha}_{\text{GMM}}$ as

$$\text{cov}(\hat{\alpha}_{\text{GMM}}) = \left\{\frac{\partial \psi'}{\partial \alpha} C_N \frac{\partial \psi}{\partial \alpha'}\right\}^{-1}. \tag{4.2}$$

4.2 Asymptotic covariance matrix of the GQL estimator

Note that since I individuals are independent, it follows from (3.2) by applying the standard central limit theorem that asymptotically ($I \to \infty$)

$$\sqrt{I}(\hat{\beta}_{\text{GQL}} - \beta) \sim N\left(0, I\left[\sum_{i=1}^{I} X_i' \Sigma_i^{-1} X_i\right]^{-1}\right). \tag{4.3}$$

Let $\hat{\xi}_{\mathrm{GQL}} = (\hat{\theta}_{\mathrm{GQL}}, \hat{\sigma}^2_{\gamma,\mathrm{GQL}}, \hat{\sigma}^2_{\epsilon,\mathrm{GQL}})'$ be the solution of (3.5). Under some mild regularity conditions, it may be shown that asymptotically ($I \to \infty$)

$$I^{1/2}(\hat{\xi}_{\mathrm{GQL}} - \xi) \sim N(0, I V^*_{\mathrm{GQL}}), \tag{4.4}$$

where V^*_{GQL} is given by

$$\begin{aligned} V^*_{\mathrm{GQL}} = {} & \left[\sum_{i=1}^{I} \frac{\partial \sigma_i'}{\partial \xi} \Omega_{iN}^{-1} \frac{\partial \sigma_i}{\partial \xi}\right]^{-1} \left[\sum_{i=1}^{I} \frac{\partial \sigma_i'}{\partial \xi} \Omega_{iN}^{-1} \Omega_i \Omega_{iN}^{-1} \frac{\partial \sigma_i'}{\partial \xi}\right] \\ & \times \left[\sum_{i=1}^{I} \frac{\partial \sigma_i'}{\partial \xi} \Omega_{iN}^{-1} \frac{\partial \sigma_i}{\partial \xi}\right]^{-1} \end{aligned} \tag{4.5}$$

with Ω_i as the true covariance matrix of s_i, as in (3.4). Note that if the true distributions of the model (1.1) errors are normal, then the asymptotic covariance matrix V^*_{GQL} in (4.5) reduces to

$$V^*_{\mathrm{GQL}} = \left[\sum_{i=1}^{I} \frac{\partial \sigma_i'}{\partial \xi} \Omega_{iN}^{-1} \frac{\partial \sigma_i}{\partial \xi}\right]^{-1}. \tag{4.6}$$

4.3 Relative efficiency of the GQL versus GMM estimators under true normal distributions: an empirical study

To reflect the asymptotic case, we consider $I = 500$. Furthermore, since the panel data is usually collected over a small period of time, we consider $T = 4$, for example. As far as the covariates are concerned, we choose two time dependent covariates. The first covariate is considered to be

$$x_{it1} = \begin{cases} 0 & \text{for } i = 1, \ldots, I/2;\, t = 1, 2, \\ 1 & \text{for } i = 1, \ldots, I/2;\, t = 3, 4, \\ 1 & \text{for } i = K/2 + 1, \ldots, I;\, t = 1, \ldots, 4, \end{cases}$$

whereas the second covariate is chosen to be

$$x_{it2} = \begin{cases} 1 & \text{for } i = 1, \ldots, I/2;\, t = 1, 2, \\ 1.5 & \text{for } i = 1, \ldots, I/2;\, t = 3, 4, \\ 0 & \text{for } i = I/2 + 1, \ldots, I;\, t = 1, 2, \\ 1 & \text{for } i = I/2 + 1, \ldots, I;\, t = 3, 4. \end{cases}$$

By using $\beta_1 = \beta_2 = 1.0$; $\theta = 0.3$ and 0.8, $\sigma^2_\gamma = 0.5$, 1.0, 1.5 and 2.0 and $\sigma^2_\epsilon = 1.0$, and the covariates given above, we have computed the diagonal elements (variances) of the covariance matrices from (4.2), and (4.3) and (4.6).

The asymptotic variances are shown in Table 1. The results of the table show that the variances of the estimators for all five parameters $\beta_1, \beta_2, \theta, \sigma^2_\gamma$ and σ^2_ϵ, under the GQL approach are uniformly smaller than the corresponding variances

Table 1 *Comparison of asymptotic variances (var) of the GQL and GMM estimators for the estimation of the regression parameters (β_1 and β_2), dynamic dependence parameter θ, and the variance components (σ_γ^2 and σ_ϵ^2), of a longitudinal mixed model for the normal panel data, with $T = 4$ and $K = 500$, $\beta_1 = \beta_2 = 1.0$ and $\sigma_\epsilon^2 = 1.0$*

			Asymptotic variances			
θ	Method	Quantity	$\sigma_\gamma^2 = 0.5$	1.0	1.5	2.0
0.3	GQL	var($\hat\beta_1$)	1.99×10.0^{-3}	1.50×10.0^{-3}	3.68×10.0^{-3}	2.45×10.0^{-3}
		var($\hat\beta_2$)	1.47×10.0^{-3}	1.18×10.0^{-3}	2.66×10.0^{-3}	1.65×10.0^{-3}
		var($\hat\theta$)	2.53×10.0^{-4}	4.85×10.0^{-5}	8.70×10.0^{-5}	9.01×10.0^{-7}
		var($\hat\sigma_\gamma^2$)	9.33×10.0^{-4}	1.50×10.0^{-3}	4.95×10.0^{-3}	4.15×10.0^{-3}
		var($\hat\sigma_\epsilon^2$)	1.08×10.0^{-3}	1.05×10.0^{-3}	1.22×10.0^{-3}	1.00×10.0^{-3}
	GMM	var($\hat\beta_1$)	2.52×10.0^{-3}	3.12×10.0^{-3}	3.71×10.0^{-3}	4.31×10.0^{-3}
		var($\hat\beta_2$)	1.86×10.0^{-3}	2.34×10.0^{-3}	2.81×10.0^{-3}	3.29×10.0^{-3}
		var($\hat\theta$)	2.53×10.0^{-3}	2.81×10.0^{-3}	3.20×10.0^{-3}	3.63×10.0^{-3}
		var($\hat\sigma_\gamma^2$)	5.27×10.0^{-2}	0.119	0.225	0.385
		var($\hat\sigma_\epsilon^2$)	9.75×10.0^{-3}	2.99×10.0^{-2}	6.82×10.0^{-2}	0.128
0.8	GQL	var($\hat\beta_1$)	2.36×10.0^{-3}	1.06×10.0^{-3}	2.79×10.0^{-2}	2.76×10.0^{-2}
		var($\hat\beta_2$)	2.70×10.0^{-3}	1.60×10.0^{-3}	6.08×10.0^{-4}	0.160
		var($\hat\theta$)	2.76×10.0^{-6}	9.60×10.0^{-8}	1.96×10.0^{-7}	1.48×10.0^{-7}
		var($\hat\sigma_\gamma^2$)	2.53×10.0^{-4}	1.01×10.0^{-3}	2.26×10.0^{-3}	4.04×10.0^{-3}
		var($\hat\sigma_\epsilon^2$)	1.00×10.0^{-3}	1.00×10.0^{-3}	1.00×10.0^{-3}	1.00×10.0^{-3}
	GMM	var($\hat\beta_1$)	6.82×10.0^{-3}	9.57×10.0^{-3}	1.23×10.0^{-2}	1.51×10.0^{-2}
		var($\hat\beta_2$)	5.53×10.0^{-3}	7.97×10.0^{-3}	1.04×10.0^{-2}	1.28×10.0^{-2}
		var($\hat\theta$)	0.799	0.229	0.161	0.136
		var($\hat\sigma_\gamma^2$)	1.108	0.447	0.421	0.459
		var($\hat\sigma_\epsilon^2$)	9.601	9.723	14.621	21.392

under the GMM approach, indicating that the GQL approach produces the same or more efficient estimates than the GMM approach for all 5 parameters of the model. For example, when $\theta = 0.3$ and $\sigma_\gamma^2 = 1.5$, the GQL estimates of β_1 and β_2 are found to be $\frac{3.71\times 10.0^{-3}}{3.68\times 10.0^{-3}} = 1.008$ and $\frac{2.81\times 10.0^{-3}}{2.66\times 10.0^{-3}} = 1.056$ times more efficient than the corresponding GMM estimates. For the estimation of θ, σ_γ^2 and σ_ϵ^2, the GQL approach is found to outperform the GMM approach. For example, for the same set of parameters, that is, when $\theta = 0.3$ and $\sigma_\gamma^2 = 1.5$, the GQL estimates of θ, σ_γ^2 and σ_ϵ^2 were found to be $\frac{3.20\times 10.0^{-3}}{8.70\times 10.0^{-5}} = 36.78$, $\frac{0.225}{4.95\times 10.0^{-3}} = 45.45$ and $\frac{6.82\times 10.0^{-2}}{1.22\times 10.0^{-3}} = 55.90$, times more efficient than the corresponding GMM estimates. It is also seen from the table that for larger dynamic dependence parameter $\theta = 0.8$, the GMM approach performs much worse as compared to the GQL approach.

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References

Ahn, S. and Schmidt, P. (1995). Efficient estimation of models for dynamic panel data. *Journal of Econometrics* **68**, 5–27. MR1345704

Arellano, M. and Bond, S. (1991). Some tests of specification for panel data: Monte Carlo evidence and an application to employment equations. *Review of Economics and Statistics* **58**, 277–298.

Blundell, R. and Bond, S. (1998). Initial conditions and moment restrictions in dynamic panel data models. *Journal of Econometrics* **87**, 115–143.

Bond, S., Bowsher, C. and Windmeijer, F. (2001). *Criterion-Based Inference for GMM in Linear Dynamic Panel Data Models*. London: IFS.

Bun, M. J. G. and Carree, M. A. (2005). Bias correction estimation in dynamic panel data models. *Journal of Business and Economics Statistics* **23**, 200–210. MR2157271

Chamberlain, G. (1992). Sequential moment restrictions in panel data-comment. *Journal of Business and Economic Statistics* **10**, 20–26.

Imbens, G. W. (2002). Generalized method of moments and empirical likelihood. *Journal of Business and Economic Statistics* **20**, 493–506. MR1973800

Jiang, J. and Zhang, W. (2001). Robust estimation in generalized linear mixed models. *Biometrika* **88**, 753–765. MR1859407

Jowaheer, V. and Sutradhar, B. C. (2009). GMM versus GQL inferences for panel count data. *Statistics and Probability Letters* **79**, 1928–1934.

Hansen, L.-P. (1982). Large sample properties of generalized method of moment estimators. *Econometrica* **50**, 1029–1054. MR0666123

Keane, M. and Runkle, D. (1992). On the estimation of panel data models with serial correlation when instruments are not strictly exogenous. *Journal of Business and Economic Statistics* **10**, 1–9.

Sutradhar, B. C. (2004). On exact quasilikelihood inference in generalized linear mixed models. *Sankhya* **66**, 261–289. MR2090978

Sutradhar, B. C., Rao, R. P. and Pandit, V. N. (2008). Generalized method of moments versus generalized quasilikelihood inferences in binary panel data models. *Sankhya B* **70**, 34–62. MR2507474

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