Golden Ratio Base Expansions of the Logarithm and Inverse Tangent of Fibonacci and Lucas Numbers

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Abstract

Let $\alpha = (1 + \sqrt{5})/2$, the golden ratio, and $\beta = -1/\alpha = (1 - \sqrt{5})/2$. Let F_n and L_n be the Fibonacci and Lucas numbers, defined by $F_n = (\alpha^n - \beta^n)/\sqrt{5}$ and $L_n = \alpha^n + \beta^n$, for all non-negative integers. We derive base α expansions of $\log F_n$, $\log L_n$, $\arctan \frac{1}{F_n}$ and $\arctan \frac{1}{L_n}$ for all positive integers n.

Keywords: Fibonacci number, Lucas number, logarithm, arctangent, inverse tangent, golden ratio, non-integer base expansion, BBP-type formula.

1 Introduction

Let α denote the golden ratio; that is $\alpha = (1 + \sqrt{5})/2$. Let $\beta = -1/\alpha = (1 - \sqrt{5})/2$. Thus $\alpha\beta = -1$ and $\alpha + \beta = 1$. Let F_n and L_n be the Fibonacci and Lucas numbers, defined by $F_n = (\alpha^n - \beta^n)/\sqrt{5}$ and $L_n = \alpha^n + \beta^n$, for all non-negative integers n.

Let b be any non-zero number whose magnitude is greater than unity. Let n and s be positive integers. A convergent series of the form

$$C = \sum_{k=0}^{\infty} \frac{1}{b^k} \left(\frac{a_1}{(kn+1)^s} + \frac{a_2}{(kn+2)^s} + \dots + \frac{a_n}{(kn+n)^s} \right),$$
(1.1)

where a_1, a_2, \ldots, a_n are certain numbers, defines a base b, length n and degree s expansion of the mathematical constant C.

If b is an integer and a_k are rational numbers, then (1.1) is referred to as a BBP-type formula, after the initials of the authors of the paper [4] in which such an expansion was first presented for π and some other mathematical constants. Any mathematical constant that possesses a base b BBP-type formula has the property that its n-th digit in base b could be calculated directly, without needing to compute any of the previous n - 1 digits. Although the infinite series presented in this paper have the structure of BBP-type formulas, it must be clearly stated that the series do not yield digit extraction since here the base $b = \alpha^n$ is not an integer and the a_k are not rational numbers; rather the series correspond to base α expansions of the mathematical constants concerned.

Our goal in this paper is to derive base α expansion formulas for the logarithm and the inverse tangent of all Fibonacci and Lucas numbers. We will often give the expansion using the compact P-notation for BBP-type formulas, introduced by Bailey and Crandall [5], namely,

$$C = P(s, b, n, A) = \sum_{k=0}^{\infty} \frac{1}{b^k} \sum_{j=1}^{n} \frac{a_j}{(kn+j)^s},$$

where s and n are integers and, in this present paper, b is an integer power of α and $A = (a_1, a_2, \ldots, a_n)$ is a vector of rational multiples of powers of β . For example, we will show (see (2.20)) that

$$\log F_3 = \log 2 = \sum_{k=0}^{\infty} \frac{1}{\alpha^{12k}} \left(\frac{\beta^2}{6k+1} + \frac{3\beta^4}{6k+2} + \frac{4\beta^6}{6k+3} + \frac{3\beta^8}{6k+4} + \frac{\beta^{10}}{6k+5} \right),$$

which, in the P-notation, can be written as

$$\log F_3 = \log 2 = P(1, \alpha^{12}, 6, (\beta^2, 3\beta^4, 4\beta^6, 3\beta^8, \beta^{10}, 0)).$$

Base α expansions have also been studied or reported by Bailey and Crandall [5], Chan [7, 8], Zhang [12], Borwein and Chamberland [6], Cloitre [9], Adegoke [2], Wei [11], and more recently Kristensen and Mathiasen [10].

2 Base α expansions of logarithms

The base α expansions of the logarithms of Fibonacci and Lucas numbers are presented in Theorems 2.3 and 2.4 but first we state a couple of Lemmata upon which the results are based.

Let

$$\operatorname{Li}_{1}(x) = -\log(1-x) = \sum_{k=1}^{\infty} \frac{x^{k}}{k} = \sum_{k=0}^{\infty} x^{k} \frac{x}{k+1}, \quad -1 \le x < 1.$$

Lemma 2.1. If |b| > 1, t > 0 and m and n are arbitrary positive integers, then,

$$\operatorname{Li}_{1}\left(\frac{1}{b^{t}}\right) = \sum_{k=0}^{\infty} \frac{1}{b^{tmk}} \sum_{j=1}^{m} \frac{1/b^{tj}}{(mk+j)},$$
(2.1)

$$\operatorname{Li}_{1}\left(-\frac{1}{b^{t}}\right) = \sum_{k=0}^{\infty} \frac{1}{b^{2tnk}} \sum_{j=1}^{2n} \frac{(-1)^{j}/b^{tj}}{(2nk+j)}.$$
(2.2)

Proof. We have

$$\operatorname{Li}_1\left(\frac{1}{b^t}\right) = \sum_{k=0}^{\infty} \frac{1}{b^{tk}} \frac{1/b^t}{k+1},$$

from which (2.1) follows upon using the identity

$$\sum_{k=0}^{\infty} f_k = \sum_{k=0}^{\infty} \sum_{j=1}^{m} f_{mk+j-1},$$
(2.3)

with

$$f_k = \frac{1}{b^{tk+t}} \frac{1}{k+1}.$$

The proof of (2.2) is similar, with m = 2n in (2.3).

Lemma 2.2. If r is an integer, then,

$$\log L_r = r \operatorname{Li}_1\left(\frac{1}{\alpha^2}\right) - \operatorname{Li}_1\left(\frac{(-1)^{r+1}}{\alpha^{2r}}\right),\tag{2.4}$$

$$\log F_r = (r-2)\operatorname{Li}_1\left(\frac{1}{\alpha^2}\right) + \operatorname{Li}_1\left(\frac{1}{\alpha^4}\right) - \operatorname{Li}_1\left(\frac{(-1)^r}{\alpha^{2r}}\right), \quad r \neq 0.$$
(2.5)

Proof. We have

$$\operatorname{Li}_{1}\left(-\frac{\beta^{r}}{\alpha^{r}}\right) = -\log\left(\frac{\alpha^{r}+\beta^{r}}{\alpha^{r}}\right) = -\log\left(\frac{L_{r}}{\alpha^{r}}\right) = -\log L_{r} + r\log\alpha, \quad (2.6)$$

in which setting r = 1 gives

$$\log \alpha = \operatorname{Li}_1\left(\frac{1}{\alpha^2}\right). \tag{2.7}$$

Using (2.7) in (2.6) gives (2.4).

Also,

$$\operatorname{Li}_{1}\left(\frac{\beta^{r}}{\alpha^{r}}\right) = -\log\left(\frac{\alpha^{r}-\beta^{r}}{\alpha^{r}}\right) = -\log\left(\frac{F_{r}\sqrt{5}}{\alpha^{r}}\right) = -\log F_{r} + r\log\alpha - \log\sqrt{5},\qquad(2.8)$$

in which setting r = 2 gives

$$\log \sqrt{5} = 2\log \alpha - \operatorname{Li}_1\left(\frac{1}{\alpha^4}\right) = 2\operatorname{Li}_1\left(\frac{1}{\alpha^2}\right) - \operatorname{Li}_1\left(\frac{1}{\alpha^4}\right),\tag{2.9}$$

where we used (2.7). Identity (2.5) follows from (2.8) and (2.9).

Theorem 2.3. If r is an integer, then,

$$\log F_r = \sum_{k=0}^{\infty} \frac{1}{\alpha^{4rk}} \sum_{j=1}^r \frac{\beta^{4j-2}(r-2+r\delta_{j,(r+1)/2})}{2rk+2j-1} + \sum_{k=0}^{\infty} \frac{1}{\alpha^{4rk}} \sum_{j=1}^{r-1} \frac{\beta^{4j}r}{2rk+2j}, \quad r \ odd, \quad (2.10)$$

$$\log F_r = \sum_{k=0}^{\infty} \frac{1}{\alpha^{4rk}} \sum_{j=1}^r \frac{(r-2)\beta^{4j-2}}{2rk+2j-1} + \sum_{k=0}^{\infty} \frac{1}{\alpha^{4rk}} \sum_{j=1}^{r-1} \frac{\beta^{4j}r(1-\delta_{j,r/2})}{2rk+2j}, \quad r \text{ even.}$$
(2.11)

Here and throughout this paper, δ_{mn} denotes the Kronecker delta symbol whose value is unity when m equals n and zero otherwise.

Proof. We prove (2.11). When r is even, (2.5) reads

$$\log F_r = (r-2)\operatorname{Li}_1\left(\frac{1}{\alpha^2}\right) + \operatorname{Li}_1\left(\frac{1}{\alpha^4}\right) - \operatorname{Li}_1\left(\frac{1}{\alpha^{2r}}\right), \quad r \neq 0.$$
 (2.12)

We proceed to write the three Li₁ terms in a common base α^{4r} , using (2.1) with appropriate t and m choices. Thus,

$$\operatorname{Li}_{1}\left(\frac{1}{\alpha^{2}}\right) = \sum_{k=0}^{\infty} \frac{1}{\alpha^{4rk}} \sum_{j=1}^{2r} \frac{1/\alpha^{2j}}{2rk+j},$$
(2.13)

$$\operatorname{Li}_{1}\left(\frac{1}{\alpha^{4}}\right) = \sum_{k=0}^{\infty} \frac{1}{\alpha^{4rk}} \sum_{j=1}^{r} \frac{1/\alpha^{4j}}{rk+j},$$
(2.14)

$$\operatorname{Li}_{1}\left(\frac{1}{\alpha^{2r}}\right) = \sum_{k=0}^{\infty} \frac{1}{\alpha^{4rk}} \left(\frac{1/\alpha^{2r}}{2k+1} + \frac{1/\alpha^{4r}}{2k+2}\right).$$
(2.15)

Using (2.13), (2.14) and (2.15) in (2.12) gives

$$\log F_r = \sum_{k=0}^{\infty} \frac{1}{\alpha^{4rk}} \sum_{j=1}^{2r} \frac{\beta^{2j}(r-2)}{2rk+j} + \sum_{k=0}^{\infty} \frac{1}{\alpha^{4rk}} \sum_{j=1}^{r} \frac{2\beta^{4j}}{2rk+2j} - \sum_{k=0}^{\infty} \frac{1}{\alpha^{4rk}} \left(\frac{r\beta^{2r}}{2rk+r} + \frac{r\beta^{4r}}{2rk+2r} \right).$$
(2.16)

Using the summation identity

$$\sum_{j=1}^{2r} f_j = \sum_{j=1}^r f_{2j} + \sum_{j=1}^r f_{2j-1}$$
(2.17)

to write its inner sum, the first term on the right hand side of (2.16) can be written as

$$\sum_{k=0}^{\infty} \frac{1}{\alpha^{4rk}} \sum_{j=1}^{2r} \frac{\beta^{2j}(r-2)}{2rk+j} = \sum_{k=0}^{\infty} \frac{1}{\alpha^{4rk}} \sum_{j=1}^{r} \frac{\beta^{4j}(r-2)}{2rk+2j} + \sum_{k=0}^{\infty} \frac{1}{\alpha^{4rk}} \sum_{j=1}^{r} \frac{\beta^{4j-2}(r-2)}{2rk+2j-1}.$$
 (2.18)

Using (2.18) in (2.16) yields (2.11).

Identities (2.10) and (2.11) written in the *P*-notation are

$$\log F_r = P(1, \alpha^{4r}, 2r, (a_1, a_2, \dots, a_{2r}))$$
(2.19)

where for $1 \leq j \leq r$,

$$a_{2j-1} = \beta^{4j-2}(r-2+r\delta_{j,(r+1)/2}), \quad a_{2j} = \beta^{4j}r(1-\delta_{rj}), \quad r \text{ odd};$$

and

$$a_{2j-1} = (r-2)\beta^{4j-2}, \quad a_{2j} = \beta^{4j}r(1-\delta_{j,r/2}-\delta_{j,r}), \quad r \text{ even.}$$

Examples.

$$\log F_3 = \log 2 = P(1, \alpha^{12}, 6, (\beta^2, 3\beta^4, 4\beta^6, 3\beta^8, \beta^{10}, 0)),$$
(2.20)

$$\log F_5 = \log 5 = P(1, \alpha^{20}, 10, (3\beta^2, 5\beta^4, 3\beta^6, 5\beta^8, 8\beta^{10}, 5\beta^{12}, 3\beta^{14}, 5\beta^{16}, 3\beta^{18}, 0)),$$
(2.21)

$$\log F_4 = \log 3 = P(1, \alpha^{16}, 8, (2\beta^2, 4\beta^4, 2\beta^6, 0, 2\beta^{10}, 4\beta^{12}, 2\beta^{14}, 0)),$$
(2.22)

$$\log F_8 = \log 21 = P(1, \alpha^{32}, 16, (6\beta^2, 8\beta^4, 6\beta^6, 8\beta^8, 6\beta^{10}, 8\beta^{12}, 6\beta^{14}, 0, 6\beta^{18}, 8\beta^{20}, 6\beta^{22}, 8\beta^{24}, 6\beta^{26}, 8\beta^{28}, 6\beta^{30}, 0)),$$
(2.23)

$$\log F_{12} = \log 144 = P(1, \alpha^{48}, 24, (10 \beta^2, 12 \beta^4, 10 \beta^6, 12 \beta^8, 10 \beta^{10}, 12 \beta^{12}, 10 \beta^{14}, 12 \beta^{16}, 10 \beta^{18}, 12 \beta^{20}, 10 \beta^{22}, 0, 10 \beta^{26}, 12 \beta^{28}, 10 \beta^{30}, 12 \beta^{32}, 10 \beta^{34}, 12 \beta^{36}, 10 \beta^{38}, 12 \beta^{40}, 10 \beta^{42}, 12 \beta^{44}, 10 \beta^{46}, 0)).$$

$$(2.24)$$

Theorem 2.4. If r is an integer, then,

$$\log L_r = \sum_{k=0}^{\infty} \frac{1}{\alpha^{2rk}} \sum_{j=1}^{r-1} \frac{\beta^{2j} r}{rk+j}, \ r \ odd,$$
(2.25)

$$\log L_r = \sum_{k=0}^{\infty} \frac{1}{\alpha^{4rk}} \sum_{j=1}^{2r-1} \frac{\beta^{2j} r(1+\delta_{rj})}{2rk+j}, \ r \ even.$$
(2.26)

Proof. We prove (2.25). If r is an odd integer, (2.4) gives

$$\log L_r = r \operatorname{Li}_1\left(\frac{1}{\alpha^2}\right) - \operatorname{Li}_1\left(\frac{1}{\alpha^{2r}}\right).$$
(2.27)

With

$$\operatorname{Li}_1\left(\frac{1}{\alpha^2}\right) = \sum_{k=0}^{\infty} \frac{1}{\alpha^{2rk}} \sum_{j=1}^r \frac{1/\alpha^{2j}}{rk+j}$$

and

$$\operatorname{Li}_1\left(\frac{1}{\alpha^{2r}}\right) = \sum_{k=0}^{\infty} \frac{1}{\alpha^{2rk}} \frac{r/\alpha^{2r}}{rk+r}$$

in (2.27); identity (2.25) follows.

Identities (2.25) and (2.26) in the *P*-notation are

$$\log L_r = P(1, \alpha^{2r}, r, (a_1, a_2, \dots, a_r)), \quad r \text{ odd},$$
(2.28)

with

$$a_j = r\beta^{2j}(1 - \delta_{rj}), \quad 1 \le j \le r,$$

and

$$\log L_r = P(1, \alpha^{4r}, 2r, (a_1, a_2, \dots, a_{2r})), \quad r \text{ even},$$
(2.29)

with

$$a_j = r\beta^{2j}(1 + \delta_{jr} - \delta_{j,2r}), \quad 1 \le j \le 2r.$$

Examples.

$$\log L_2 = \log 3 = \sum_{k=0}^{\infty} \frac{1}{\alpha^{8k}} \left(\frac{2\beta^2}{4k+1} + \frac{4\beta^4}{4k+2} + \frac{2\beta^6}{4k+3} \right);$$
(2.30)

that is,

$$\log 3 = P(1, \alpha^8, 4, (2\beta^2, 4\beta^4, 2\beta^6, 0)).$$
(2.31)

$$\log L_3 = \log 4 = P(1, \alpha^6, 3, (3\beta^2, 3\beta^4, 0)), \qquad (2.32)$$

$$\log L_4 = \log 7 = 4\beta^2 P(1, \alpha^{16}, 8, (1, \beta^2, \beta^4, 2\beta^6, \beta^8, \beta^{10}, \beta^{12}, 0)),$$
(2.33)

$$\log L_6 = \log 18 = P(1, \alpha^{24}, 12, (6\beta^2, 6\beta^4, 6\beta^6, 6\beta^8, 6\beta^{10}, 12\beta^{12}, 6\beta^{14}, 6\beta^{16}, 6\beta^{18}, 6\beta^{20}, 6\beta^{22}, 0)).$$
(2.34)

3 Base α expansions of inverse tangents

The base α expansions of the inverse tangent of Fibonacci and Lucas numbers are stated in Theorems 3.4–3.8 but first we collect some required identities in Lemmata 3.1–3.3.

Lemma 3.1. If r is an integer, then,

$$\alpha^{r} - \alpha^{-r} = \begin{cases} F_r \sqrt{5}, & r \text{ even}; \\ L_r, & r \text{ odd}; \end{cases}$$
(3.1)

$$\alpha^r + \alpha^{-r} = \begin{cases} L_r & r \text{ even;} \\ F_r \sqrt{5}, & r \text{ odd.} \end{cases}$$
(3.2)

Lemma 3.2. If r and m are integers, then,

$$\arctan \frac{1}{\alpha^{r-m}} - \arctan \frac{1}{\alpha^{r+m}} = \begin{cases} \arctan \left(\frac{L_m}{(F_r \sqrt{5})} \right), & m \text{ odd, } r \text{ odd,} \\ \arctan \left(\frac{L_m}{L_r} \right), & m \text{ odd, } r \text{ even,} \\ \arctan \left(\frac{F_m}{F_r} \right), & m \text{ even, } r \text{ odd,} \\ \arctan \left(\frac{F_m}{\sqrt{5}} \right), & m \text{ even, } r \text{ even;} \end{cases}$$
(3.3)

$$\arctan \frac{1}{\alpha^{r-m}} + \arctan \frac{1}{\alpha^{r+m}} = \begin{cases} \arctan(F_m\sqrt{5}/L_r), & m \text{ odd, } r \text{ odd,} \\ \arctan(F_m/F_r), & m \text{ odd, } r \text{ even,} \\ \arctan(L_m/L_r), & m \text{ even, } r \text{ odd,} \\ \arctan\left(L_m/(F_r\sqrt{5})\right), & m \text{ even, } r \text{ even.} \end{cases}$$
(3.4)

Proof. The arctangent subtraction and addition formulas give

$$\arctan \frac{1}{\alpha^{r-m}} - \arctan \frac{1}{\alpha^{r+m}} = \arctan \left(\frac{\alpha^r (\alpha^m - \alpha^{-m})}{\alpha^{2r} + 1}\right),$$
$$\arctan \frac{1}{\alpha^{r-m}} + \arctan \frac{1}{\alpha^{r+m}} = \arctan \left(\frac{\alpha^r (\alpha^m + \alpha^{-m})}{\alpha^{2r} - 1}\right);$$

and hence the stated identities upon the use of Lemma 3.1.

Lemma 3.3. If r is an integer, then,

$$\alpha^{2r} - 1 = \alpha^r L_r, \quad \beta^{2r} - 1 = \beta^r L_r, \quad r \text{ odd}, \tag{3.5}$$

$$\alpha^{2r} - 1 = \alpha^r F_r \sqrt{5}, \quad \beta^{2r} - 1 = -\beta^r F_r \sqrt{5}, \quad r \ even, \tag{3.6}$$

$$\alpha^{2r} + 1 = \alpha^r F_r \sqrt{5}, \quad \beta^{2r} + 1 = -\beta^r F_r \sqrt{5}, \quad r \ odd, \tag{3.7}$$

$$\alpha^{2r} + 1 = \alpha^r L_r, \quad \beta^{2r} + 1 = \beta^r L_r, \quad r \text{ even.}$$
(3.8)

Theorem 3.4. If r is an odd integer greater than unity, then,

$$\arctan \frac{1}{F_r} = P(1, \alpha^{4(r^2 - 4)}, 4(r^2 - 4), (a_1, a_2, \dots, a_{4(r^2 - 4)})),$$
(3.9)

where the only non-zero constants a_j are given by

$$a_{(r-2)(4j-3)} = -\beta^{(r-2)(4j-3)}(r-2), \quad j = 1, 2, \dots, r+2,$$

$$a_{(r-2)(4j-1)} = \beta^{(r-2)(4j-1)}(r-2), \quad j = 1, 2, \dots, r+2,$$

$$a_{(r+2)(4j-3)} = \beta^{(r+2)(4j-3)}(r+2), \quad j = 1, 2, \dots, r-2,$$

$$a_{(r+2)(4j-1)} = -\beta^{(r+2)(4j-1)}(r+2), \quad j = 1, 2, \dots, r-2,$$

$$a_{(r-2)(r+2)} = (-1)^{(r+1)/2} 4\beta^{r^2-4}$$

and

$$a_{3(r-2)(r+2)} = (-1)^{(r-1)/2} 4\beta^{3(r^2-4)}.$$

Proof. With r an odd number, m = 2 in (3.3) gives

$$\arctan \frac{1}{F_r} = \arctan \frac{1}{\alpha^{r-2}} - \arctan \frac{1}{\alpha^{r+2}}.$$

The following identity, proved in [1, Identity (10)]:

$$n\sqrt{n}\arctan\left(\frac{1}{\sqrt{n}}\right) = \sum_{k=0}^{\infty} \frac{1}{n^{2k}} \left(\frac{n}{4k+1} - \frac{1}{4k+3}\right)$$
 (3.10)

gives

$$\arctan\frac{1}{\alpha^{r-2}} = \sum_{k=0}^{\infty} \frac{1}{\alpha^{(r-2)(4k+3)}} \left(\frac{\alpha^{2r-4}}{4k+1} - \frac{1}{4k+3}\right)$$

and

$$\arctan\frac{1}{\alpha^{r+2}} = \sum_{k=0}^{\infty} \frac{1}{\alpha^{(r+2)(4k+3)}} \left(\frac{\alpha^{2r+4}}{4k+1} - \frac{1}{4k+3}\right),$$

or, by (2.3),

$$\arctan\frac{1}{\alpha^{r-2}} = \sum_{k=0}^{\infty} \frac{1}{\alpha^{(4r^2 - 16)k}} \sum_{j=1}^{r+2} \left(\frac{\alpha^{-(r-2)(4j-3)}}{4(r+2)k + 4j - 3} - \frac{\alpha^{-(r-2)(4j-1)}}{4(r+2)k + 4j - 1} \right)$$

and

$$\arctan \frac{1}{\alpha^{r+2}} = \sum_{k=0}^{\infty} \frac{1}{\alpha^{(4r^2 - 16)k}} \sum_{j=1}^{r-2} \left(\frac{\alpha^{-(r+2)(4j-3)}}{4(r-2)k + 4j - 3} - \frac{\alpha^{-(r+2)(4j-1)}}{4(r-2)k + 4j - 1} \right)$$

Thus,

$$\arctan \frac{1}{F_r} = \sum_{k=0}^{\infty} \frac{1}{\alpha^{(4r^2 - 16)k}} \times \left(\sum_{j=1}^{r+2} \left(\frac{-\beta^{(r-2)(4j-3)}(r-2)}{4(r^2 - 4)k + (r-2)(4j-3)} + \frac{\beta^{(r-2)(4j-1)}(r-2)}{4(r^2 - 4)k + (r-2)(4j-1)} \right) + \sum_{j=1}^{r-2} \left(\frac{\beta^{(r+2)(4j-3)}(r+2)}{4(r^2 - 4)k + (r+2)(4j-3)} - \frac{\beta^{(r+2)(4j-1)}(r+2)}{4(r^2 - 4)k + (r+2)(4j-1)} \right) \right).$$
(3.11)

Identity (3.9) is (3.11) expressed in the *P*-notation.

Examples.

$$\arctan \frac{1}{F_3} = \arctan \frac{1}{2} = P(1, \alpha^{20}, 20, (-\beta, 0, \beta^3, 0, 4\beta^5, 0, \beta^7, 0, -\beta^9, 0, \beta^{11}, 0, -\beta^{13}, 0, -4\beta^{15}, 0, -\beta^{17}, 0, \beta^{19}, 0)),$$
(3.12)

$$\arctan \frac{1}{F_5} = \arctan \frac{1}{5} = P(1, \alpha^{84}, 84, (0, 0, -3\beta^3, 0, 0, 0, 7\beta^7, 0, 3\beta^9, 0, 0, 0, 0, 0, -3\beta^{15}, 0, 0, 0, 0, 0, 0, -4\beta^{21}, 0, 0, 0, 0, 0, -3\beta^{27}, 0, 0, 0, 0, 0, 0, 3\beta^{33}, 0, 7\beta^{35}, 0, 0, 0, -3\beta^{39}, 0, 0, 0, 0, 0, 3\beta^{45}, 0, 0, 0, -7\beta^{49}, 0, -3\beta^{51}, 0, 0, 0, 0, 0, 3\beta^{57}, 0, 0, 0, 0, 0, 4\beta^{63}, 0, 0, 0, 0, 0, 3\beta^{69}, 0, 0, 0, 0, 0, -3\beta^{75}, 0, -7\beta^{77}, 0, 0, 0, 3\beta^{81}, 0, 0, 0)),$$

$$(3.13)$$

Theorem 3.5. If r is a positive even integer, then,

$$\arctan \frac{1}{F_r} = P(1, \alpha^{4(r^2 - 1)}, 4(r^2 - 1), (a_1, a_2, \dots, a_{4(r^2 - 1)})),$$
(3.15)

where the only non-zero constants a_j are given by

$$a_{(r-1)(4j-3)} = -\beta^{(r-1)(4j-3)}(r-1), \quad j = 1, 2, \dots, r+1,$$

$$a_{(r-1)(4j-1)} = \beta^{(r-1)(4j-1)}(r-1), \quad j = 1, 2, \dots, r+1,$$

$$a_{(r+1)(4j-3)} = -\beta^{(r+1)(4j-3)}(r+1), \quad j = 1, 2, \dots, r-1,$$

$$a_{(r+1)(4j-1)} = \beta^{(r+1)(4j-1)}(r+1), \quad j = 1, 2, \dots, r-1,$$

$$a_{(r-1)(r+1)} = (-1)^{r/2} 2\beta^{r^2-1}$$

and

$$a_{3(r-1)(r+1)} = (-1)^{(r+2)/2} 2\beta^{3(r^2-1)}.$$

Proof. Setting m = 1 in (3.4) gives

$$\arctan \frac{1}{F_r} = \arctan \frac{1}{\alpha^{r-1}} + \arctan \frac{1}{\alpha^{r+1}}, \quad r \text{ even}$$

The proof now proceeds as in that of Theorem 3.4.

Examples.

$$\arctan\frac{1}{F_2} = \frac{\pi}{4} = P(1, \alpha^{12}, 12, (-\beta, 0, -2\beta^3, 0, -\beta^5, 0, \beta^7, 0, 2\beta^9, 0, \beta^{11}, 0)),$$
(3.16)

$$\arctan \frac{1}{F_4} = \arctan \frac{1}{3} = P(1, \alpha^{60}, 60, (0, 0, -3\beta^3, 0, -5\beta^5, 0, 0, 0, 3\beta^9, 0, 0, 0, 0, 0, 2\beta^{15}, 0, 0, 0, 0, 0, 0, 3\beta^{21}, 0, 0, 0, -5\beta^{25}, 0, -3\beta^{27}, 0, 0, 0, 0, 0, 3\beta^{33}, 0, 5\beta^{35}, 0, 0, 0, -3\beta^{39}, 0, 0, 0, 0, 0, -2\beta^{45}, 0, 0, 0, 0, 0, -3\beta^{51}, 0, 0, 0, 0, 5\beta^{55}, 0, 3\beta^{57}, 0, 0, 0)).$$

$$(3.17)$$

Theorem 3.6. If r is a positive even integer, then,

$$\arctan \frac{1}{L_r} = P(1, \alpha^{4(r^2 - 1)}, 4(r^2 - 1), (a_1, a_2, \dots, a_{4(r^2 - 1)})),$$
(3.18)

where the only non-zero constants a_j are given by

$$a_{(r-1)(4j-3)} = -\beta^{(r-1)(4j-3)}(r-1), \quad j = 1, 2, \dots, r+1,$$

$$a_{(r-1)(4j-1)} = \beta^{(r-1)(4j-1)}(r-1), \quad j = 1, 2, \dots, r+1,$$

$$a_{(r+1)(4j-3)} = \beta^{(r+1)(4j-3)}(r+1), \quad j = 1, 2, \dots, r-1,$$

$$a_{(r+1)(4j-1)} = -\beta^{(r+1)(4j-1)}(r+1), \quad j = 1, 2, \dots, r-1,$$

$$a_{(r-1)(r+1)} = (-1)^{(r+2)/2} 2r \beta^{r^2-1}$$

and

$$a_{3(r-1)(r+1)} = (-1)^{r/2} 2r \beta^{3(r^2-1)}.$$

Proof. Setting m = 1 in (3.3) gives

$$\arctan \frac{1}{L_r} = \arctan \frac{1}{\alpha^{r-1}} - \arctan \frac{1}{\alpha^{r+1}}, \quad r \text{ even.}$$

The proof now proceeds as in that of Theorem 3.4.

Examples.

$$\arctan \frac{1}{L_2} = \arctan \frac{1}{3} = (1, \alpha^{12}, 12, (-\beta, 0, 4\beta^3, 0, -\beta^5, 0, \beta^7, 0, -4\beta^9, 0, \beta^{11}, 0)), \quad (3.19)$$

$$\arctan \frac{1}{L_4} = \arctan \frac{1}{7} = (1, \alpha^{60}, 60, (0, 0, -3\beta^3, 0, 5\beta^5, 0, 0, 0, 3\beta^9, 0, 0, 0, 0, 0, -8\beta^{15}, 0, 0, 0, 0, 0, 0, 3\beta^{21}, 0, 0, 0, 5\beta^{25}, 0, -3\beta^{27}, 0, 0, 0, 0, 0, 3\beta^{33}, 0, -5\beta^{35}, 0, 0, 0, -3\beta^{39}, 0, 0, 0, 0, 0, 8\beta^{45}, 0, 0, 0, 0, 0, 0, -3\beta^{51}, 0, 0, 0, -5\beta^{55}, 0, 3\beta^{57}, 0, 0, 0)). \quad (3.20)$$

Theorem 3.7. If r is an integer, then,

$$\sum_{k=0}^{\infty} \frac{1}{\alpha^{12rk}} \left(\frac{\beta^r}{12k+1} + \frac{2\beta^{3r}}{12k+3} + \frac{\beta^{5r}}{12k+5} - \frac{\beta^{7r}}{12k+7} - \frac{2\beta^{9r}}{12k+9} - \frac{\beta^{11r}}{12k+11} \right) = \begin{cases} -\arctan\left(\frac{1}{L_r}\right), & r \ odd, \\ \arctan\left(\frac{1}{F_r\sqrt{5}}\right), & r \ even; \end{cases}$$
(3.21)

that is,

$$P(1, \alpha^{12r}, 12, (\beta^{r}, 0, 2\beta^{3r}, 0, \beta^{5r}, 0, -\beta^{7r}, 0, -2\beta^{9r}, 0, -\beta^{11r}, 0)) = \begin{cases} -\arctan\left(\frac{1}{L_{r}}\right), & r \text{ odd}, \\ \arctan\left(\frac{1}{F_{r}\sqrt{5}}\right), & r \text{ even.} \end{cases}$$
(3.22)

Proof. In [1, Identity (27)], it was shown that

$$n^2 \sqrt{n} \arctan\left(\frac{\sqrt{n}}{n-1}\right) = \sum_{k=0}^{\infty} \frac{1}{(-n^3)^k} \left(\frac{n^2}{6k+1} + \frac{2n}{6k+3} + \frac{1}{6k+5}\right).$$

In base n^6 , length 12, this is

$$n^{2}\sqrt{n}\arctan\left(\frac{\sqrt{n}}{n-1}\right) = \sum_{k=0}^{\infty} \frac{1}{n^{6k}} \left(\frac{n^{2}}{12k+1} + \frac{2n}{12k+3} + \frac{1}{12k+5} - \frac{1/n}{12k+7} - \frac{2/n^{2}}{12k+9} - \frac{1/n^{3}}{12k+11}\right).$$
(3.23)

Identity (3.21) follows upon setting $n = \alpha^{2r}$ in (3.23) and making use of (3.5) and (3.6).

Examples.

$$\frac{\pi}{4} = P(1, \alpha^{12}, 12, (-\beta, 0, -2\beta^3, 0, -\beta^5, 0, \beta^7, 0, 2\beta^9, 0, \beta^{11}, 0)),$$
(3.24)

$$\arctan\left(\frac{1}{4}\right) = P(1, \alpha^{12}, 12, (-\beta^3, 0, -2\beta^9, 0, -\beta^{15}, 0, \beta^{21}, 0, 2\beta^{27}, 0, \beta^{33}, 0)),$$
(3.25)

$$\arctan\left(\frac{1}{\sqrt{5}}\right) = P(1, \alpha^{12}, 12, (\beta^2, 0, 2\beta^6, 0, \beta^{10}, 0, -\beta^{14}, 0, -2\beta^{18}, 0, -\beta^{22}, 0)), \qquad (3.26)$$

$$\arctan\left(\frac{1}{3\sqrt{5}}\right) = P(1,\alpha^{12}, 12, (\beta^4, 0, 2\beta^{12}, 0, \beta^{20}, 0, -\beta^{28}, 0, -2\beta^{36}, 0, -\beta^{44}, 0)).$$
(3.27)

Remark. Identity (3.24) is the same golden ratio base expansion of π that was obtained in Theorem 3.5.

Theorem 3.8. If r is an integer, then,

$$\sum_{k=0}^{\infty} \frac{1}{\alpha^{4rk}} \left(\frac{2\beta^r}{4k+1} - \frac{2\beta^{3r}}{4k+3} \right) = \begin{cases} -\arctan\left(\frac{2}{L_r}\right), & r \ odd, \\ \arctan\left(\frac{2}{F_r\sqrt{5}}\right), & r \ even; \end{cases}$$
(3.28)

 $that \ is$

$$P(1, \alpha^{4r}, 4, (2\beta^r, 0, -2\beta^{3r}, 0)) = \begin{cases} -\arctan\left(\frac{2}{L_r}\right), & r \text{ odd,} \\ \arctan\left(\frac{2}{F_r\sqrt{5}}\right), & r \text{ even.} \end{cases}$$
(3.29)

Proof. Setting $x = \beta^r$ in the identity

$$2\arctan x = \arctan\left(\frac{2x}{1-x^2}\right)$$

and using (3.5) and (3.6), we have

$$2 \arctan \frac{1}{\alpha^r} = \begin{cases} \arctan\left(\frac{2}{L_r}\right), & r \text{ odd,} \\ \arctan\left(\frac{2}{F_r\sqrt{5}}\right), & r \text{ even.} \end{cases}$$
(3.30)

Setting $n = \alpha^{2r}$ in (3.10) and comparing with (3.30), we obtain (3.28). \Box Examples.

$$\arctan \frac{2}{L_3} = \arctan \frac{1}{2} = P(1, \alpha^{12}, 4, (-2\beta^3, 0, 2\beta^9, 0)), \tag{3.31}$$

$$\arctan\left(\frac{2}{F_2\sqrt{5}}\right) = \arctan\frac{2}{\sqrt{5}} = P(1,\alpha^8,4,(2\beta^2,0,-2\beta^6,0)).$$
(3.32)

4 Zero relations

Zero relations are expansion formulas that evaluate to zero. They are useful in the determination and classification of new expansion formulas. A base α expansion is not considered new if it can be written as a linear combination of existing formulas and known zero relations.

4.1 Zero relations arising from the logarithm formulas

Zero relation from $\log(F_3^2/L_3) = 0$

Theorem 4.1. We have

$$\sum_{k=0}^{\infty} \frac{1}{\alpha^{12k}} \left(\frac{1}{6k+1} - \frac{3\beta^2}{6k+2} - \frac{8\beta^4}{6k+3} - \frac{3\beta^6}{6k+4} + \frac{\beta^8}{6k+5} \right) = 0$$

that is,

$$0 = P(1, \alpha^{12}, 6, (1, -3\beta^2, -8\beta^4, -3\beta^6, \beta^8, 0)).$$

Proof. We have

$$2\log F_3 - \log L_3 = 0. \tag{4.1}$$

The expansion of log L_3 given in (2.32) has the following base α^{12} , length 12 version:

$$\log L_3 = P(1, \alpha^{12}, 6, (3\beta^2, 3\beta^4, 0, 3\beta^8, 3\beta^{10}, 0)).$$
(4.2)

Use of (2.20) and (4.2) in (4.1) yields the zero relation stated in Theorem 4.1.

Zero relation from $\log(L_6/(F_4^2F_3)) = 0$

Theorem 4.2. We have

$$0 = P(1, \alpha^{48}, 24, (1, -5\beta^2, -2\beta^4, 3\beta^6, \beta^8, 4\beta^{10}, \beta^{12}, 3\beta^{14}, -2\beta^{16}, -5\beta^{18}, \beta^{20}, 0, \beta^{24}, -5\beta^{26}, -2\beta^{28}, 3\beta^{30}, \beta^{32}, 4\beta^{34}, \beta^{36}, 3\beta^{38}, -2\beta^{40}, -5\beta^{42}, \beta^{44}, 0)).$$

Proof. Write log F_3 , log F_4 and log L_6 , that is, identities (2.20), (2.22) and (2.34), respectively, in the common base α^{48} and common length 24 and use

$$\log L_6 - 2\log F_4 - \log F_3 = 0$$

Zero relation from $\log(F_{12}/(F_3^4L_2^2)) = 0$

Theorem 4.3. We have

$$0 = P(1, \alpha^{48}, 24, (1, -4\beta^2, -5\beta^4, 0, \beta^8, 2\beta^{10}, \beta^{12}, 0, -5\beta^{16}, -4\beta^{18}, \beta^{20}, 0, \beta^{24}, -4\beta^{26}, -5\beta^{28}, 0, \beta^{32}, 2\beta^{34}, \beta^{36}, 0, -5\beta^{40}, -4\beta^{42}, \beta^{44}, 0)).$$

Proof. Write log F_3 , log F_{12} and log L_2 from (2.20), (2.24) and (2.31), in common base α^{48} and consider

$$\log F_{12} - 4 \, \log F_3 - 2 \, \log L_2 = 0.$$

4.2 Zero relations arising from the inverse tangent formulas

Zero relation from $2 \arctan(2/L_3) + \arctan(2/L_5) - \arctan(2/L_1) = 0$

Theorem 4.4. We have

$$\begin{split} 0 &= P(1, \alpha^{60}, 60, (1, 0, -7\beta^2, 0, -4\beta^4, 0, -\beta^6, 0, 7\beta^8, 0, -\beta^{10}, 0, b^{12}, 0, -2\beta^{14}, \\ &0, \beta^{16}, 0, -\beta^{18}, 0, 7\beta^{20}, 0, -\beta^{22}, 0, -4\beta^{24}, 0, -7\beta^{26}, 0, \beta^{28}, 0, -\beta^{30}, 0, \\ &7\beta^{32}, 0, 4\beta^{34}, 0, \beta^{36}, 0, -7\beta^{38}, 0, \beta^{40}, 0, -\beta^{42}, 0, 2\beta^{44}, 0, -\beta^{46}, 0, \\ &\beta^{48}, 0, -7\beta^{50}, 0, \beta^{52}, 0, 4\beta^{54}, 0, 7\beta^{56}, 0, -\beta^{58}, 0)). \end{split}$$

Proof. Using the addition and subtraction formulas for inverse tangents, it is readily verified that

$$\arctan\left(\frac{2}{L_1}\right) - \arctan\left(\frac{2}{L_3}\right) = \arctan\left(\frac{3}{4}\right)$$
$$\arctan\left(\frac{2}{L_5}\right) + \arctan\left(\frac{2}{L_3}\right) = \arctan\left(\frac{3}{4}\right);$$

and

so that

$$2 \arctan\left(\frac{2}{L_3}\right) + \arctan\left(\frac{2}{L_5}\right) - \arctan\left(\frac{2}{L_1}\right) = 0,$$

from which the zero relation follows upon use of (3.29).

Remark. The zero relation stated in Theorem 4.4 can also be obtained directly from

$$\arctan \frac{1}{F_3} = \arctan \frac{2}{L_3}$$

by writing (3.12) and (3.31) in the common base α^{60} and common length 60; or from

$$\arctan \frac{1}{F_4} = \arctan \frac{1}{L_2},$$

using (3.17) and (3.19).

Zero relation from

 $2 \arctan(1/L_1) - 2 \arctan(2/(F_2\sqrt{5})) - \arctan(2/(F_6\sqrt{5})) = 0$

Theorem 4.5. We have

$$\begin{split} 0 &= P(1, \alpha^{24}, 24, (1, 4\beta, 2\beta^2, 0, \beta^4, 2\beta^5, -\beta^6, 0, -2\beta^8, 4\beta^9, -\beta^{10}, 0, \beta^{12}, -4\beta^{13}, \\ &2\beta^{14}, 0, \beta^{16}, -2\beta^{17}, -\beta^{18}, 0, -2\beta^{20}, -4\beta^{21}, -\beta^{22}, 0)). \end{split}$$

Proof. The identity

$$\frac{\pi}{2} - \arctan\left(\frac{2}{\sqrt{5}}\right) = \arctan\left(\frac{1}{4\sqrt{5}}\right) + \arctan\left(\frac{2}{\sqrt{5}}\right)$$
$$= \arctan\left(\frac{\sqrt{5}}{2}\right)$$

can be arranged as

$$2 \arctan\left(\frac{1}{L_1}\right) - 2 \arctan\left(\frac{2}{F_2\sqrt{5}}\right) - \arctan\left(\frac{2}{F_6\sqrt{5}}\right) = 0$$

which, on account of (3.29), gives the stated zero relation.

5 Other degree 1 base α expansions and zero relations

Base α expansions of $\log \alpha$

Theorem 5.1.

$$\log \alpha = P(1, \alpha, 2, (0, -\beta)).$$
(5.1)

Proof. We have

$$\log \alpha = \frac{1}{2} \operatorname{Li}_1\left(\frac{1}{\alpha}\right) = \frac{1}{2} \sum_{k=0}^{\infty} \frac{1}{\alpha^k} \frac{1/\alpha}{k+1} = \sum_{k=0}^{\infty} \frac{1}{\alpha^k} \frac{-\beta}{2k+2}.$$

Theorem 5.2.

$$\log \alpha = P(1, \alpha^2, 2, (0, 2\beta^2)).$$
(5.2)

Proof. We have

$$\log \alpha = \text{Li}_1\left(\frac{1}{\alpha^2}\right) = \sum_{k=0}^{\infty} \frac{1}{\alpha^{2k}} \frac{1/\alpha^2}{k+1} = \sum_{k=0}^{\infty} \frac{1}{\alpha^{2k}} \frac{2\beta^2}{2k+2}.$$

Another base α expansion of $\log 2$

Theorem 5.3.

$$\log 2 = P(1, \alpha^3, 3, (-\beta, \beta^2, 2\beta^3)).$$
(5.3)

Proof. A straightforward consequence of the identity

$$\log 2 = \operatorname{Li}_1\left(\frac{1}{\alpha}\right) - \operatorname{Li}_1\left(\frac{1}{\alpha^3}\right).$$

Another base α expansion of $\log 5$

Theorem 5.4.

$$\log 5 = P(1, \alpha^4, 2, (4\beta^2, 0)).$$
(5.4)

Proof. A consequence of the identity

$$\log 5 = 2 \operatorname{Li}_1\left(\frac{1}{\alpha^2}\right) - 2 \operatorname{Li}_1\left(-\frac{1}{\alpha^2}\right).$$

A length 2, base α zero relation

Theorem 5.5.

$$P(1, \alpha^2, 2, (1, 3\beta)) = 0.$$
(5.5)

Proof. Follows from

$$\operatorname{Li}_1\left(\frac{1}{\alpha^2}\right) + \operatorname{Li}_1\left(-\frac{1}{\alpha}\right) = 0.$$

Remark. Relation (5.5) also follows from (5.1) and (5.2).

A length 12, base α zero relation

Theorem 5.6.

$$P(1, \alpha^{12}, 12, (1, \beta, -2\beta^2, 5\beta^3, \beta^4, 10\beta^5, \beta^6, 5\beta^7, -2\beta^8, \beta^9, \beta^{10}, 2\beta^{11})) = 0.$$

Proof. Follows from (2.20) and (5.3).

A length 10, base α zero relation

Theorem 5.7.

$$P(1, \alpha^{20}, 10, (1, -5\beta^2, \beta^4, -5\beta^6, -4\beta^8, -5\beta^{10}, \beta^{12}, -5\beta^{14}, \beta^{16}, 0)) = 0.$$

Proof. Ensues from (2.21) and (5.4).

A length 5, base α zero relation

Theorem 5.8.

$$P(1, \alpha^5, 5, (\beta, 1, -\beta, -\beta^4, -2\beta^4)) = 0.$$
(5.6)

Proof. Setting $p = 2 \cos x$ in the identity

$$\sum_{k=1}^{\infty} \frac{p^k \cos(kx)}{k} = -\frac{1}{2} \log(1 - 2p \cos x + p^2)$$

produces

$$\sum_{k=1}^{\infty} \frac{(2\cos x)^k \cos(kx)}{k} = 0.$$
(5.7)

Now $2\cos(2\pi/5) = -\beta$.

Thus, setting $x = 2\pi/5$ in (5.7) gives

$$\sum_{k=0}^{\infty} \frac{1}{\alpha^{5k}} \left(\frac{\beta}{5k+1} + \frac{1}{5k+2} - \frac{\beta}{5k+3} - \frac{\beta^4}{5k+4} - \frac{2\beta^4}{5k+5} \right) = 0;$$

since

$$\cos\left(\frac{2\pi}{5}(5j-4)\right) = \frac{-\beta}{2} = \cos\left(\frac{2\pi}{5}(5j-1)\right), \quad j = 1, 2, \dots$$

and

$$\cos\left(\frac{2\pi}{5}(5j-2)\right) = \frac{1}{2\beta} = \cos\left(\frac{2\pi}{5}(5j-5)\right), \quad j = 1, 2, \dots$$

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