GORENSTEIN CATEGORIES, SINGULAR EQUIVALENCES AND FINITE GENERATION OF COHOMOLOGY RINGS IN RECOLLEMENTS

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ABSTRACT. Given an artin algebra Λ with an idempotent element a we compare the algebras Λ and $a\Lambda a$ with respect to Gorensteinness, singularity categories and the finite generation condition Fg for the Hochschild cohomology. In particular, we identify assumptions on the idempotent element a which ensure that Λ is Gorenstein if and only if $a\Lambda a$ is Gorenstein, that the singularity categories of Λ and $a\Lambda a$ are equivalent and that Fg holds for Λ if and only if Fg holds for $a\Lambda a$. We approach the problem by using recollements of abelian categories and we prove the results concerning Gorensteinness and singularity categories in this general setting. The results are applied to stable categories of Cohen–Macaulay modules and classes of triangular matrix algebras and quotients of path algebras.

CONTENTS

1. INTRODUCTION

This paper deals with Gorenstein algebras and categories, singularity categories and a finiteness condition ensuring existence of a useful theory of support for modules over finite dimensional algebras. First we give some background and indicate how these subjects are linked for us. Then we discuss the common framework for our investigations and give a sample of the main results in the paper. Finally we

describe the structure of the paper. For related work see Green–Madsen–Marcos [34] and Nagase [47]. In Subsection 8.4, we compare our results to those of Nagase.

For a group algebra of a finite group G over a field k there is a theory of support varieties of modules introduced by Jon Carlson in the seminal paper [13]. This theory has proven useful and powerful, where the support of a module is defined in terms of the maximal ideal spectrum of the group cohomology ring $H^*(G, k)$. Crucial facts here are that the group cohomology ring is graded commutative and noetherian, and for any finitely generated kG-module M, the Yoneda algebra $\operatorname{Ext}^*_{kG}(M, M)$ is a finitely generated module over the group cohomology ring (see [29, 31, 62]). For a finitely generated kG-module M the support variety is defined as the variety associated to the annihilator ideal of the action of the group cohomology ring $H^*(G, k)$ on $\operatorname{Ext}^*_{kG}(M, M)$. This construction is based on the Hopf algebra structure of the group algebra kG, and until recently a theory of support was not available for finite dimensional algebras in general.

Snashall and Solberg [59] have extended the theory of support varieties from group algebras to finite dimensional algebras by replacing the group cohomology $H^*(G, k)$ with the Hochschild cohomology ring of the algebra. Whenever similar properties as for group algebras are satisfied, that is, (i) the Hochschild cohomology ring is noetherian and (ii) all Yoneda algebras $\operatorname{Ext}^*_\Lambda(M, M)$ for a finitely generated Λ-module M are finitely generated modules over the Hochschild cohomology ring, then many of the same results as for group algebras of finite groups are still true when Λ is a selfinjective algebra [26]. The above set of conditions is referred to as Fg (see [26, 60]).

Triangulated categories of singularities or for simplicity *singularity categories* have been introduced and studied by Buchweitz [12], under the name *stable derived categories*, and later they have been considered by Orlov [50]. For an algebraic variety $\mathbb{X}$, Orlov introduced the *singularity category* of $\mathbb{X}$, as the Verdier quotient $\mathsf{D}_{\mathsf{sg}}(\mathbb{X}) = \mathsf{D}^{\mathsf{b}}(\operatorname{coh}\mathbb{X})/\operatorname{perf}(\mathbb{X})$, where $\mathsf{D}^{\mathsf{b}}(\operatorname{coh}\mathbb{X})$ is the bounded derived category of coherent sheaves on $\mathbb{X}$ and $\operatorname{perf}(\mathbb{X})$ is the full subcategory consisting of perfect complexes on $\mathbb{X}$. The singularity category $\mathsf{D}_{\mathsf{sg}}(\mathbb{X})$ captures many geometric properties of $\mathbb{X}$. For instance, if the variety $\mathbb{X}$ is smooth, then the singularity category $\mathsf{D}_{\mathsf{sg}}(\mathbb{X})$ is trivial but this is not true in general [50]. It should be noted that the singularity category is not only related to the study of the singularities of a given variety $\mathbb{X}$ but is also related to the Homological Mirror Symmetry Conjecture due to Kontsevich [42]. For more information we refer to [50–52].

Similarly, the singularity category over a noetherian ring R is defined [12] to be the Verdier quotient of the bounded derived category $\mathsf{D}^{\mathsf{b}}(\operatorname{mod} R)$ of the finitely generated R-modules by the full subcategory $\operatorname{perf}(R)$ of perfect complexes and is denoted by

$$\mathsf{D}_{\mathsf{sg}}(R) = \mathsf{D}^{\mathsf{b}}(\operatorname{mod} R)/\operatorname{perf}(R).$$

In this case the singularity category $\mathsf{D}_{\mathsf{sg}}(R)$ can be viewed as a categorical measure of the singularities of the spectrum $\operatorname{Spec}(R)$. Moreover, by a fundamental result of Buchweitz [12], and independently by Happel [37], the singularity category of a Gorenstein ring is equivalent to the stable category of (maximal) Cohen–Macaulay modules $\underline{\operatorname{CM}}(R)$, where the latter is well known to be a triangulated category [38]. Note that this equivalence generalizes the well known equivalence between the singularity category of a selfinjective algebra and the stable module category, a result

due to Rickard [56]. If there exists a triangle equivalence between the singularity categories of two rings R and S, then such an equivalence is called a *singular equivalence* between R and S. Singular equivalences were introduced by Chen, who studied singularity categories of non-Gorenstein algebras and investigated when there is a singular equivalence between certain extensions of rings [15, 17, 19, 20].

Next, from the perspective of support varieties, we describe some links between the above topics. Support varieties for $\mathsf{D}^{\mathsf{b}}(\operatorname{mod}\Lambda)$ using the Hochschild cohomology ring of Λ were considered in [60] for a finite dimensional algebra Λ over a field k, where all the perfect complexes $\operatorname{perf}(\Lambda)$ were shown to have trivial support variety. Hence the theory of support via the Hochschild cohomology ring naturally only says something about the Verdier quotient $\mathsf{D}^{\mathsf{b}}(\operatorname{mod}\Lambda)/\operatorname{perf}(\Lambda)$ – the singularity category. Furthermore, in [65], Hochschild (co)homology is discussed in connection with a particular class of singular equivalences.

To have an interesting theory of support, the finiteness condition Fg is pivotal. When Fg is satisfied for an algebra Λ, then Λ is Gorenstein [26, Proposition 1.2], or equivalently, $\operatorname{mod}\Lambda$ is a Gorenstein category.

As we pointed out above, when Λ is Gorenstein, then by Buchweitz–Happel the singularity category $\mathsf{D}^{\mathsf{b}}(\operatorname{mod}\Lambda)/\operatorname{perf}(\Lambda)$ is triangle equivalent to $\underline{\mathrm{CM}}(\Lambda)$, the stable category of Cohen–Macaulay modules. When Λ is a selfinjective algebra, then Λ^{e} is selfinjective and $\underline{\mathrm{CM}}(\Lambda^{\mathrm{e}}) = \underline{\operatorname{mod}}\Lambda^{\mathrm{e}}$ is a tensor triangulated category with Λ as a tensor identity. Let $\mathcal{B}$ be the full subcategory of $\underline{\mathrm{CM}}(\Lambda^{\mathrm{e}})$ consisting of all bimodules which are projective as a left and as a right Λ-module. Then $\mathcal{B}$ is also a tensor triangulated category with tensor identity Λ. The strictly positive part of the graded endomorphism ring

$$\operatorname{End}^*_{\mathcal{B}}(\Lambda) = \bigoplus_{i \in \mathbb{Z}} \operatorname{Hom}_{\mathcal{B}}(\Lambda, \Omega^i_{\Lambda^{\mathrm{e}}}(\Lambda)),$$

of the tensor identity Λ in $\underline{\mathrm{CM}}(\Lambda^{\mathrm{e}})$ is isomorphic to the strictly positive part $\mathrm{HH}^{\geqslant 1}(\Lambda)$ of the Hochschild cohomology ring of Λ. This is the relevant part for the theory of support varieties via the Hochschild cohomology ring. In addition $\mathcal{B}$ is a tensor triangulated category acting on the triangulated category $\underline{\mathrm{CM}}(\Lambda)$, and we can consider a theory of support varieties for $\underline{\mathrm{CM}}(\Lambda)$ based on the framework described in the forthcoming paper [11]. Therefore the singularity category of the enveloping algebra Λ^{e} encodes the geometric object for support varieties of modules and complexes over the algebra Λ.

Next we describe the categorical framework for our work. There has recently been a lot of interest around recollements of abelian (and triangulated) categories. These are exact sequences of abelian categories

$$0 \longrightarrow \mathscr{A} \overset{i}{\longrightarrow} \mathscr{B} \overset{e}{\longrightarrow} \mathscr{C} \longrightarrow 0,$$

where both the inclusion functor $i\colon \mathscr{A} \longrightarrow \mathscr{B}$ and the quotient functor $e\colon \mathscr{B} \longrightarrow \mathscr{C}$ have left and right adjoints. They have been introduced by Beilinson, Bernstein and Deligne [8] first in the context of triangulated categories in their study of derived categories of sheaves on singular spaces.

Properties of recollements of abelian categories were studied by Franjou and Pirashvili in [32], motivated by the MacPherson–Vilonen construction for the category of perverse sheaves [45], and recently homological properties of recollements of abelian and triangulated categories have also been studied in [54]. Recollements

of abelian categories were used by Cline, Parshall and Scott in the context of representation theory (see [24, 53]), and later Kuhn used recollements in his study of polynomial functors (see [44]). Recently, recollements of triangulated categories have appeared in the work of Angeleri Hügel, Koenig and Liu in connection with tilting theory, homological conjectures and stratifications of derived categories of rings (see [1–4]). Also, Chen and Xi have investigated recollements in relation with tilting theory [21] and algebraic K-theory [22, 23]. Furthermore, Han [35] has studied the relations between recollements of derived categories of algebras, smoothness and Hochschild cohomology of algebras.

It should be noted that module recollements, i.e. recollements of abelian categories whose terms are categories of modules, appear quite naturally in various settings. For instance any idempotent element e in a ring R induces a recollement situation between the module categories over the rings $R/\langle e \rangle$, R and eRe. In fact recollements of module categories are now well understood since every such recollement is equivalent, in an appropriate sense, to one induced by an idempotent element [55].

We want to compare the Fg condition for Hochschild cohomology, Gorensteinness and the singularity categories of two algebras. Our aim in this paper is to present a common context where we can compare these properties for an algebra Λ and $a\Lambda a$, where a is an idempotent of Λ. This is achieved using recollements of abelian categories. To summarize our main results we introduce the following notion. Given a functor $f\colon \mathscr{B} \longrightarrow \mathscr{C}$ between abelian categories, the functor f is called an **eventually homological isomorphism** if there is an integer t such that for every pair of objects B and B' in $\mathscr{B}$, and every $j > t$, there is an isomorphism

$$\operatorname{Ext}^j_{\mathscr{B}}(B, B') \cong \operatorname{Ext}^j_{\mathscr{C}}(f(B), f(B'))$$

of abelian groups (the isomorphism is not necessarily induced by the functor f). Our main results, stated in the context of artin algebras, are summarized in the following theorem. The four parts of the theorem are proved in Corollary 3.12, Corollary 5.4, Corollary 4.7 and Theorem 7.10, respectively. More general versions of the first three parts, in the setting of abelian categories, are given in Corollary 3.6 and Proposition 3.7, Theorem 5.2 and Theorem 4.3.

Main Theorem. *Let Λ be an artin algebra over a commutative ring k and let a be an idempotent element of Λ. Let e be the functor $a-\colon \operatorname{mod}\Lambda \longrightarrow \operatorname{mod} a\Lambda a$ given by multiplication by a. Consider the following conditions*:

$$(\alpha)\ \ \operatorname{id}_\Lambda\left(\frac{\Lambda/\langle a\rangle}{\operatorname{rad}\Lambda/\langle a\rangle}\right) < \infty \qquad\qquad (\beta)\ \ \operatorname{pd}_{a\Lambda a} a\Lambda < \infty$$

$$(\gamma)\ \ \operatorname{pd}_\Lambda\left(\frac{\Lambda/\langle a\rangle}{\operatorname{rad}\Lambda/\langle a\rangle}\right) < \infty \qquad\qquad (\delta)\ \ \operatorname{pd}_{(a\Lambda a)^{\operatorname{op}}} \Lambda a < \infty$$

Then the following hold.

- (i) *The following are equivalent*:
    - (a) *(α) and (β) hold.*
    - (b) *(γ) and (δ) hold.*
    - (c) *The functor e is an eventually homological isomorphism.*
- (ii) *The functor e induces a singular equivalence between Λ and $a\Lambda a$ if and only if conditions (β) and (γ) hold.*

(iii) *Assume that e is an eventually homological isomorphism. Then Λ is Gorenstein if and only if $a\Lambda a$ is Gorenstein.*

(iv) *Assume that e is an eventually homological isomorphism. Assume also that k is a field and that $(\Lambda/\operatorname{rad}\Lambda)\otimes_k(\Lambda^{\mathrm{op}}/\operatorname{rad}\Lambda^{\mathrm{op}})$ is a semisimple Λ^{e}-module (for instance, this is true if k is algebraically closed). Then Λ satisfies Fg if and only if $a\Lambda a$ satisfies Fg.*

Now we describe the contents of the paper section by section. In Section 2, we recall notions and results on recollements of abelian categories and Hochschild cohomology that are used throughout the paper.

In Section 3, we study extension groups in a recollement of abelian categories $(\mathscr{A},\mathscr{B},\mathscr{C})$. More precisely, we investigate when the exact functor $e\colon\mathscr{B}\longrightarrow\mathscr{C}$ is an eventually homological isomorphism. It turns out that the answer to this problem is closely related to the characterization given in [54] of when the functor e induces isomorphisms between extension groups in all degrees below some bound n. In Corollary 3.6 and Proposition 3.7 we give sufficient and necessary conditions, respectively, for the functor e to be an eventually homological isomorphism. We specialize these results to recollements of artin algebras and characterize when the functor e is an eventually homological isomorphism in Corollary 3.12. The results of this section are used in Section 4 and Section 7.

In Section 4, we study Gorenstein categories, introduced by Beligiannis and Reiten [9]. Assuming that we have an eventually homological isomorphism $f\colon\mathscr{D}\longrightarrow\mathscr{F}$ between abelian categories, we investigate when Gorensteinness is transferred between $\mathscr{D}$ and $\mathscr{F}$. Among other things, we prove that if f is an essentially surjective eventually homological isomorphism, then $\mathscr{D}$ is Gorenstein if and only if $\mathscr{F}$ is (see Theorem 4.3). We apply this to recollements of abelian categories and recollements of module categories.

In Section 5, we investigate singularity categories, in the sense of Buchweitz [12] and Orlov [50], in a recollement $(\mathscr{A},\mathscr{B},\mathscr{C})$ of abelian categories. In fact, we give necessary and sufficient conditions for the quotient functor $e\colon\mathscr{B}\longrightarrow\mathscr{C}$ to induce a triangle equivalence between the singularity categories of $\mathscr{B}$ and $\mathscr{C}$; see Theorem 5.2. This result generalizes earlier results by Chen [15]. We obtain the results of Chen in Corollary 5.4 by applying Theorem 5.2 to rings with idempotents. Finally, for an artin algebra Λ with an idempotent element a, we give a sufficient condition for the stable categories of Cohen–Macaulay modules of Λ and $a\Lambda a$ to be triangle equivalent; see Corollary 5.6.

In Section 6 and Section 7, which form a unit, we investigate the finite generation condition Fg for the Hochschild cohomology of a finite dimensional algebra over a field. In particular, in Section 6 we show how we can compare the Fg condition for two different algebras. This is achieved by showing, for two graded rings and graded modules over them, that if we have isomorphisms in all but finitely many degrees, then the noetherian property of the rings and the finite generation of the modules is preserved; see Propositions 6.3 and 6.4. In Section 7, we use this result to show that Fg holds for a finite dimensional algebra Λ over a field if and only if Fg holds for the algebra $a\Lambda a$, where a is an idempotent element of Λ which satisfies certain assumptions (see Theorem 7.10). As part of this, we show that under the same assumptions, the Hochschild cohomology rings of Λ and $a\Lambda a$ are the same in almost all degrees (Proposition 7.9).

The final Section 8 is devoted to examples and applications of our main results. We first discuss conditions (α)–(δ) of the Main Theorem. Then we consider some special cases where these conditions are related, and provide an interpretation for quotients of path algebras. Then we apply our results to triangular matrix algebras. For a triangular matrix algebra $\Lambda = \left(\begin{smallmatrix} \Sigma & 0 \\ {}_\Gamma M_\Sigma & \Gamma \end{smallmatrix} \right)$, we compare Λ to the algebras Σ and Γ with respect to the Fg condition, Gorensteinness and singularity categories. In particular, we recover a result by Chen [15] concerning the singularity categories of Λ and Σ. Finally, we compare our results to those of Nagase [47].

Conventions and notation. For a ring R we usually work with left R-modules and the corresponding category is denoted by $\mathrm{Mod}\,R$. The full subcategory of finitely presented R-modules is denoted by $\mathrm{mod}\,R$. Our additive categories are assumed to have finite direct sums and our subcategories are assumed to be closed under isomorphisms and direct summands. The Jacobson radical of a ring R is denoted by $\mathrm{rad}\,R$. By a module over an artin algebra Λ, we mean a finitely presented (generated) left Λ-module.

2. Preliminaries

In this section we recall notions and results on recollements of abelian categories and Hochschild cohomology.

2.1. **Recollements of abelian categories.** In this subsection we recall the definition of a recollement situation in the context of abelian categories (see for instance [32, 36, 44]), we fix notation and recall some well known properties of recollements which are used later in the paper. We also include our basic source of examples of recollements. For an additive functor $F \colon \mathscr{A} \longrightarrow \mathscr{B}$ between additive categories, the **essential image** $\{B \in \mathscr{B} \mid B \cong F(A) \text{ for some } A \in \mathscr{A}\}$ of F is denoted by $\mathrm{Im}\,F$ and the **kernel** $\{A \in \mathscr{A} \mid F(A) = 0\}$ of F is denoted by $\mathrm{Ker}\,F$.

Definition 2.1. A **recollement situation** between abelian categories $\mathscr{A}, \mathscr{B}$ and $\mathscr{C}$ is a diagram

$$
\begin{array}{ccc}
\mathscr{A} & \xrightarrow{\;\;i\;\;} & \mathscr{B} & \xrightarrow{\;\;e\;\;} & \mathscr{C}
\end{array}
$$
with q, p over $\mathscr{A}\to\mathscr{B}$ and l, r over $\mathscr{B}\to\mathscr{C}$

henceforth denoted by $(\mathscr{A}, \mathscr{B}, \mathscr{C})$, satisfying the following conditions:

  1. (l, e, r) is an adjoint triple.
  2. (q, i, p) is an adjoint triple.
  3. The functors i, l and r are fully faithful.
  4. $\mathrm{Im}\,i = \mathrm{Ker}\,e$.

In the next result we collect some basic properties of a recollement situation of abelian categories that can be derived easily from Definition 2.1. For more details, see [32, 54].

Proposition 2.2. *Let $(\mathscr{A}, \mathscr{B}, \mathscr{C})$ be a recollement of abelian categories. Then the following hold.*

  (i) *The functors $i \colon \mathscr{A} \longrightarrow \mathscr{B}$ and $e \colon \mathscr{B} \longrightarrow \mathscr{C}$ are exact.*
  (ii) *The compositions ei, ql and pr are zero.*
  (iii) *The functor $e \colon \mathscr{B} \longrightarrow \mathscr{C}$ is essentially surjective.*

(iv) *The units of the adjoint pairs (i,p) and (l,e) and the counits of the adjoint pairs (q,i) and (e,r) are isomorphisms*:

$$\mathrm{Id}_{\mathscr{A}} \xrightarrow{\cong} pi, \qquad \mathrm{Id}_{\mathscr{C}} \xrightarrow{\cong} el, \qquad qi \xrightarrow{\cong} \mathrm{Id}_{\mathscr{A}}, \qquad er \xrightarrow{\cong} \mathrm{Id}_{\mathscr{C}}.$$

(v) *The functors $l\colon \mathscr{C} \longrightarrow \mathscr{B}$ and $q\colon \mathscr{B} \longrightarrow \mathscr{A}$ preserve projective objects and the functors $r\colon \mathscr{C} \longrightarrow \mathscr{B}$ and $p\colon \mathscr{B} \longrightarrow \mathscr{A}$ preserve injective objects.*

(vi) *The functor $i\colon \mathscr{A} \longrightarrow \mathscr{B}$ induces an equivalence between $\mathscr{A}$ and the Serre subcategory $\mathrm{Ker}\, e = \mathrm{Im}\, i$ of $\mathscr{B}$. Moreover, $\mathscr{A}$ is a localizing and colocalizing subcategory of $\mathscr{B}$ and there is an equivalence of categories $\mathscr{B}/\mathscr{A} \simeq \mathscr{C}$.*

(vii) *For every B in $\mathscr{B}$ there are A and A' in $\mathscr{A}$ such that the units and counits of the adjunctions induce the following exact sequences*:

$$0 \longrightarrow i(A) \longrightarrow le(B) \longrightarrow B \longrightarrow iq(B) \longrightarrow 0$$

and

$$0 \longrightarrow ip(B) \longrightarrow B \longrightarrow re(B) \longrightarrow i(A') \longrightarrow 0.$$

Throughout the paper, we apply our results to recollements of module categories, and in particular to recollements of module categories over artin algebras as described in the following example.

Example 2.3. Let Λ be an artin k-algebra, where k is a commutative artin ring, and let a be an idempotent element in Λ.

(i) We have the following recollement of abelian categories:

$$\mathrm{mod}\,\Lambda/\langle a\rangle \xrightarrow{\;\;\mathrm{inc}\;\;} \mathrm{mod}\,\Lambda \xrightarrow{\;e=a(-)\;} \mathrm{mod}\,a\Lambda a$$

with top arrows $\Lambda/\langle a\rangle \otimes_\Lambda -$ and $\Lambda a \otimes_{a\Lambda a} -$, and bottom arrows $\mathrm{Hom}_\Lambda(\Lambda/\langle a\rangle, -)$ and $\mathrm{Hom}_{a\Lambda a}(a\Lambda, -)$.

The functor $e\colon \mathrm{mod}\,\Lambda \longrightarrow \mathrm{mod}\,a\Lambda a$ can also be described as follows: $e = a(-) \cong \mathrm{Hom}_\Lambda(\Lambda a, -) \cong a\Lambda \otimes_\Lambda -$. We write $\langle a\rangle$ for the ideal of Λ generated by the idempotent element a. Then every left $\Lambda/\langle a\rangle$-module is annihilated by $\langle a\rangle$ and thus the category $\mathrm{mod}\,\Lambda/\langle a\rangle$ is the kernel of the functor $a(-)$.

(ii) Let $\Lambda^{\mathrm{e}} = \Lambda \otimes_k \Lambda^{\mathrm{op}}$ be the enveloping algebra of Λ. The element $\varepsilon = a \otimes a^{\mathrm{op}}$ is an idempotent element of Λ^{e}. Therefore as above we have the following recollement of abelian categories:

$$\mathrm{mod}\,\Lambda^{\mathrm{e}}/\langle\varepsilon\rangle \xrightarrow{\;\;\mathrm{inc}\;\;} \mathrm{mod}\,\Lambda^{\mathrm{e}} \xrightarrow{\;E=\varepsilon(-)\;} \mathrm{mod}\,(a\Lambda a)^{\mathrm{e}}$$

with top arrows $\Lambda^{\mathrm{e}}/\langle\varepsilon\rangle \otimes_{\Lambda^{\mathrm{e}}} -$ and $\Lambda^{\mathrm{e}}\varepsilon \otimes_{(a\Lambda a)^{\mathrm{e}}} -$, and bottom arrows $\mathrm{Hom}_{\Lambda^{\mathrm{e}}}(\Lambda^{\mathrm{e}}/\langle\varepsilon\rangle, -)$ and $\mathrm{Hom}_{(a\Lambda a)^{\mathrm{e}}}(\varepsilon\Lambda^{\mathrm{e}}, -)$.

Note that $(a\Lambda a)^{\mathrm{e}} \cong \varepsilon\Lambda^{\mathrm{e}}\varepsilon$ as k-algebras.

Remark 2.4. As in Example 2.3, any idempotent element e in a ring R induces a recollement situation between the module categories over the rings $R/\langle e\rangle$, R and eRe. This should be considered as the universal example for recollements of abelian categories whose terms are categories of modules. Indeed, in [55] it is proved that any recollement of module categories is equivalent, in an appropriate sense, to one induced by an idempotent element.

2.2. **Hochschild cohomology rings.** We briefly explain the terminology we need regarding Hochschild cohomology and the finite generation condition Fg, and recall some important results. For a more detailed exposition of these topics, see sections 2–5 of [60].

Let Λ be an artin algebra over a commutative ring k. We define the **Hochschild cohomology ring** $\mathrm{HH}^*(\Lambda)$ of Λ by

$$\mathrm{HH}^*(\Lambda) = \mathrm{Ext}^*_{\Lambda^e}(\Lambda, \Lambda) = \bigoplus_{i=0}^{\infty} \mathrm{Ext}^i_{\Lambda^e}(\Lambda, \Lambda).$$

This is a graded k-algebra with multiplication given by Yoneda product. Hochschild cohomology was originally defined by Hochschild in [39], using the bar resolution. It was shown in [14, IX, §6] that our definition coincides with the original definition when Λ is projective over k.

Gerstenhaber showed in [33] that the Hochschild cohomology ring as originally defined is graded commutative. This implies that the Hochschild cohomology ring as defined above is graded commutative when Λ is projective over k. The following more general result was shown in [59, Theorem 1.1] (see also [61], which proves graded commutativity of several cohomology theories in a uniform way).

Theorem 2.5. *Let Λ be an algebra over a commutative ring k such that Λ is flat as a module over k. Then the Hochschild cohomology ring $\mathrm{HH}^*(\Lambda)$ is graded commutative.*

To describe the finite generation condition Fg, we first need to define a $\mathrm{HH}^*(\Lambda)$-module structure on the direct sum of all extension groups of a Λ-module with itself (for more details about this module structure, see [59]). Assume that Λ is flat as a k-module, and let M be a Λ-module. The direct sum

$$\mathrm{Ext}^*_{\Lambda}(M, M) = \bigoplus_{i=0}^{\infty} \mathrm{Ext}^i_{\Lambda}(M, M)$$

of all extension groups of M with itself is a graded k-algebra with multiplication given by Yoneda product. We give it a graded $\mathrm{HH}^*(\Lambda)$-module structure by the graded ring homomorphism

$$\varphi_M \colon \mathrm{HH}^*(\Lambda) \longrightarrow \mathrm{Ext}^*_{\Lambda}(M, M),$$

which is defined in the following way. Any homogeneous element of positive degree in $\mathrm{HH}^*(\Lambda)$ can be represented by an exact sequence

$$\eta \colon 0 \longrightarrow \Lambda \longrightarrow X \longrightarrow P_n \longrightarrow \cdots \longrightarrow P_1 \longrightarrow P_0 \longrightarrow \Lambda \longrightarrow 0$$

of Λ^e-modules, where every P_i is projective. Tensoring this sequence throughout with M gives an exact sequence

$$0 \longrightarrow \Lambda \otimes_{\Lambda} M \longrightarrow X \otimes_{\Lambda} M \longrightarrow P_n \otimes_{\Lambda} M \longrightarrow \cdots$$
$$\longrightarrow P_1 \otimes_{\Lambda} M \longrightarrow P_0 \otimes_{\Lambda} M \longrightarrow \Lambda \otimes_{\Lambda} M \longrightarrow 0$$

of Λ-modules (the exactness of this sequence follows from the facts that Λ is flat as a k-module and that the modules P_i are projective Λ^e-modules). Using the isomorphism $\Lambda \otimes_{\Lambda} M \cong M$, we get an exact sequence of Λ-modules starting and ending in M; we define $\varphi_M([\eta])$ to be the element of $\mathrm{Ext}^*_{\Lambda}(M, M)$ represented by this sequence. For elements of degree zero in $\mathrm{HH}^*(\Lambda)$, the map φ_M is defined by tensoring with M and using the identification $\Lambda \otimes_{\Lambda} M \cong M$.

In [26], Erdmann–Holloway–Snashall–Solberg–Taillefer identified certain assumptions about an algebra Λ, which are sufficient in order for the theory of support varieties to have good properties. They called these assumptions **Fg1** and **Fg2**. We say that an algebra satisfies **Fg** if it satisfies both **Fg1** and **Fg2**. We use the following definition of **Fg**, which is equivalent (by [60, Proposition 5.7]) to the definition of **Fg1** and **Fg2** given in [26].

Definition 2.6. Let Λ be an algebra over a commutative ring k such that Λ is flat as a module over k. We say that Λ satisfies the **Fg condition** if the following is true:

(i) The ring $\mathrm{HH}^*(\Lambda)$ is noetherian.
(ii) The $\mathrm{HH}^*(\Lambda)$-module $\mathrm{Ext}^*_\Lambda(\Lambda/\operatorname{rad}\Lambda, \Lambda/\operatorname{rad}\Lambda)$ is finitely generated.

The following result states that in our definition of **Fg** we could have replaced part (ii) by the same requirement for all Λ-modules. It can be proved in a similar way as [26, Proposition 1.4].

Theorem 2.7. *If an artin algebra Λ satisfies the* **Fg** *condition, then* $\mathrm{Ext}^*_\Lambda(M, M)$ *is a finitely generated* $\mathrm{HH}^*(\Lambda)$-*module for every* Λ-*module* M.

We end this section by describing a connection between the **Fg** condition and Gorensteinness.

Theorem 2.8 ([26, Theorem 1.5 (a)])**.** *If an artin algebra Λ satisfies the* **Fg** *condition, then Λ is Gorenstein.*

3. Eventually homological isomorphisms in recollements

Given a functor $f \colon \mathscr{D} \longrightarrow \mathscr{F}$ between abelian categories and an integer t, the functor f is called a t-**homological isomorphism** if there is a group isomorphism

$$\mathrm{Ext}^j_{\mathscr{B}}(B, B') \cong \mathrm{Ext}^j_{\mathscr{C}}(f(B), f(B'))$$

for every pair of objects B and B' in $\mathscr{B}$, and every $j > t$. Note that we do not require these isomorphisms to be induced by the functor f. If f is a t-homological isomorphism for some t, then it is an **eventually homological isomorphism**. In this section, we investigate when the functor e in a recollement

of abelian categories is an eventually homological isomorphism.

The functor e induces maps

(3.1) $$\mathrm{Ext}^j_{\mathscr{B}}(X, Y) \longrightarrow \mathrm{Ext}^j_{\mathscr{C}}(e(X), e(Y))$$

of extension groups for all objects X and Y in $\mathscr{B}$ and for every $j \geq 0$. With one argument fixed and the other one varying over all objects we study when these maps are isomorphisms in almost all degrees, that is, for every degree j greater than some bound n (see Theorem 3.4 and Theorem 3.5). We use this to find two sets of sufficient conditions for the functor $e \colon \mathscr{B} \longrightarrow \mathscr{C}$ to be an eventually homological isomorphism (Corollary 3.6), and we find a partial converse (Proposition 3.7). Finally, we specialize these results to artin algebras, using the recollement

$(\operatorname{mod} \Lambda/\langle a \rangle, \operatorname{mod} \Lambda, \operatorname{mod} a \Lambda a)$ of Example 2.3 (i). In particular, we characterize when the functor $e \colon \operatorname{mod} \Lambda \longrightarrow \operatorname{mod} a \Lambda a$ is an eventually homological isomorphism (Corollary 3.12).

These results are used in Section 4 for comparing Gorensteinness of the categories in a recollement, and in Section 7 for comparing the Fg condition of the algebras Λ and $a \Lambda a$, where a is an idempotent in Λ.

We start by fixing some notation. For an injective coresolution $0 \longrightarrow B \longrightarrow I^0 \longrightarrow I^1 \longrightarrow \cdots$ of B in $\mathscr{B}$, we say that the image of the morphism $I^{n-1} \longrightarrow I^n$ is an n-**th cosyzygy** of B, and we denote it by $\Sigma^n(B)$. Dually, if $\cdots \longrightarrow P_1 \longrightarrow P_0 \longrightarrow B \longrightarrow 0$ is a projective resolution of B in $\mathscr{B}$, then we say that the kernel of the morphism $P^{n-1} \longrightarrow P^{n-2}$ is an n-**th syzygy** of B, and we denote it by $\Omega^n(B)$. Also, if $\mathfrak{X}$ is a class of objects in $\mathscr{B}$, then the **right orthogonal subcategory** $\{B \in \mathscr{B} \mid \operatorname{Hom}_{\mathscr{B}}(\mathfrak{X}, B) = 0\}$ of $\mathfrak{X}$ is denoted by $\mathfrak{X}^{\perp}$ and the **left orthogonal subcategory** $\{B \in \mathscr{B} \mid \operatorname{Hom}_{\mathscr{B}}(B, \mathfrak{X}) = 0\}$ of $\mathfrak{X}$ is denoted by ${}^{\perp}\mathfrak{X}$.

We now describe precisely how the maps (3.1) induced by the functor e in a recollement are defined. Let $\mathscr{D}$ and $\mathscr{F}$ be abelian categories and let $f \colon \mathscr{D} \longrightarrow \mathscr{F}$ be an additive exact functor. If

$$\xi \colon \ 0 \longrightarrow X_n \xrightarrow{d_n} X_{n-1} \longrightarrow \cdots \longrightarrow X_1 \xrightarrow{d_1} X_0 \longrightarrow 0$$

is an exact sequence in $\mathscr{D}$, then we denote by $f(\xi)$ the exact sequence

$$f(\xi) \colon \ 0 \longrightarrow f(X_n) \xrightarrow{f(d_n)} f(X_{n-1}) \longrightarrow \cdots \longrightarrow f(X_1) \xrightarrow{f(d_1)} f(X_0) \longrightarrow 0$$

in $\mathscr{F}$. It is clear that this operation commutes with Yoneda product; that is, if ξ and ζ are composable exact sequences in $\mathscr{D}$, then $f(\xi\zeta) = f(\xi) \cdot f(\zeta)$. For every pair of objects X and Y in $\mathscr{D}$ and every nonnegative integer j, we define a group homomorphism

$$f^j_{X,Y} \ : \ \operatorname{Ext}^j_{\mathscr{D}}(X, Y) \ \longrightarrow \ \operatorname{Ext}^j_{\mathscr{F}}(f(X), f(Y))$$

by

$$f^0_{X,Y}(d) = f(d) \qquad \text{for a morphism } d \colon X \longrightarrow Y;$$
$$f^j_{X,Y}([\eta]) = [f(\eta)] \qquad \text{for a } j\text{-fold extension } \eta \text{ of } X \text{ by } Y, \text{ where } j > 0.$$

For an object X in $\mathscr{D}$, the direct sum $\operatorname{Ext}^*_{\mathscr{D}}(X, X) = \bigoplus_{j=0}^{\infty} \operatorname{Ext}^j_{\mathscr{D}}(X, X)$ is a graded ring with multiplication given by Yoneda product, and taking the maps $f^j_{X,X}$ in all degrees j gives a graded ring homomorphism

$$f^*_{X,X} \colon \operatorname{Ext}^*_{\mathscr{D}}(X, X) \longrightarrow \operatorname{Ext}^*_{\mathscr{F}}(f(X), f(X)).$$

Remark 3.1. We explain briefly why the maps $f^j_{X,Y}$ and $f^*_{X,X}$ defined above are homomorphisms.

(i) Since the functor f is exact, it preserves pullbacks and pushouts. Thus the map $f^j_{X,Y}$ preserves the Baer sum between two extensions.

(ii) For checking that the map $f^*_{X,X}$ is a graded ring homomorphism, the only nontrivial case to consider is the product of a morphism and an extension. For this case, we again use that the functor f preserves pullbacks and pushouts.

We now consider the maps

$$e^j_{B,B'} \colon \operatorname{Ext}^j_{\mathscr{B}}(B, B') \longrightarrow \operatorname{Ext}^j_{\mathscr{C}}(e(B), e(B'))$$

induced by the functor $e \colon \mathscr{B} \longrightarrow \mathscr{C}$ in a recollement, where we let one argument be fixed and the other vary over all objects of $\mathscr{B}$. In [54], the first author studied when these maps are isomorphisms for all degrees up to some bound n, that is, for $0 \leq j \leq n$. This immediately leads to a description of when these maps are isomorphisms in all degrees, which we state as the following theorem.

Theorem 3.2 ([54, Propositions 3.3 and 3.4, Theorem 3.10]). *Let $(\mathscr{A}, \mathscr{B}, \mathscr{C})$ be a recollement of abelian categories and assume that $\mathscr{B}$ and $\mathscr{C}$ have enough projective and injective objects. Let B be an object in $\mathscr{B}$.*

   (i) *The following statements are equivalent*:

      (a) *The map $e^j_{B,B'} \colon \operatorname{Ext}^j_{\mathscr{B}}(B, B') \longrightarrow \operatorname{Ext}^j_{\mathscr{C}}(e(B), e(B'))$ is an isomorphism for every object B' in $\mathscr{B}$ and every nonnegative integer j.*

      (b) *The object B has a projective resolution of the form*

$$\cdots \longrightarrow l(P_2) \longrightarrow l(P_1) \longrightarrow l(P_0) \longrightarrow B \longrightarrow 0,$$

        where P_j is a projective object in $\mathscr{C}$.

      (c) *$\operatorname{Ext}^j_{\mathscr{B}}(B, i(A)) = 0$ for every $A \in \mathscr{A}$ and $j \geq 0$.*

      (d) *$\operatorname{Ext}^j_{\mathscr{B}}(B, i(I)) = 0$ for every $I \in \operatorname{Inj} \mathscr{A}$ and $j \geq 0$.*

   (ii) *The following statements are equivalent*:

      (a) *The map $e^j_{B',B} \colon \operatorname{Ext}^j_{\mathscr{B}}(B', B) \longrightarrow \operatorname{Ext}^j_{\mathscr{C}}(e(B'), e(B))$ is an isomorphism for every object B' in $\mathscr{B}$ and every nonnegative integer j.*

      (b) *The object B has an injective coresolution of the form*

$$0 \longrightarrow B \longrightarrow r(I^0) \longrightarrow r(I^1) \longrightarrow r(I^2) \longrightarrow \cdots,$$

        where I^j is an injective object in $\mathscr{C}$.

      (c) *$\operatorname{Ext}^j_{\mathscr{B}}(i(A), B) = 0$ for every $A \in \mathscr{A}$ and $j \geq 0$.*

      (d) *$\operatorname{Ext}^j_{\mathscr{B}}(i(P), B) = 0$ for every $P \in \operatorname{Proj} \mathscr{A}$ and $j \geq 0$.*

The above theorem describes when the maps $e^j_{B,B'}$ induced by the functor e are isomorphisms in all degrees j. Our aim in this section is to give a similar description of when these maps are isomorphisms in almost all degrees. The basic idea is to translate the conditions in the above theorem to similar conditions stated for almost all degrees, and show the equivalence of these conditions by using the above theorem and dimension shifting. In order for this to work, however, we need to modify the conditions somewhat. We obtain Theorem 3.4 which is stated below and generalizes parts of Theorem 3.2 (i) (and the dual Theorem 3.5 which generalizes parts of Theorem 3.2 (ii)). In order to prove the theorem, we need a general version of dimension shifting as stated in the following lemma.

Lemma 3.3. *Let $\mathscr{A}$ be an abelian category, let n be an integer, and let*

$$\epsilon \colon \; 0 \longrightarrow X \longrightarrow E_{m-1} \longrightarrow \cdots \longrightarrow E_0 \longrightarrow Y \longrightarrow 0$$

be an exact sequence in $\mathscr{A}$ with $\operatorname{pd}_{\mathscr{A}} E_i \leq n$ for every i. Then for every $i > n$ and $Z \in \mathscr{A}$, the map given by $\epsilon^*([\eta]) = [\eta\epsilon]$ is an isomorphism:

$$\epsilon^* \; \colon \; \operatorname{Ext}^i_{\mathscr{A}}(X, Z) \longrightarrow \operatorname{Ext}^{i+m}_{\mathscr{A}}(Y, Z).$$

Now we are ready to show our characterization of when the functor e in a recollement induces isomorphisms of extension groups in almost all degrees.

Theorem 3.4. *Let $(\mathscr{A}, \mathscr{B}, \mathscr{C})$ be a recollement of abelian categories and assume that $\mathscr{B}$ and $\mathscr{C}$ have enough projective and injective objects. Consider the following statements for an object B of $\mathscr{B}$ and two integers n and m:*

(a) *The map $e_{B,B'}^j \colon \operatorname{Ext}_{\mathscr{B}}^j(B, B') \longrightarrow \operatorname{Ext}_{\mathscr{C}}^j(e(B), e(B'))$ is an isomorphism for every object B' in $\mathscr{B}$ and every integer $j > m + n$.*

(b) *The object B has a projective resolution of the form*

$$\cdots \longrightarrow l(Q_1) \longrightarrow l(Q_0) \longrightarrow P_{n-1} \longrightarrow \cdots \longrightarrow P_0 \longrightarrow B \longrightarrow 0,$$

where each Q_j is a projective object in $\mathscr{C}$.

(c) *$\operatorname{Ext}_{\mathscr{B}}^j(B, i(A)) = 0$ for every $A \in \mathscr{A}$ and $j > n$, and there exists an n-th syzygy of B lying in ${}^{\perp} i(\mathscr{A})$.*

(d) *$\operatorname{Ext}_{\mathscr{B}}^j(B, i(I)) = 0$ for every $I \in \operatorname{Inj} \mathscr{A}$ and $j > n$, and there exists an n-th syzygy of B lying in ${}^{\perp} i(\operatorname{Inj} \mathscr{A})$.*

We have the following relations between these statements:

(i) (b) $\iff$ (c) $\iff$ (d).

(ii) *If $\operatorname{pd}_{\mathscr{C}} e(P) \leq m$ for every projective object P in $\mathscr{B}$, then (b) $\implies$ (a).*

Proof. (i) By dimension shift, statement (c) is equivalent to

$$\operatorname{Ext}_{\mathscr{B}}^j(\Omega^n(B), i(A)) = 0 \qquad \text{for every } j \geq 0 \text{ and every } A \in \mathscr{A},$$

and statement (d) is equivalent to

$$\operatorname{Ext}_{\mathscr{B}}^j(\Omega^n(B), i(I)) = 0 \qquad \text{for every } j \geq 0 \text{ and every } I \in \operatorname{Inj} \mathscr{A},$$

where in both cases $\Omega^n(B)$ is a suitably chosen n-th syzygy of B. The equivalence of statements (b), (c) and (d) now follows from the equivalence of (b), (c) and (d) in Theorem 3.2 (i).

(ii) Let

$$\pi \colon \ 0 \longrightarrow K \longrightarrow P_{n-1} \longrightarrow \cdots \longrightarrow P_1 \longrightarrow P_0 \longrightarrow B \longrightarrow 0$$

be the beginning of the chosen projective resolution of B, where $K = \Omega^n(B)$ is the n-th syzygy of B. Consider the following group homomorphisms:

$$(3.2) \quad \operatorname{Ext}_{\mathscr{B}}^j(B, B') \xleftarrow{\ \pi^* \ } \operatorname{Ext}_{\mathscr{B}}^{j-n}(K, B') \xrightarrow{\ e_{K,B'}^{j-n} \ } \operatorname{Ext}_{\mathscr{C}}^{j-n}(e(K), e(B'))$$

$$\xrightarrow{\ (e(\pi))^* \ } \operatorname{Ext}_{\mathscr{C}}^j(e(B), e(B')).$$

Here, the maps π^* and $(e(\pi))^*$ are isomorphisms by Lemma 3.3. Note that for $(e(\pi))^*$ we use the fact that $\operatorname{pd}_{\mathscr{C}} e(P) \leq m$ for every projective object P in $\mathscr{B}$. The map $e_{K,B'}^{j-n}$ is an isomorphism by Theorem 3.2 (i). Thus, we have an isomorphism

$$(e(\pi))^* \circ e_{K,B'}^{j-n} \circ (\pi^*)^{-1} \colon \operatorname{Ext}_{\mathscr{B}}^j(B, B') \longrightarrow \operatorname{Ext}_{\mathscr{C}}^j(e(B), e(B'))$$

for every $j \geq m + n + 1$ and $B' \in \mathscr{B}$. We want to show that this is the same map as $e_{B,B'}^j$. We consider an element $[\eta] \in \operatorname{Ext}_{\mathscr{B}}^{j-n}(K, B')$, and follow it through the

maps (3.2). We then get the following elements:

$$\mathrm{Ext}^j_{\mathscr{B}}(B,B') \xleftarrow[\cong]{\pi^*} \mathrm{Ext}^{j-n}_{\mathscr{B}}(K,B') \xrightarrow[\cong]{e^{j-n}_{K,B'}} \mathrm{Ext}^{j-n}_{\mathscr{C}}(e(K),e(B')) \xrightarrow[\cong]{(e(\pi))^*} \mathrm{Ext}^j_{\mathscr{C}}(e(B),e(B'))$$

$$[\eta\pi] \longleftarrow\!\!\mid [\eta] \longmapsto [e(\eta)] \longmapsto [e(\eta)\cdot e(\pi)]$$

$$\|$$

$$[e(\eta\pi)]$$

This shows that our isomorphism takes any element $[\zeta] \in \mathrm{Ext}^j_{\mathscr{B}}(B,B')$ to the element $[e(\zeta)] \in \mathrm{Ext}^j_{\mathscr{C}}(e(B),e(B'))$. Thus, our isomorphism is $e^j_{B,B'}$. □

Dually to the above theorem, we have the following generalization of some of the implications in Theorem 3.2 (ii).

Theorem 3.5. *Let $(\mathscr{A},\mathscr{B},\mathscr{C})$ be a recollement of abelian categories and assume that $\mathscr{B}$ and $\mathscr{C}$ have enough projective and injective objects. Consider the following statements for an object B of $\mathscr{B}$ and two integers n and m:*

    (a) *The map $e^j_{B',B}\colon \mathrm{Ext}^j_{\mathscr{B}}(B',B) \longrightarrow \mathrm{Ext}^j_{\mathscr{C}}(e(B'),e(B))$ is an isomorphism for every object B' in $\mathscr{B}$ and every integer $j > m+n$.*

    (b) *The object B has an injective coresolution of the form*

$$0 \longrightarrow B \longrightarrow I^0 \longrightarrow \cdots \longrightarrow I^{n-1} \longrightarrow r(J^0) \longrightarrow r(J^1) \longrightarrow \cdots ,$$

        where each J^j is a projective object in $\mathscr{C}$.

    (c) $\mathrm{Ext}^j_{\mathscr{B}}(i(A),B) = 0$ *for every $A \in \mathscr{A}$ and $j > n$, and there exists an n-th cosyzygy of B lying in $i(\mathscr{A})^{\perp}$.*

    (d) $\mathrm{Ext}^j_{\mathscr{B}}(i(P),B) = 0$ *for every $P \in \mathrm{Proj}\,\mathscr{A}$ and $j > n$, and there exists an n-th cosyzygy of B lying in $i(\mathrm{Proj}\,\mathscr{A})^{\perp}$.*

We have the following relations between these statements:

    (i) (b) $\Longleftrightarrow$ (c) $\Longleftrightarrow$ (d).

    (ii) *If $\mathsf{id}_{\mathscr{C}}\,e(I) \le m$ for every injective object I in $\mathscr{B}$, then* (b) $\Longrightarrow$ (a).

In the above results, we fixed an object B of the category $\mathscr{B}$, and considered the maps $e^j_{B,B'}$ or $e^j_{B',B}$ for all objects B' in $\mathscr{B}$. With certain conditions on the object B, we found that these maps are isomorphisms for almost all degrees j. We now describe some conditions on the recollement which are sufficient to ensure that the maps $e^j_{B,B'}$ are isomorphisms in almost all degrees j for all objects B and B' of $\mathscr{B}$, in other words, that the functor e is an eventually homological isomorphism. These conditions are given in the following corollary, which follows directly from Theorem 3.4 and Theorem 3.5.

Corollary 3.6. *Let $(\mathscr{A},\mathscr{B},\mathscr{C})$ be a recollement and assume that $\mathscr{B}$ and $\mathscr{C}$ have enough projective and injective objects. Let m and n be two integers. Assume that one of the following conditions holds:*

    (i) (α') $\sup\{\mathsf{id}_{\mathscr{B}}\,i(I) \mid I \in \mathrm{Inj}\,\mathscr{A}\} < m$.

        (ϵ) *Every object of $\mathscr{B}$ has an m-th syzygy which lies in $^{\perp}i(\mathrm{Inj}\,\mathscr{A})$.*

        (β) $\sup\{\mathsf{pd}_{\mathscr{C}}\,e(P) \mid P \in \mathrm{Proj}\,\mathscr{B}\} \le n$.

    (ii) (γ') $\sup\{\mathsf{pd}_{\mathscr{B}}\,i(P) \mid P \in \mathrm{Proj}\,\mathscr{A}\} < m$.

        (ϵ^{op}) *Every object of $\mathscr{B}$ has an m-th cosyzygy which lies in $i(\mathrm{Proj}\,\mathscr{A})^{\perp}$.*

        (δ) $\sup\{\mathsf{id}_{\mathscr{C}}\,e(I) \mid I \in \mathrm{Inj}\,\mathscr{B}\} \le n$.

Then the functor e is an $(m + n)$-homological isomorphism, and in particular the map

$$e^j_{B,B'} \; : \; \operatorname{Ext}^j_{\mathscr{B}}(B, B') \xrightarrow{\;\cong\;} \operatorname{Ext}^j_{\mathscr{C}}(e(B), e(B'))$$

is an isomorphism for all objects B and B' of $\mathscr{B}$ and for every $j > m + n$.

We now show a partial converse of the above result.

Proposition 3.7. *Let $(\mathscr{A}, \mathscr{B}, \mathscr{C})$ be a recollement and assume that $\mathscr{B}$ and $\mathscr{C}$ have enough projective and injective objects. Assume that the functor e is an eventually homological isomorphism. Then the following hold:*

    (α) $\sup\{\operatorname{id}_{\mathscr{B}} i(A) \mid A \in \mathscr{A}\} < \infty.$
    (β) $\sup\{\operatorname{pd}_{\mathscr{C}} e(P) \mid P \in \operatorname{Proj} \mathscr{B}\} < \infty.$
    (γ) $\sup\{\operatorname{pd}_{\mathscr{B}} i(A) \mid A \in \mathscr{A}\} < \infty.$
    (δ) $\sup\{\operatorname{id}_{\mathscr{C}} e(I) \mid I \in \operatorname{Inj} \mathscr{B}\} < \infty.$

In particular, if e is a t-homological isomorphism for a nonnegative integer t, then each of the above dimensions is bounded by t.

Proof. (α) Let A be an object of $\mathscr{A}$. For every B in $\mathscr{B}$ and $j > t$, we get

$$\operatorname{Ext}^j_{\mathscr{B}}(B, i(A)) \cong \operatorname{Ext}^j_{\mathscr{C}}(e(B), ei(A)) \cong \operatorname{Ext}^j_{\mathscr{C}}(e(B), 0) = 0,$$

since $ei = 0$ by Proposition 2.2, and thus $\operatorname{id}_{\mathscr{B}} i(A) \leq t$. The proof of (γ) is similar.
    (β) Let P be a projective object of $\mathscr{B}$. For every C in $\mathscr{C}$ and $j > t$, we get

$$\operatorname{Ext}^j_{\mathscr{C}}(e(P), C) \cong \operatorname{Ext}^j_{\mathscr{C}}(e(P), el(C)) \cong \operatorname{Ext}^j_{\mathscr{B}}(P, l(C)) = 0,$$

since $el \cong \operatorname{Id}_{\mathscr{C}}$ by Proposition 2.2, and thus $\operatorname{pd}_{\mathscr{C}} e(P) \leq t$. The proof of (δ) is similar. $\qquad\square$

Remark 3.8. Recall from [54] that $\sup\{\operatorname{pd}_{\mathscr{B}} i(A) \mid A \in \mathscr{A}\} < \infty$, which appears in statement (γ) above, is called the $\mathscr{A}$-relative global dimension of $\mathscr{B}$, and denoted by $\operatorname{gl.dim}_{\mathscr{A}} \mathscr{B}$.

We close this section by interpreting Theorem 3.4, Theorem 3.5 and Corollary 3.6 for artin algebras. To this end, for an artin algebra Λ and $a \in \Lambda$ an idempotent element, we denote by

$$e = (a\Lambda \otimes_\Lambda -)\colon \operatorname{mod} \Lambda \longrightarrow \operatorname{mod} a\Lambda a$$

the quotient functor of the recollement $(\operatorname{mod} \Lambda/\langle a \rangle, \operatorname{mod} \Lambda, \operatorname{mod} a\Lambda a)$; see Example 2.3.
We first need the following well known observation.

Lemma 3.9. *Let Λ be an artin algebra, let M be a Λ-module and let S be a simple Λ-module. Then for every $n \geq 1$ we have:*

$$\operatorname{Ext}^n_\Lambda(M, S) \cong \operatorname{Hom}_\Lambda(\Omega^n(M), S) \qquad and \qquad \operatorname{Ext}^n_\Lambda(S, M) \cong \operatorname{Hom}_\Lambda(S, \Sigma^n(M)),$$

where $\Omega^n(M)$ is the n-th syzygy in a minimal projective resolution of M, and $\Sigma^n(M)$ is the n-th cosyzygy in a minimal injective coresolution of M.

We also need the next easy result whose proof is left to the reader.

Lemma 3.10. *Let Λ be an artin algebra and let a be an idempotent element of Λ. Then the following inequalities hold*:

(i) $\mathsf{pd}_{a\Lambda a}\, e(P) \leq \mathsf{pd}_{a\Lambda a}\, a\Lambda$, *for every $P \in \mathrm{proj}\,\Lambda$.*
(ii) $\mathsf{id}_{a\Lambda a}\, e(I) \leq \mathsf{pd}_{(a\Lambda a)^{\mathrm{op}}} \Lambda a$, *for every $I \in \mathrm{inj}\,\Lambda$.*

The following is a consequence of Theorem 3.4 and Theorem 3.5 for artin algebras.

Corollary 3.11. *Let Λ be an artin algebra, let a be an idempotent element in Λ and let m and n be integers.*

(i) *Let M be a Λ-module such that $\mathrm{Ext}_\Lambda^j\big(M, (\Lambda/\langle a\rangle)/(\mathrm{rad}\,\Lambda/\langle a\rangle)\big) = 0$ for every $j \geq m$. Assume that $\mathsf{pd}_{a\Lambda a}\, a\Lambda \leq n$. Then the map*

$$e_{M,N}^j \;:\; \mathrm{Ext}_\Lambda^j(M, N) \;\xrightarrow{\;\cong\;}\; \mathrm{Ext}_{a\Lambda a}^j(e(M), e(N))$$

is an isomorphism for every Λ-module N, and for every integer $j > m+n$.

(ii) *Let M be a Λ-module such that $\mathrm{Ext}_\Lambda^j\big((\Lambda/\langle a\rangle)/(\mathrm{rad}\,\Lambda/\langle a\rangle), M\big) = 0$ for every $j \geq m$. Assume that $\mathsf{pd}_{(a\Lambda a)^{\mathrm{op}}} \Lambda a \leq n$. Then the map*

$$e_{N,M}^j \;:\; \mathrm{Ext}_\Lambda^j(N, M) \;\xrightarrow{\;\cong\;}\; \mathrm{Ext}_{a\Lambda a}^j(e(N), e(M))$$

is an isomorphism for every Λ-module N, and for every integer $j > m+n$.

Proof. (i) Consider the recollement $(\mathrm{mod}\,\Lambda/\langle a\rangle, \mathrm{mod}\,\Lambda, \mathrm{mod}\,a\Lambda a)$ of Example 2.3. Since every simple $\Lambda/\langle a\rangle$-module is also simple as a Λ-module it follows from Lemma 3.9 that

$$\mathrm{Hom}_\Lambda\big(\Omega^m(M), (\Lambda/\langle a\rangle)/(\mathrm{rad}\,\Lambda/\langle a\rangle)\big) = 0.$$

This implies that $\mathrm{Hom}_\Lambda(\Omega^m(M), N) = 0$ for every $\Lambda/\langle a\rangle$-module N since every module has a finite composition series. Then the result is a consequence of Theorem 3.4.

(ii) The result follows similarly as in (i), using Theorem 3.5 and the second isomorphism of Lemma 3.9. $\qquad\square$

As an immediate consequence of the above results we have the following characterization of when the functor $e\colon \mathrm{mod}\,\Lambda \longrightarrow \mathrm{mod}\,a\Lambda a$ is an eventually homological isomorphism. This constitutes the first part of the Main Theorem presented in the Introduction.

Corollary 3.12. *Let Λ be an artin algebra and let a be an idempotent element in Λ. The following are equivalent:*

(i) *There is an integer s such that for every pair of Λ-modules M and N, and every $j > s$, the map*

$$e_{M,N}^j \;:\; \mathrm{Ext}_\Lambda^j(M, N) \;\xrightarrow{\;\cong\;}\; \mathrm{Ext}_{a\Lambda a}^j(e(M), e(N))$$

is an isomorphism.

(ii) *The functor e is an eventually homological isomorphism.*

(iii) (α) $\mathsf{id}_\Lambda\big((\Lambda/\langle a\rangle)/(\mathrm{rad}\,\Lambda/\langle a\rangle)\big) < \infty$ *and* (β) $\mathsf{pd}_{a\Lambda a}\, a\Lambda < \infty$.

(iv) (γ) $\mathsf{pd}_\Lambda\big((\Lambda/\langle a\rangle)/(\mathrm{rad}\,\Lambda/\langle a\rangle)\big) < \infty$ *and* (δ) $\mathsf{pd}_{(a\Lambda a)^{\mathrm{op}}} \Lambda a < \infty$.

In particular, if the functor e is a t-homological isomorphism, then each of the dimensions in (iii) *and* (iv) *are at most t. The bound s in* (i) *is bounded by the sum of the dimensions occurring in* (iii)*, and also bounded by the sum of the dimensions occurring in* (iv)*.*

Proof. The implications (ii) $\implies$ (iii) and (ii) $\implies$ (iv) follow from Proposition 3.7. The implications (iii) $\implies$ (i) and (iv) $\implies$ (i) follow from Corollary 3.11. □

The above corollary shows that e being an eventually homological isomorphism is equivalent to the two conditions (α) and (β), and also equivalent to the two conditions (γ) and (δ). In Subsection 8.1, we give examples showing that no other pair of the four conditions (α)–(δ) implies that e is an eventually homological isomorphism.

4. Gorenstein categories and eventually homological isomorphisms

Our aim in this section is to study Gorenstein categories, introduced by Beligiannis–Reiten [9]. The main objective is to study when a functor $f\colon \mathscr{D} \longrightarrow \mathscr{F}$ between abelian categories preserves Gorensteinness. A central property here is whether the functor f is an eventually homological isomorphism. We prove that for an essentially surjective eventually homological isomorphism $f\colon \mathscr{D} \longrightarrow \mathscr{F}$, then $\mathscr{D}$ is Gorenstein if and only if $\mathscr{F}$ is. The results are applied to recollements of abelian categories, and recollements of module categories.

We start by briefly recalling the notion of Gorenstein categories introduced in [9]. Let $\mathscr{A}$ be an abelian category with enough projective and injective objects. We consider the following invariants associated to $\mathscr{A}$:

$$\operatorname{\mathsf{spli}} \mathscr{A} = \sup\{\operatorname{\mathsf{pd}}_{\mathscr{A}} I \mid I \in \operatorname{Inj} \mathscr{A}\} \quad \text{and} \quad \operatorname{\mathsf{silp}} \mathscr{A} = \sup\{\operatorname{\mathsf{id}}_{\mathscr{A}} P \mid P \in \operatorname{Proj} \mathscr{A}\}.$$

Then we have the following notion of Gorensteinness for abelian categories.

Definition 4.1 ([9])**.** An abelian category $\mathscr{A}$ with enough projective and injective objects is called **Gorenstein** if $\operatorname{\mathsf{spli}} \mathscr{A} < \infty$ and $\operatorname{\mathsf{silp}} \mathscr{A} < \infty$.

Note that the above notion is a common generalization of Gorensteinness in the commutative and in the noncommutative setting. We refer to [9, Chapter VII] for a thorough discussion on Gorenstein categories and connections with Cohen–Macaulay objects and cotorsion pairs.

We start with the following useful observation whose direct proof is left to the reader.

Lemma 4.2. *Let $\mathscr{A}$ be an abelian category with enough projective and injective objects and let X be an object of $\mathscr{A}$.*

    (i) *If $\operatorname{\mathsf{pd}}_{\mathscr{A}} X < \infty$, then $\operatorname{\mathsf{id}}_{\mathscr{A}} X \leq \operatorname{\mathsf{silp}} \mathscr{A}$.*
    (ii) *If $\operatorname{\mathsf{id}}_{\mathscr{A}} X < \infty$, then $\operatorname{\mathsf{pd}}_{\mathscr{A}} X \leq \operatorname{\mathsf{spli}} \mathscr{A}$.*

In the main result of this section we study eventually homological isomorphisms between abelian categories with enough projective and injective objects. In particular we show that an essentially surjective eventually homological isomorphism preserves Gorensteinness. This is a general version of the third part of the Main Theorem presented in the Introduction.

Theorem 4.3. *Let $f \colon \mathscr{D} \longrightarrow \mathscr{F}$ be a functor, where $\mathscr{D}$ and $\mathscr{F}$ are abelian categories with enough projective and injective objects, and let t be a nonnegative integer. Consider the following four statements*:

(a) *For every D in $\mathscr{D}$*:
$$\begin{cases} \mathsf{pd}_{\mathscr{D}} D \leq \sup\{\mathsf{pd}_{\mathscr{F}} f(D), t\} \\ \mathsf{id}_{\mathscr{D}} D \leq \sup\{\mathsf{id}_{\mathscr{F}} f(D), t\} \end{cases}$$

(c)
$$\begin{cases} \mathsf{spli}\,\mathscr{D} \leq \sup\{\mathsf{spli}\,\mathscr{F}, t\} \\ \mathsf{silp}\,\mathscr{D} \leq \sup\{\mathsf{silp}\,\mathscr{F}, t\} \end{cases}$$

(b) *For every D in $\mathscr{D}$*:
$$\begin{cases} \mathsf{pd}_{\mathscr{F}} f(D) \leq \sup\{\mathsf{pd}_{\mathscr{D}} D, t\} \\ \mathsf{id}_{\mathscr{F}} f(D) \leq \sup\{\mathsf{id}_{\mathscr{D}} D, t\} \end{cases}$$

(d)
$$\begin{cases} \mathsf{spli}\,\mathscr{F} \leq \sup\{\mathsf{spli}\,\mathscr{D}, t\} \\ \mathsf{silp}\,\mathscr{F} \leq \sup\{\mathsf{silp}\,\mathscr{D}, t\} \end{cases}$$

We have the following.

- (i) *If f is a t-homological isomorphism, then* (a) *holds.*
- (ii) *If f is an essentially surjective t-homological isomorphism, then* (a) *and* (b) *hold.*
- (iii) *If* (a) *and* (b) *hold, then* (c) *holds.*
- (iv) *If* (a) *and* (b) *hold and f is essentially surjective, then* (c) *and* (d) *hold.*

In particular, we obtain the following.

- (v) *If f is an essentially surjective eventually homological isomorphism, then $\mathscr{D}$ is Gorenstein if and only if $\mathscr{F}$ is Gorenstein.*
- (vi) *If f is an eventually homological isomorphism and* (b) *holds, then $\mathscr{F}$ being Gorenstein implies that $\mathscr{D}$ is Gorenstein.*

Proof. We first assume that f is an essentially surjective t-homological isomorphism and show the inequality $\mathsf{pd}_{\mathscr{F}} f(D) \leq \sup\{\mathsf{pd}_{\mathscr{D}} D, t\}$; the other inequalities in parts (i) and (ii) are proved similarly. The inequality clearly holds if D has infinite projective dimension. Assume that D has finite projective dimension, and let $n = \max\{\mathsf{pd}_{\mathscr{D}} D, t\} + 1$. For any object X in $\mathscr{F}$, there is an object X' in $\mathscr{D}$ with $f(X') \cong X$, since the functor f is essentially surjective. By using that f is a t-homological isomorphism, we get

$$\mathrm{Ext}_{\mathscr{F}}^{n}(f(D), X) \cong \mathrm{Ext}_{\mathscr{F}}^{n}(f(D), f(X')) \cong \mathrm{Ext}_{\mathscr{D}}^{n}(D, X') = 0.$$

This means that we have $\mathsf{pd}_{\mathscr{F}} f(D) < n$, and therefore $\mathsf{pd}_{\mathscr{F}} f(D) \leq \sup\{\mathsf{pd}_{\mathscr{D}} D, t\}$.

We now assume that (a) and (b) hold and f is essentially surjective, and show the inequality $\mathsf{spli}\,\mathscr{F} \leq \sup\{\mathsf{spli}\,\mathscr{D}, t\}$; the other inequalities in parts (iii) and (iv) are proved similarly. Let I be an injective object of $\mathscr{F}$. Since f is essentially surjective, we can choose an object D in $\mathscr{D}$ such that $f(D) \cong I$. By (a), the object D has finite injective dimension, and then by Lemma 4.2, its projective dimension is at most $\mathsf{spli}\,\mathscr{D}$. Using (b), we get

$$\mathsf{pd}_{\mathscr{F}} I \leq \sup\{\mathsf{pd}_{\mathscr{D}} D, t\} \leq \sup\{\mathsf{spli}\,\mathscr{D}, t\}.$$

Since this holds for any injective object I in $\mathscr{F}$, we have $\mathsf{spli}\,\mathscr{F} \leq \sup\{\mathsf{spli}\,\mathscr{D}, t\}$.

Parts (v) and (vi) follow by combining parts (i)–(iv). $\qquad\square$

Now we return to the setting of a recollement $(\mathscr{A}, \mathscr{B}, \mathscr{C})$. We use Theorem 4.3 to study the functors $i \colon \mathscr{A} \longrightarrow \mathscr{B}$ and $e \colon \mathscr{B} \longrightarrow \mathscr{C}$ with respect to Gorensteinness.

Corollary 4.4. *Let $(\mathscr{A}, \mathscr{B}, \mathscr{C})$ be a recollement of abelian categories.*

(i) *Assume that the categories $\mathscr{B}$ and $\mathscr{C}$ have enough projective and injective objects, and that the functor e is an eventually homological isomorphism. Then $\mathscr{B}$ is Gorenstein if and only if $\mathscr{C}$ is Gorenstein.*

(ii) *Assume that the category $\mathscr{B}$ has enough projective and injective objects, and that we have either*

$$\left.\begin{array}{c} \sup\{\mathsf{pd}_{\mathscr{B}} i(P) \mid P \in \operatorname{Proj}\mathscr{A}\} \leq 1 \\ \sup\{\mathsf{id}_{\mathscr{B}} i(I) \mid I \in \operatorname{Inj}\mathscr{A}\} < \infty \end{array}\right\} \ or \ \left\{\begin{array}{c} \sup\{\mathsf{pd}_{\mathscr{B}} i(P) \mid P \in \operatorname{Proj}\mathscr{A}\} < \infty \\ \sup\{\mathsf{id}_{\mathscr{B}} i(I) \mid I \in \operatorname{Inj}\mathscr{A}\} \leq 1 \end{array}\right.$$

If $\mathscr{B}$ is Gorenstein, then $\mathscr{A}$ is Gorenstein.

Proof. Part (i) follows directly from Theorem 4.3 (v), noting that e is essentially surjective by Proposition 2.2.

We now show part (ii). By Proposition 2.2 (iv) and (v), $\mathscr{A}$ has enough projective and injective objects since $\mathscr{B}$ does (see [54, Remark 2.5]).

It follows from [54, Proposition 4.15] (or its dual) that the functor $i\colon \mathscr{A} \longrightarrow \mathscr{B}$ is a homological embedding, i.e. the map $i^n_{X,Y}$ is an isomorphism for all objects X and Y in $\mathscr{A}$ and every $n \geq 0$. In particular, this means that i is a 0-homological isomorphism. By Theorem 4.3 (i), we have

$$(4.1) \qquad \mathsf{pd}_{\mathscr{A}} A \leq \mathsf{pd}_{\mathscr{B}} i(A) \qquad \text{and} \qquad \mathsf{id}_{\mathscr{A}} A \leq \mathsf{id}_{\mathscr{B}} i(A)$$

for every object A in $\mathscr{A}$.

We show that $\mathsf{spli}\,\mathscr{A} \leq \mathsf{spli}\,\mathscr{B}$. Let I be an injective object in A. By assumption, we have $\mathsf{id}_{\mathscr{B}} i(I) < \infty$, and then by the first inequality in (4.1) and Lemma 4.2, we have

$$\mathsf{pd}_{\mathscr{A}} I \leq \mathsf{pd}_{\mathscr{B}} i(I) \leq \mathsf{spli}\,\mathscr{B}.$$

Hence we have $\mathsf{spli}\,\mathscr{A} \leq \mathsf{spli}\,\mathscr{B}$. By a similar argument, we have $\mathsf{silp}\,\mathscr{A} \leq \mathsf{silp}\,\mathscr{B}$. The result follows. □

In a recollement $(\mathscr{A}, \mathscr{B}, \mathscr{C})$ we have seen that the implications (i) $\mathscr{B}$ Gorenstein if and only if $\mathscr{C}$ Gorenstein and (ii) $\mathscr{B}$ Gorenstein implies $\mathscr{A}$ Gorenstein hold under various additional assumptions. It is then natural to ask if the categories $\mathscr{A}$ and $\mathscr{C}$ being Gorenstein could imply that $\mathscr{B}$ is Gorenstein. The next example shows that this is not true in general.

Example 4.5. Let k be a field and consider the algebra $k[x]/\langle x^2 \rangle$. Then from the triangular matrix algebra

$$\Lambda = \begin{pmatrix} k & k \\ 0 & k[x]/\langle x^2 \rangle \end{pmatrix}$$

we have the recollement of module categories $(\operatorname{mod} k[x]/\langle x^2 \rangle, \operatorname{mod}\Lambda, \operatorname{mod} k)$, where $\operatorname{mod} k[x]/\langle x^2 \rangle$ and $\operatorname{mod} k$ are Gorenstein categories but $\operatorname{mod}\Lambda$ is not Gorenstein. We refer to [15, Example 4.3 (2)] for more details about the algebra Λ.

Recall from [9] that a ring R is called **left Gorenstein** if the category $\operatorname{Mod} R$ of left R-modules is a Gorenstein category. Applying Corollary 4.4 to the module recollement $(\operatorname{Mod} R/\langle e \rangle, \operatorname{Mod} R, \operatorname{Mod} eRe)$ from Example 2.3, we have the following result.

Corollary 4.6. *Let R be a ring and e an idempotent element of R.*

    (i) *If the functor $e-\colon \operatorname{Mod} R \longrightarrow \operatorname{Mod} eRe$ is an eventually homological isomorphism, then the ring R is left Gorenstein if and only if the ring eRe is left Gorenstein.*

    (ii) *Assume that we have either*

$$\left. \begin{array}{r} \mathsf{pd}_R R/\langle e\rangle \leq 1 \\ \sup\{\mathsf{id}_{\mathscr{B}}\, i(I) \mid I \in \operatorname{Inj} R/\langle e\rangle\} < \infty \end{array} \right\} \quad or \quad \left\{ \begin{array}{l} \mathsf{pd}_R R/\langle e\rangle < \infty \\ \sup\{\mathsf{id}_{\mathscr{B}}\, i(I) \mid I \in \operatorname{Inj} R/\langle e\rangle\} \leq 1 \end{array} \right.$$

    If the ring R is left Gorenstein, then the ring $R/\langle e\rangle$ is left Gorenstein.

Recall that an artin algebra Λ is called **Gorenstein** if $\mathsf{id}\,_{\Lambda}\Lambda < \infty$ and $\mathsf{id}\,\Lambda_{\Lambda} < \infty$ (see [5,6]). Note that $\operatorname{mod}\Lambda$ is a Gorenstein category if and only if Λ is a Gorenstein algebra. We close this section with the following consequence for artin algebras, whose first part constitutes the third part of the Main Theorem presented in the Introduction.

Corollary 4.7. *Let Λ be an artin algebra and a an idempotent element of Λ.*

    (i) *Assume that the functor $a-\colon \operatorname{mod}\Lambda \longrightarrow \operatorname{mod} a\Lambda a$ is an eventually homological isomorphism. Then the algebra Λ is Gorenstein if and only if the algebra $a\Lambda a$ is Gorenstein.*

    (ii) *Assume that we have either*

$$\left. \begin{array}{l} \mathsf{pd}_\Lambda \Lambda/\langle a\rangle \leq 1 \\ \mathsf{pd}_{\Lambda^{\mathrm{op}}} \Lambda/\langle a\rangle < \infty \end{array} \right\} \quad or \quad \left\{ \begin{array}{l} \mathsf{pd}_\Lambda \Lambda/\langle a\rangle < \infty \\ \mathsf{pd}_{\Lambda^{\mathrm{op}}} \Lambda/\langle a\rangle \leq 1 \end{array} \right.$$

If the algebra Λ is Gorenstein, then the algebra $\Lambda/\langle a\rangle$ is Gorenstein.

5. Singular equivalences in recollements

Our purpose in this section is to study singularity categories, in the sense of Buchweitz [12] and Orlov [50], in a recollement of abelian categories $(\mathscr{A}, \mathscr{B}, \mathscr{C})$. In particular we are interested in finding necessary and sufficient conditions such that the singularity categories of $\mathscr{B}$ and $\mathscr{C}$ are triangle equivalent. We start by recalling some well known facts about singularity categories.

Let $\mathscr{B}$ be an abelian category with enough projective objects. We denote by $\mathsf{D}^{\mathsf{b}}(\mathscr{B})$ the derived category of bounded complexes of objects of $\mathscr{B}$ and by $\mathsf{K}^{\mathsf{b}}(\operatorname{Proj}\mathscr{B})$ the homotopy category of bounded complexes of projective objects of $\mathscr{B}$. Then the **singularity category** of $\mathscr{B}$ ([12,50]) is defined to be the Verdier quotient:

$$\mathsf{D}_{\mathsf{sg}}(\mathscr{B}) = \mathsf{D}^{\mathsf{b}}(\mathscr{B})/\mathsf{K}^{\mathsf{b}}(\operatorname{Proj}\mathscr{B}).$$

See [18] for a discussion of more general quotients of $\mathsf{D}^{\mathsf{b}}(\mathscr{B})$ by $\mathsf{K}^{\mathsf{b}}(\mathfrak{X})$, where $\mathfrak{X}$ is a selforthogonal subcategory of $\mathscr{B}$.

It is well known that the singularity category $\mathsf{D}_{\mathsf{sg}}(\mathscr{B})$ carries a unique triangulated structure such that the quotient functor $Q_{\mathscr{B}}\colon \mathsf{D}^{\mathsf{b}}(\mathscr{B}) \longrightarrow \mathsf{D}_{\mathsf{sg}}(\mathscr{B})$ is triangulated; see [43,49,63]. Recall that the objects of the singularity category $\mathsf{D}_{\mathsf{sg}}(\mathscr{B})$ are the objects of the bounded derived category $\mathsf{D}^{\mathsf{b}}(\mathscr{B})$, the morphisms between two objects $X^{\bullet} \longrightarrow Y^{\bullet}$ are equivalence classes of fractions $(X^{\bullet} \leftarrow L^{\bullet} \to Y^{\bullet})$ such that the cone of the morphism $L^{\bullet} \longrightarrow X^{\bullet}$ belongs to $\mathsf{K}^{\mathsf{b}}(\operatorname{Proj}\mathscr{B})$ and the exact triangles in $\mathsf{D}_{\mathsf{sg}}(\mathscr{B})$ are all the triangles which are isomorphic to images of exact triangles of $\mathsf{D}^{\mathsf{b}}(\mathscr{B})$ via the quotient functor $Q_{\mathscr{B}}$. Note that a complex $X^{\bullet}$ is zero in $\mathsf{D}_{\mathsf{sg}}(\mathscr{B})$

if and only if $X^\bullet \in \mathsf{K}^\mathsf{b}(\operatorname{Proj}\mathscr{B})$. Following Chen [19, 20], we say that two abelian categories $\mathscr{A}$ and $\mathscr{B}$ are **singularly equivalent** if there is a triangle equivalence between the singularity categories $\mathsf{D}_\mathsf{sg}(\mathscr{A})$ and $\mathsf{D}_\mathsf{sg}(\mathscr{B})$. This triangle equivalence is called a **singular equivalence** between $\mathscr{A}$ and $\mathscr{B}$.

To proceed further we need the following well known result for exact triangles in derived categories. For a complex $X^\bullet$ in an abelian category $\mathscr{A}$ we denote by $\sigma_{>n}(X^\bullet)$ the truncation complex $\cdots \longrightarrow 0 \longrightarrow \operatorname{Im} d^n \longrightarrow X^{n+1} \overset{d^{n+1}}{\longrightarrow} X^{n+2} \longrightarrow \cdots$, and by $H^n(X^\bullet)$ the n-th homology of $X^\bullet$.

Lemma 5.1. *Let $\mathscr{A}$ be an abelian category and let $X^\bullet$ be a complex in $\mathscr{A}$. Then we have the following triangle in $\mathsf{D}(\mathscr{A})$:*

$$H^n(X^\bullet)[-n] \longrightarrow \sigma_{>n-1}(X^\bullet) \longrightarrow \sigma_{>n}(X^\bullet) \longrightarrow H^n(X^\bullet)[1-n].$$

Now we are ready to prove the main result of this section which gives necessary and sufficient conditions for the quotient functor $e\colon \mathscr{B} \longrightarrow \mathscr{C}$ to induce a triangle equivalence between the singularity categories of $\mathscr{B}$ and $\mathscr{C}$. This is a general version of the second part of the Main Theorem presented in the Introduction.

Theorem 5.2. *Let $(\mathscr{A}, \mathscr{B}, \mathscr{C})$ be a recollement of abelian categories. Then the following statements are equivalent*:

(i) *We have $\mathsf{pd}_\mathscr{B} i(A) < \infty$ and $\mathsf{pd}_\mathscr{C} e(P) < \infty$ for every $A \in \mathscr{A}$ and $P \in \operatorname{Proj}\mathscr{B}$.*

(ii) *The functor $e\colon \mathscr{B} \longrightarrow \mathscr{C}$ induces a singular equivalence between $\mathscr{B}$ and $\mathscr{C}$*:

$$\mathsf{D}_\mathsf{sg}(e)\colon \mathsf{D}_\mathsf{sg}(\mathscr{B}) \overset{\simeq}{\longrightarrow} \mathsf{D}_\mathsf{sg}(\mathscr{C}).$$

Proof. (i) $\implies$ (ii) First note that we have a well defined derived functor $\mathsf{D}^\mathsf{b}(e)\colon \mathsf{D}^\mathsf{b}(\mathscr{B}) \longrightarrow \mathsf{D}^\mathsf{b}(\mathscr{C})$ since the quotient functor $e\colon \mathscr{B} \longrightarrow \mathscr{C}$ is exact. Also the recollement situation $(\mathscr{A}, \mathscr{B}, \mathscr{C})$ implies that $0 \longrightarrow \mathscr{A} \longrightarrow \mathscr{B} \longrightarrow \mathscr{C} \longrightarrow 0$ is an exact sequence of abelian categories; see Proposition 2.2. Then it follows from [46, Theorem 3.2] (see also [40]) that $0 \longrightarrow \mathsf{D}^\mathsf{b}_\mathscr{A}(\mathscr{B}) \longrightarrow \mathsf{D}^\mathsf{b}(\mathscr{B}) \longrightarrow \mathsf{D}^\mathsf{b}(\mathscr{C}) \longrightarrow 0$ is an exact sequence of triangulated categories, where $\mathsf{D}^\mathsf{b}_\mathscr{A}(\mathscr{B})$ is the full sub-category of $\mathsf{D}^\mathsf{b}(\mathscr{B})$ consisting of complexes whose homology lie in $\mathscr{A}$. Hence $\mathsf{D}^\mathsf{b}(e)$ is a quotient functor, i.e. $\mathsf{D}^\mathsf{b}(\mathscr{B})/\mathsf{D}^\mathsf{b}_\mathscr{A}(\mathscr{B}) \simeq \mathsf{D}^\mathsf{b}(\mathscr{C})$. Next we claim that $\mathsf{D}^\mathsf{b}(e)(\mathsf{K}^\mathsf{b}(\operatorname{Proj}\mathscr{B})) \subseteq \mathsf{K}^\mathsf{b}(\operatorname{Proj}\mathscr{C})$. Let $P^\bullet \in \mathsf{K}^\mathsf{b}(\operatorname{Proj}\mathscr{B})$. Suppose first that $P^\bullet$ is concentrated in degree zero, so we deal with a projective object P of $\mathscr{B}$. Since the object $e(P)$ has finite projective dimension it follows that there is a quasi-isomorphism $Q^\bullet \longrightarrow e(P)[0]$, where $Q^\bullet \in \mathsf{K}^\mathsf{b}(\operatorname{Proj}\mathscr{C})$ is a projective resolution of $e(P)$. Then the object $e(P)[0]$ is isomorphic with $Q^\bullet$ in $\mathsf{D}^\mathsf{b}(\mathscr{C})$ and therefore $e(P) \in \mathsf{K}^\mathsf{b}(\operatorname{Proj}\mathscr{C})$. Now let $P^\bullet = (0 \longrightarrow P_0 \longrightarrow P_1 \longrightarrow 0) \in \mathsf{K}^\mathsf{b}(\operatorname{Proj}\mathscr{B})$. Then we have the triangle $P_0[0] \longrightarrow P_1[0] \longrightarrow P^\bullet \longrightarrow P_0[1]$ and if we apply the functor $\mathsf{D}^\mathsf{b}(e)$ we infer that $\mathsf{D}^\mathsf{b}(e)(P^\bullet) \in \mathsf{K}^\mathsf{b}(\operatorname{Proj}\mathscr{C})$ since $\mathsf{K}^\mathsf{b}(\operatorname{Proj}\mathscr{C})$ is a triangulated subcategory. Continuing inductively on the length of the complex $P^\bullet$ we infer that the object $\mathsf{D}^\mathsf{b}(e)(P^\bullet)$ lies in $\mathsf{K}^\mathsf{b}(\operatorname{Proj}\mathscr{C})$ and so our claim follows. Then since the triangulated functor $\mathsf{D}^\mathsf{b}(e) \circ Q_\mathscr{C}\colon \mathsf{D}^\mathsf{b}(\mathscr{B}) \longrightarrow \mathsf{D}_\mathsf{sg}(\mathscr{C})$ annihilates $\mathsf{K}^\mathsf{b}(\operatorname{Proj}\mathscr{B})$ it follows that $\mathsf{D}^\mathsf{b}(e) \circ Q_\mathscr{C}$ factors uniquely through $Q_\mathscr{B}$ via a triangulated functor

$\mathsf{D}_{\mathsf{sg}}(e)\colon \mathsf{D}_{\mathsf{sg}}(\mathscr{B}) \longrightarrow \mathsf{D}_{\mathsf{sg}}(\mathscr{C})$, that is the following diagram is commutative:

$$
\begin{array}{ccc}
\mathsf{D}^{\mathrm{b}}(\mathscr{B}) & \xrightarrow{\ Q_{\mathscr{B}}\ } & \mathsf{D}_{\mathsf{sg}}(\mathscr{B}) \\
\scriptstyle \mathsf{D}^{\mathrm{b}}(e) \downarrow & & \downarrow \scriptstyle \mathsf{D}_{\mathsf{sg}}(e) \\
\mathsf{D}^{\mathrm{b}}(\mathscr{C}) & \xrightarrow{\ Q_{\mathscr{C}}\ } & \mathsf{D}_{\mathsf{sg}}(\mathscr{C})
\end{array}
$$

Next we show that $\mathsf{D}^{\mathrm{b}}_{\mathscr{A}}(\mathscr{B}) \subseteq \mathsf{K}^{\mathrm{b}}(\operatorname{Proj}\mathscr{B})$ in $\mathsf{D}^{\mathrm{b}}(\mathscr{B})$. Since the projective dimension of $i(A)$ is finite for all A in $\mathscr{A}$, it follows that $i(\mathscr{A}) \subseteq \mathsf{K}^{\mathrm{b}}(\operatorname{Proj}\mathscr{B})$ in $\mathsf{D}^{\mathrm{b}}(\mathscr{B})$. Let $B^{\bullet}$ be an object of $\mathsf{D}^{\mathrm{b}}_{\mathscr{A}}(\mathscr{B})$. Assume first that $B^{\bullet}$ is concentrated in degree zero. Hence we deal with an object $B \in \mathscr{B}$ such that $B \cong i(A)$ for some $A \in \mathscr{A}$, and therefore our claim follows. Now consider a complex

$$
B^{\bullet}\colon\ 0 \longrightarrow B^{0} \xrightarrow{\ d^{0}\ } B^{1} \longrightarrow 0,
$$

where $H^{0}(B^{\bullet})$ and $H^{1}(B^{\bullet})$ lie in $\mathscr{A}$. From Lemma 5.1 we have the triangles

$$
H^{0}(B^{\bullet}) \longrightarrow \sigma_{>-1}(B^{\bullet}) \longrightarrow \sigma_{>0}(B^{\bullet}) \longrightarrow H^{0}(B^{\bullet})[1]
$$

and

$$
H^{1}(B^{\bullet})[-1] \longrightarrow \sigma_{>0}(B^{\bullet}) \longrightarrow \sigma_{>1}(B^{\bullet}) \longrightarrow H^{1}(B^{\bullet})
$$

in $\mathsf{D}^{\mathrm{b}}(\mathscr{B})$. Then from the second triangle it follows that $\sigma_{>0}(B^{\bullet}) \in \mathsf{K}^{\mathrm{b}}(\operatorname{Proj}\mathscr{B})$ and therefore from the first triangle we get that $\sigma_{>-1}(B^{\bullet}) = B^{\bullet} \in \mathsf{K}^{\mathrm{b}}(\operatorname{Proj}\mathscr{B})$. Continuing inductively on the length of the complex $B^{\bullet}$, we infer that $\mathsf{D}^{\mathrm{b}}_{\mathscr{A}}(\mathscr{B}) \subseteq \mathsf{K}^{\mathrm{b}}(\operatorname{Proj}\mathscr{B})$ in $\mathsf{D}^{\mathrm{b}}(\mathscr{B})$. Using this we can form the quotient $\mathsf{K}^{\mathrm{b}}(\operatorname{Proj}\mathscr{B})/\mathsf{D}^{\mathrm{b}}_{\mathscr{A}}(\mathscr{B})$, and then we have the following exact commutative diagram:

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \mathsf{K}^{\mathrm{b}}(\operatorname{Proj}\mathscr{B})/\mathsf{D}^{\mathrm{b}}_{\mathscr{A}}(\mathscr{B}) & \longrightarrow & \mathsf{D}^{\mathrm{b}}(\mathscr{B})/\mathsf{D}^{\mathrm{b}}_{\mathscr{A}}(\mathscr{B}) & \longrightarrow & \mathsf{D}_{\mathsf{sg}}(\mathscr{B}) & \longrightarrow & 0 \\
& & \downarrow & & \downarrow{\scriptstyle \simeq} & & \downarrow{\scriptstyle \mathsf{D}_{\mathsf{sg}}(e)} & & \\
0 & \longrightarrow & \mathsf{K}^{\mathrm{b}}(\operatorname{Proj}\mathscr{C}) & \longrightarrow & \mathsf{D}^{\mathrm{b}}(\mathscr{C}) & \longrightarrow & \mathsf{D}_{\mathsf{sg}}(\mathscr{C}) & \longrightarrow & 0
\end{array}
$$

We show that the functor $\mathsf{K}^{\mathrm{b}}(\operatorname{Proj}\mathscr{B})/\mathsf{D}^{\mathrm{b}}_{\mathscr{A}}(\mathscr{B}) \longrightarrow \mathsf{K}^{\mathrm{b}}(\operatorname{Proj}\mathscr{C})$ is an equivalence, where we denote it by $\mathsf{K}^{\mathrm{b}}(e)$. First from the above commutative diagram and since there is an equivalence $\mathsf{D}^{\mathrm{b}}(\mathscr{B})/\mathsf{D}^{\mathrm{b}}_{\mathscr{A}}(\mathscr{B}) \simeq \mathsf{D}^{\mathrm{b}}(\mathscr{C})$, it follows that the functor $\mathsf{K}^{\mathrm{b}}(e)$ is fully faithful. Let $P^{\bullet}\colon 0 \longrightarrow P_{n} \longrightarrow \cdots \longrightarrow P_{1} \longrightarrow P_{0} \longrightarrow 0$ be an object of $\mathsf{K}^{\mathrm{b}}(\operatorname{Proj}\mathscr{C})$. Each P_{i} is a projective object in $\mathscr{C}$ and from Proposition 2.2 we have $el(P_{i}) \cong P_{i}$ with $l(P_{i}) \in \operatorname{Proj}\mathscr{B}$. Then the complex $l(P^{\bullet})\colon 0 \longrightarrow l(P_{n}) \longrightarrow \cdots \longrightarrow l(P_{1}) \longrightarrow l(P_{0}) \longrightarrow 0$ is such that $\mathsf{K}^{\mathrm{b}}(e)(l(P^{\bullet})) = P^{\bullet}$. This implies that the functor $\mathsf{K}^{\mathrm{b}}(e)$ is essentially surjective. Hence the functor $\mathsf{K}^{\mathrm{b}}(e)$ is an equivalence.

In conclusion, from the above exact commutative diagram we infer that the singularity categories of $\mathscr{B}$ and $\mathscr{C}$ are triangle equivalent.

(ii) $\implies$ (i) Suppose that there is a triangle equivalence $\mathsf{D}_{\mathsf{sg}}(e)\colon \mathsf{D}_{\mathsf{sg}}(\mathscr{B}) \xrightarrow{\ \simeq\ } \mathsf{D}_{\mathsf{sg}}(\mathscr{C})$. Let P be a projective object of $\mathscr{B}$. Then $P[0] \in \mathsf{K}^{\mathrm{b}}(\operatorname{Proj}\mathscr{B})$ and $\mathsf{D}^{\mathrm{b}}(e)(P[0]) \in \mathsf{K}^{\mathrm{b}}(\operatorname{Proj}\mathscr{C})$. Thus the object $e(P)$ has finite projective dimension. Let $A \in \mathscr{A}$ and consider the object $i(A)$ of $\mathscr{B}$. Then from Proposition 2.2 we have $e(i(A)) = 0$. Since $\mathsf{D}_{\mathsf{sg}}(e)$ is an equivalence, the object $i(A)$ is zero in $\mathsf{D}_{\mathsf{sg}}(\mathscr{B})$, and therefore $i(A) \in \mathsf{K}^{\mathrm{b}}(\operatorname{Proj}\mathscr{B})$. We infer that $i(A)$ has finite projective dimension. $\square$

Remark 5.3. If the functor $e\colon \mathscr{B} \longrightarrow \mathscr{C}$ is an eventually homological isomorphism, then statement (i) in Theorem 5.2 is true by Proposition 3.7. Thus Theorem 5.2 in particular says that if the functor $e\colon \mathscr{B} \longrightarrow \mathscr{C}$ in a recollement $(\mathscr{A}, \mathscr{B}, \mathscr{C})$ is an eventually homological isomorphism, then it induces a singular equivalence between $\mathscr{B}$ and $\mathscr{C}$. Example 5.5 below shows that the converse does not hold.

Note that statement (i) in Theorem 5.2 only states that each object of the form $i(A)$ or $e(P)$ has finite projective dimension, and not that there exists a finite bound for the projective dimensions of all such objects. In other words, the supremums

$$\sup\{\mathsf{pd}_{\mathscr{B}}\, i(A) \mid A \in \mathscr{A}\} \qquad \text{and} \qquad \sup\{\mathsf{pd}_{\mathscr{C}}\, e(P) \mid P \in \operatorname{Proj}\mathscr{B}\}$$

(which are used in other parts of the paper) may be infinite even if statement (i) is true.

Applying Theorem 5.2 to the recollement of module categories $(\operatorname{mod} R/\langle e\rangle, \operatorname{mod} R, \operatorname{mod} eRe)$ (see Example 2.3), we have the following consequence due to Chen; see [15, Theorem 2.1] and [16, Corollary 3.3]. Note that our version is somewhat stronger; the difference is that Chen takes $\mathsf{pd}_{eRe}\, eR < \infty$ as an assumption instead of including it in one of the equivalent statements. This result constitutes the second part of the Main Theorem presented in the Introduction.

Corollary 5.4. *Let R be a left Noetherian ring and e an idempotent element of R. Then the following statements are equivalent*:

(i) *For every $R/\langle e\rangle$-module X we have $\mathsf{pd}_R X < \infty$, and $\mathsf{pd}_{eRe}\, eR < \infty$.*
(ii) *The functor $e(-)\colon \operatorname{mod} R \longrightarrow \operatorname{mod} eRe$ induces a singular equivalence between $\operatorname{mod} R$ and $\operatorname{mod} eRe$:*

$$\mathsf{D}_{\mathsf{sg}}(e(-))\colon \mathsf{D}_{\mathsf{sg}}(\operatorname{mod} R) \xrightarrow{\;\simeq\;} \mathsf{D}_{\mathsf{sg}}(\operatorname{mod} eRe).$$

Now we give an example where the functor $e = a(-)\colon \operatorname{mod} \Lambda \to \operatorname{mod} a\Lambda a$ induces a singular equivalence, but is not an eventually homological isomorphism.

Example 5.5. We recycle the algebra from Example 4.5, that is, let Λ be the triangular matrix algebra

$$\begin{pmatrix} k & k \\ 0 & k[x]/\langle x^2\rangle \end{pmatrix},$$

where k is a field. We have the recollement of module categories $(\operatorname{mod} k, \operatorname{mod} \Lambda, \operatorname{mod} k[x]/\langle x^2\rangle)$, with $a = \left(\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ as the chosen idempotent. Then $(\Lambda/\langle a\rangle)/(\operatorname{rad}\Lambda/\langle a\rangle) \cong \Lambda/\langle a\rangle \cong \Lambda\left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)$ as Λ-modules, which implies that all $\Lambda/\langle a\rangle$-modules are projective as Λ-modules. Furthermore, $a\Lambda \cong a\Lambda a$ as $a\Lambda a$-modules. Then it follows from Corollary 5.4 that the functor $e = a(-)$ induces a singular equivalence between $\operatorname{mod} \Lambda$ and $\operatorname{mod} a\Lambda a$. However, condition (α) in Corollary 3.12 is not satisfied, hence e is not an eventually homological isomorphism.

The algebra $k[x]/\langle x^2\rangle$ is a Gorenstein algebra, but the algebra Λ is not Gorenstein. This example therefore shows that singular equivalences do not in general preserve Gorensteinness.

We end this section with an application to stable categories of Cohen–Macaulay modules. Let Λ be a Gorenstein artin algebra. We denote by $\operatorname{CM}(\Lambda)$ the category of (maximal) **Cohen–Macaulay modules** defined as follows:

$$\operatorname{CM}(\Lambda) = \{X \in \operatorname{mod} \Lambda \mid \operatorname{Ext}_\Lambda^n(X, \Lambda) = 0 \text{ for all } n \geq 1\}.$$

Then it is known that the stable category $\underline{\mathrm{CM}}(\Lambda)$ modulo projectives is a triangulated category (see [38]), and moreover there is a triangle equivalence between the singularity category $\mathsf{D}_{\mathsf{sg}}(\mathrm{mod}\,\Lambda)$ and the stable category $\underline{\mathrm{CM}}(\Lambda)$; see [12, Theorem 4.4.1] and [37, Theorem 4.6]. As a consequence of Corollary 3.12, Corollary 4.7 and Corollary 5.4 we get the following.

Corollary 5.6. *Let Λ be a Gorenstein artin algebra and a an idempotent element of Λ. Assume that the functor $a-\colon \mathrm{mod}\,\Lambda \longrightarrow \mathrm{mod}\,a\Lambda a$ is an eventually homological isomorphism. Then there is a triangle equivalence between the stable categories of Cohen–Macaulay modules of Λ and $a\Lambda a$:*

$$\underline{\mathrm{CM}}(\Lambda) \xrightarrow{\ \simeq\ } \underline{\mathrm{CM}}(a\Lambda a).$$

6. Finite generation of cohomology rings

In this section, we describe a way to compare the Fg condition (see Definition 2.6) for two different algebras. This is used in the next section for the algebras Λ and $a\Lambda a$, where Λ is a finite dimensional algebra over a field and a is an idempotent in Λ.

Let Λ and Γ be two artin algebras over a commutative ring k, and assume that they are flat as k-modules. Let $M = \Lambda/(\mathrm{rad}\,\Lambda)$ and $N = \Gamma/(\mathrm{rad}\,\Gamma)$. Assume that we have graded ring isomorphisms f and g making the diagram

(6.1)
$$\begin{array}{ccc} \mathrm{HH}^*(\Lambda) & \xrightarrow{\ \varphi_M\ } & \mathrm{Ext}^*_\Lambda(M, M) \\ {\scriptstyle f}\downarrow{\scriptstyle \cong} & & {\scriptstyle g}\downarrow{\scriptstyle \cong} \\ \mathrm{HH}^*(\Gamma) & \xrightarrow[\ \varphi_N\ ]{} & \mathrm{Ext}^*_\Gamma(N, N) \end{array}$$

commute, where the maps φ_M and φ_N are defined in Subsection 2.2. Then it is clear that Fg for Λ is exactly the same as Fg for Γ, since all the relevant data for the Fg condition is exactly the same for the two algebras.

However, we can come to the same conclusion even if the homology groups for Λ and Γ are different in some degrees, as long as they are the same in all but finitely many degrees. In other words, if the maps f and g above are just graded ring homomorphisms such that f_n and g_n are group isomorphisms for almost all degrees n, then the Fg condition holds for Λ if and only if it holds for Γ. The goal of this section is to show this.

We first prove the result in a more general setting, where we replace the rings in (6.1) by arbitrary graded rings satisfying appropriate assumptions. This is done in Proposition 6.3, after we have shown a part of the result (corresponding to part (i) of the Fg condition) in Proposition 6.2. Finally, we state the result for Fg in Proposition 6.4.

We now introduce some terminology and notation which is used in this section and the next. By **graded ring** we always mean a ring of the form

$$R = \bigoplus_{i=0}^{\infty} R_i$$

graded over the nonnegative integers. We denote the set of nonnegative integers by $\mathbb{N}_0$. If R is a graded ring and n a nonnegative integer, we use the notation $R_{\geq n}$ for

the graded ideal

$$R_{\geq n} = \bigoplus_{i=n}^{\infty} R_i$$

in R. We use the term **rng** for a "ring without identity", that is, an object which satisfies all the axioms for a ring except having a multiplicative identity element.

We use the following characterization of noetherianness for graded rings.

Theorem 6.1. *Let R be a graded ring. Then R is noetherian if and only if it satisfies the ascending chain condition on graded ideals.*

Proof. This follows directly from [48, Theorem 5.4.7]. $\square$

We now begin the main work of this section by showing that an isomorphism in all but finitely many degrees between two sufficiently nice graded rings preserves noetherianness. This implies that such a map between Hochschild cohomology rings preserves part (i) of the Fg condition, and thus gives one half of the result we want.

Proposition 6.2. *Let R and S be graded rings. Assume that R_0 and S_0 are noetherian, that every R_i is finitely generated as left and as right R_0-modules, and that every S_i is finitely generated as left and as right S_0-modules. Let n be a nonnegative integer, and assume that there exists an isomorphism $\phi\colon R_{\geq n} \longrightarrow S_{\geq n}$ of graded rngs. Then R is noetherian if and only if S is noetherian.*

Proof. We prove (by showing the contrapositive) that R is left noetherian if S is left noetherian. The corresponding result with right noetherian is proved in the same way. This gives one of the implications we need. The opposite implication is proved in the same way by interchanging R and S and using ϕ^{-1} instead of ϕ.

Assume that R is not left noetherian. Let

$$\mathbb{I}:\quad I^{(0)} \subset I^{(1)} \subset \cdots$$

be an infinite strictly ascending sequence of graded left ideals in R (this is possible by Theorem 6.1). For every index i in this sequence, we can write the ideal $I^{(i)}$ as a direct sum

$$I^{(i)} = \bigoplus_{d \in \mathbb{N}_0} I_d^{(i)}$$

of abelian groups, where $I_d^{(i)} \subseteq R_d$ is the degree d part of $I^{(i)}$. For any degree d, we can make an ascending sequence

$$I_d^{(0)} \subseteq I_d^{(1)} \subseteq \cdots$$

of R_0-submodules of R_d by taking the degree d part of each ideal in $\mathbb{I}$. But R_d is a noetherian R_0-module (since R_0 is noetherian and R_d is a finitely generated R_0-module), and hence this sequence must stabilize at some point. Let $s(d)$ be the point where it stabilizes, that is, the smallest integer such that $I_d^{(s(d))} = I_d^{(i)}$ for every $i > s(d)$.

We now define two functions $\sigma\colon \mathbb{N}_0 \longrightarrow \mathbb{N}_0$ and $\delta\colon \mathbb{N}_0 \longrightarrow \mathbb{N}_0$. For $d \in \mathbb{N}_0$, we define

$$\sigma(d) = \max\{s(0), s(1), \ldots, s(d)\}.$$

For $i \in \mathbb{N}_0$, we define $\delta(i)$ as the smallest number such that

$$I_{\delta(i)}^{(i)} \neq I_{\delta(i)}^{(i+1)}.$$

These functions have the following interpretation. For a degree d, the number $\sigma(d)$ is the index in the sequence $\mathbb{I}$ where the ideals in the sequence have stabilized up to degree d. For an index i, the number $\delta(i)$ is the lowest degree at which there is a difference from the ideal $I^{(i)}$ to the ideal $I^{(i+1)}$.

We now define a sequence $(i_j)_{j \in \mathbb{N}_0}$ of indices and a sequence $(d_j)_{j \in \mathbb{N}_0}$ of degrees by

$$i_j = \begin{cases} \sigma(n) & \text{if } j = 0, \\ \sigma(d_{j-1} + n) & \text{otherwise,} \end{cases}$$
$$d_j = \delta(i_j).$$

We observe that for every positive integer j, we have

$$i_j > i_{j-1} \quad \text{and} \quad d_j > d_{j-1} + n.$$

We now construct a sequence $\mathbb{J}$ of graded left ideals in S. For every nonnegative integer j, we choose an element

$$x_j \in I_{d_j}^{(i_j+1)} - I_{d_j}^{(i_j)}$$

(this is possible because $d_j = \delta(i_j)$). Note that the degree of x_j is d_j, which is greater than n. We then define $J^{(j)}$ to be the left ideal of S generated by the set

$$\{\phi(x_0), \dots, \phi(x_j)\}.$$

We let $\mathbb{J}$ be the sequence of these ideals:

$$\mathbb{J}: \quad J^{(0)} \subseteq J^{(1)} \subseteq \cdots.$$

We want to show that each inclusion here is strict. This means that we must show, for every positive integer j, that $\phi(x_j)$ is not an element of $J^{(j-1)}$.

We show this by contradiction. Assume that there is a j such that $\phi(x_j) \in J^{(j-1)}$. Then we can write $\phi(x_j)$ as a sum

$$\phi(x_j) = \sum_{m=0}^{j-1} s_m \cdot \phi(x_m),$$

where each s_m is an element of S. Since $\phi(x_j)$ and every $\phi(x_m)$ are homogeneous elements, we can choose every s_m to be homogeneous. For each m, we have that if s_m is nonzero, then its degree is

$$|s_m| = |\phi(x_j)| - |\phi(x_m)| = |x_j| - |x_m| = d_j - d_m > n.$$

Thus s_m is either zero or in the image of ϕ. We use this to find corresponding elements in R. Let, for each $m \in \{1, \dots, j-1\}$,

$$r_m = \begin{cases} 0 & \text{if } s_m = 0, \\ \phi^{-1}(s_m) & \text{otherwise.} \end{cases}$$

Now we have

$$\phi(x_j) = \sum_{m=0}^{j-1} s_m \cdot \phi(x_m) = \phi\left(\sum_{m=0}^{j-1} r_m \cdot x_m \right).$$

Applying ϕ^{-1} gives

$$x_j = \sum_{m=0}^{j-1} r_m \cdot x_m.$$

Since we have $x_m \in I^{(i_m+1)} \subseteq I^{(i_j)}$ for every m, this means that $x_j \in I^{(i_j)}$. This is a contradiction, since x_j is chosen so that it does not lie in $I^{(i_j)}$.

We have shown that the sequence $\mathbb{J}$ is a strictly ascending sequence of graded left ideals in S. Thus S is not left noetherian. $\square$

We now complete the picture by considering two graded rings and a graded module over each ring, and showing that isomorphisms in all but finitely many degrees preserve both noetherianness of the rings and finite generation of the modules (given that certain assumptions are satisfied).

Proposition 6.3. *Let R and M be graded rings, and $\theta \colon R \longrightarrow M$ a graded ring homomorphism. View M as a graded left R-module with scalar multiplication given by θ. Assume that R_0 is noetherian, that every R_i is finitely generated as left and as right R_0-modules, and that every M_i is finitely generated as a left R_0-module.*

Similarly, let R' and M' be graded rings, and $\theta' \colon R' \longrightarrow M'$ a graded ring homomorphism. View M' as a graded left R'-module with scalar multiplication given by θ'. Assume that R'_0 is noetherian, that every R'_i is finitely generated as left and as right R'_0-modules, and that every M'_i is finitely generated as a left R'_0-module.

Assume that there are graded rng isomorphisms $\phi \colon R_{\geq n} \longrightarrow R'_{\geq n}$ and $\psi \colon M_{\geq n} \longrightarrow M'_{\geq n}$ (for some nonnegative integer n) such that the diagram

$$
\begin{array}{ccc}
R_{\geq n} & \xrightarrow{\;\theta_{\geq n}\;} & M_{\geq n} \\
\phi \downarrow & & \downarrow \psi \\
R'_{\geq n} & \xrightarrow{\;\theta'_{\geq n}\;} & M'_{\geq n}
\end{array}
$$

commutes. Then the following two conditions are equivalent.

  (i) *R is noetherian and M is finitely generated as a left R-module.*
  (ii) *R' is noetherian and M' is finitely generated as a left R'-module.*

Proof. We prove that condition (i) implies condition (ii). The opposite implication is proved in exactly the same way by using ϕ^{-1} and ψ^{-1} instead of ϕ and ψ.

Assume that condition (i) holds. Then by Proposition 6.2, R' is noetherian. We need to show that M' is finitely generated as a left R'-module.

We begin with choosing generating sets for things we know to be finitely generated. Note that the ideal $R_{\geq n}$ of R is finitely generated, since R is noetherian. Let A be a finite homogeneous generating set for $R_{\geq n}$. Let G be a finite homogeneous generating set for M as a left R-module. For every i, let B_i be a finite generating set for M'_i as a left R'_0-module.

Let
$$
b_R = \max\big\{\, |a| \,\big|\, a \in A \big\} \qquad \text{and} \qquad b_M = \max\big\{\, |g| \,\big|\, g \in G \big\}
$$
be the maximal degrees of elements in our chosen generating sets for R and M, respectively. Let
$$
b = b_R + b_M + n.
$$
Define the set G' to be
$$
G' = \bigcup_{i=0}^{b} B_i.
$$
We want to show that G' generates M' as a left R'-module.

Let N' be the R'-submodule of M' generated by G'. It is clear that N' contains every homogeneous element of M' with degree at most b. Let $m' \in M'$ be a homogeneous element with $|m'| > b$. Let $m = \psi^{-1}(m')$. We can write m as a sum

$$m = \sum_i \theta(r_i) \cdot g_i,$$

where every r_i is a homogeneous nonzero element of R and every g_i is an element of the generating set G for M. For every r_i, we have

$$|r_i| = |m| - |g_i| = |m'| - |g_i| > b - b_M = b_R + n.$$

Thus r_i lies in the ideal $R_{\geq n}$, so we can write it as a sum

$$r_i = \sum_j u_{i,j} \cdot a_{i,j},$$

where every $u_{i,j}$ is a homogeneous nonzero element of R, and every $a_{i,j}$ is an element of the generating set A for $R_{\geq n}$. For every $u_{i,j}$, we have

$$|u_{i,j}| = |r_i| - |a_{i,j}| > (b_R + n) - b_R = n.$$

Now we can write the element m as

$$m = \sum_{i,j} \theta(u_{i,j} \cdot a_{i,j}) \cdot g_i = \sum_{i,j} \theta(u_{i,j}) \cdot \theta(a_{i,j}) \cdot g_i = \sum_{i,j} \theta(u_{i,j}) \cdot (a_{i,j} \cdot g_i).$$

If we have $a_{i,j} \cdot g_i = 0$ for some terms in the sum, we ignore these terms. For every pair (i, j), we have

$$|\theta(u_{i,j})| = |u_{i,j}| > n \qquad \text{and} \qquad |a_{i,j} \cdot g_i| \geq |a_{i,j}| \geq n.$$

This means that when applying ψ to a term in the above sum for m, we have

$$\psi(\theta(u_{i,j}) \cdot (a_{i,j} \cdot g_i)) = \psi(\theta(u_{i,j})) \cdot \psi(a_{i,j} \cdot g_i).$$

Using this, we can write our element m' of M' in the following way:

$$m' = \psi(m) = \psi\Big(\sum_{i,j} \theta(u_{i,j}) \cdot (a_{i,j} \cdot g_i) \Big)$$

$$= \sum_{i,j} \psi(\theta(u_{i,j})) \cdot \psi(a_{i,j} \cdot g_i)$$

$$= \sum_{i,j} \theta'(\phi(u_{i,j})) \cdot \psi(a_{i,j} \cdot g_i).$$

For every pair (i, j), we have

$$|\psi(a_{i,j} \cdot g_i)| = |a_{i,j} \cdot g_i| = |a_{i,j}| + |g_i| \leq b_R + b_M \leq b,$$

so $\psi(a_{i,j} \cdot g_i)$ lies in the module N' generated by G'. Thus m' also lies in N'. Since every homogeneous element of M' lies in N', we have $M' = N'$, and hence M' is finitely generated. $\qquad \square$

Finally, we apply the above result to the rings which are involved in the Fg condition, and obtain the main result of this section.

Proposition 6.4. *Let Λ and Γ be artin algebras over a commutative ring k, and assume that they are flat as k-modules. Let M and M' be Λ-modules, and let N and N' be Γ-modules, such that $M \cong \Lambda/(\operatorname{rad}\Lambda)$ and $N' \cong \Gamma/(\operatorname{rad}\Gamma)$. Let n be some nonnegative integer, and assume that there are graded rng isomorphisms f, g, f' and g' making the following two diagrams commute:*

$$
\begin{array}{ccc}
\operatorname{HH}^{\geq n}(\Lambda) & \xrightarrow{\varphi_M^{\geq n}} & \operatorname{Ext}_{\Lambda}^{\geq n}(M, M) \\
f \downarrow \cong & & g \downarrow \cong \\
\operatorname{HH}^{\geq n}(\Gamma) & \xrightarrow[\varphi_N^{\geq n}]{} & \operatorname{Ext}_{\Gamma}^{\geq n}(N, N)
\end{array}
\quad and \quad
\begin{array}{ccc}
\operatorname{HH}^{\geq n}(\Lambda) & \xrightarrow{\varphi_{M'}^{\geq n}} & \operatorname{Ext}_{\Lambda}^{\geq n}(M', M') \\
f' \downarrow \cong & & g' \downarrow \cong \\
\operatorname{HH}^{\geq n}(\Gamma) & \xrightarrow[\varphi_{N'}^{\geq n}]{} & \operatorname{Ext}_{\Gamma}^{\geq n}(N', N')
\end{array}
$$

Then Λ satisfies Fg if and only if Γ satisfies Fg.

Proof. We first check that the conditions on the graded rings in Proposition 6.3 are satisfied in this case. For every degree i, we have that $\operatorname{HH}^i(\Lambda)$, $\operatorname{Ext}_{\Lambda}^i(M, M)$ and $\operatorname{Ext}_{\Lambda}^i(M', M')$ are finitely generated as k-modules. Therefore, they are also finitely generated as $\operatorname{HH}^0(\Lambda)$-modules. The ring $\operatorname{HH}^0(\Lambda)$ is noetherian since it is an artin algebra. Similarly, we see that $\operatorname{HH}^i(\Gamma)$, $\operatorname{Ext}_{\Gamma}^i(N, N)$ and $\operatorname{Ext}_{\Gamma}^i(N', N')$ are finitely generated $\operatorname{HH}^0(\Gamma)$-modules, and that the ring $\operatorname{HH}^0(\Gamma)$ is noetherian.

Assume that Λ satisfies Fg. Then $\operatorname{HH}^*(\Lambda)$ is noetherian, and by Theorem 2.7, $\operatorname{Ext}_{\Lambda}^*(M', M')$ is a finitely generated $\operatorname{HH}^*(\Lambda)$-module. By applying Proposition 6.3 to the commutative diagram with f' and g', we see that Γ satisfies Fg.

The opposite inclusion is proved in the same way by using the other commutative diagram. $\square$

7. Finite generation of cohomology rings in module recollements

We now investigate the relationship between the Fg condition (see Definition 2.6) for an algebra Λ and the algebra $a\Lambda a$, where a is an idempotent of Λ. We show that, given some conditions on the idempotent a, the algebra Λ satisfies Fg if and only if the algebra $a\Lambda a$ satisfies Fg. We prove this result only for finite dimensional algebras over a field, and not more general artin algebras.

Throughout this section, we let k be a field, Λ a finite dimensional k-algebra and a an idempotent in Λ. We denote by e and E the exact functors

$$
e = (a-) \colon \operatorname{mod}\Lambda \longrightarrow \operatorname{mod} a\Lambda a,
$$
$$
E = (a - a) \colon \operatorname{mod}\Lambda^{\mathrm{e}} \longrightarrow \operatorname{mod}(a\Lambda a)^{\mathrm{e}}.
$$

These functors fit into the recollements described in Example 2.3.

For a Λ-module M, we can construct the diagram

$$
\begin{array}{ccc}
\operatorname{HH}^*(\Lambda) & \xrightarrow{\varphi_M} & \operatorname{Ext}_{\Lambda}^*(M, M) \\
E_{\Lambda,\Lambda}^* \downarrow & & \downarrow e_{M,M}^* \\
\operatorname{HH}^*(a\Lambda a) & \xrightarrow[\varphi_{e(M)}]{} & \operatorname{Ext}_{a\Lambda a}^*(e(M), e(M))
\end{array}
$$

where the maps φ_M and $\varphi_{e(M)}$ are defined in Subsection 2.2, and the maps $E_{\Lambda,\Lambda}^*$ and $e_{M,M}^*$ are defined in Section 3. We show that this diagram commutes, and that under certain conditions on a, the vertical maps are isomorphisms in almost

all degrees. We then use Proposition 6.4 to show that Λ satisfies Fg if and only if $a\Lambda a$ satisfies Fg.

Let us consider what kind of conditions we need to put on the choice of the idempotent a. From Corollary 3.12, we know that the map $e_{M,M}^*$ in the above diagram is an isomorphism in all but finitely many degrees if the two dimensions

$$\mathsf{id}_\Lambda \left(\frac{\Lambda/\langle a \rangle}{\operatorname{rad} \Lambda/\langle a \rangle} \right) \qquad \text{and} \qquad \mathsf{pd}_{a\Lambda a}(a\Lambda)$$

are finite, or, equivalently, that the two dimensions

$$\mathsf{pd}_\Lambda \left(\frac{\Lambda/\langle a \rangle}{\operatorname{rad} \Lambda/\langle a \rangle} \right) \qquad \text{and} \qquad \mathsf{pd}_{(a\Lambda a)^{\mathrm{op}}}(\Lambda a)$$

are finite. We show (given an additional technical assumption about the algebra Λ) that this is in fact also sufficient for the map $E_{\Lambda,\Lambda}^*$ to be an isomorphism in all but finitely many degrees.

This section is structured as follows. The first part considers the commutativity of the above diagram, concluding with Proposition 7.2. The second part considers when the map $E_{\Lambda,\Lambda}^*$ is an isomorphism in high degrees, concluding with Proposition 7.9. Finally, the main result of this section is stated as Theorem 7.10.

We now show that the above diagram is commutative. The maps φ_M and $\varphi_{e(M)}$ are defined by using tensor functors. It is convenient to have short names for these functors. For every Λ-module M, we define t_M and T_M to be the tensor functors

$$t_M = (- \otimes_\Lambda M) \colon \operatorname{mod} \Lambda^{\mathrm{e}} \longrightarrow \operatorname{mod} \Lambda,$$
$$T_M = (- \otimes_{a\Lambda a} aM) \colon \operatorname{mod}(a\Lambda a)^{\mathrm{e}} \longrightarrow \operatorname{mod} a\Lambda a.$$

Together with the functors e and E from above, these functors fit into the following diagram of categories and functors:

$$
\begin{array}{ccc}
\operatorname{mod} \Lambda^{\mathrm{e}} & \xrightarrow{\ t_M\ } & \operatorname{mod} \Lambda \\
{\scriptstyle E}\big\downarrow & & \big\downarrow{\scriptstyle e} \\
\operatorname{mod}(a\Lambda a)^{\mathrm{e}} & \xrightarrow[\ T_M\ ]{} & \operatorname{mod} a\Lambda a
\end{array}
$$

We begin by showing that the two possible compositions of maps from upper left to lower right in this diagram are related by a natural transformation.

Lemma 7.1. *For every Λ-module M, there is a natural transformation $\tau^M \colon T_M \circ E \longrightarrow e \circ t_M$.*

Proof. Note that we have

$$T_M E(N) = aNa \otimes_{a\Lambda a} aM \qquad \text{and} \qquad et_M(N) = aN \otimes_\Lambda M$$

for every Λ^{e}-module N. We define the maps τ_N^M of the natural transformation τ^M by

$$\tau_N^M(n \otimes m) = n \otimes m$$

for an element $n \otimes m$ of $T_M E(N)$. This gives well defined maps since $a\Lambda a \subseteq \Lambda$. It is easy to check that the compositions $et_M(f) \circ \tau_N^M$ and $\tau_{N'}^M \circ T_M E(f)$ are equal for a homomorphism $f \colon N \longrightarrow N'$ of Λ^{e}-modules, so τ^M is a natural transformation. $\square$

We are now able to show that the diagrams we consider are commutative.

Proposition 7.2. *For any Λ-module M, the following diagram of graded rings commutes*:

$$
\begin{array}{ccc}
\mathrm{HH}^*(\Lambda) & \xrightarrow{\ \varphi_M\ } & \mathrm{Ext}^*_\Lambda(M, M) \\
{\scriptstyle E^*_{\Lambda,\Lambda}} \downarrow & & \downarrow {\scriptstyle e^*_{M,M}} \\
\mathrm{HH}^*(a\Lambda a) & \xrightarrow[\varphi_{e(M)}]{} & \mathrm{Ext}^*_{a\Lambda a}(e(M), e(M))
\end{array}
$$

Proof. We show that the result holds in the positive degrees of the graded rings and graded ring homomorphisms in the diagram. Showing that it also holds in degree zero can be done in a similar way, by looking at elements given by homomorphisms instead of extensions.

Let μ and ν be the natural isomorphisms

$$
\mu \colon \Lambda \otimes_\Lambda M \longrightarrow M \qquad \text{and} \qquad \nu \colon a\Lambda a \otimes_{a\Lambda a} e(M) \longrightarrow e(M)
$$

given by multiplication.

Consider, for some positive integer i, an element $[\eta] \in \mathrm{Ext}^i_{\Lambda^e}(\Lambda, \Lambda)$ which is represented by the exact sequence

$$
\eta \colon \quad 0 \longrightarrow \Lambda \longrightarrow X \longrightarrow P_{i-2} \longrightarrow \cdots \longrightarrow P_0 \longrightarrow \Lambda \longrightarrow 0,
$$

where each P_j is a projective Λ^e-module. We apply the compositions of maps $\varphi_{e(M)} \circ E^*_{\Lambda,\Lambda}$ and $e^*_{M,M} \circ \varphi_M$ to $[\eta]$, and show that we get the same result in both cases.

We first consider the map $\varphi_{e(M)} \circ E^*_{\Lambda,\Lambda}$. If we apply the functor E to η, then we get the exact sequence

$$
E(\eta) \colon 0 \longrightarrow E(\Lambda) \longrightarrow E(X) \longrightarrow E(P_{i-2}) \longrightarrow \cdots \longrightarrow E(P_0) \longrightarrow E(\Lambda) \longrightarrow 0
$$

of $(a\Lambda a)^e$-modules, and we have that $E^*_{\Lambda,\Lambda}([\eta]) = [E(\eta)]$. Since the objects $E(P_j)$ are not necessarily projective, we may need to find a different representative of the element $[E(\eta)]$ in order to apply the map $\varphi_{e(M)}$. We construct the following commutative diagram with exact rows, where each Q_j is a projective $(a\Lambda a)^e$-module and the bottom row is $E(\eta)$:

$$
\begin{array}{ccccccccccc}
0 & \longrightarrow & a\Lambda a & \longrightarrow & Y & \longrightarrow & Q_{i-2} & \longrightarrow & \cdots & \longrightarrow & Q_0 & \longrightarrow & a\Lambda a & \longrightarrow & 0 \\
& & \| & & {\scriptstyle f_{i-1}}\downarrow & & {\scriptstyle f_{i-2}}\downarrow & & & & {\scriptstyle f_0}\downarrow & & \| & & \\
0 & \longrightarrow & E(\Lambda) & \longrightarrow & E(X) & \longrightarrow & E(P_{i-2}) & \longrightarrow & \cdots & \longrightarrow & E(P_0) & \longrightarrow & E(\Lambda) & \longrightarrow & 0
\end{array}
$$

Note that both rows represent the same element in $\mathrm{Ext}^i_{(a\Lambda a)^e}(a\Lambda a, a\Lambda a)$. Applying the functor T_M to this diagram gives the two lower rows in the following commutative diagram of $a\Lambda a$-modules, where the two upper rows are exact:

$$
\begin{array}{ccccccccccc}
0 \longrightarrow & e(M) & \longrightarrow & T_M(Y) & \longrightarrow & T_M(Q_{i-2}) & \longrightarrow \cdots \longrightarrow & T_M(Q_0) & \longrightarrow & e(M) & \longrightarrow 0 \\
& {\scriptstyle \nu^{-1}}\downarrow{\scriptstyle \cong} & & \| & & \| & & \| & & {\scriptstyle \nu^{-1}}\downarrow{\scriptstyle \cong} & \\
0 \longrightarrow & T_M(a\Lambda a) & \longrightarrow & T_M(Y) & \longrightarrow & T_M(Q_{i-2}) & \longrightarrow \cdots \longrightarrow & T_M(Q_0) & \longrightarrow & T_M(a\Lambda a) & \longrightarrow 0 \\
& \| & & {\scriptstyle T_M(f_{i-1})}\downarrow & & {\scriptstyle T_M(f_{i-2})}\downarrow & & {\scriptstyle T_M(f_0)}\downarrow & & \| & \\
& T_M E(\Lambda) & \longrightarrow & T_M E(X) & \longrightarrow & T_M E(P_{i-2}) & \longrightarrow \cdots \longrightarrow & T_M E(P_0) & \longrightarrow & T_M E(\Lambda) &
\end{array}
$$

The top row in this diagram is a representative for the element $(\varphi_{e(M)} \circ E^*_{\Lambda,\Lambda})([\eta])$.

We now consider the map $e_{M,M}^* \circ \varphi_M$. Applying the functor $e \circ t_M$ to the exact sequence η gives the top row in the following commutative diagram of $a\Lambda a$-modules with exact rows, where the bottom row is a representative of the element $(e_{M,M}^* \circ \varphi_M)([\eta])$:

$$0 \longrightarrow et_M(\Lambda) \longrightarrow et_M(X) \longrightarrow et_M(P_{i-2}) \longrightarrow \cdots \longrightarrow et_M(P_0) \longrightarrow et_M(\Lambda) \longrightarrow 0$$

$$\downarrow{e(\mu)}\cong \qquad \qquad \parallel \qquad \qquad \parallel \qquad \qquad \parallel \qquad \qquad \downarrow{e(\mu)}\cong$$

$$0 \longrightarrow e(M) \longrightarrow et_M(X) \longrightarrow et_M(P_{i-2}) \longrightarrow \cdots \longrightarrow et_M(P_0) \longrightarrow e(M) \longrightarrow 0$$

Finally, we use the natural transformation τ^M from Lemma 7.1 to combine the two above diagrams into the following commutative diagram of $a\Lambda a$-modules:

$$0 \longrightarrow e(M) \longrightarrow T_M(Y) \longrightarrow T_M(Q_{i-2}) \longrightarrow \cdots \longrightarrow T_M(Q_0) \longrightarrow e(M) \longrightarrow 0$$

$$\downarrow{\nu^{-1}}\cong \qquad \parallel \qquad \qquad \parallel \qquad \qquad \parallel \qquad \qquad \downarrow{\nu^{-1}}\cong$$

$$0 \longrightarrow T_M(a\Lambda a) \longrightarrow T_M(Y) \longrightarrow T_M(Q_{i-2}) \longrightarrow \cdots \longrightarrow T_M(Q_0) \longrightarrow T_M(a\Lambda a) \longrightarrow 0$$

$$\parallel \qquad T_M(f_{i-1})\downarrow \qquad T_M(f_{i-2})\downarrow \qquad \qquad T_M(f_0)\downarrow \qquad \parallel$$

$$T_M E(\Lambda) \longrightarrow T_M E(X) \longrightarrow T_M E(P_{i-2}) \longrightarrow \cdots \longrightarrow T_M E(P_0) \longrightarrow T_M E(\Lambda)$$

$$\tau_\Lambda^M \downarrow \qquad \tau_X^M \downarrow \qquad \tau_{P_{i-2}}^M \downarrow \qquad \qquad \tau_{P_0}^M \downarrow \qquad \tau_\Lambda^M \downarrow$$

$$0 \longrightarrow et_M(\Lambda) \longrightarrow et_M(X) \longrightarrow et_M(P_{i-2}) \longrightarrow \cdots \longrightarrow et_M(P_0) \longrightarrow et_M(\Lambda) \longrightarrow 0$$

$$\downarrow{e(\mu)}\cong \qquad \parallel \qquad \qquad \parallel \qquad \qquad \parallel \qquad \qquad \downarrow{e(\mu)}\cong$$

$$0 \longrightarrow e(M) \longrightarrow et_M(X) \longrightarrow et_M(P_{i-2}) \longrightarrow \cdots \longrightarrow et_M(P_0) \longrightarrow e(M) \longrightarrow 0$$

It is easy to check that the composition of maps along the leftmost column is the identity map on $e(M)$, and the same holds for the composition of maps along the rightmost column. Thus the top and bottom rows in this diagram represent the same element in $\operatorname{Ext}_{a\Lambda a}^i(e(M), e(M))$. Since the top row is a representative of the element $(\varphi_{e(M)} \circ E_{\Lambda,\Lambda}^*)([\eta])$ and the bottom row is a representative of the element $(e_{M,M}^* \circ \varphi_M)([\eta])$, this means that $\varphi_{e(M)} \circ E_{\Lambda,\Lambda}^* = e_{M,M}^* \circ \varphi_M$. $\qquad\square$

We thank the referee for pointing out an alternative proof of Proposition 7.2, which we sketch next.

Alternative proof of Proposition 7.2. Consider the following diagram of functors between derived categories:

$$\begin{array}{ccc}
\mathsf{D}^-(\operatorname{mod}\Lambda^{\mathrm{e}}) & \xrightarrow{\;-\otimes_\Lambda^{\mathbb{L}} M\;} & \mathsf{D}^-(\operatorname{mod}\Lambda) \\
{\scriptstyle (a\otimes a)\Lambda^{\mathrm{e}}\otimes_{\Lambda^{\mathrm{e}}}-}\downarrow & & \downarrow{\scriptstyle a\Lambda\otimes_\Lambda -} \\
\mathsf{D}^-(\operatorname{mod} a\Lambda a^{\mathrm{e}}) & \xrightarrow{\;-\otimes_{a\Lambda a}^{\mathbb{L}} aM\;} & \mathsf{D}^-(\operatorname{mod} a\Lambda a)
\end{array}$$

Proposition 7.2 really amounts, starting with a morphism $\eta: \Lambda \to \Lambda[n]$ in

$$\operatorname{Hom}_{\mathsf{D}^-(\operatorname{mod}\Lambda^{\mathrm{e}})}(\Lambda, \Lambda[n]) \cong \operatorname{HH}^n(\Lambda)$$

with $n \geqslant 0$, to showing that the images with the two different compositions in the above diagram coincide in

$$\mathrm{Hom}_{\mathsf{D}^-(\mathrm{mod}\, a\Lambda a)}(e(M), e(M)[n]) \cong \mathrm{Ext}^n_{a\Lambda a}(e(M), e(M)).$$

Let $\eta\colon \Lambda \to \Lambda[n]$ be a morphism in $\mathsf{D}^-(\mathrm{mod}\,\Lambda^{\mathrm{e}})$, which we represent as

$$\begin{array}{ccc}
 & \mathbb{P} & \\
{}^{s}\nearrow & & \searrow{}^{f} \\
\Lambda & & \Lambda[n]
\end{array}$$

where $s\colon \mathbb{P} \to \Lambda$ is a projective resolution of Λ over Λ^{e}.

The image of the composition via the upper right corner is

$$\begin{array}{ccc}
 & a\Lambda \otimes_\Lambda \mathbb{P} \otimes_\Lambda M & \\
{}^{1\otimes s\otimes 1}\swarrow & & \searrow{}^{1\otimes f\otimes 1} \\
a\Lambda \otimes_\Lambda \Lambda \otimes_\Lambda M & & a\Lambda \otimes_\Lambda \Lambda[n] \otimes_\Lambda \Lambda
\end{array}$$

since $\mathbb{P} \to \Lambda \to 0$ as a sequence of right Λ-modules splits.

The image of the composition via the lower left corner we compute in two steps. The image of the first functor is:

$$\begin{array}{ccc}
a\Lambda a & & a\Lambda[n]a \\
\nwarrow{}^{asa} & & \nearrow{}^{afa} \\
 & a\mathbb{P}a &
\end{array}$$

Let $\pi\colon \mathbb{P}_{a\Lambda a} \to a\Lambda a \to 0$ be a projective resolution of $a\Lambda a$ as a $a\Lambda a^{\mathrm{e}}$-module. Since $asa\colon a\mathbb{P}a \to a\Lambda a$ is a quasi-isomorphism, there exists a homomorphism of chain complexes $\varphi\colon \mathbb{P}_{a\Lambda a} \to a\mathbb{P}a$ such that we have the following commutative diagram:

$$\begin{array}{ccc}
a\Lambda a \xleftarrow{\;asa\;} & a\mathbb{P}a & \xrightarrow{\;afa\;} a\Lambda[n]a \\
{}_{\pi}\nwarrow \quad & \uparrow{}_{\varphi} & \quad \nearrow{}_{afa\varphi} \\
 & \mathbb{P}_{a\Lambda a} &
\end{array}$$

The image of the functor $-\otimes^{\mathbb{L}}_{a\Lambda a} aM$ is computed applying $-\otimes_{a\Lambda a} aM$ to the lower roof in the above diagram, and we obtain the following diagram:

$$\begin{array}{ccc}
a\Lambda a \otimes_{a\Lambda a} aM \xleftarrow{\;asa\otimes 1\;} & a\mathbb{P}a \otimes_{a\Lambda a} aM & \xrightarrow{\;afa\otimes 1\;} a\Lambda[n]a \otimes_{a\Lambda a} aM \\
{}_{\pi\otimes 1}\nwarrow \quad & \uparrow{}_{\varphi\otimes 1} & \quad \nearrow{}_{afa\varphi\otimes 1} \\
 & \mathbb{P}_{a\Lambda a} \otimes_{a\Lambda a} aM &
\end{array}$$

From this we can construct the following commutative diagram:

$$a\Lambda \otimes_\Lambda \mathbb{P} \otimes_\Lambda M$$

(commutative diagram with top vertex $a\Lambda \otimes_\Lambda \mathbb{P} \otimes_\Lambda M$; left arm $1 \otimes s \otimes 1$ to $a\Lambda \otimes_\Lambda \Lambda \otimes_\Lambda M$; right arm $1 \otimes f \otimes 1$ to $a\Lambda \otimes_\Lambda \Lambda[n] \otimes_\Lambda M$; central vertical $\tau_{\mathbb{P}[n]}^M$; left vertical τ_Λ^M from $a\Lambda a \otimes_{a\Lambda a} aM$; right vertical $\tau_{\Lambda[n]}^M$ from $a\Lambda[n]a \otimes_{a\Lambda a} aM$; middle row $a\Lambda a \otimes_{a\Lambda a} aM \xleftarrow{asa \otimes 1} a\mathbb{P}a \otimes_{a\Lambda a} aM \xrightarrow{afa \otimes 1} a\Lambda[n]a \otimes_{a\Lambda a} aM$; bottom vertex $\mathbb{P}_{a\Lambda a} \otimes_{a\Lambda a} aM$ with maps $\pi \otimes 1$, $\varphi \otimes 1$, $afa\varphi \otimes 1$)

where the extreme vertical morphisms are isomorphisms and $asa \otimes 1 \colon a\mathbb{P}a \otimes_{a\Lambda a} aM \to a\Lambda a \otimes_{a\Lambda a} aM$ is a quasi-isomorphism (since $a\Lambda \otimes_\Lambda \mathbb{P} \to a\Lambda \otimes_\Lambda \Lambda \to 0$ as a sequence of right Λ-modules splits). This shows that the two images of $\eta \colon \Lambda \to \Lambda[n]$ in $\mathrm{Hom}_{\mathsf{D}^-(\mathrm{mod}\,\Lambda^e)}(\Lambda, \Lambda[n])$ are the same. $\qquad\square$

Having shown that our diagrams are commutative, we now move on to describing when the map $E_{\Lambda,\Lambda}^*$ is an isomorphism in almost all degrees. For this, we use Corollary 3.11 (i) on the algebras Λ^e and $(a \otimes a^{\mathrm{op}})\Lambda^e(a \otimes a^{\mathrm{op}})$ and the Λ^e-module Λ. We let ε denote the element $a \otimes a^{\mathrm{op}}$ of Λ^e, so that we can write the algebra $(a \otimes a^{\mathrm{op}})\Lambda^e(a \otimes a^{\mathrm{op}})$ more simply as $\varepsilon\Lambda^e\varepsilon$. Note that Corollary 3.11 uses a recollement situation; in this case, the recollement is like the one in Example 2.3 (ii).

In order to use Corollary 3.11 (i) in this situation, we need to show the following:

$$\mathsf{pd}_{\varepsilon\Lambda^e\varepsilon}\, \varepsilon\Lambda^e < \infty \qquad \text{and} \qquad \mathrm{Ext}_{\Lambda^e}^j\left(\Lambda, \frac{\Lambda^e/\langle\varepsilon\rangle}{\mathrm{rad}\,\Lambda^e/\langle\varepsilon\rangle}\right) = 0 \quad \text{for } j \gg 0.$$

We show the first of these conditions in Lemma 7.4, and the second one in Lemma 7.8 (here we need an additional technical assumption on Λ to be able to describe the simple modules over Λ^e), and finally tie it together in Proposition 7.9, where we show that $E_{\Lambda,\Lambda}^*$ is an isomorphism in sufficiently high degrees.

First, we show how the projective dimension of the tensor product $M \otimes_k N$ is related to the projective dimensions of M and N, when M and N are modules over k-algebras. In particular, the following result implies that if a left and a right Λ-module ${}_\Lambda M$ and N_Λ both have finite projective dimension, then their tensor product $M \otimes_k N$ has finite projective dimension as a Λ^e-module.

Lemma 7.3. *Let Σ and Γ be k-algebras, and let M be a Σ-module and N a Γ-module. If M has finite projective dimension as a Σ-module and N has finite projective dimension as a Γ-module, then $M \otimes_k N$ has finite projective dimension as a $(\Sigma \otimes_k \Gamma)$-module, and*

$$\mathsf{pd}_{\Sigma\otimes_k\Gamma}(M \otimes_k N) \le \mathsf{pd}_\Sigma M + \mathsf{pd}_\Gamma N.$$

Proof. Assume that $\mathsf{pd}_\Sigma M = m$ and $\mathsf{pd}_\Gamma N = n$. Then we have finite projective resolutions

$$0 \to P_m \to \cdots \to P_0 \to M \to 0 \qquad \text{and} \qquad 0 \to Q_n \to \cdots \to Q_0 \to N \to 0$$

of M and N, respectively. Let P and Q denote the corresponding deleted resolutions. Consider the tensor product

$$P \otimes_k Q \colon \cdots \to (P_0 \otimes_k Q_2) \oplus (P_1 \otimes_k Q_1) \oplus (P_2 \otimes_k Q_0)$$
$$\to (P_0 \otimes_k Q_1) \oplus (P_1 \otimes_k Q_0) \to P_0 \otimes_k Q_0 \to 0$$

of the complexes P and Q. This is a bounded complex of projective $(\Sigma \otimes_k \Gamma)$-modules. We want to show that it is in fact a deleted projective resolution of the $(\Sigma \otimes_k \Gamma)$-module $M \otimes_k N$, which completes the proof.

We need to show that the complex $P \otimes_k Q$ is exact in all positive degrees and has homology $M \otimes_k N$ in degree zero. Let us temporarily forget the Σ- and Γ-structures, and view P as a complex of right k-modules, Q as a complex of left k-modules, and $P \otimes_k Q$ as a complex of abelian groups. Then by the Künneth formula for homology (see [57, Corollary 11.29]), we have an isomorphism

$$\alpha \colon \bigoplus_{i+j=n} H_i(P) \otimes_k H_j(Q) \xrightarrow{\cong} H_n(P \otimes_k Q)$$

of abelian groups, given by $\alpha([p] \otimes [q]) = [p \otimes q]$, for $p \in P_i$ and $q \in Q_j$. Observe that α preserves a $(\Sigma \otimes_k \Gamma)$-module structure. Thus, α is a $(\Sigma \otimes_k \Gamma)$-module isomorphism, and we get

$$H_n(P \otimes_k Q) \cong \bigoplus_{i+j=n} H_i(P) \otimes_k H_j(Q) \cong \begin{cases} M \otimes_k N & \text{if } n = 0, \\ 0 & \text{if } n > 0. \end{cases}$$

This means that the complex $P \otimes_k Q$ is a deleted projective resolution of the $(\Sigma \otimes_k \Gamma)$-module $M \otimes_k N$. Since the complex $P \otimes_k Q$ is zero in all degrees above $m + n$, we get

$$\mathsf{pd}_{\Sigma \otimes_k \Gamma}(M \otimes_k N) \le m + n = \mathsf{pd}_\Sigma M + \mathsf{pd}_\Gamma N,$$

and the proof is complete. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Using the above result, we find that the assumptions we make about the left and right $a\Lambda a$-modules $a\Lambda$ and Λa having finite projective dimension imply the first condition we need for applying Corollary 3.11 (i), namely that the $\varepsilon \Lambda^{\mathrm{e}} \varepsilon$-module $\varepsilon \Lambda^{\mathrm{e}}$ has finite projective dimension. We state this as the following result.

Lemma 7.4. *We have the following inequality*:

$$\mathsf{pd}_{\varepsilon \Lambda^{\mathrm{e}} \varepsilon} \varepsilon \Lambda^{\mathrm{e}} \le \mathsf{pd}_{a\Lambda a} a\Lambda + \mathsf{pd}_{(a\Lambda a)^{\mathrm{op}}} \Lambda a.$$

Proof. Note that $\varepsilon \Lambda^{\mathrm{e}}$ is isomorphic to $(a\Lambda \otimes_k \Lambda a)$ as left $(a\Lambda a)^{\mathrm{e}}$-modules and that the rings $(a\Lambda a)^{\mathrm{e}}$ and $\varepsilon \Lambda^{\mathrm{e}} \varepsilon$ are isomorphic. By using these isomorphisms and Lemma 7.3, we get that

$$\mathsf{pd}_{\varepsilon \Lambda^{\mathrm{e}} \varepsilon} \varepsilon \Lambda^{\mathrm{e}} = \mathsf{pd}_{(a\Lambda a)^{\mathrm{e}}} \varepsilon \Lambda^{\mathrm{e}} = \mathsf{pd}_{(a\Lambda a)^{\mathrm{e}}}(a\Lambda \otimes_k \Lambda a) \le \mathsf{pd}_{a\Lambda a} a\Lambda + \mathsf{pd}_{(a\Lambda a)^{\mathrm{op}}} \Lambda a. \quad \square$$

Now we show how we get the second condition needed for applying Corollary 3.11 (i). We begin with a general result which relates extension groups over Λ^{e} to extension groups over Λ.

Lemma 7.5. *Let M and N be Λ-modules. Let D be the duality* $\mathrm{Hom}_k(-, k) \colon \mathrm{mod}\,\Lambda \longrightarrow \mathrm{mod}\,\Lambda^{\mathrm{op}}$. *Then*

$$\mathrm{Ext}^j_{\Lambda^{\mathrm{e}}}(\Lambda, M \otimes_k D(N)) \cong \mathrm{Ext}^j_\Lambda(N, M)$$

for every nonnegative integer j.

Proof. This follows from [14, Corollary 4.4, Chapter IX] by using the isomorphism $M \otimes_k D(N) \cong \mathrm{Hom}_k(N, M)$ of Λ^e-modules.   □

Furthermore, we need to be able to describe the simple Λ^e-modules in terms of simple Λ-modules. It is reasonable to expect that taking the tensor product

$$(\Lambda/\operatorname{rad}\Lambda) \otimes_k (\Lambda^{\mathrm{op}}/\operatorname{rad}\Lambda^{\mathrm{op}})$$

should produce all the simple Λ^e-modules. This is, however, not true for all finite dimensional algebras, as Example 7.7 shows. The following result describes when it is true.

Lemma 7.6. *We have an isomorphism*

$$\Lambda^e/\operatorname{rad}\Lambda^e \cong (\Lambda/\operatorname{rad}\Lambda) \otimes_k (\Lambda^{\mathrm{op}}/\operatorname{rad}\Lambda^{\mathrm{op}})$$

of Λ^e-modules if and only if the Λ^e-module

$$(\Lambda/\operatorname{rad}\Lambda) \otimes_k (\Lambda^{\mathrm{op}}/\operatorname{rad}\Lambda^{\mathrm{op}})$$

is semisimple.

Proof. It is easy to show that

$$(\Lambda/\operatorname{rad}\Lambda) \otimes_k (\Lambda^{\mathrm{op}}/\operatorname{rad}\Lambda^{\mathrm{op}}) \cong \frac{\Lambda^e}{\Lambda \otimes_k (\operatorname{rad}\Lambda^{\mathrm{op}}) + (\operatorname{rad}\Lambda) \otimes_k \Lambda^{\mathrm{op}}}$$

as Λ^e-modules, and that the ideal $\Lambda \otimes_k (\operatorname{rad}\Lambda^{\mathrm{op}}) + (\operatorname{rad}\Lambda) \otimes_k \Lambda^{\mathrm{op}}$ of Λ^e is nilpotent. This means that if $(\Lambda/\operatorname{rad}\Lambda) \otimes_k (\Lambda^{\mathrm{op}}/\operatorname{rad}\Lambda^{\mathrm{op}})$ is a semisimple Λ^e-module, then it is isomorphic to $\Lambda^e/\operatorname{rad}\Lambda^e$. The opposite implication is obvious.   □

Now we give an example showing that $(\Lambda/\operatorname{rad}\Lambda) \otimes_k (\Lambda^{\mathrm{op}}/\operatorname{rad}\Lambda^{\mathrm{op}})$ is not necessarily semisimple for a finite dimensional algebra Λ over a field k.

Example 7.7. Let $k = \mathbb{Z}_2(x)$ be the field of rational functions in one indeterminant x over $\mathbb{Z}_2$, and let Λ be the 2-dimensional k-algebra $k[y]/\langle y^2 - x \rangle$. Then Λ is a field, so that $\operatorname{rad}\Lambda = (0)$. The element $\alpha = y \otimes 1 + 1 \otimes y$ satisfies $\alpha^2 = 0$. Hence $\langle \alpha \rangle$ is a nilpotent nonzero ideal in Λ^e, and therefore Λ^e is not semisimple.

We assume that $(\Lambda/\operatorname{rad}\Lambda) \otimes_k (\Lambda^{\mathrm{op}}/\operatorname{rad}\Lambda^{\mathrm{op}})$ is semisimple whenever we need it. In particular, this assumption is included in the main result at the end of this section. Note that this assumption is satisfied in many cases, for example if $\Lambda/\operatorname{rad}\Lambda$ is separable as a k-algebra (by [25, Corollary 7.8 (i)]), if k is algebraically closed (this can be shown by using the Wedderburn–Artin Theorem), or if Λ is a quotient of a path algebra by an admissible ideal.

Now we can show how to get the second condition we need for applying Corollary 3.11 (i).

Lemma 7.8. *Assume that $(\Lambda/\operatorname{rad}\Lambda) \otimes_k (\Lambda^{\mathrm{op}}/\operatorname{rad}\Lambda^{\mathrm{op}})$ is a semisimple Λ^e-module, and that we have*

$$(\alpha) \ \ \mathsf{id}_\Lambda \left(\frac{\Lambda/\langle a \rangle}{\operatorname{rad}\Lambda/\langle a \rangle} \right) < \infty \qquad and \qquad (\gamma) \ \ \mathsf{pd}_\Lambda \left(\frac{\Lambda/\langle a \rangle}{\operatorname{rad}\Lambda/\langle a \rangle} \right) < \infty.$$

Then

$$\mathrm{Ext}^j_{\Lambda^e} \left(\Lambda, \frac{\Lambda^e/\langle \varepsilon \rangle}{\operatorname{rad}\Lambda^e/\langle \varepsilon \rangle} \right) = 0 \quad for \quad j > \max \left\{ \mathsf{pd}_\Lambda \left(\frac{\Lambda/\langle a \rangle}{\operatorname{rad}\Lambda/\langle a \rangle} \right), \mathsf{id}_\Lambda \left(\frac{\Lambda/\langle a \rangle}{\operatorname{rad}\Lambda/\langle a \rangle} \right) \right\}.$$

Proof. By Lemma 7.6, every simple Λ^{e}-module is a direct summand of a module of the form $S \otimes_k D(T)$ for some simple Λ-modules S and T, where D is the duality $\mathrm{Hom}_k(-, k) \colon \mathrm{mod}\,\Lambda \longrightarrow \mathrm{mod}\,\Lambda^{\mathrm{op}}$. If neither of the modules S or T is annihilated by the ideal $\langle a \rangle$, then we have

$$\langle \varepsilon \rangle (S \otimes_k D(T)) = \langle a \otimes a^{\mathrm{op}} \rangle (S \otimes_k D(T)) = (\langle a \rangle S) \otimes_k D(\langle a \rangle T) = S \otimes_k D(T),$$

which means that no nonzero direct summand of the Λ^{e}-module $S \otimes_k D(T)$ is a $\Lambda^{\mathrm{e}}/\langle \varepsilon \rangle$-module.

Let j be an integer such that

$$j > \max \Big\{ \mathsf{pd}_\Lambda \Big(\frac{\Lambda/\langle a \rangle}{\mathrm{rad}\,\Lambda/\langle a \rangle} \Big), \, \mathsf{id}_\Lambda \Big(\frac{\Lambda/\langle a \rangle}{\mathrm{rad}\,\Lambda/\langle a \rangle} \Big) \Big\}.$$

In order to prove the result, it is sufficient to show that $\mathrm{Ext}^j_{\Lambda^{\mathrm{e}}}(\Lambda, U) = 0$ for every simple $\Lambda^{\mathrm{e}}/\langle \varepsilon \rangle$-module U. By the above reasoning, every such U is a direct summand of a module $S \otimes_k D(T)$ for some simple Λ-modules S and T, where at least one of S and T is annihilated by $\langle a \rangle$ and is thus a simple $\Lambda/\langle a \rangle$-module. Using Lemma 7.5, we get

$$\mathrm{Ext}^j_{\Lambda^{\mathrm{e}}}(\Lambda, S \otimes_k D(T)) \cong \mathrm{Ext}^j_\Lambda(T, S) = 0,$$

since we have $\mathsf{pd}_\Lambda T < j$ or $\mathsf{id}_\Lambda S < j$. It follows that $\mathrm{Ext}^j_{\Lambda^{\mathrm{e}}}(\Lambda, U) = 0$. $\qquad\square$

The following result summarizes the above work and shows that, with the assumptions we have indicated for the algebra Λ and the idempotent a, the functor E gives isomorphisms $E^j_{\Lambda,\Lambda} \colon \mathrm{HH}^j(\Lambda) \longrightarrow \mathrm{HH}^j(a\Lambda a)$ in almost all degrees j.

Proposition 7.9. *Assume that $(\Lambda/\mathrm{rad}\,\Lambda) \otimes_k (\Lambda^{\mathrm{op}}/\mathrm{rad}\,\Lambda^{\mathrm{op}})$ is a semisimple Λ^{e}-module, and that the functor e is an eventually homological isomorphism. Then the map*

$$E^j_{\Lambda,M} \colon \mathrm{Ext}^j_{\Lambda^{\mathrm{e}}}(\Lambda, M) \longrightarrow \mathrm{Ext}^j_{(a\Lambda a)^{\mathrm{e}}}(E(\Lambda), E(M))$$

is an isomorphism for every Λ^{e}-module M and every integer j such that

$$j > \max \Big\{ \mathsf{pd}_\Lambda \Big(\frac{\Lambda/\langle a \rangle}{\mathrm{rad}\,\Lambda/\langle a \rangle} \Big), \, \mathsf{id}_\Lambda \Big(\frac{\Lambda/\langle a \rangle}{\mathrm{rad}\,\Lambda/\langle a \rangle} \Big) \Big\} + \mathsf{pd}_{a\Lambda a}\, a\Lambda + \mathsf{pd}_{(a\Lambda a)^{\mathrm{op}}}\, \Lambda a + 1 < \infty.$$

In particular, we have isomorphisms

$$\mathrm{HH}^j(\Lambda) \cong \mathrm{HH}^j(a\Lambda a)$$

for almost all degrees j.

Proof. We use Corollary 3.11 (i) on the algebra Λ^{e}, the idempotent $\varepsilon = a \otimes a^{\mathrm{op}}$ and the Λ^{e}-module Λ. Let m and n be the integers

$$m = \max \Big\{ \mathsf{pd}_\Lambda \Big(\frac{\Lambda/\langle a \rangle}{\mathrm{rad}\,\Lambda/\langle a \rangle} \Big), \, \mathsf{id}_\Lambda \Big(\frac{\Lambda/\langle a \rangle}{\mathrm{rad}\,\Lambda/\langle a \rangle} \Big) \Big\} + 1$$

and

$$n = \mathsf{pd}_{a\Lambda a}\, a\Lambda + \mathsf{pd}_{(a\Lambda a)^{\mathrm{op}}}\, \Lambda a.$$

Note that m and n are finite by Corollary 3.12. By Lemma 7.8, we have

$$\mathrm{Ext}^j_{\Lambda^{\mathrm{e}}} \Big(\Lambda, \frac{\Lambda^{\mathrm{e}}/\langle \varepsilon \rangle}{\mathrm{rad}\,\Lambda^{\mathrm{e}}/\langle \varepsilon \rangle} \Big) = 0 \quad \text{for} \quad j \geq m,$$

and by Lemma 7.4, we have

$$\mathsf{pd}_{\varepsilon \Lambda^{\mathrm{e}} \varepsilon}\, \varepsilon \Lambda^{\mathrm{e}} \leq n.$$

Now the result follows from Corollary 3.11 (i) by noting that $(a\Lambda a)^{\mathrm{e}}$ is the same algebra as $\varepsilon\Lambda^{\mathrm{e}}\varepsilon$ and that our functor $E = a - a$ is the same as the functor $\varepsilon-$ given by left multiplication with the idempotent ε. $\qquad\square$

Finally, we conclude this section by showing that the assumptions we have indicated imply that Fg holds for Λ if and only if Fg holds for $a\Lambda a$. The following theorem is the main result of this section and constitutes the fourth part of the Main Theorem presented in the Introduction.

Theorem 7.10. *Let Λ be a finite dimensional algebra over a field k, and let a be an idempotent in Λ. Assume that $(\Lambda/\operatorname{rad}\Lambda) \otimes_k (\Lambda^{\mathrm{op}}/\operatorname{rad}\Lambda^{\mathrm{op}})$ is a semisimple Λ^{e}-module, and that the functor $a-\colon \operatorname{mod}\Lambda \longrightarrow \operatorname{mod} a\Lambda a$ is an eventually homological isomorphism. Then Λ satisfies Fg if and only if $a\Lambda a$ satisfies Fg.*

Proof. For every Λ-module M, we can make a diagram

$$
\begin{array}{ccc}
\mathrm{HH}^*(\Lambda) & \xrightarrow{\;\;\varphi_M\;\;} & \mathrm{Ext}_\Lambda^*(M,M) \\[2pt]
{\scriptstyle E_{\Lambda,\Lambda}^*}\Big\downarrow & & \Big\downarrow{\scriptstyle e_{M,M}^*} \\[2pt]
\mathrm{HH}^*(a\Lambda a) & \xrightarrow[\;\;\varphi_{e(M)}\;\;]{} & \mathrm{Ext}_{a\Lambda a}^*(e(M), e(M))
\end{array}
$$

of graded rings and graded ring homomorphisms. This diagram commutes by Proposition 7.2, and the maps $E_{\Lambda,\Lambda}^*$ and $e_{M,M}^*$ are isomorphisms in almost all degrees by Proposition 7.9 and Corollary 3.12, respectively.

Since we have such diagrams for every Λ-module M and the functor e is essentially surjective (see Proposition 2.2), we can make one diagram with $M = \Lambda/\operatorname{rad}\Lambda$ and another with $e(M) \cong a\Lambda a/\operatorname{rad} a\Lambda a$. Then, by Proposition 6.4, it follows that Λ satisfies Fg if and only if $a\Lambda a$ satisfies Fg. $\qquad\square$

8. Applications and examples

In this section we provide applications of our Main Theorem (stated in the Introduction), and examples illustrating its use. For ease of reference, we restate the Main Theorem here.

Theorem 8.1. *Let Λ be an artin algebra over a commutative ring k and let a be an idempotent element of Λ. Let e be the functor $a-\colon \operatorname{mod}\Lambda \longrightarrow \operatorname{mod} a\Lambda a$ given by multiplication by a. Consider the following conditions:*

$$(\alpha)\;\; \mathsf{id}_\Lambda\left(\frac{\Lambda/\langle a\rangle}{\operatorname{rad}\Lambda/\langle a\rangle}\right) < \infty \qquad\qquad (\beta)\;\; \mathsf{pd}_{a\Lambda a}\, a\Lambda < \infty$$

$$(\gamma)\;\; \mathsf{pd}_\Lambda\left(\frac{\Lambda/\langle a\rangle}{\operatorname{rad}\Lambda/\langle a\rangle}\right) < \infty \qquad\qquad (\delta)\;\; \mathsf{pd}_{(a\Lambda a)^{\mathrm{op}}}\, \Lambda a < \infty$$

Then the following hold.

* (i) *The following are equivalent*:
  * (a) *(α) and (β) hold.*
  * (b) *(γ) and (δ) hold.*
  * (c) *The functor e is an eventually homological isomorphism.*
* (ii) *The functor $a-\colon \operatorname{mod}\Lambda \longrightarrow \operatorname{mod} a\Lambda a$ induces a singular equivalence between Λ and $a\Lambda a$ if and only if conditions (β) and (γ) hold.*
* (iii) *Assume that e is an eventually homological isomorphism. Then Λ is Gorenstein if and only if $a\Lambda a$ is Gorenstein.*

(iv) *Assume that e is an eventually homological isomorphism, that k is a field and that $(\Lambda/\operatorname{rad}\Lambda) \otimes_k (\Lambda^{\mathrm{op}}/\operatorname{rad}\Lambda^{\mathrm{op}})$ is a semisimple Λ^{e}-module. Then Λ satisfies* Fg *if and only if $a\Lambda a$ satisfies* Fg.

This section is divided into four subsections. In the first subsection, we discuss dependencies between conditions (α)–(δ) in Theorem 8.1. In the second subsection, we consider some cases where these conditions are related. As a consequence, we find sufficient conditions, stated in terms of the quiver and relations, for applying Theorem 8.1 to a quotient of a path algebra. In the third subsection, we apply Theorem 8.1 to the class of triangular matrix algebras. In the last subsection, we compare our work to that of Nagase in [47].

8.1. **Conditions in the Main Theorem.** The first part of Theorem 8.1 says that if conditions (α) and (β) hold, or conditions (γ) and (δ) hold, then all four of these conditions hold, and the functor e is an essentially homological isomorphism. We now give examples which show that assuming any other combination of two of the conditions is not sufficient to get the same conclusion.

Our first example shows that conditions (α) and (δ) do not imply conditions (β) and (γ).

Example 8.2. Let k be a field. Let the k-algebra $\Lambda = kQ/\langle\rho\rangle$ be given by the following quiver and relations:

$$Q\colon\; 1 \xrightarrow{\;\alpha\;} 2\, \circlearrowright \beta \qquad\qquad \rho = \{\beta^2, \beta\alpha\}.$$

Let $a = e_2$. Then $(\Lambda/\langle a\rangle)/(\operatorname{rad}\Lambda/\langle a\rangle) \cong S_1$ as Λ-modules. We have $\operatorname{id}_\Lambda S_1 = 0$ and $\operatorname{pd}_\Lambda S_1 = \infty$, which means that (α) holds and (γ) does not hold. Furthermore, we have $a\Lambda a \cong k[x]/\langle x^2\rangle$ as algebras, $a\Lambda \cong a\Lambda a \oplus S$ as left $a\Lambda a$-modules, where S is the simple $a\Lambda a$-module, and $\Lambda a \cong a\Lambda a$ as right $a\Lambda a$-modules. We therefore have $\operatorname{pd}_{a\Lambda a} a\Lambda = \infty$ and $\operatorname{pd}_{(a\Lambda a)^{\mathrm{op}}} \Lambda a = 0$, which means that (β) does not hold, while (δ) holds.

By considering the opposite algebra of the algebra in the above example, we see that conditions (β) and (γ) do not imply conditions (α) and (δ).

The next example shows that conditions (α) and (γ) do not imply conditions (β) and (δ).

Example 8.3. Let k be a field. Let the k-algebra $\Lambda = kQ/\langle\rho\rangle$ be given by the following quiver and relations:

$$Q\colon\; 1 \underset{\beta}{\overset{\alpha}{\rightleftarrows}} 2 \qquad\qquad \rho = \{\alpha\beta\}.$$

Let $a = e_1$. Let S_2 be the simple Λ-module associated to the vertex 2. Conditions (α) and (γ) are satisfied since $\operatorname{id}_\Lambda S_2 = 2$ and $\operatorname{pd}_\Lambda S_2 = 2$, respectively. Conditions (β) and (δ) are not satisfied, since $\operatorname{pd}_{a\Lambda a} a\Lambda = \infty$ and $\operatorname{pd}_{(a\Lambda a)^{\mathrm{op}}} \Lambda a = \infty$.

Finally, the next example shows that conditions (β) and (δ) do not imply conditions (α) and (γ).

Example 8.4. Let $\Lambda = k(\, \alpha \circlearrowleft 1 \underset{\gamma}{\overset{\beta}{\rightleftarrows}} 2 \,)/\langle\alpha^2 - \gamma\beta, \beta\gamma, \alpha\gamma, \beta\alpha\rangle$, which is Example 2.3 from [37]. Let $a = e_2$. Then $a\Lambda a \cong k$, so both (β) and (δ) are satisfied.

However $\mathsf{id}_\Lambda \Lambda/\langle a \rangle = \mathsf{id}_\Lambda S_1 = \infty$ and $\mathsf{pd}_\Lambda \Lambda/\langle a \rangle = \mathsf{pd}_\Lambda S_1 = \infty$. Hence (α) and (γ) are not satisfied.

We have now seen that in general, there are no more relations between conditions (α)–(δ) than that described in Theorem 8.1. In the next subsection, we consider some special cases where there are more relations between the conditions.

8.2. **Algebras with ordered simples.** In this subsection, we apply Theorem 8.1 to cases where there exists a total order $\preceq$ of the simple $\Lambda/\langle a \rangle$-modules with the property that

$$(8.1) \qquad S \preceq S' \implies \mathrm{Ext}_\Lambda^{>0}(S, S') = 0$$

for every pair S and S' of simple $\Lambda/\langle a \rangle$-modules. With this assumption, we show that we have the implications $(\alpha) \implies (\delta)$ and $(\gamma) \implies (\beta)$ between the conditions in Theorem 8.1. We then consider some special cases where such orderings appear.

We need the following preliminary results.

Lemma 8.5. *Let Λ be an artin algebra, let M be a Λ-module with minimal projective resolution $\cdots \longrightarrow P_1 \longrightarrow P_0 \longrightarrow M \longrightarrow 0$, and let S be a simple Λ-module. Then, for every nonnegative integer n, we have $\mathrm{Ext}_\Lambda^n(M, S) = 0$ if and only if the projective cover of S is not a direct summand of P_n.*

Lemma 8.6. *Let Λ be an artin algebra, and let a be an idempotent in Λ. Let S be a simple Λ-module which is not annihilated by the ideal $\langle a \rangle$, and let P be the projective cover of S. Then aP is a projective $a\Lambda a$-module.*

Proof. We have

$$\mathrm{Hom}_\Lambda(\Lambda a, S) \cong aS \neq 0,$$

so there exists a nonzero morphism $f \colon \Lambda a \longrightarrow S$. Decomposing the idempotent a into a sum $a = a_1 + \cdots + a_t$ of orthogonal primitive idempotents gives a decomposition $\Lambda a \cong \Lambda a_1 \oplus \cdots \oplus \Lambda a_t$ of Λa into indecomposable projective modules. For some i, we must then have a nonzero morphism $f_i \colon \Lambda a_i \longrightarrow S$. Since S is simple, this means that Λa_i is its projective cover. Since $a \cdot a_i = a_i$, we get

$$aP \cong a\Lambda a_i = (a\Lambda a)a_i.$$

Therefore aP is a projective $a\Lambda a$-module. $\qquad\square$

Now we show that the conditions of Theorem 8.1 are related when we have an ordering of the simple $\Lambda/\langle a \rangle$-modules.

Proposition 8.7. *Let Λ be an artin algebra, and let a be an idempotent in Λ. Assume that there is a total order $\preceq$ on the simple $\Lambda/\langle a \rangle$-modules satisfying condition (8.1). Then we have the following implications between the conditions of Theorem 8.1:*

  (i) *$(\alpha) \implies (\delta)$.*
  (ii) *$(\gamma) \implies (\beta)$.*

In particular, we have that the functor $a- \colon \mathrm{mod}\,\Lambda \longrightarrow \mathrm{mod}\,a\Lambda a$ is an eventually homological isomorphism if and only if conditions (α) and (γ) hold.

Proof. We show the second implication; the first can be shown in a similar way. The last claim follows directly from these two implications by Theorem 8.1 (i).

Assume that (γ) holds, that is, every $\Lambda/\langle a \rangle$-module has finite projective dimension as a Λ-module. We want to show that (β) holds, that is, the $a\Lambda a$-module $a\Lambda$ has finite projective dimension.

As in Section 7, we let e be the exact functor $e = (a-) \colon \operatorname{mod}\Lambda \longrightarrow \operatorname{mod}a\Lambda a$ given by multiplication with a. Then what we need to show is that $e(\Lambda)$ has finite projective dimension as a $a\Lambda a$-module.

Let $S_1 \preceq \cdots \preceq S_s$ be all the simple $\Lambda/\langle a \rangle$-modules (up to isomorphism), ordered by the total order $\preceq$. Let $T_1, \ldots, T_t$ be all the other simple Λ-modules (up to isomorphism). Let Q_i be the projective cover of S_i (considered as a Λ-module) and Q_j' the projective cover of T_j, for every i and j. These are all the indecomposable projective Λ-modules up to isomorphism, so it is sufficient to show that $e(Q_i)$ and $e(Q_j')$ have finite projective dimension as $a\Lambda a$-modules for every i and j.

For each of the modules Q_j', we have that $e(Q_j')$ is a projective $a\Lambda a$-module by Lemma 8.6. We need to check that $e(Q_i)$ has finite projective dimension for every i.

Consider the module S_1. By our assumptions, every simple $\Lambda/\langle a \rangle$-module has finite projective dimension over Λ. Let

$$0 \longrightarrow P_{n_1}^{(1)} \longrightarrow \cdots \longrightarrow P_2^{(1)} \longrightarrow P_1^{(1)} \longrightarrow Q_1 \longrightarrow S_1 \longrightarrow 0$$

be a minimal projective resolution of S_1. Applying the functor e to this sequence gives the exact sequence

$$(8.2) \quad 0 \longrightarrow e(P_{n_1}^{(1)}) \longrightarrow \cdots \longrightarrow e(P_2^{(1)}) \longrightarrow e(P_1^{(1)}) \longrightarrow e(Q_1) \longrightarrow 0$$

of $a\Lambda a$-modules, since $e(S_1) = 0$. Since we have $\operatorname{Ext}_\Lambda^{>0}(S_1, S_i) = 0$ for every i, it follows from Lemma 8.5 that the only indecomposable projective Λ-modules which can occur as direct summands of the modules $P_1^{(1)}, \ldots, P_{n_1}^{(1)}$ are the modules Q_j'. Since we know that these are mapped to projective modules by e, the sequence (8.2) is a projective resolution of the $a\Lambda a$-module $e(Q_1)$.

We continue inductively. For every i, we apply the functor e to a minimal projective resolution

$$0 \longrightarrow P_{n_i}^{(i)} \longrightarrow \cdots \longrightarrow P_2^{(i)} \longrightarrow P_1^{(i)} \longrightarrow Q_i \longrightarrow S_i \longrightarrow 0$$

and obtain the sequence

$$0 \longrightarrow e(P_{n_i}^{(i)}) \longrightarrow \cdots \longrightarrow e(P_2^{(i)}) \longrightarrow e(P_1^{(i)}) \longrightarrow e(Q_i) \longrightarrow 0$$

of $a\Lambda a$-modules. Each of the modules $P_1^{(i)}, \ldots, P_{n_i}^{(i)}$ has only the indecomposable projective modules $Q_1', \ldots, Q_t', Q_1, \ldots, Q_{i-1}$ as direct summands. Therefore (by the induction assumption), all the modules $e(P_1^{(i)}), \ldots, e(P_{n_i}^{(i)})$ have finite projective dimension, and thus the module $e(Q_i)$ has finite projective dimension. $\qquad\square$

By combining Theorem 8.1 with Proposition 8.7, we get the following result.

Corollary 8.8. *Let Λ be an artin algebra over a commutative ring k, and let a be an idempotent in Λ. Assume that there is a total order $\preceq$ on the simple $\Lambda/\langle a \rangle$-modules satisfying condition (8.1). Then the following hold, where (α), (β), (γ) and (δ) refer to the conditions in Theorem 8.1.*

    (i) *The functor $a- \colon \operatorname{mod}\Lambda \longrightarrow \operatorname{mod}a\Lambda a$ induces a singular equivalence between Λ and $a\Lambda a$ if and only if (γ) holds.*

(ii) *Assume that (α) and (γ) hold. Then Λ is Gorenstein if and only if $a\Lambda a$ is Gorenstein.*

(iii) *Assume that (α) and (γ) hold, that k is a field and $(\Lambda/\operatorname{rad}\Lambda) \otimes_k (\Lambda^{\mathrm{op}}/\operatorname{rad}\Lambda^{\mathrm{op}})$ is a semisimple Λ^{e}-module. Then Λ satisfies Fg if and only if $a\Lambda a$ satisfies Fg.*

We now consider special cases of conditions (α) and (γ) where the dimensions are not only finite, but at most one. We show that if one of these dimensions is at most one, then we have an ordering of the simple $\Lambda/\langle a\rangle$-modules as assumed in Proposition 8.7 and Corollary 8.8.

Lemma 8.9. *Let Λ be an artin algebra, and let a be an idempotent in Λ. Assume that we have either*

$$(\alpha_1) \ \operatorname{id}_\Lambda\left(\frac{\Lambda/\langle a\rangle}{\operatorname{rad}\Lambda/\langle a\rangle}\right) \leq 1 \qquad or \qquad (\gamma_1) \ \operatorname{pd}_\Lambda\left(\frac{\Lambda/\langle a\rangle}{\operatorname{rad}\Lambda/\langle a\rangle}\right) \leq 1.$$

Then there exists a total order $\preceq$ on the simple $\Lambda/\langle a\rangle$-modules satisfying condition (8.1).

Proof. Assume that (γ_1) holds (the proof using (α_1) is similar). Let $S_1, \ldots, S_s$ be all the simple $\Lambda/\langle a\rangle$-modules (up to isomorphism), and let $P_1, \ldots, P_s$ be their projective covers as Λ-modules, such that $P_i/((\operatorname{rad}\Lambda)P_i) \cong S_i$ for every i. Assume that we have ordered these by increasing length of the projective covers, that is,

$$\operatorname{length}(P_1) \leq \operatorname{length}(P_2) \leq \cdots \leq \operatorname{length}(P_s).$$

For any i, the module S_i has a projective resolution of the form

$$0 \longrightarrow Q \longrightarrow P_i \longrightarrow S_i \longrightarrow 0.$$

Since the module Q has shorter length than the module P_i, it cannot have any of the modules $P_i, \ldots, P_s$ as direct summands. Then Lemma 8.5 implies that $\operatorname{Ext}^{>0}_\Lambda(S_i, S_j) = 0$ for $i \leq j$. $\qquad\square$

By using Proposition 8.7, Lemma 8.9 and Theorem 8.1, we have the following.

Corollary 8.10. *Let Λ be an artin algebra over a commutative ring k, and let a be an idempotent in Λ. Then the following hold, where (α), (β), (γ) and (δ) refer to the conditions in Theorem 8.1, and (α_1) and (γ_1) refer to the conditions in Lemma 8.9.*

(i) *If (γ_1) holds, then the functor $a-\colon \operatorname{mod}\Lambda \longrightarrow \operatorname{mod} a\Lambda a$ induces a singular equivalence between Λ and $a\Lambda a$.*

(ii) *Assume either that (α_1) and (γ) hold, or that (α) and (γ_1) hold. Then Λ is Gorenstein if and only if $a\Lambda a$ is Gorenstein.*

(iii) *Assume either that (α_1) and (γ) hold, or that (α) and (γ_1) hold. Furthermore, assume that k is a field and $(\Lambda/\operatorname{rad}\Lambda) \otimes_k (\Lambda^{\mathrm{op}}/\operatorname{rad}\Lambda^{\mathrm{op}})$ is a semisimple Λ^{e}-module. Then Λ satisfies Fg if and only if $a\Lambda a$ satisfies Fg.*

For the following results, we let $\Lambda = kQ/\langle\rho\rangle$ be a quotient of a path algebra, where k is a field, Q is a quiver, and ρ is a minimal set of relations in kQ generating an admissible ideal $\langle\rho\rangle$.

First we describe how conditions (α_1) and (γ_1) can be interpreted for quotients of path algebras. The result follows directly from [10, Corollary, Section 1.1].

Lemma 8.11. *Let S be the simple Λ-module corresponding to a vertex v in the quiver Q.*

    (i) *We have $\mathsf{pd}_\Lambda S \leq 1$ if and only if no relation starts in the vertex v.*
    (ii) *We have $\mathsf{id}_\Lambda S \leq 1$ if and only if no relation ends in the vertex v.*

As a consequence of Lemma 8.11 and Corollary 8.10, we get the following results for path algebras.

Corollary 8.12. *Let $\Lambda = kQ/\langle\rho\rangle$ be a quotient of a path algebra as above. Choose some vertices in Q where no relations start, and let a be the sum of all vertices except these. Then the functor $a-\colon \operatorname{mod}\Lambda \longrightarrow \operatorname{mod} a\Lambda a$ induces a singular equivalence between Λ and $a\Lambda a$:*

$$\mathsf{D}_{\mathsf{sg}}(a-)\colon \mathsf{D}_{\mathsf{sg}}(\operatorname{mod}\Lambda) \xrightarrow{\simeq} \mathsf{D}_{\mathsf{sg}}(\operatorname{mod} a\Lambda a).$$

Corollary 8.13. *Let $\Lambda = kQ/\langle\rho\rangle$ be a quotient of a path algebra as above. Choose some vertices in Q where no relations start and no relations end, and let a be the sum of all vertices except these. Then the following hold*:

    (i) *Λ is Gorenstein if and only if $a\Lambda a$ is Gorenstein.*
    (ii) *Λ satisfies Fg if and only if $a\Lambda a$ satisfies Fg.*

We apply the above result in the following example.

Example 8.14. Let Q be the quiver with relations ρ given by

$$Q\colon 1 \underset{\alpha_m}{\overset{\alpha_1}{\rightleftarrows}} 2 \xrightarrow{\alpha_2} \cdots \xrightarrow{\alpha_{m-1}} m \qquad \text{and} \qquad \rho = \{(\alpha_m\cdots\alpha_1)^n\},$$

for some integers $m \geq 2$ and $n \geq 2$. Let $\Lambda = kQ/\langle\rho\rangle$, and let $a = e_1$ (the only vertex where a relation starts and ends). Then $a\Lambda a \cong k[x]/\langle x^n\rangle$, so $a\Lambda a$ satisfies Fg by [27, 28]. By Corollary 8.13, the algebra Λ also satisfies Fg. By Corollary 8.12, the algebras Λ and $k[x]/\langle x^n\rangle$ are singularly equivalent. See [59] for a general discussion of the Hochschild cohomology ring of the path algebra kQ modulo one relation.

8.3. Triangular matrix algebras.

Let Σ and Γ be two artin algebras over a commutative ring k, and let $_\Gamma M_\Sigma$ be a Γ-Σ-bimodule such that M is finitely generated over k, and k acts centrally on M. Then we have the artin triangular matrix algebra

$$\Lambda = \begin{pmatrix} \Sigma & 0 \\ _\Gamma M_\Sigma & \Gamma \end{pmatrix},$$

where the addition and the multiplication are given by the ordinary operations on matrices.

The module category of Λ has a well known description; see [7, 30]. In fact, a module over Λ is described as a triple (X, Y, f), where X is a Σ-module, Y is a Γ-module and $f\colon M \otimes_\Sigma X \longrightarrow Y$ is a Γ-homomorphism. A morphism between two triples (X, Y, f) and (X', Y', f') is a pair of homomorphisms (a, b), where $a \in \operatorname{Hom}_\Sigma(X, X')$ and $b \in \operatorname{Hom}_\Gamma(Y, Y')$, such that the following diagram commutes:

$$
\begin{array}{ccc}
M \otimes_\Sigma X & \xrightarrow{\ f\ } & Y \\
{\scriptstyle \mathrm{Id}_M \otimes a}\downarrow & & \downarrow{\scriptstyle b} \\
M \otimes_\Sigma X' & \xrightarrow{\ f'\ } & Y'
\end{array}
$$

We define the following functors:

(i) The functor $T_\Sigma \colon \operatorname{mod}\Sigma \longrightarrow \operatorname{mod}\Lambda$ is defined on Σ-modules X by $T_\Sigma(X) = (X, M \otimes_\Sigma X, \operatorname{Id}_{M\otimes X})$ and given a Σ-homomorphism $a \colon X \longrightarrow X'$ then $T_\Sigma(a) = (a, \operatorname{Id}_M \otimes a)$.

(ii) The functor $U_\Sigma \colon \operatorname{mod}\Lambda \longrightarrow \operatorname{mod}\Sigma$ is defined on Λ-modules (X, Y, f) by $U_\Sigma(X, Y, f) = X$ and given a Λ-homomorphism $(a, b) \colon (X, Y, f) \longrightarrow (X', Y', f')$ then $U_\Sigma(a, b) = a$. Similarly we define the functor $U_\Gamma \colon \operatorname{mod}\Lambda \longrightarrow \operatorname{mod}\Gamma$.

(iii) The functor $Z_\Sigma \colon \operatorname{mod}\Sigma \longrightarrow \operatorname{mod}\Lambda$ is defined on Σ-modules X by $Z_\Sigma(X) = (X, 0, 0)$ and given a Σ-homomorphism $a \colon X \longrightarrow X'$ then $Z_\Sigma(a) = (a, 0)$. Similarly we define the functor $Z_\Gamma \colon \operatorname{mod}\Gamma \longrightarrow \operatorname{mod}\Lambda$.

(iv) The functor $H_\Gamma \colon \operatorname{mod}\Gamma \longrightarrow \operatorname{mod}\Lambda$ is defined by $H_\Gamma(Y) = (\operatorname{Hom}_\Gamma(M, Y), Y, \epsilon_X)$ on Γ-modules Y and given a Γ-homomorphism $b \colon Y \longrightarrow Y'$ then $H_\Gamma(b) = (\operatorname{Hom}_\Gamma(M, b), b)$.

Then from Example 2.3 (see also [54, Example 2.12]), using the idempotent elements $e_1 = \left(\begin{smallmatrix} 1_\Sigma & 0 \\ 0 & 0 \end{smallmatrix}\right)$ and $e_2 = \left(\begin{smallmatrix} 0 & 0 \\ 0 & 1_\Gamma \end{smallmatrix}\right)$, we have the following recollements of abelian categories:

(8.3)
$$\operatorname{mod}\Gamma \xrightarrow{\quad Z_\Gamma \quad} \operatorname{mod}\Lambda \xrightarrow{\quad U_\Sigma \quad} \operatorname{mod}\Sigma$$

with upper arrow q and lower arrow U_Γ on the left, and upper arrow T_Σ and lower arrow Z_Σ on the right.

and

(8.4)
$$\operatorname{mod}\Sigma \xrightarrow{\quad Z_\Sigma \quad} \operatorname{mod}\Lambda \xrightarrow{\quad U_\Gamma \quad} \operatorname{mod}\Gamma$$

with upper arrow U_Σ and lower arrow p on the left, and upper arrow Z_Γ and lower arrow H_Γ on the right.

The functors q and p are induced from the adjoint pairs (T_Σ, U_Σ) and (U_Γ, H_Γ), respectively; see [54, Remark 2.3] for more details.

We want to use Theorem 8.1 to compare the triangular matrix algebra Λ with the algebras Σ and Γ. First consider the case where we compare Λ with Σ. We then take the idempotent a in the theorem to be e_1, and we can reformulate conditions (α), (β), (γ) and (δ) as follows:

(α) The functor Z_Γ sends every Γ-module to a Λ-module with finite injective dimension.

(β) The functor U_Σ sends every projective Λ-module to a Σ-module with finite projective dimension.

(γ) The functor Z_Γ sends every Γ-module to a Λ-module with finite projective dimension.

(δ) The functor U_Σ sends every injective Λ-module to a Σ-module with finite injective dimension.

By interchanging Σ and Γ, we get a similar reformulation of the conditions for the case where we compare Λ with Γ.

The next result clarifies when the above hold for the recollement (8.4) of a triangular matrix algebra Λ.

Lemma 8.15. *Let* $\Lambda = \left(\begin{smallmatrix} \Sigma & 0 \\ {}_\Gamma M_\Sigma & \Gamma \end{smallmatrix} \right)$ *be a triangular matrix algebra. The following hold.*

(i) *If* $\mathsf{pd}_\Gamma M < \infty$, *then the functor* U_Γ *sends projective* Λ-*modules to* Γ-*modules of finite projective dimension.*

(ii) *The functor* U_Γ *preserves injectives.*

(iii) *Assume that* $\mathsf{gl.\,dim}\,\Sigma < \infty$. *Then* $\mathsf{id}_\Lambda Z_\Sigma(X) < \infty$ *for every* Σ-*module* X.

(iv) *Assume that* $\mathsf{gl.\,dim}\,\Sigma < \infty$ *and* $\mathsf{pd}_\Gamma M < \infty$. *Then we have* $\mathsf{pd}_\Lambda Z_\Sigma(X) < \infty$ *for all* Σ-*modules* X.

Proof. (i) It is known (see [7]) that the indecomposable projective Λ-modules are of the forms $T_\Sigma(P)$, where P is an indecomposable projective Σ-module, and $Z_\Gamma(Q)$, where Q is an indecomposable projective Γ-module. Hence it is enough to consider modules of these forms. We have $U_\Gamma Z_\Gamma(Q) = Q$, and since $\mathsf{pd}_\Gamma M < \infty$ it follows that $\mathsf{pd}_\Gamma U_\Gamma T_\Sigma(P) = \mathsf{pd}_\Gamma(M \otimes_\Sigma P) < \infty$.

(ii) Since (Z_Γ, U_Γ) is an adjoint pair and Z_Γ is exact it follows that the functor U_Γ preserves injectives.

(iii) Let $0 \longrightarrow X \longrightarrow I^0 \longrightarrow \cdots \longrightarrow I^n \longrightarrow 0$ be a finite injective resolution of a Σ-module X. Then applying the functor Z_Σ we get the exact sequence $0 \longrightarrow Z_\Sigma(X) \longrightarrow Z_\Sigma(I^0) \longrightarrow \cdots \longrightarrow Z_\Sigma(I^n) \longrightarrow 0$, where every $Z_\Sigma(I^i)$ is an injective Λ-module since we have the adjoint pair (U_Σ, Z_Σ) and U_Σ is exact. Hence the injective dimension of $Z_\Sigma(X)$ is finite.

(iv) This follows from [58, Lemma 2.4], which says that, if $\mathsf{pd}_\Gamma M < \infty$, then a Λ-module (X, Y, f) has finite projective dimension if and only if the projective dimensions of X and Y are finite. $\qquad\square$

Using now the recollement (8.3) we have the following dual result of Lemma 8.15. The proof is left to the reader.

Lemma 8.16. *Let* $\Lambda = \left(\begin{smallmatrix} \Sigma & 0 \\ {}_\Gamma M_\Sigma & \Gamma \end{smallmatrix} \right)$ *be a triangular matrix algebra. The following hold.*

(i) *The functor* U_Σ *preserves projectives.*

(ii) *If* $\mathsf{pd}_\Sigma M_\Sigma < \infty$, *then the functor* U_Σ *sends injective* Λ-*modules to* Σ-*modules of finite injective dimension.*

(iii) *Assume that* $\mathsf{gl.\,dim}\,\Gamma < \infty$. *Then* $\mathsf{pd}_\Lambda Z_\Gamma(Y) < \infty$ *for every* Γ-*module* Y.

(iv) *Assume that* $\mathsf{gl.\,dim}\,\Gamma < \infty$ *and* $\mathsf{pd}_\Sigma M_\Sigma < \infty$. *Then for every* Γ-*module* Y *we have* $\mathsf{id}_\Lambda Z_\Gamma(Y) < \infty$.

As a consequence of Lemma 8.15 and Theorem 8.1 we have the following result. For similar characterizations with (ii) see [64].

Corollary 8.17. *Let* $\Lambda = \left(\begin{smallmatrix} \Sigma & 0 \\ {}_\Gamma M_\Sigma & \Gamma \end{smallmatrix} \right)$ *be an artin triangular matrix algebra over a commutative ring* k *such that* $\mathsf{gl.\,dim}\,\Sigma < \infty$ *and* $\mathsf{pd}_\Gamma M < \infty$. *Then the following hold.*

(i) *The singularity categories of* Λ *and* Γ *are triangle equivalent:*

$$\mathsf{D}_{\mathsf{sg}}(U_\Gamma) \colon \mathsf{D}_{\mathsf{sg}}(\mathrm{mod}\,\Lambda) \xrightarrow{\;\simeq\;} \mathsf{D}_{\mathsf{sg}}(\mathrm{mod}\,\Gamma).$$

(ii) Λ *is Gorenstein if and only if* Γ *is Gorenstein.*

(iii) *Assume that* k *is a field and that* $(\Lambda/\operatorname{rad}\Lambda) \otimes_k (\Lambda^{\mathrm{op}}/\operatorname{rad}\Lambda^{\mathrm{op}})$ *is a semisimple* Λ^{e}-*module. Then* Λ *satisfies* Fg *if and only if* Γ *satisfies* Fg.

Remark 8.18. The algebra $(\Lambda/\operatorname{rad}\Lambda) \otimes_k (\Lambda^{\mathrm{op}}/\operatorname{rad}\Lambda^{\mathrm{op}})$ being semisimple (as required in part (iii) above) can be shown to be equivalent to the following three algebras being semisimple: $(\Sigma/\operatorname{rad}\Sigma) \otimes_k (\Sigma^{\mathrm{op}}/\operatorname{rad}\Sigma^{\mathrm{op}})$, $(\Sigma/\operatorname{rad}\Sigma) \otimes_k (\Gamma^{\mathrm{op}}/\operatorname{rad}\Gamma^{\mathrm{op}})$ and $(\Gamma/\operatorname{rad}\Gamma) \otimes_k (\Gamma^{\mathrm{op}}/\operatorname{rad}\Gamma^{\mathrm{op}})$.

We also have the following consequence, obtained now from Lemma 8.16 and Theorem 8.1. Note that in the first statement we recover a theorem by Chen [15].

Corollary 8.19. *Let* $\Lambda = \left(\begin{smallmatrix} \Sigma & 0 \\ {}_\Gamma M_\Sigma & \Gamma \end{smallmatrix} \right)$ *be an artin triangular matrix algebra over a commutative ring* k.

(i) [15, Theorem 4.1] *Assume that* $\mathsf{gl.dim}\,\Gamma < \infty$. *Then there is a triangle equivalence*:

$$\mathsf{D}_{\mathsf{sg}}(\operatorname{mod}\Lambda) \xrightarrow[\mathsf{D}_{\mathsf{sg}}(U_\Sigma)]{\simeq} \mathsf{D}_{\mathsf{sg}}(\operatorname{mod}\Sigma).$$

(ii) *Assume that* $\mathsf{gl.dim}\,\Gamma < \infty$ *and* $\mathsf{pd}_\Sigma M_\Sigma < \infty$. *Then the following hold.*
   (a) Λ *is Gorenstein if and only if* Σ *is Gorenstein.*
   (b) *Assume that* k *is a field and that* $(\Lambda/\operatorname{rad}\Lambda) \otimes_k (\Lambda^{\mathrm{op}}/\operatorname{rad}\Lambda^{\mathrm{op}})$ *is a semisimple* Λ^{e}-*module. Then* Λ *satisfies* Fg *if and only if* Σ *satisfies* Fg.

From the above corollaries and the classical result of Buchweitz–Happel (see the text before Corollary 5.6) we have the following result for stable categories of Cohen–Macaulay modules.

Corollary 8.20. *Let* $\Lambda = \left(\begin{smallmatrix} \Sigma & 0 \\ {}_\Gamma M_\Sigma & \Gamma \end{smallmatrix} \right)$ *be an artin triangular matrix algebra.*

(i) [15, Corollary 4.2] *Assume that* $\mathsf{gl.dim}\,\Gamma < \infty$ *and* Σ *is Gorenstein. Then there is a triangle equivalence*:

$$\mathsf{D}_{\mathsf{sg}}(\operatorname{mod}\Lambda) \xrightarrow{\simeq} \underline{\mathrm{CM}}(\Sigma).$$

(ii) *Assume that* $\mathsf{gl.dim}\,\Gamma < \infty$ *and* $\mathsf{pd}_\Sigma M_\Sigma < \infty$. *If* Σ *is Gorenstein, then there is a triangle equivalence between the stable categories of Cohen–Macaulay modules of* Λ *and* Σ:

$$\underline{\mathrm{CM}}(\Lambda) \xrightarrow{\simeq} \underline{\mathrm{CM}}(\Sigma).$$

(iii) *Assume that* $\mathsf{gl.dim}\,\Sigma < \infty$ *and* $\mathsf{pd}_\Gamma M < \infty$. *If* Γ *is Gorenstein, then there is a triangle equivalence between the stable categories of Cohen–Macaulay modules of* Λ *and* Γ:

$$\underline{\mathrm{CM}}(\Lambda) \xrightarrow{\simeq} \underline{\mathrm{CM}}(\Gamma).$$

8.4. **Comparison to work by Nagase.** In this subsection we recall a result of Hiroshi Nagase [47] and relate his set of assumptions to ours.

In [47] Hiroshi Nagase proves the following result.

Proposition 8.21. *Let* Λ *be a finite dimensional algebra over an algebraically closed field with a stratifying ideal* $\langle a \rangle$ *for an idempotent* a *in* Λ. *Suppose* $\mathsf{pd}_{\Lambda^{\mathrm{e}}} \Lambda/\langle a \rangle < \infty$. *Then we have*

(1) $\mathrm{HH}^{\geq n}(\Lambda) \cong \mathrm{HH}^{\geq n}(a\Lambda a)$ *as graded algebras, where* $n = \mathsf{pd}_{\Lambda^{\mathrm{e}}} \Lambda/\langle a \rangle + 1$.
(2) Λ *satisfies* Fg *if and only if so does* $a\Lambda a$.
(3) Λ *is Gorenstein if and only if so is* $a\Lambda a$.

This work is based on the paper [41], where stratifying ideals $\langle a \rangle$ in a finite dimensional algebra Λ were used to show that the Hochschild cohomology groups of Λ and $a\Lambda a$ are isomorphic in almost all degrees.

We start by giving an example of a recollement $(\mathrm{mod}\,\Lambda/\langle a \rangle, \mathrm{mod}\,\Lambda, \mathrm{mod}\,a\Lambda a)$, where the ideal $\langle a \rangle$ is not a stratifying ideal but it satisfies our conditions from Theorem 7.10.

Example 8.22. Let Q be the quiver with relations ρ given by
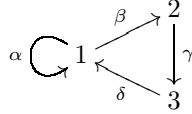
and $\rho = \{\alpha^2, \gamma\beta, \beta\alpha\delta\}$. Let $\Lambda = kQ/\langle\rho\rangle$ for some field k, and let $a = e_1$. We want to study the relationship between Λ and $a\Lambda a$. Let S_i denote the simple Λ-module associated to the vertex i for $i = 1, 2, 3$. Then $\mathrm{pd}_\Lambda S_2 = 1$, $\mathrm{pd}_\Lambda S_3 = 3$, $\mathrm{id}_\Lambda S_2 = 2$ and $\mathrm{id}_\Lambda S_3 = 3$. Furthermore, the left and right $a\Lambda a$-modules $a\Lambda$ and Λa have finite projective dimension (they are projective) as $a\Lambda a$-modules. Hence, according to Theorem 7.10 Λ satisfies Fg if and only if $a\Lambda a \cong k[x]/\langle x^2 \rangle$ does. We infer from this that Λ satisfies Fg. Moreover, the Hochschild cohomology groups of Λ and $a\Lambda a$ are isomorphic in almost all degrees by Proposition 7.9.

We claim that $\langle a \rangle$ is not a stratifying ideal. Recall that $\langle a \rangle$ is stratifying if (i) the multiplication map $\Lambda a \otimes_{a\Lambda a} a\Lambda \longrightarrow \Lambda a\Lambda$ is an isomorphism and (ii) $\mathrm{Tor}_i^{a\Lambda a}(\Lambda a, a\Lambda) = (0)$ for $i > 0$. Using that $(1 - a)\Lambda a \cong a\Lambda a$ as a right $a\Lambda a$-module, direct computations show that $\Lambda a \otimes_{a\Lambda a} a\Lambda$ has dimension 12, while $\langle a \rangle$ has dimension 10. Consequently $\langle a \rangle$ is not a stratifying ideal in Λ. However, the condition (ii) is satisfied since Λa is a projective $a\Lambda a$-module.

Next we show that, when $\langle a \rangle$ is a stratifying ideal, then the property $\mathrm{pd}_{\Lambda^e} \Lambda/\langle a \rangle < \infty$ is equivalent to the functor $e \colon \mathrm{mod}\,\Lambda \longrightarrow \mathrm{mod}\,a\Lambda a$ being an eventually homological isomorphism. We thank Hiroshi Nagase for pointing out that (a) implies (b) in the second part of the following result. This led to a much better understanding of the conditions occurring in the main results.

Lemma 8.23. *Let Λ be a finite dimensional algebra over an algebraically closed field k.*

   (i) *Assume that (α) $\mathrm{id}_\Lambda\left(\frac{\Lambda/\langle a \rangle}{\mathrm{rad}\,\Lambda/\langle a \rangle}\right) < \infty$ and (γ) $\mathrm{pd}_\Lambda\left(\frac{\Lambda/\langle a \rangle}{\mathrm{rad}\,\Lambda/\langle a \rangle}\right) < \infty$. Then $\mathrm{pd}_{\Lambda^e} \Lambda/\langle a \rangle < \infty$.*
   (ii) *Assume that $\langle a \rangle$ is a stratifying ideal in Λ. Then the following are equivalent.*
      (a) $\mathrm{pd}_{\Lambda^e} \Lambda/\langle a \rangle < \infty$.
      (b) *The functor $e \colon \mathrm{mod}\,\Lambda \longrightarrow \mathrm{mod}\,a\Lambda a$ is an eventually homological isomorphism.*

Proof. (i) For two primitive idempotents u and v in Λ, we have that

$$\mathrm{Hom}_{\Lambda^e}(\Lambda^e(u \otimes v), \Lambda/\langle a \rangle) \cong u(\Lambda/\langle a \rangle)v.$$

Then, if u or v occurs in a, then this homomorphism set is zero. Consequently we infer that the composition factors of $\Lambda/\langle a \rangle$ are direct summands of the semisimple

module $\left(\frac{\Lambda/\langle a\rangle}{\operatorname{rad}\Lambda/\langle a\rangle}\right)\otimes_k\left(\frac{\Lambda^{\mathrm{op}}/\langle a\rangle}{\operatorname{rad}\Lambda^{\mathrm{op}}/\langle a\rangle}\right)$. By Lemma 7.3 $\operatorname{pd}_{\Lambda^e}\left(\frac{\Lambda/\langle a\rangle}{\operatorname{rad}\Lambda/\langle a\rangle}\right)\otimes_k\left(\frac{\Lambda^{\mathrm{op}}/\langle a\rangle}{\operatorname{rad}\Lambda^{\mathrm{op}}/\langle a\rangle}\right)$ is finite, hence the claim follows.

(ii) By Corollary 3.12 and part (i), statement (b) implies (a).

Conversely, assume (a). For $j > \operatorname{pd}_{\Lambda^e}\Lambda/\langle a\rangle$ and any Λ-modules M and N we have that

$$\operatorname{Ext}^j_{\Lambda^e}(\Lambda,\operatorname{Hom}_k(M,N))\cong\operatorname{Ext}^j_{\Lambda^e}(\langle a\rangle,\operatorname{Hom}_k(M,N)).$$

Using the isomorphism in the proof of Proposition 3.3 in [41],

$$\operatorname{Ext}^i_{\Lambda^e}(\langle a\rangle,X)\cong\operatorname{Ext}^i_{a\Lambda a^e}(a\Lambda a,aXa),$$

we obtain that

$$\begin{aligned}
\operatorname{Ext}^i_{\Lambda^e}(\langle a\rangle,\operatorname{Hom}_k(M,N)) &\cong\operatorname{Ext}^i_{a\Lambda a^e}(a\Lambda a,a\operatorname{Hom}_k(M,N)a)\\
&\cong\operatorname{Ext}^i_{a\Lambda a^e}(a\Lambda a,\operatorname{Hom}_k(aM,aN))\\
&\cong\operatorname{Ext}^i_{a\Lambda a}(aM,aN))
\end{aligned}$$

for all Λ-modules M and N. Since $\operatorname{Ext}^i_{\Lambda^e}(\Lambda,\operatorname{Hom}_k(M,N))\cong\operatorname{Ext}^i_\Lambda(M,N)$, we obtain the isomorphism

$$\operatorname{Ext}^j_\Lambda(M,N)\cong\operatorname{Ext}^j_{a\Lambda a}(aM,aN)$$

for all $j > \operatorname{pd}_{\Lambda^e}\Lambda/\langle a\rangle$ and all Λ-modules M and N. Hence e is an eventually homological isomorphism. $\qquad\square$

The following result gives a characterization of condition (γ) when $\langle a\rangle$ is a stratifying ideal.

Lemma 8.24. *Let Λ be an artin algebra and a an idempotent in Λ. Assume that $\langle a\rangle$ is a stratifying ideal in Λ. Then we have (γ) $\operatorname{pd}_\Lambda\left(\frac{\Lambda/\langle a\rangle}{\operatorname{rad}\Lambda/\langle a\rangle}\right) < \infty$ if and only if $\operatorname{gl.dim}\Lambda/\langle a\rangle < \infty$ and $\operatorname{pd}_\Lambda\langle a\rangle < \infty$. Moreover, if (γ) holds, then (β) holds.*

Proof. Assume that (γ) $\operatorname{pd}_\Lambda\left(\frac{\Lambda/\langle a\rangle}{\operatorname{rad}\Lambda/\langle a\rangle}\right) < \infty$. It is clear that $\operatorname{pd}_\Lambda\langle a\rangle < \infty$ if and only if $\operatorname{pd}_\Lambda\Lambda/\langle a\rangle < \infty$. Since $\Lambda/\langle a\rangle$ as a Λ-module is filtered in simple modules occurring as direct summands in $(\Lambda/\langle a\rangle)/(\operatorname{rad}\Lambda/\langle a\rangle)$, we infer that $\operatorname{pd}_\Lambda\Lambda/\langle a\rangle < \infty$ by the property (γ). Since $\langle a\rangle$ is a stratifying ideal in Λ, we have that

$$\operatorname{Ext}^j_{\Lambda/\langle a\rangle}(X,Y)\cong\operatorname{Ext}^j_\Lambda(X,Y)$$

for all $j\geq 0$ and all modules X and Y in $\operatorname{mod}\Lambda/\langle a\rangle$. Using the above isomorphism and property (γ) again, we obtain that $\operatorname{id}_{\Lambda/\langle a\rangle}Y\leq\operatorname{pd}_\Lambda(\Lambda/\langle a\rangle)/(\operatorname{rad}\Lambda/\langle a\rangle)$ for all Y in $\operatorname{mod}\Lambda/\langle a\rangle$. Hence $\operatorname{gl.dim}\Lambda/\langle a\rangle < \infty$.

Assume conversely that $\operatorname{gl.dim}\Lambda/\langle a\rangle < \infty$ and $\operatorname{pd}_\Lambda\langle a\rangle < \infty$. From [54, Theorem 3.9] we have a finite projective resolution $0\longrightarrow\Lambda a\otimes_{a\Lambda a}Q_n\longrightarrow\cdots\longrightarrow\Lambda a\otimes_{a\Lambda a}Q_0\longrightarrow\langle a\rangle\longrightarrow 0$, where Q_i are projective $a\Lambda a$-modules. Then applying the exact functor $e = a-$, it follows from Proposition 2.2 that the sequence $0\longrightarrow Q_n\longrightarrow\cdots\longrightarrow Q_0\longrightarrow a(\langle a\rangle)\longrightarrow 0$ is exact. We infer that (β) $\operatorname{pd}_{a\Lambda a}a\Lambda < \infty$, since $a\langle a\rangle\cong a\Lambda$. Since $\operatorname{gl.dim}\Lambda/\langle a\rangle < \infty$ and $\operatorname{pd}_\Lambda\Lambda/\langle a\rangle < \infty$, we have that $\operatorname{pd}_\Lambda X\leq\operatorname{pd}_{\Lambda/\langle a\rangle}X+\operatorname{pd}_\Lambda\Lambda/\langle a\rangle$. We infer that (γ) $\operatorname{pd}_\Lambda\left(\frac{\Lambda/\langle a\rangle}{\operatorname{rad}\Lambda/\langle a\rangle}\right) < \infty$ holds.

The last claim follows immediately from the above. $\qquad\square$

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