# Golden Ratio Base Expansions of the Logarithm and Inverse Tangent of Fibonacci and Lucas Numbers 

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#### Abstract

Let $\alpha=(1+\sqrt{5}) / 2$, the golden ratio, and $\beta=-1 / \alpha=(1-\sqrt{5}) / 2$. Let $F_{n}$ and $L_{n}$ be the Fibonacci and Lucas numbers, defined by $F_{n}=\left(\alpha^{n}-\beta^{n}\right) / \sqrt{5}$ and $L_{n}=\alpha^{n}+\beta^{n}$, for all non-negative integers. We derive base $\alpha$ expansions of $\log F_{n}, \log L_{n}, \arctan \frac{1}{F_{n}}$ and $\arctan \frac{1}{L_{n}}$ for all positive integers $n$.


Keywords: Fibonacci number, Lucas number, logarithm, arctangent, inverse tangent, golden ratio, non-integer base expansion, BBP-type formula.

## 1 Introduction

Let $\alpha$ denote the golden ratio; that is $\alpha=(1+\sqrt{5}) / 2$. Let $\beta=-1 / \alpha=(1-\sqrt{5}) / 2$. Thus $\alpha \beta=-1$ and $\alpha+\beta=1$. Let $F_{n}$ and $L_{n}$ be the Fibonacci and Lucas numbers, defined by $F_{n}=\left(\alpha^{n}-\beta^{n}\right) / \sqrt{5}$ and $L_{n}=\alpha^{n}+\beta^{n}$, for all non-negative integers $n$.

Let $b$ be any non-zero number whose magnitude is greater than unity. Let $n$ and $s$ be positive integers. A convergent series of the form

$$
\begin{equation*}
C=\sum_{k=0}^{\infty} \frac{1}{b^{k}}\left(\frac{a_{1}}{(k n+1)^{s}}+\frac{a_{2}}{(k n+2)^{s}}+\cdots+\frac{a_{n}}{(k n+n)^{s}}\right) \tag{1.1}
\end{equation*}
$$

where $a_{1}, a_{2}, \ldots, a_{n}$ are certain numbers, defines a base $b$, length $n$ and degree $s$ expansion of the mathematical constant $C$.

If $b$ is an integer and $a_{k}$ are rational numbers, then (1.1) is referred to as a BBP-type formula, after the initials of the authors of the paper [4] in which such an expansion was first presented for $\pi$ and some other mathematical constants. Any mathematical constant that possesses a base $b$ BBP-type formula has the property that its $n$-th digit in base $b$ could be calculated directly, without needing to compute any of the previous $n-1$ digits. Although the infinite series presented in this paper have the structure of BBP-type formulas, it must be clearly stated that the series do not yield digit extraction since here the base $b=\alpha^{n}$ is not an integer and the $a_{k}$ are not rational numbers; rather the series correspond to base $\alpha$ expansions of the mathematical constants concerned.

Our goal in this paper is to derive base $\alpha$ expansion formulas for the logarithm and the inverse tangent of all Fibonacci and Lucas numbers. We will often give the expansion using the compact $P$-notation for BBP-type formulas, introduced by Bailey and Crandall [5], namely,

$$
C=P(s, b, n, A)=\sum_{k=0}^{\infty} \frac{1}{b^{k}} \sum_{j=1}^{n} \frac{a_{j}}{(k n+j)^{s}},
$$

where $s$ and $n$ are integers and, in this present paper, $b$ is an integer power of $\alpha$ and $A=\left(a_{1}, a_{2}, \ldots, a_{n}\right)$ is a vector of rational multiples of powers of $\beta$. For example, we will show (see (2.20)) that

$$
\log F_{3}=\log 2=\sum_{k=0}^{\infty} \frac{1}{\alpha^{12 k}}\left(\frac{\beta^{2}}{6 k+1}+\frac{3 \beta^{4}}{6 k+2}+\frac{4 \beta^{6}}{6 k+3}+\frac{3 \beta^{8}}{6 k+4}+\frac{\beta^{10}}{6 k+5}\right)
$$

which, in the $P$-notation, can be written as

$$
\log F_{3}=\log 2=P\left(1, \alpha^{12}, 6,\left(\beta^{2}, 3 \beta^{4}, 4 \beta^{6}, 3 \beta^{8}, \beta^{10}, 0\right)\right)
$$

Base $\alpha$ expansions have also been studied or reported by Bailey and Crandall [5], Chan [7, 8], Zhang [12], Borwein and Chamberland [6], Cloitre [9], Adegoke [2], Wei [11], and more recently Kristensen and Mathiasen [10].

## 2 Base $\alpha$ expansions of logarithms

The base $\alpha$ expansions of the logarithms of Fibonacci and Lucas numbers are presented in Theorems 2.3 and 2.4 but first we state a couple of Lemmata upon which the results are based.

Let

$$
\operatorname{Li}_{1}(x)=-\log (1-x)=\sum_{k=1}^{\infty} \frac{x^{k}}{k}=\sum_{k=0}^{\infty} x^{k} \frac{x}{k+1}, \quad-1 \leq x<1 .
$$

Lemma 2.1. If $|b|>1, t>0$ and $m$ and $n$ are arbitrary positive integers, then,

$$
\begin{align*}
\operatorname{Li}_{1}\left(\frac{1}{b^{t}}\right) & =\sum_{k=0}^{\infty} \frac{1}{b^{t m k}} \sum_{j=1}^{m} \frac{1 / b^{t j}}{(m k+j)}  \tag{2.1}\\
\operatorname{Li}_{1}\left(-\frac{1}{b^{t}}\right) & =\sum_{k=0}^{\infty} \frac{1}{b^{2 t n k}} \sum_{j=1}^{2 n} \frac{(-1)^{j} / b^{t j}}{(2 n k+j)} . \tag{2.2}
\end{align*}
$$

Proof. We have

$$
\mathrm{Li}_{1}\left(\frac{1}{b^{t}}\right)=\sum_{k=0}^{\infty} \frac{1}{b^{t k}} \frac{1 / b^{t}}{k+1},
$$

from which (2.1) follows upon using the identity

$$
\begin{equation*}
\sum_{k=0}^{\infty} f_{k}=\sum_{k=0}^{\infty} \sum_{j=1}^{m} f_{m k+j-1} \tag{2.3}
\end{equation*}
$$

with

$$
f_{k}=\frac{1}{b^{t k+t}} \frac{1}{k+1} .
$$

The proof of (2.2) is similar, with $m=2 n$ in (2.3).
Lemma 2.2. If $r$ is an integer, then,

$$
\begin{gather*}
\log L_{r}=r \operatorname{Li}_{1}\left(\frac{1}{\alpha^{2}}\right)-\operatorname{Li}_{1}\left(\frac{(-1)^{r+1}}{\alpha^{2 r}}\right),  \tag{2.4}\\
\log F_{r}=(r-2) \operatorname{Li}_{1}\left(\frac{1}{\alpha^{2}}\right)+\mathrm{Li}_{1}\left(\frac{1}{\alpha^{4}}\right)-\mathrm{Li}_{1}\left(\frac{(-1)^{r}}{\alpha^{2 r}}\right), \quad r \neq 0 . \tag{2.5}
\end{gather*}
$$

Proof. We have

$$
\begin{equation*}
\operatorname{Li}_{1}\left(-\frac{\beta^{r}}{\alpha^{r}}\right)=-\log \left(\frac{\alpha^{r}+\beta^{r}}{\alpha^{r}}\right)=-\log \left(\frac{L_{r}}{\alpha^{r}}\right)=-\log L_{r}+r \log \alpha \tag{2.6}
\end{equation*}
$$

in which setting $r=1$ gives

$$
\begin{equation*}
\log \alpha=\operatorname{Li}_{1}\left(\frac{1}{\alpha^{2}}\right) \tag{2.7}
\end{equation*}
$$

Using (2.7) in (2.6) gives (2.4).

Also,

$$
\begin{equation*}
\operatorname{Li}_{1}\left(\frac{\beta^{r}}{\alpha^{r}}\right)=-\log \left(\frac{\alpha^{r}-\beta^{r}}{\alpha^{r}}\right)=-\log \left(\frac{F_{r} \sqrt{5}}{\alpha^{r}}\right)=-\log F_{r}+r \log \alpha-\log \sqrt{5}, \tag{2.8}
\end{equation*}
$$

in which setting $r=2$ gives

$$
\begin{equation*}
\log \sqrt{5}=2 \log \alpha-\operatorname{Li}_{1}\left(\frac{1}{\alpha^{4}}\right)=2 \operatorname{Li}_{1}\left(\frac{1}{\alpha^{2}}\right)-\operatorname{Li}_{1}\left(\frac{1}{\alpha^{4}}\right) \tag{2.9}
\end{equation*}
$$

where we used (2.7). Identity (2.5) follows from (2.8) and (2.9).

Theorem 2.3. If $r$ is an integer, then,

$$
\begin{gather*}
\log F_{r}=\sum_{k=0}^{\infty} \frac{1}{\alpha^{4 r k}} \sum_{j=1}^{r} \frac{\beta^{4 j-2}\left(r-2+r \delta_{j,(r+1) / 2}\right)}{2 r k+2 j-1}+\sum_{k=0}^{\infty} \frac{1}{\alpha^{4 r k}} \sum_{j=1}^{r-1} \frac{\beta^{4 j} r}{2 r k+2 j}, \quad r \text { odd, }  \tag{2.10}\\
\log F_{r}=\sum_{k=0}^{\infty} \frac{1}{\alpha^{4 r k}} \sum_{j=1}^{r} \frac{(r-2) \beta^{4 j-2}}{2 r k+2 j-1}+\sum_{k=0}^{\infty} \frac{1}{\alpha^{4 r k}} \sum_{j=1}^{r-1} \frac{\beta^{4 j} r\left(1-\delta_{j, r / 2}\right)}{2 r k+2 j}, \quad r \text { even. } \tag{2.11}
\end{gather*}
$$

Here and throughout this paper, $\delta_{m n}$ denotes the Kronecker delta symbol whose value is unity when $m$ equals $n$ and zero otherwise.
Proof. We prove (2.11). When $r$ is even, (2.5) reads

$$
\begin{equation*}
\log F_{r}=(r-2) \operatorname{Li}_{1}\left(\frac{1}{\alpha^{2}}\right)+\operatorname{Li}_{1}\left(\frac{1}{\alpha^{4}}\right)-\operatorname{Li}_{1}\left(\frac{1}{\alpha^{2 r}}\right), \quad r \neq 0 \tag{2.12}
\end{equation*}
$$

We proceed to write the three $\mathrm{Li}_{1}$ terms in a common base $\alpha^{4 r}$, using (2.1) with appropriate $t$ and $m$ choices. Thus,

$$
\begin{gather*}
\mathrm{Li}_{1}\left(\frac{1}{\alpha^{2}}\right)=\sum_{k=0}^{\infty} \frac{1}{\alpha^{4 r k}} \sum_{j=1}^{2 r} \frac{1 / \alpha^{2 j}}{2 r k+j},  \tag{2.13}\\
\mathrm{Li}_{1}\left(\frac{1}{\alpha^{4}}\right)=\sum_{k=0}^{\infty} \frac{1}{\alpha^{4 r k}} \sum_{j=1}^{r} \frac{1 / \alpha^{4 j}}{r k+j},  \tag{2.14}\\
\mathrm{Li}_{1}\left(\frac{1}{\alpha^{2 r}}\right)=\sum_{k=0}^{\infty} \frac{1}{\alpha^{4 r k}}\left(\frac{1 / \alpha^{2 r}}{2 k+1}+\frac{1 / \alpha^{4 r}}{2 k+2}\right) . \tag{2.15}
\end{gather*}
$$

Using (2.13), (2.14) and (2.15) in (2.12) gives

$$
\begin{align*}
\log F_{r}=\sum_{k=0}^{\infty} & \frac{1}{\alpha^{4 r k}} \sum_{j=1}^{2 r} \frac{\beta^{2 j}(r-2)}{2 r k+j}+\sum_{k=0}^{\infty} \frac{1}{\alpha^{4 r k}} \sum_{j=1}^{r} \frac{2 \beta^{4 j}}{2 r k+2 j}  \tag{2.16}\\
& -\sum_{k=0}^{\infty} \frac{1}{\alpha^{4 r k}}\left(\frac{r \beta^{2 r}}{2 r k+r}+\frac{r \beta^{4 r}}{2 r k+2 r}\right) .
\end{align*}
$$

Using the summation identity

$$
\begin{equation*}
\sum_{j=1}^{2 r} f_{j}=\sum_{j=1}^{r} f_{2 j}+\sum_{j=1}^{r} f_{2 j-1} \tag{2.17}
\end{equation*}
$$

to write its inner sum, the first term on the right hand side of (2.16) can be written as

$$
\begin{equation*}
\sum_{k=0}^{\infty} \frac{1}{\alpha^{4 r k}} \sum_{j=1}^{2 r} \frac{\beta^{2 j}(r-2)}{2 r k+j}=\sum_{k=0}^{\infty} \frac{1}{\alpha^{4 r k}} \sum_{j=1}^{r} \frac{\beta^{4 j}(r-2)}{2 r k+2 j}+\sum_{k=0}^{\infty} \frac{1}{\alpha^{4 r k}} \sum_{j=1}^{r} \frac{\beta^{4 j-2}(r-2)}{2 r k+2 j-1} . \tag{2.18}
\end{equation*}
$$

Using (2.18) in (2.16) yields (2.11).

Identities (2.10) and (2.11) written in the $P$-notation are

$$
\begin{equation*}
\log F_{r}=P\left(1, \alpha^{4 r}, 2 r,\left(a_{1}, a_{2}, \ldots, a_{2 r}\right)\right) \tag{2.19}
\end{equation*}
$$

where for $1 \leq j \leq r$,

$$
a_{2 j-1}=\beta^{4 j-2}\left(r-2+r \delta_{j,(r+1) / 2}\right), \quad a_{2 j}=\beta^{4 j} r\left(1-\delta_{r j}\right), \quad r \text { odd; }
$$

and

$$
a_{2 j-1}=(r-2) \beta^{4 j-2}, \quad a_{2 j}=\beta^{4 j} r\left(1-\delta_{j, r / 2}-\delta_{j, r}\right), \quad r \text { even. }
$$

Examples.

$$
\begin{gather*}
\log F_{3}=\log 2=P\left(1, \alpha^{12}, 6,\left(\beta^{2}, 3 \beta^{4}, 4 \beta^{6}, 3 \beta^{8}, \beta^{10}, 0\right)\right),  \tag{2.20}\\
\log F_{5}=\log 5=P\left(1, \alpha^{20}, 10,\left(3 \beta^{2}, 5 \beta^{4}, 3 \beta^{6}, 5 \beta^{8}, 8 \beta^{10},\right.\right. \\
\left.\left.5 \beta^{12}, 3 \beta^{14}, 5 \beta^{16}, 3 \beta^{18}, 0\right)\right),  \tag{2.21}\\
\log F_{4}=\log 3=P\left(1, \alpha^{16}, 8,\left(2 \beta^{2}, 4 \beta^{4}, 2 \beta^{6}, 0,2 \beta^{10}, 4 \beta^{12}, 2 \beta^{14}, 0\right)\right),  \tag{2.22}\\
\log F_{8}=\log 21=P\left(1, \alpha^{32}, 16,\left(6 \beta^{2}, 8 \beta^{4}, 6 \beta^{6}, 8 \beta^{8}, 6 \beta^{10}, 8 \beta^{12}, 6 \beta^{14},\right.\right. \\
\left.\left.0,6 \beta^{18}, 8 \beta^{20}, 6 \beta^{22}, 8 \beta^{24}, 6 \beta^{26}, 8 \beta^{28}, 6 \beta^{30}, 0\right)\right),  \tag{2.23}\\
\log F_{12}=\log 144=P\left(1, \alpha^{48}, 24,\left(10 \beta^{2}, 12 \beta^{4}, 10 \beta^{6}, 12 \beta^{8}, 10 \beta^{10},\right.\right. \\
\\
12 \beta^{12}, 10 \beta^{14}, 12 \beta^{16}, 10 \beta^{18}, 12 \beta^{20}, 10 \beta^{22}, 0,  \tag{2.24}\\
10 \beta^{26}, 12 \beta^{28}, 10 \beta^{30}, 12 \beta^{32}, 10 \beta^{34}, 12 \beta^{36}, \\
\left.\left.10 \beta^{38}, 12 \beta^{40}, 10 \beta^{42}, 12 \beta^{44}, 10 \beta^{46}, 0\right)\right) .
\end{gather*}
$$

Theorem 2.4. If $r$ is an integer, then,

$$
\begin{gather*}
\log L_{r}=\sum_{k=0}^{\infty} \frac{1}{\alpha^{2 r k}} \sum_{j=1}^{r-1} \frac{\beta^{2 j} r}{r k+j}, r \text { odd },  \tag{2.25}\\
\log L_{r}=\sum_{k=0}^{\infty} \frac{1}{\alpha^{4 r k}} \sum_{j=1}^{2 r-1} \frac{\beta^{2 j} r\left(1+\delta_{r j}\right)}{2 r k+j}, r \text { even. } \tag{2.26}
\end{gather*}
$$

Proof. We prove (2.25). If $r$ is an odd integer, (2.4) gives

$$
\begin{equation*}
\log L_{r}=r \operatorname{Li}_{1}\left(\frac{1}{\alpha^{2}}\right)-\operatorname{Li}_{1}\left(\frac{1}{\alpha^{2 r}}\right) \tag{2.27}
\end{equation*}
$$

With

$$
\operatorname{Li}_{1}\left(\frac{1}{\alpha^{2}}\right)=\sum_{k=0}^{\infty} \frac{1}{\alpha^{2 r k}} \sum_{j=1}^{r} \frac{1 / \alpha^{2 j}}{r k+j}
$$

and

$$
\operatorname{Li}_{1}\left(\frac{1}{\alpha^{2 r}}\right)=\sum_{k=0}^{\infty} \frac{1}{\alpha^{2 r k}} \frac{r / \alpha^{2 r}}{r k+r}
$$

in (2.27); identity (2.25) follows.
Identities (2.25) and (2.26) in the $P$-notation are

$$
\begin{equation*}
\log L_{r}=P\left(1, \alpha^{2 r}, r,\left(a_{1}, a_{2}, \ldots, a_{r}\right)\right), \quad r \text { odd } \tag{2.28}
\end{equation*}
$$

with

$$
a_{j}=r \beta^{2 j}\left(1-\delta_{r j}\right), \quad 1 \leq j \leq r
$$

and

$$
\begin{equation*}
\log L_{r}=P\left(1, \alpha^{4 r}, 2 r,\left(a_{1}, a_{2}, \ldots, a_{2 r}\right)\right), \quad r \text { even } \tag{2.29}
\end{equation*}
$$

with

$$
a_{j}=r \beta^{2 j}\left(1+\delta_{j r}-\delta_{j, 2 r}\right), \quad 1 \leq j \leq 2 r .
$$

Examples.

$$
\begin{equation*}
\log L_{2}=\log 3=\sum_{k=0}^{\infty} \frac{1}{\alpha^{8 k}}\left(\frac{2 \beta^{2}}{4 k+1}+\frac{4 \beta^{4}}{4 k+2}+\frac{2 \beta^{6}}{4 k+3}\right) \tag{2.30}
\end{equation*}
$$

that is,

$$
\begin{gather*}
\log 3=P\left(1, \alpha^{8}, 4,\left(2 \beta^{2}, 4 \beta^{4}, 2 \beta^{6}, 0\right)\right)  \tag{2.31}\\
\log L_{3}=\log 4=P\left(1, \alpha^{6}, 3,\left(3 \beta^{2}, 3 \beta^{4}, 0\right)\right)  \tag{2.32}\\
\log L_{4}=\log 7=4 \beta^{2} P\left(1, \alpha^{16}, 8,\left(1, \beta^{2}, \beta^{4}, 2 \beta^{6}, \beta^{8}, \beta^{10}, \beta^{12}, 0\right)\right),  \tag{2.33}\\
\log L_{6}=\log 18=P\left(1, \alpha^{24}, 12,\left(6 \beta^{2}, 6 \beta^{4}, 6 \beta^{6}, 6 \beta^{8}, 6 \beta^{10}\right.\right.  \tag{2.34}\\
\left.\left.12 \beta^{12}, 6 \beta^{14}, 6 \beta^{16}, 6 \beta^{18}, 6 \beta^{20}, 6 \beta^{22}, 0\right)\right)
\end{gather*}
$$

## 3 Base $\alpha$ expansions of inverse tangents

The base $\alpha$ expansions of the inverse tangent of Fibonacci and Lucas numbers are stated in Theorems 3.4-3.8 but first we collect some required identities in Lemmata 3.1-3.3.

Lemma 3.1. If $r$ is an integer, then,

$$
\begin{align*}
\alpha^{r}-\alpha^{-r} & = \begin{cases}F_{r} \sqrt{5}, & r \text { even } ; \\
L_{r}, & r \\
\text { odd } ;\end{cases}  \tag{3.1}\\
\alpha^{r}+\alpha^{-r} & =\left\{\begin{array}{lll}
L_{r} & r & \text { even } ; \\
F_{r} \sqrt{5}, & r & \text { odd } .
\end{array}\right. \tag{3.2}
\end{align*}
$$

Lemma 3.2. If $r$ and $m$ are integers, then,

$$
\begin{gather*}
\arctan \frac{1}{\alpha^{r-m}}-\arctan \frac{1}{\alpha^{r+m}}=\left\{\begin{array}{l}
\arctan \left(L_{m} /\left(F_{r} \sqrt{5}\right)\right), \quad m \text { odd, } r \text { odd, }, \\
\arctan \left(L_{m} / L_{r}\right), \quad m \text { odd, } r \text { even, } \\
\arctan \left(F_{m} / F_{r}\right), \quad m \text { evev, } r \text { odd, } \\
\arctan \left(F_{m} \sqrt{5} / L_{r}\right), \quad m \text { even, } r \text { even },
\end{array}\right.  \tag{3.3}\\
\arctan \frac{1}{\alpha^{r-m}}+\arctan \frac{1}{\alpha^{r+m}}=\left\{\begin{array}{l}
\arctan \left(F_{m} \sqrt{5} / L_{r}\right), \quad m \text { odd, } r \text { odd, } \\
\arctan \left(F_{m} / F_{r}\right), \quad m \text { odd, } r \text { even, } \\
\arctan \left(L_{m} / L_{r}\right), \quad m \text { even, } r \text { odd, } \\
\arctan \left(L_{m} /\left(F_{r} \sqrt{5}\right)\right), \quad m \text { even, } r \text { even. }
\end{array}\right. \tag{3.4}
\end{gather*}
$$

Proof. The arctangent subtraction and addition formulas give

$$
\begin{aligned}
& \arctan \frac{1}{\alpha^{r-m}}-\arctan \frac{1}{\alpha^{r+m}}=\arctan \left(\frac{\alpha^{r}\left(\alpha^{m}-\alpha^{-m}\right)}{\alpha^{2 r}+1}\right), \\
& \arctan \frac{1}{\alpha^{r-m}}+\arctan \frac{1}{\alpha^{r+m}}=\arctan \left(\frac{\alpha^{r}\left(\alpha^{m}+\alpha^{-m}\right)}{\alpha^{2 r}-1}\right) ;
\end{aligned}
$$

and hence the stated identities upon the use of Lemma 3.1.
Lemma 3.3. If $r$ is an integer, then,

$$
\begin{array}{cl}
\alpha^{2 r}-1=\alpha^{r} L_{r}, & \beta^{2 r}-1=\beta^{r} L_{r}, \quad r \quad \text { odd }, \\
\alpha^{2 r}-1=\alpha^{r} F_{r} \sqrt{5}, & \beta^{2 r}-1=-\beta^{r} F_{r} \sqrt{5}, \quad r \text { even }, \\
\alpha^{2 r}+1=\alpha^{r} F_{r} \sqrt{5}, & \beta^{2 r}+1=-\beta^{r} F_{r} \sqrt{5}, \quad r \text { odd }, \\
\alpha^{2 r}+1=\alpha^{r} L_{r}, & \beta^{2 r}+1=\beta^{r} L_{r}, \quad r \text { even. } \tag{3.8}
\end{array}
$$

Theorem 3.4. If $r$ is an odd integer greater than unity, then,

$$
\begin{equation*}
\arctan \frac{1}{F_{r}}=P\left(1, \alpha^{4\left(r^{2}-4\right)}, 4\left(r^{2}-4\right),\left(a_{1}, a_{2}, \ldots, a_{4\left(r^{2}-4\right)}\right)\right), \tag{3.9}
\end{equation*}
$$

where the only non-zero constants $a_{j}$ are given by

$$
\begin{gathered}
a_{(r-2)(4 j-3)}=-\beta^{(r-2)(4 j-3)}(r-2), \quad j=1,2, \ldots, r+2, \\
a_{(r-2)(4 j-1)}=\beta^{(r-2)(4 j-1)}(r-2), \quad j=1,2, \ldots, r+2, \\
a_{(r+2)(4 j-3)}=\beta^{(r+2)(4 j-3)}(r+2), \quad j=1,2, \ldots, r-2, \\
a_{(r+2)(4 j-1)}=-\beta^{(r+2)(4 j-1)}(r+2), \quad j=1,2, \ldots, r-2, \\
a_{(r-2)(r+2)}=(-1)^{(r+1) / 2} 4 \beta^{r^{2}-4}
\end{gathered}
$$

and

$$
a_{3(r-2)(r+2)}=(-1)^{(r-1) / 2} 4 \beta^{3\left(r^{2}-4\right)} .
$$

Proof. With $r$ an odd number, $m=2$ in (3.3) gives

$$
\arctan \frac{1}{F_{r}}=\arctan \frac{1}{\alpha^{r-2}}-\arctan \frac{1}{\alpha^{r+2}} .
$$

The following identity, proved in [1, Identity (10)]:

$$
\begin{equation*}
n \sqrt{n} \arctan \left(\frac{1}{\sqrt{n}}\right)=\sum_{k=0}^{\infty} \frac{1}{n^{2 k}}\left(\frac{n}{4 k+1}-\frac{1}{4 k+3}\right) \tag{3.10}
\end{equation*}
$$

gives

$$
\arctan \frac{1}{\alpha^{r-2}}=\sum_{k=0}^{\infty} \frac{1}{\alpha^{(r-2)(4 k+3)}}\left(\frac{\alpha^{2 r-4}}{4 k+1}-\frac{1}{4 k+3}\right)
$$

and

$$
\arctan \frac{1}{\alpha^{r+2}}=\sum_{k=0}^{\infty} \frac{1}{\alpha^{(r+2)(4 k+3)}}\left(\frac{\alpha^{2 r+4}}{4 k+1}-\frac{1}{4 k+3}\right)
$$

or, by (2.3),

$$
\arctan \frac{1}{\alpha^{r-2}}=\sum_{k=0}^{\infty} \frac{1}{\alpha^{\left(4 r^{2}-16\right) k}} \sum_{j=1}^{r+2}\left(\frac{\alpha^{-(r-2)(4 j-3)}}{4(r+2) k+4 j-3}-\frac{\alpha^{-(r-2)(4 j-1)}}{4(r+2) k+4 j-1}\right)
$$

and

$$
\arctan \frac{1}{\alpha^{r+2}}=\sum_{k=0}^{\infty} \frac{1}{\alpha^{\left(4 r^{2}-16\right) k}} \sum_{j=1}^{r-2}\left(\frac{\alpha^{-(r+2)(4 j-3)}}{4(r-2) k+4 j-3}-\frac{\alpha^{-(r+2)(4 j-1)}}{4(r-2) k+4 j-1}\right)
$$

Thus,

$$
\begin{align*}
& \arctan \frac{1}{F_{r}}=\sum_{k=0}^{\infty} \frac{1}{\alpha^{\left(4 r^{2}-16\right) k}} \times \\
&\left(\sum_{j=1}^{r+2}\left(\frac{-\beta^{(r-2)(4 j-3)}(r-2)}{4\left(r^{2}-4\right) k+(r-2)(4 j-3)}+\frac{\beta^{(r-2)(4 j-1)}(r-2)}{4\left(r^{2}-4\right) k+(r-2)(4 j-1)}\right)\right. \\
&\left.\quad+\sum_{j=1}^{r-2}\left(\frac{\beta^{(r+2)(4 j-3)}(r+2)}{4\left(r^{2}-4\right) k+(r+2)(4 j-3)}-\frac{\beta^{(r+2)(4 j-1)}(r+2)}{4\left(r^{2}-4\right) k+(r+2)(4 j-1)}\right)\right) . \tag{3.11}
\end{align*}
$$

Identity (3.9) is (3.11) expressed in the $P$-notation.

## Examples.

$$
\begin{gather*}
\arctan \frac{1}{F_{3}}=\arctan \frac{1}{2}=P\left(1, \alpha^{20}, 20,\left(-\beta, 0, \beta^{3}, 0,4 \beta^{5}, 0, \beta^{7}, 0,-\beta^{9}, 0, \beta^{11}\right.\right.  \tag{3.12}\\
\left.\left.0,-\beta^{13}, 0,-4 \beta^{15}, 0,-\beta^{17}, 0, \beta^{19}, 0\right)\right)
\end{gather*}
$$

$\arctan \frac{1}{F_{5}}=\arctan \frac{1}{5}=P\left(1, \alpha^{84}, 84,\left(0,0,-3 \beta^{3}, 0,0,0,7 \beta^{7}, 0,3 \beta^{9}, 0,0,0,0,0,-3 \beta^{15}\right.\right.$,

$$
\begin{align*}
& 0,0,0,0,0,-4 \beta^{21}, 0,0,0,0,0,-3 \beta^{27}, 0,0,0,0,0,3 \beta^{33} \\
& 0,7 \beta^{35}, 0,0,0,-3 \beta^{39}, 0,0,0,0,0,3 \beta^{45}, 0,0,0,-7 \beta^{49} \\
& 0,-3 \beta^{51}, 0,0,0,0,0,3 \beta^{57}, 0,0,0,0,0,4 \beta^{63}, 0,0,0,0,0 \\
& \left.\left.3 \beta^{69}, 0,0,0,0,0,-3 \beta^{75}, 0,-7 \beta^{77}, 0,0,0,3 \beta^{81}, 0,0,0\right)\right) \tag{3.13}
\end{align*}
$$

$\arctan \frac{1}{F_{7}}=\arctan \frac{1}{13}=P\left(1, \alpha^{180}, 180,\left(0,0,0,0,-5 \beta^{5}, 0,0,0,9 \beta^{9}, 0,0,0,0,0,5 \beta^{15}\right.\right.$,

$$
\begin{align*}
& 0,0,0,0,0,0,0,0,0,-5 \beta^{25}, 0,-9 \beta^{27}, 0,0,0,0,0,0,0 \\
& \quad 5 \beta^{35}, 0,0,0,0,0,0,0,0,0,4 \beta^{45}, 0,0,0,0,0,0,0,0,0 \\
& \quad 5 \beta^{55}, 0,0,0,0,0,0,0,-9 \beta^{63}, 0,-5 \beta^{65}, 0,0,0,0,0 \\
& \quad 0,0,0,0,5 \beta^{75}, 0,0,0,0,0,9 \beta^{81}, 0,0,0,-5 \beta^{85}, 0,0 \\
& \quad 0,0,0,0,0,0,0,5 \beta^{95}, 0,0,0,-9 \beta^{99}, 0,0,0,0,0 \\
& \quad-5 \beta^{105}, 0,0,0,0,0,0,0,0,0,5 \beta^{115}, 0,9 \beta^{117}, 0,0,0 \\
& \quad 0,0,0,0,-5 \beta^{125}, 0,0,0,0,0,0,0,0,0,-4 \beta^{135} \\
& \quad 0,0,0,0,0,0,0,0,0,-5 \beta^{145}, 0,0,0,0,0,0,0,9 \beta^{153}, 0 \\
& \quad 5 \beta^{155}, 0,0,0,0,0,0,0,0,0,-5 \beta^{165}, 0,0,0,0,0,-9 \beta^{171} \\
& \left.\left.\quad 0,0,0,5 \beta^{175}, 0,0,0,0,0\right)\right) \tag{3.14}
\end{align*}
$$

Theorem 3.5. If $r$ is a positive even integer, then,

$$
\begin{equation*}
\arctan \frac{1}{F_{r}}=P\left(1, \alpha^{4\left(r^{2}-1\right)}, 4\left(r^{2}-1\right),\left(a_{1}, a_{2}, \ldots, a_{4\left(r^{2}-1\right)}\right)\right), \tag{3.15}
\end{equation*}
$$

where the only non-zero constants $a_{j}$ are given by

$$
\begin{gathered}
a_{(r-1)(4 j-3)}=-\beta^{(r-1)(4 j-3)}(r-1), \quad j=1,2, \ldots, r+1, \\
a_{(r-1)(4 j-1)}=\beta^{(r-1)(4 j-1)}(r-1), \quad j=1,2, \ldots, r+1, \\
a_{(r+1)(4 j-3)}=-\beta^{(r+1)(4 j-3)}(r+1), \quad j=1,2, \ldots, r-1, \\
a_{(r+1)(4 j-1)}=\beta^{(r+1)(4 j-1)}(r+1), \quad j=1,2, \ldots, r-1, \\
a_{(r-1)(r+1)}=(-1)^{r / 2} 2 \beta^{r^{2}-1}
\end{gathered}
$$

and

$$
a_{3(r-1)(r+1)}=(-1)^{(r+2) / 2} 2 \beta^{3\left(r^{2}-1\right)}
$$

Proof. Setting $m=1$ in (3.4) gives

$$
\arctan \frac{1}{F_{r}}=\arctan \frac{1}{\alpha^{r-1}}+\arctan \frac{1}{\alpha^{r+1}}, \quad r \text { even } .
$$

The proof now proceeds as in that of Theorem 3.4.
Examples.

$$
\begin{equation*}
\arctan \frac{1}{F_{2}}=\frac{\pi}{4}=P\left(1, \alpha^{12}, 12,\left(-\beta, 0,-2 \beta^{3}, 0,-\beta^{5}, 0, \beta^{7}, 0,2 \beta^{9}, 0, \beta^{11}, 0\right)\right) \tag{3.16}
\end{equation*}
$$

$$
\begin{gather*}
\arctan \frac{1}{F_{4}}=\arctan \frac{1}{3}=P\left(1, \alpha^{60}, 60,\left(0,0,-3 \beta^{3}, 0,-5 \beta^{5}, 0,0,0,3 \beta^{9}, 0,0,0,0,0,2 \beta^{15}\right.\right. \\
0,0,0,0,0,3 \beta^{21}, 0,0,0,-5 \beta^{25}, 0,-3 \beta^{27}, 0,0,0,0,0,3 \beta^{33} \\
0,5 \beta^{35}, 0,0,0,-3 \beta^{39}, 0,0,0,0,0,-2 \beta^{45}, 0,0,0,0,0,-3 \beta^{51} \\
\left.\left.0,0,0,5 \beta^{55}, 0,3 \beta^{57}, 0,0,0\right)\right) \tag{3.17}
\end{gather*}
$$

Theorem 3.6. If $r$ is a positive even integer, then,

$$
\begin{equation*}
\arctan \frac{1}{L_{r}}=P\left(1, \alpha^{4\left(r^{2}-1\right)}, 4\left(r^{2}-1\right),\left(a_{1}, a_{2}, \ldots, a_{4\left(r^{2}-1\right)}\right)\right) \tag{3.18}
\end{equation*}
$$

where the only non-zero constants $a_{j}$ are given by

$$
a_{(r-1)(4 j-3)}=-\beta^{(r-1)(4 j-3)}(r-1), \quad j=1,2, \ldots, r+1,
$$

$$
\begin{aligned}
& a_{(r-1)(4 j-1)}=\beta^{(r-1)(4 j-1)}(r-1), \\
& a_{(r+1)(4 j-3)}=\beta^{(r+1)(4 j-3)}(r+1), \\
& a_{(r+1)(4 j-1)}=-\beta^{(r+1)(4 j-1)}(r+1), \quad j=1,2, \ldots, r-1, \\
& a_{(r-1)(r+1)}=(-1)^{(r+2) / 2} 2 r \beta^{r^{2}-1}
\end{aligned}
$$

and

$$
a_{3(r-1)(r+1)}=(-1)^{r / 2} 2 r \beta^{3\left(r^{2}-1\right)}
$$

Proof. Setting $m=1$ in (3.3) gives

$$
\arctan \frac{1}{L_{r}}=\arctan \frac{1}{\alpha^{r-1}}-\arctan \frac{1}{\alpha^{r+1}}, \quad r \text { even. }
$$

The proof now proceeds as in that of Theorem 3.4.
Examples.

$$
\begin{gather*}
\arctan \frac{1}{L_{2}}=\arctan \frac{1}{3}=\left(1, \alpha^{12}, 12,\left(-\beta, 0,4 \beta^{3}, 0,-\beta^{5}, 0, \beta^{7}, 0,-4 \beta^{9}, 0, \beta^{11}, 0\right)\right)  \tag{3.19}\\
\begin{array}{c}
\arctan \frac{1}{L_{4}}=\arctan \frac{1}{7}=\left(1, \alpha^{60},\right.
\end{array} \begin{array}{c}
60,\left(0,0,-3 \beta^{3}, 0,5 \beta^{5}, 0,0,0,3 \beta^{9}, 0,0,0,0,0,-8 \beta^{15}\right. \\
0,0,0,0,0,3 \beta^{21}, 0,0,0,5 \beta^{25}, 0,-3 \beta^{27}, 0,0,0,0,0,3 \beta^{33} \\
0,-5 \beta^{35}, 0,0,0,-3 \beta^{39}, 0,0,0,0,0,8 \beta^{45}, 0,0,0,0,0 \\
\\
\left.\left.-3 \beta^{51}, 0,0,0,-5 \beta^{55}, 0,3 \beta^{57}, 0,0,0\right)\right)
\end{array}
\end{gather*}
$$

Theorem 3.7. If $r$ is an integer, then,

$$
\begin{align*}
& \sum_{k=0}^{\infty} \frac{1}{\alpha^{12 r k}}\left(\frac{\beta^{r}}{12 k+1}+\frac{2 \beta^{3 r}}{12 k+3}+\frac{\beta^{5 r}}{12 k+5}-\frac{\beta^{7 r}}{12 k+7}-\frac{2 \beta^{9 r}}{12 k+9}-\frac{\beta^{11 r}}{12 k+11}\right) \\
& \quad= \begin{cases}-\arctan \left(\frac{1}{L_{r}}\right), & r \text { odd }, \\
\arctan \left(\frac{1}{F_{r} \sqrt{5}}\right), & r \text { even } ;\end{cases} \tag{3.21}
\end{align*}
$$

that is,

$$
\begin{align*}
& P\left(1, \alpha^{12 r}, 12,\left(\beta^{r}, 0,2 \beta^{3 r}, 0, \beta^{5 r}, 0,-\beta^{7 r}, 0,-2 \beta^{9 r}, 0,-\beta^{11 r}, 0\right)\right) \\
& \quad= \begin{cases}-\arctan \left(\frac{1}{L_{r}}\right), & r \text { odd } \\
\arctan \left(\frac{1}{F_{r} \sqrt{5}}\right), & r \text { even. }\end{cases} \tag{3.22}
\end{align*}
$$

Proof. In [1, Identity (27)], it was shown that

$$
n^{2} \sqrt{n} \arctan \left(\frac{\sqrt{n}}{n-1}\right)=\sum_{k=0}^{\infty} \frac{1}{\left(-n^{3}\right)^{k}}\left(\frac{n^{2}}{6 k+1}+\frac{2 n}{6 k+3}+\frac{1}{6 k+5}\right)
$$

In base $n^{6}$, length 12 , this is

$$
\begin{align*}
& n^{2} \sqrt{n} \arctan \left(\frac{\sqrt{n}}{n-1}\right) \\
& \quad=\sum_{k=0}^{\infty} \frac{1}{n^{6 k}}\left(\frac{n^{2}}{12 k+1}+\frac{2 n}{12 k+3}+\frac{1}{12 k+5}-\frac{1 / n}{12 k+7}-\frac{2 / n^{2}}{12 k+9}-\frac{1 / n^{3}}{12 k+11}\right) . \tag{3.23}
\end{align*}
$$

Identity (3.21) follows upon setting $n=\alpha^{2 r}$ in (3.23) and making use of (3.5) and (3.6).

Examples.

$$
\begin{align*}
\frac{\pi}{4} & =P\left(1, \alpha^{12}, 12,\left(-\beta, 0,-2 \beta^{3}, 0,-\beta^{5}, 0, \beta^{7}, 0,2 \beta^{9}, 0, \beta^{11}, 0\right)\right)  \tag{3.24}\\
\arctan \left(\frac{1}{4}\right) & =P\left(1, \alpha^{12}, 12,\left(-\beta^{3}, 0,-2 \beta^{9}, 0,-\beta^{15}, 0, \beta^{21}, 0,2 \beta^{27}, 0, \beta^{33}, 0\right)\right)  \tag{3.25}\\
\arctan \left(\frac{1}{\sqrt{5}}\right) & =P\left(1, \alpha^{12}, 12,\left(\beta^{2}, 0,2 \beta^{6}, 0, \beta^{10}, 0,-\beta^{14}, 0,-2 \beta^{18}, 0,-\beta^{22}, 0\right)\right)  \tag{3.26}\\
\arctan \left(\frac{1}{3 \sqrt{5}}\right) & =P\left(1, \alpha^{12}, 12,\left(\beta^{4}, 0,2 \beta^{12}, 0, \beta^{20}, 0,-\beta^{28}, 0,-2 \beta^{36}, 0,-\beta^{44}, 0\right)\right) \tag{3.27}
\end{align*}
$$

Remark. Identity (3.24) is the same golden ratio base expansion of $\pi$ that was obtained in Theorem 3.5.

Theorem 3.8. If $r$ is an integer, then,

$$
\sum_{k=0}^{\infty} \frac{1}{\alpha^{4 r k}}\left(\frac{2 \beta^{r}}{4 k+1}-\frac{2 \beta^{3 r}}{4 k+3}\right)= \begin{cases}-\arctan \left(\frac{2}{L_{r}}\right), & r \text { odd }  \tag{3.28}\\ \arctan \left(\frac{2}{F_{r} \sqrt{5}}\right), & r \text { even }\end{cases}
$$

that is

$$
P\left(1, \alpha^{4 r}, 4,\left(2 \beta^{r}, 0,-2 \beta^{3 r}, 0\right)\right)= \begin{cases}-\arctan \left(\frac{2}{L_{r}}\right), & r \text { odd },  \tag{3.29}\\ \arctan \left(\frac{2}{F_{r} \sqrt{5}}\right), & r \text { even } .\end{cases}
$$

Proof. Setting $x=\beta^{r}$ in the identity

$$
2 \arctan x=\arctan \left(\frac{2 x}{1-x^{2}}\right)
$$

and using (3.5) and (3.6), we have

$$
2 \arctan \frac{1}{\alpha^{r}}= \begin{cases}\arctan \left(\frac{2}{L_{r}}\right), & r \text { odd }  \tag{3.30}\\ \arctan \left(\frac{2}{F_{r} \sqrt{5}}\right), & r \text { even }\end{cases}
$$

Setting $n=\alpha^{2 r}$ in (3.10) and comparing with (3.30), we obtain (3.28).
Examples.

$$
\begin{gather*}
\arctan \frac{2}{L_{3}}=\arctan \frac{1}{2}=P\left(1, \alpha^{12}, 4,\left(-2 \beta^{3}, 0,2 \beta^{9}, 0\right)\right),  \tag{3.31}\\
\arctan \left(\frac{2}{F_{2} \sqrt{5}}\right)=\arctan \frac{2}{\sqrt{5}}=P\left(1, \alpha^{8}, 4,\left(2 \beta^{2}, 0,-2 \beta^{6}, 0\right)\right) . \tag{3.32}
\end{gather*}
$$

## 4 Zero relations

Zero relations are expansion formulas that evaluate to zero. They are useful in the determination and classification of new expansion formulas. A base $\alpha$ expansion is not considered new if it can be written as a linear combination of existing formulas and known zero relations.

### 4.1 Zero relations arising from the logarithm formulas

Zero relation from $\log \left(F_{3}^{2} / L_{3}\right)=0$
Theorem 4.1. We have

$$
\sum_{k=0}^{\infty} \frac{1}{\alpha^{12 k}}\left(\frac{1}{6 k+1}-\frac{3 \beta^{2}}{6 k+2}-\frac{8 \beta^{4}}{6 k+3}-\frac{3 \beta^{6}}{6 k+4}+\frac{\beta^{8}}{6 k+5}\right)=0
$$

that is,

$$
0=P\left(1, \alpha^{12}, 6,\left(1,-3 \beta^{2},-8 \beta^{4},-3 \beta^{6}, \beta^{8}, 0\right)\right)
$$

Proof. We have

$$
\begin{equation*}
2 \log F_{3}-\log L_{3}=0 \tag{4.1}
\end{equation*}
$$

The expansion of $\log L_{3}$ given in (2.32) has the following base $\alpha^{12}$, length 12 version:

$$
\begin{equation*}
\log L_{3}=P\left(1, \alpha^{12}, 6,\left(3 \beta^{2}, 3 \beta^{4}, 0,3 \beta^{8}, 3 \beta^{10}, 0\right)\right) \tag{4.2}
\end{equation*}
$$

Use of (2.20) and (4.2) in (4.1) yields the zero relation stated in Theorem 4.1.

Zero relation from $\log \left(L_{6} /\left(F_{4}^{2} F_{3}\right)\right)=0$
Theorem 4.2. We have

$$
\begin{gathered}
0=P\left(1, \alpha^{48}, 24,\left(1,-5 \beta^{2},-2 \beta^{4}, 3 \beta^{6}, \beta^{8}, 4 \beta^{10}, \beta^{12}, 3 \beta^{14},\right.\right. \\
-2 \beta^{16},-5 \beta^{18}, \beta^{20}, 0, \beta^{24},-5 \beta^{26},-2 \beta^{28}, 3 \beta^{30}, \beta^{32}, \\
\left.\left.4 \beta^{34}, \beta^{36}, 3 \beta^{38},-2 \beta^{40},-5 \beta^{42}, \beta^{44}, 0\right)\right) .
\end{gathered}
$$

Proof. Write $\log F_{3}, \log F_{4}$ and $\log L_{6}$, that is, identities (2.20), (2.22) and (2.34), respectively, in the common base $\alpha^{48}$ and common length 24 and use

$$
\log L_{6}-2 \log F_{4}-\log F_{3}=0
$$

Zero relation from $\log \left(F_{12} /\left(F_{3}^{4} L_{2}^{2}\right)\right)=0$
Theorem 4.3. We have

$$
\begin{gathered}
0=P\left(1, \alpha^{48}, 24,\left(1,-4 \beta^{2},-5 \beta^{4}, 0, \beta^{8}, 2 \beta^{10}, \beta^{12}, 0,-5 \beta^{16},-4 \beta^{18}, \beta^{20}, 0\right.\right. \\
\left.\left.\beta^{24},-4 \beta^{26},-5 \beta^{28}, 0, \beta^{32}, 2 \beta^{34}, \beta^{36}, 0,-5 \beta^{40},-4 \beta^{42}, \beta^{44}, 0\right)\right) .
\end{gathered}
$$

Proof. Write $\log F_{3}, \log F_{12}$ and $\log L_{2}$ from (2.20), (2.24) and (2.31), in common base $\alpha^{48}$ and consider

$$
\log F_{12}-4 \log F_{3}-2 \log L_{2}=0
$$

### 4.2 Zero relations arising from the inverse tangent formulas

Zero relation from $2 \arctan \left(2 / L_{3}\right)+\arctan \left(2 / L_{5}\right)-\arctan \left(2 / L_{1}\right)=0$
Theorem 4.4. We have

$$
\begin{aligned}
0= & P\left(1, \alpha^{60}, 60,\left(1,0,-7 \beta^{2}, 0,-4 \beta^{4}, 0,-\beta^{6}, 0,7 \beta^{8}, 0,-\beta^{10}, 0, b^{12}, 0,-2 \beta^{14},\right.\right. \\
& 0, \beta^{16}, 0,-\beta^{18}, 0,7 \beta^{20}, 0,-\beta^{22}, 0,-4 \beta^{24}, 0,-7 \beta^{26}, 0, \beta^{28}, 0,-\beta^{30}, 0 \\
& 7 \beta^{32}, 0,4 \beta^{34}, 0, \beta^{36}, 0,-7 \beta^{38}, 0, \beta^{40}, 0,-\beta^{42}, 0,2 \beta^{44}, 0,-\beta^{46}, 0 \\
& \left.\left.\beta^{48}, 0,-7 \beta^{50}, 0, \beta^{52}, 0,4 \beta^{54}, 0,7 \beta^{56}, 0,-\beta^{58}, 0\right)\right)
\end{aligned}
$$

Proof. Using the addition and subtraction formulas for inverse tangents, it is readily verified that

$$
\arctan \left(\frac{2}{L_{1}}\right)-\arctan \left(\frac{2}{L_{3}}\right)=\arctan \left(\frac{3}{4}\right)
$$

and

$$
\arctan \left(\frac{2}{L_{5}}\right)+\arctan \left(\frac{2}{L_{3}}\right)=\arctan \left(\frac{3}{4}\right) ;
$$

so that

$$
2 \arctan \left(\frac{2}{L_{3}}\right)+\arctan \left(\frac{2}{L_{5}}\right)-\arctan \left(\frac{2}{L_{1}}\right)=0,
$$

from which the zero relation follows upon use of (3.29).
Remark. The zero relation stated in Theorem 4.4 can also be obtained directly from

$$
\arctan \frac{1}{F_{3}}=\arctan \frac{2}{L_{3}},
$$

by writing (3.12) and (3.31) in the common base $\alpha^{60}$ and common length 60; or from

$$
\arctan \frac{1}{F_{4}}=\arctan \frac{1}{L_{2}},
$$

using (3.17) and (3.19).

## Zero relation from

$2 \arctan \left(1 / L_{1}\right)-2 \arctan \left(2 /\left(F_{2} \sqrt{5}\right)\right)-\arctan \left(2 /\left(F_{6} \sqrt{5}\right)\right)=0$
Theorem 4.5. We have

$$
\begin{aligned}
0= & P\left(1, \alpha^{24}, 24,\left(1,4 \beta, 2 \beta^{2}, 0, \beta^{4}, 2 \beta^{5},-\beta^{6}, 0,-2 \beta^{8}, 4 \beta^{9},-\beta^{10}, 0, \beta^{12},-4 \beta^{13}\right.\right. \\
& \left.\left.2 \beta^{14}, 0, \beta^{16},-2 \beta^{17},-\beta^{18}, 0,-2 \beta^{20},-4 \beta^{21},-\beta^{22}, 0\right)\right)
\end{aligned}
$$

Proof. The identity

$$
\begin{aligned}
\frac{\pi}{2}-\arctan \left(\frac{2}{\sqrt{5}}\right) & =\arctan \left(\frac{1}{4 \sqrt{5}}\right)+\arctan \left(\frac{2}{\sqrt{5}}\right) \\
& =\arctan \left(\frac{\sqrt{5}}{2}\right)
\end{aligned}
$$

can be arranged as

$$
2 \arctan \left(\frac{1}{L_{1}}\right)-2 \arctan \left(\frac{2}{F_{2} \sqrt{5}}\right)-\arctan \left(\frac{2}{F_{6} \sqrt{5}}\right)=0
$$

which, on account of (3.29), gives the stated zero relation.

## 5 Other degree 1 base $\alpha$ expansions and zero relations

Base $\alpha$ expansions of $\log \alpha$
Theorem 5.1.

$$
\begin{equation*}
\log \alpha=P(1, \alpha, 2,(0,-\beta)) \tag{5.1}
\end{equation*}
$$

Proof. We have

$$
\log \alpha=\frac{1}{2} \operatorname{Li}_{1}\left(\frac{1}{\alpha}\right)=\frac{1}{2} \sum_{k=0}^{\infty} \frac{1}{\alpha^{k}} \frac{1 / \alpha}{k+1}=\sum_{k=0}^{\infty} \frac{1}{\alpha^{k}} \frac{-\beta}{2 k+2} .
$$

Theorem 5.2.

$$
\begin{equation*}
\log \alpha=P\left(1, \alpha^{2}, 2,\left(0,2 \beta^{2}\right)\right) \tag{5.2}
\end{equation*}
$$

Proof. We have

$$
\log \alpha=\operatorname{Li}_{1}\left(\frac{1}{\alpha^{2}}\right)=\sum_{k=0}^{\infty} \frac{1}{\alpha^{2 k}} \frac{1 / \alpha^{2}}{k+1}=\sum_{k=0}^{\infty} \frac{1}{\alpha^{2 k}} \frac{2 \beta^{2}}{2 k+2} .
$$

## Another base $\alpha$ expansion of $\log 2$

Theorem 5.3.

$$
\begin{equation*}
\log 2=P\left(1, \alpha^{3}, 3,\left(-\beta, \beta^{2}, 2 \beta^{3}\right)\right) \tag{5.3}
\end{equation*}
$$

Proof. A straightforward consequence of the identity

$$
\log 2=\operatorname{Li}_{1}\left(\frac{1}{\alpha}\right)-\mathrm{Li}_{1}\left(\frac{1}{\alpha^{3}}\right) .
$$

Another base $\alpha$ expansion of $\log 5$
Theorem 5.4.

$$
\begin{equation*}
\log 5=P\left(1, \alpha^{4}, 2,\left(4 \beta^{2}, 0\right)\right) \tag{5.4}
\end{equation*}
$$

Proof. A consequence of the identity

$$
\log 5=2 \operatorname{Li}_{1}\left(\frac{1}{\alpha^{2}}\right)-2 \operatorname{Li}_{1}\left(-\frac{1}{\alpha^{2}}\right)
$$

## A length 2 , base $\alpha$ zero relation

Theorem 5.5.

$$
\begin{equation*}
P\left(1, \alpha^{2}, 2,(1,3 \beta)\right)=0 \tag{5.5}
\end{equation*}
$$

Proof. Follows from

$$
\mathrm{Li}_{1}\left(\frac{1}{\alpha^{2}}\right)+\mathrm{Li}_{1}\left(-\frac{1}{\alpha}\right)=0
$$

Remark. Relation (5.5) also follows from (5.1) and (5.2).

## A length 12, base $\alpha$ zero relation

Theorem 5.6.

$$
P\left(1, \alpha^{12}, 12,\left(1, \beta,-2 \beta^{2}, 5 \beta^{3}, \beta^{4}, 10 \beta^{5}, \beta^{6}, 5 \beta^{7},-2 \beta^{8}, \beta^{9}, \beta^{10}, 2 \beta^{11}\right)\right)=0
$$

Proof. Follows fron (2.20) and (5.3).

## A length 10, base $\alpha$ zero relation

## Theorem 5.7.

$$
P\left(1, \alpha^{20}, 10,\left(1,-5 \beta^{2}, \beta^{4},-5 \beta^{6},-4 \beta^{8},-5 \beta^{10}, \beta^{12},-5 \beta^{14}, \beta^{16}, 0\right)\right)=0
$$

Proof. Ensues from (2.21) and (5.4).

## A length 5, base $\alpha$ zero relation

Theorem 5.8.

$$
\begin{equation*}
P\left(1, \alpha^{5}, 5,\left(\beta, 1,-\beta,-\beta^{4},-2 \beta^{4}\right)\right)=0 . \tag{5.6}
\end{equation*}
$$

Proof. Setting $p=2 \cos x$ in the identity

$$
\sum_{k=1}^{\infty} \frac{p^{k} \cos (k x)}{k}=-\frac{1}{2} \log \left(1-2 p \cos x+p^{2}\right)
$$

produces

$$
\begin{equation*}
\sum_{k=1}^{\infty} \frac{(2 \cos x)^{k} \cos (k x)}{k}=0 \tag{5.7}
\end{equation*}
$$

Now $2 \cos (2 \pi / 5)=-\beta$.

Thus, setting $x=2 \pi / 5$ in (5.7) gives

$$
\sum_{k=0}^{\infty} \frac{1}{\alpha^{5 k}}\left(\frac{\beta}{5 k+1}+\frac{1}{5 k+2}-\frac{\beta}{5 k+3}-\frac{\beta^{4}}{5 k+4}-\frac{2 \beta^{4}}{5 k+5}\right)=0
$$

since

$$
\cos \left(\frac{2 \pi}{5}(5 j-4)\right)=\frac{-\beta}{2}=\cos \left(\frac{2 \pi}{5}(5 j-1)\right), \quad j=1,2, \ldots
$$

and

$$
\cos \left(\frac{2 \pi}{5}(5 j-2)\right)=\frac{1}{2 \beta}=\cos \left(\frac{2 \pi}{5}(5 j-5)\right), \quad j=1,2, \ldots
$$

## References

[1] K. Adegoke, A non-PSLQ route to BBP-type formulas, Journal of Mathematics Research 2:2 (2010), 56-54.
[2] K. Adegoke, The golden ratio, Fibonacci numbers and BBP-type formulas, The Fibonacci Quarterly 52:2 (2014), 129-138.
[3] D. H. Bailey, A compendium of BBP-type formulas for mathematical constants, Preprint available at http://www.davidhbailey.com/dhbpapers/bbp-formulas.pdf (2017).
[4] D. H. Bailey and P. B. Borwein and S. Plouffe, On the rapid computation of various polylogarithmic constants, Mathematics of Computation 66:218 (1997), 903-913.
[5] D. H. Bailey and R. E. Crandall, On the random character of fundamental constant expansions, Experimental Mathematics 10:2 (2001), 175-190.
[6] J. Borwein and M. Chamberland, A Golden Example Solved. Undated manuscript.
[7] H. C. Chan, $\pi$ in terms of $\phi$, The Fibonacci Quarterly 44:2 (2006), 141-144.
[8] H. C. Chan, Machin-type formulas expressing $\pi$ in terms of $\phi$, The Fibonacci Quarterly 46/47:1 (2008/2009), 32-37.
[9] B. Cloitre, A BBP formula for $\pi^{2}$, Undated manuscript, last seen at https://les-mathematiques.net/phorum/file.php/2/3190/BBPbasePHI.pdf
[10] S. Kristensen and O. Mathiasen, BBP-type formulas - an elementary approach, Journal of Number Theory 244 (2023), 251-263.
[11] C. Wei, Several BBP-type formulas for $\pi$, Integral Transforms and Special Functions 26:5 (2015), 315-324.
[12] W. Zhang, New formulae of BBP-type with different moduli, Journal of Mathematical Analysis and Applications 398 (2013), 46-60.

