

# Gossip Based Centroid and Common Reference Frame Estimation in Multi-Agent Systems

Mauro Franceschelli and Andrea Gasparri

**Abstract**—In this work the decentralized common reference frame estimation problem for multi-agent systems in absence of any common coordinate system is investigated. Each agent is deployed in a 2-D space and can only measure the relative distance of neighboring agents and the angle of their line of sight in its local reference frame, no relative attitude measurement is available. Only asynchronous and random pairwise communications are allowed between neighboring agents. The convergence properties of the proposed algorithm are characterized and its sensitiveness against additive noise on the relative distance measurements is investigated. An experimental validation of the effectiveness of the proposed algorithm is provided.

**Index Terms**—Multi-Agent Systems, Sensor Network Localization, Distributed Randomized Algorithms, Consensus, Gossip.

## I. INTRODUCTION

The development of decentralized motion coordination algorithms for networked multi-agent systems has drawn the attention of a large part of the control systems community. In this framework, coordination algorithms have been developed making use of relative distance measurements between agents to perform the most various tasks such as aggregation and dispersion under topological constraints [1], rendezvous for nonholonomic agents [2], leader following with switching topologies [3], attitude control [4], attitude tracking [5], connectivity maintenance [6] and formation control [7].

When dealing with decentralized motion coordination problems, the greatest limitation is that a common assumption forbids agents to have access to absolute position information (GPS) and thus have a common reference frame that makes it easy to interpret the information passed by other agents. In some cases the agents are not supposed to know their absolute position, but share a common attitude reference to exchange motion information. This can be achieved by using a compass and gravity as common reference for their coordinate system or other external references. In all these scenarios several technological countermeasures have to be undertaken for the implementation of coordination algorithms increasing the cost and complexity of the agents design.

In our opinion, having a common reference frame greatly simplifies the design of coordination algorithms. Furthermore, not requiring hardware components like GPS could significantly advance the technological feasibility of mobile swarms

of agents, reducing their dependence on the global positioning system in the low level control loops thus improving their robustness against GPS denial of service (DoS) attacks.

In this work, we propose a decentralized solution to the centroid and common reference frame estimation problem based on a gossip communication scheme [8], [9], [10], [11], [12], i.e., it requires only asynchronous and random pairwise communications between pairs of agents. The proposed algorithm does not require the agents to share any common reference frame nor to measure their attitude, or their relative orientation but only to detect and measure their relative position. A preliminary version of this work can be found in [13]. Furthermore we quantify the amplitude of the worst case errors in the estimated common reference frame.

The contributions of this paper are the following:

- A decentralized algorithm based on gossip that makes the multi-agent system achieve agreement on a common point and on a common reference frame in a 2-dimensional scenario.
- A theoretical analysis of the convergence properties of the algorithm and its sensitivity against additive noise on the relative distance measurements.
- Analytical bounds on worst case performance regarding errors in the estimated common reference frame.
- Experimental results to validate the noise model assumptions and verify the effectiveness of the proposed algorithm in a real-world scenario.

## II. RELATED WORK

The network centroid and common reference frame estimation problems are related to the network localization problem. In this field, several decentralized techniques have been proposed.

In [14] the problem of estimating the absolute positions of sensors in a sensor network is addressed. The authors propose DILOC, a distributed iterative algorithm based only on relative distance measurements. They assume that there exists a set of anchor nodes that know their positions and that the rest of the nodes are inside the convex hull spanned by the anchor nodes. They also assume that each node has a sufficient number of neighbors so that it can triangulate its position with respect to its neighbors.

In [15] the authors propose a distributed algorithm based on the information version of the Kalman Filter for the relative localization problem in sensor networks. The algorithm distribution is achieved by neglecting the coupling terms in the Information matrix. This allows to run an independent reduced-order filter onboard each node. An interlacement technique is proposed to cope with the error introduced by this approximation.

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In [16], the authors investigate under which conditions the sensor network localization problem is well-posed in the presence of noise. The authors show that given the graph theoretic conditions that would guarantee unique localizability in the noiseless case, localization in the noisy case can be stated as a minimization problem, the solution of which approaches the true sensor positions continuously as the noise perturbations on the measurements approach zero.

In [17], the authors investigate how to minimize the effects of noisy distance measurements on localization of multi-agent formations. In particular, they introduce a criterion to measure the effect of distance measurement error on the localization of agents. Then, they introduce a methodology for selection of anchors among the agents to minimize that error.

In [18], the formation stabilization problem for relative sensing networks is proposed. By assuming all the agents reference frames to be equally oriented, the authors propose a distributed coordination algorithm for the shape stabilization of the relative sensing network to a desired formation.

In [19], [20], [21], decentralized algorithms for the attitude synchronization problem in a 3-d space are given. Compared to our result the authors focus their analysis on the estimation of a common attitude in a 3d-space rather than a common reference frame and they assume the measurements to be noise-free while we consider noise in the relative distance measurements. Their approach is developed in continuous time and based on a synchronous communication mechanism while we consider asynchronous communications in discrete time, furthermore we only allow to measure the direction of the line-of-sight instead of the attitude.

In [22] the authors solve the localization problem under noisy measurements by estimating the relative positions of the sensors with respect to the network centroid. In particular, the authors do not require the availability of anchors but they assume all the agents to have their reference frames equally aligned by exploiting a compass or an attitude synchronization algorithm and adopt a synchronous communication mechanism. In this work we do not exploit these assumptions.

In [23] the synchronization problem for a multi-agent system where the state of the agents is represented by a phase in the unit circle is addressed. This method could be used in the framework of multi-vehicle systems to perform distributed agreement on a common heading if the information about the relative attitude of the agents is available. Our work differs from [23] in that to compute a common heading for the networked system we perform agreement on two distinct points of the plane and consider one of these as the origin of the estimated common reference frame while the other fixes a common heading. Our approach allows to compute a common heading without knowledge of the relative attitude for any possible initial condition.

In [24], [25] the decentralized centroid estimation problem in absence of reference frames is investigated in a 3-D space. The authors show that the algorithm converges if the network topology is described by a rigid graph.

Finally, examples of gossip based algorithms which exploit pairwise averaging can be found in [8] with application to the distributed average problem, in [10] for the distributed

averaging with quantized states and in [26], [12] for distributed load balancing, discrete consensus and heterogeneous multi-vehicle routing problem. Furthermore in [9], [11] algorithms for the distributed average problems based on a broadcast gossip communication scheme are presented.

### III. PROBLEM DESCRIPTION

Let graph  $\mathcal{G} = (V, E)$  describe the network topology of a multi-agent system, where  $V = \{1, \dots, n\}$  is the set of agents and  $E \subseteq \{V \times V\}$  is the set of edges representing possible communication channels. Let us denote with a time varying graph  $\mathcal{G}(t) = (V, E(t))$  the point-to-point interactions at time  $t$ , an edge  $(i, j) \in E(t)$  exists only if there is an interaction between agent  $i$  and  $j$  at time  $t$ . A position  $p_i = [p_{i,x} \ p_{i,y}] \in \mathbb{R}^2$  in a plane is associated to each agent  $i \in V$ . We assume that agents can not overlap position, that is  $p_i \neq p_j, \forall i, j \in V$ . Each agent has a local orthonormal reference frame  $\Sigma^i(p_i, \theta_i) = [\hat{x}_i, \hat{y}_i]^T$  where  $p_i$  is its origin. Let  $\theta_i$  be the angle between the  $x$ -axis of  $\Sigma^i$  and the  $x$ -axis of a common reference frame  $\Sigma$  unknown to every agent. The generic estimate of agent  $i$  of the agreement point is denoted as  $s_i$  in the common frame  $\Sigma$  while  $s_j^i$  denotes the estimate  $s_j$  in the reference frame  $\Sigma^i$ .

Let us consider a pair of agents  $i$  and  $j$  for which  $(i, j) \in E(t)$ , this implies that the agents are able to sense their relative position reciprocally with respect to their local orthonormal reference frames. To this end, let us define the direction of the line of sight by which agent  $i$  is able to sense agent  $j$  with respect to a common reference frame  $\Sigma$  as  $c_{ij} = \frac{(p_j - p_i)}{\|p_j - p_i\|}$  where  $\|\cdot\|$  is the Euclidean norm. The direction of the line of sight can be expressed with respect to the local frame  $\Sigma^i$  of agent  $i$  as  $c_{ij}^i = R(\theta_i)^T c_{ij}$ , where  $R(\theta_i)$  is a rotation matrix that aligns the local frame  $\Sigma^i$  to the common one  $\Sigma$ . Note that, it holds  $R(\theta_i)c_{ij}^i = -R(\theta_j)c_{ji}^j$ . Let us define the orthogonal vector  $c_{ij}^{\perp} = R(\theta_i)c_{ij}^i{}^{\perp}$  so that a right handed frame is built. In addition, let the relative distance between two agents  $i$  and  $j$  be  $d_{ij} = d_{ji} = \|p_i - p_j\|$ . Finally, graph  $\mathbb{G}(t, t+T)$ , represents the union of all the active edges during the interval of time  $[t, t+T]$  as  $\mathbb{G}(t, t+T) = \bigcup_{\tau=t}^{t+T} \mathcal{G}(\tau)$ . We now state the fundamental working assumptions that characterize this work:

- Communications are asynchronous based on gossip.
- Each agent can sense the relative position (distance and angle) of neighboring agents with respect to its local reference frame.
- The network topology can be described by a connected undirected time-varying graph.

Note that, for each agent  $i$  it is possible to express its estimate of the agreement point  $s_i$  with respect to a common reference frame as follows:

$$s_i = R(\theta_i)s_i^i + p_i. \quad (1)$$

We point out that in our framework the agents do not know parameters  $\theta_i$  and  $p_i$ , therefore they can not exploit eq. (1).

Our objective is to build a reference frame common to every agent in the network by making each agent estimate locally two common reference points in their own local set of coordinates.

#### IV. DECENTRALIZED ESTIMATION OF A COMMON POINT WITH NOISY MEASUREMENTS

In this section, we present an iterative algorithm to achieve an agreement on a common point in a 2-dimensional space under the working assumptions given in Section III.

The proposed algorithm consists in iteratively choosing at random pairs of agents  $i, j$  according to an edge selection process  $\epsilon$  that in turn perform the next four simple operations:

- 1) Estimation of the relative noisy distance  $\|p_i - p_j\|$  and direction of the line of sight ( $c_{ij}^i$  and  $c_{ji}^j$ ) in the respective local reference frames.
- 2) Exchange of the projection of the local estimations over their line of sight ( $s_i^i(t)^T c_{ij}^i$  and  $s_j^j(t)^T c_{ji}^j$ ) and the perpendicular to it ( $s_i^i(t)^T c_{ij}^{\perp}$  and  $s_j^j(t)^T c_{ji}^{\perp}$ ).
- 3) Conversion of the estimations of the location of the agreement point ( $s_i^i$  and  $s_j^j$ ) in local coordinates according to eq. (2).
- 4) Update of the current estimation ( $s_i^i(t+1)$  and  $s_j^j(t+1)$ ) according to the local update rule given in eq. (3).

In particular, agent  $i$  can compute  $s_j^j$  by exploiting the common line of sight  $c_{ij}^i$  as follows [13]:

$$s_j^j = (d_{ij} - s_j^{jT} c_{ji}^j) c_{ij}^i - (s_j^{jT} c_{ji}^{\perp}) c_{ij}^{\perp}, \quad (2)$$

while the local state update rule  $\mathcal{R}$  is:

$$s_i^i(t+1) = \frac{s_i^i(t) + s_j^j(t)}{2}, \quad s_j^j(t+1) = \frac{s_j^j(t) + s_i^i(t)}{2}. \quad (3)$$

Note that, communications are sequential, not parallel, therefore this locals state update rule can be implemented with a single communication channel used in alternating directions at different times (half-duplex). Indeed, the update does not need to be carried out at the same time by both agents. Furthermore, although an actual implementation takes a finite interval of time to apply the state update, the state trajectory can be analyzed without loss of generality by considering this interval of time as an instant in discrete time.

Let us now characterize how the local interaction rule is affected by the noisy measurement of the relative distance and of the angle of the line of sight between a pair of agents. Let  $\bar{d}_{ij} = \frac{\bar{d}_{ij} + \bar{d}_{ji}}{2} = \frac{d_{ij} + \bar{\delta}_{ij} + d_{ji} + \bar{\delta}_{ji}}{2} = d_{ij} + \frac{\bar{\delta}_{ij} + \bar{\delta}_{ji}}{2} = d_{ij} + \bar{\delta}_{ij}$  be the average between  $\bar{d}_{ij}, \bar{d}_{ji}$  which are respectively the inter-agent distance measured by agent  $i$  and agent  $j$  and  $\bar{\delta}_{ij}$  is the average of the relative distance measurement errors. It is clear that despite the fact that the two measurements are different, the agents use the same value of  $\bar{d}_{ij}$  in their respective computations.

Let  $R(\phi_i) c_{ij}^i$  ( $R(\phi_j) c_{ji}^j$ ) represent the line of sight between agent  $i$  and  $j$  measured by agent  $i$  (agent  $j$ ) where  $\phi_i$  ( $\phi_j$ ) represent its angular errors. Therefore, agent  $i$  computes the noisy estimate  $s_j^j$  by exploiting eq. (2) as follows:

$$\tilde{s}_j^j = (d_{ij} + \bar{\delta}_{ij} - s_j^{jT} R(\phi_j) c_{ji}^j) R(\phi_i) c_{ij}^i - (s_j^{jT} R(\phi_j) c_{ji}^{\perp}) R(\phi_i) c_{ij}^{\perp}. \quad (4)$$

Let us now consider the angular errors to be small so that the approximation of  $\sin(\phi_i) \approx \phi_i$  and  $\cos(\phi_i) \approx 1$  gives an error of less than 1%, i.e.,  $\phi_i \leq 0.145$  radians which

corresponds to about 8.3 degrees. The rotation matrix  $R(\phi_i)$  can be approximated as:

$$R(\phi_i) = \begin{bmatrix} \cos(\phi_i) & -\sin(\phi_i) \\ \sin(\phi_i) & \cos(\phi_i) \end{bmatrix} = I + \phi_i R(\pi/2). \quad (5)$$

Since  $R(\pi/2) c_{ij}^i = c_{ij}^{\perp}$  and  $R(\pi/2) c_{ji}^{\perp} = -c_{ji}^i$ , we get:

$$\tilde{s}_j^j = (d_{ij} + \bar{\delta}_{ij} - s_j^{jT} c_{ji}^j - \phi_j s_j^{jT} c_{ji}^{\perp}) (c_{ij}^i + \phi_i c_{ij}^{\perp}) - (s_j^{jT} c_{ji}^{\perp} - \phi_j s_j^{jT} c_{ji}^j) (c_{ij}^{\perp} - \phi_i c_{ij}^i). \quad (6)$$

By some manipulations eq. (6) can be put in the following form:

$$\tilde{s}_j^j = s_j^j + \bar{\delta}_{ij} c_{ij}^i - \phi_j \left( s_j^{jT} c_{ji}^{\perp} c_{ij}^i - s_j^{jT} c_{ji}^j c_{ij}^{\perp} \right) + \phi_i R(\pi/2) s_j^j + \phi_i \bar{\delta}_{ij} c_{ij}^{\perp} - \phi_j \phi_i \left( s_j^{jT} c_{ji}^j c_{ij}^i + s_j^{jT} c_{ji}^{\perp} c_{ij}^{\perp} \right). \quad (7)$$

It is clear from eq. (7) that angular errors have a disruptive potential for the convergence properties of the proposed algorithm since the terms proportional to it are also proportional to the norm of the current estimation  $\|s_j^j\|$ . Therefore, for the proposed algorithm it is mandatory to implement sensory systems that give precise angular measurements while errors in the distance measurements can be tolerated. In Section VII we performed experiments with our mobile robotic platform and show that even cheap vision-based systems can achieve sufficiently accurate angular measurements and therefore implement successfully the proposed algorithm. We analyze the effect of errors in the distance measurements assuming angular measurements to be accurate.

The following proposition shows how the proposed gossip algorithm can be stated with respect to a common reference frame.

**Lemma 1:** The gossip algorithm with set of states  $\mathcal{S}$  and local update rule  $\mathcal{R}$  as in eq. (3) can be equivalently stated with respect to a common reference frame  $\Sigma$  as follows:

$$\begin{aligned} x(t+1) &= W(t)x(t) + \frac{\bar{\delta}_{ij}}{d_{ij}} \left( \frac{(e_i - e_j)(e_i - e_j)^T}{2} p_x \right), \\ y(t+1) &= W(t)y(t) + \frac{\bar{\delta}_{ij}}{d_{ij}} \left( \frac{(e_i - e_j)(e_i - e_j)^T}{2} p_y \right), \end{aligned} \quad (8)$$

where the vectors  $x(t) = [x_1(t), \dots, x_n(t)]^T \in \mathbb{R}^n$  and  $y(t) = [y_1(t), \dots, y_n(t)]^T \in \mathbb{R}^n$  are a compact representation for the agents estimate  $s_i = [x_i(t), y_i(t)]^T$  with  $i = 1, \dots, n$ ,  $W(t)$  for a given time  $t$  such that  $\epsilon(t) = (i, j)$  is defined as  $W(t) = I - \frac{(e_i - e_j)(e_i - e_j)^T}{2}$ , with  $e_i = [0 \dots, 1, \dots, 0]^T$  a  $n \times 1$  vector with all elements equal to 0 but the  $i$ -th element equal to 1, and  $\bar{\delta}_{ij} = \frac{\bar{\delta}_{ij} + \bar{\delta}_{ji}}{2}$  is the average of the distance measurements errors.

*Proof:* See Appendix A. ■

The next Lemma shows that the average of the estimates is time-invariant despite noisy measurements.

**Lemma 2:** Let us consider a multi-agent system that executes the local update rule in eq. (3) and consider only noisy distance measurements. Then, the average  $\bar{s}$  of the agents' estimations is constant with respect to a common reference frame:

$$\bar{s}(t) = \frac{1}{n} \sum_{i=1}^n s_i(t) = \bar{s}, \quad \forall t \geq 0. \quad (9)$$

*Proof:* See appendix B. ■

Let us now introduce a Theorem to characterize the effect of noise or errors in the relative distance measurements.

**Theorem 3:** Consider the dynamics given in eq. (8) for a system of  $n$  agents and the noise affecting the distance measurements between each pair of agents  $i$  and  $j$ , to be upper bounded by  $\bar{\delta}$ . If  $\epsilon$  is such that  $\forall t, \exists T : \mathbb{G}(t, t+T)$  is connected, then the agents' estimates converge inside a ball of fixed radius around the point located at the average of the estimates  $\bar{s}$ , that is:

$$\forall i \in V, s_i \in \mathbb{B}_r, \quad \mathbb{B}_r = \left\{ z \in \mathbb{R}^2 : \|z - \bar{s}\| \leq d(\mathcal{G}) \bar{\delta} \sqrt{2} \right\}.$$

with  $d(\mathcal{G})$  the diameter of the graph  $\mathcal{G}$ .

*Proof:* See Appendix C ■

**Remark 1:** If each agent initializes its estimate  $s_i^i(0)$  to zero for all  $i = 1, \dots, n$ , and there are no measurement errors, then all the agents' estimates converge towards the network centroid, i.e.,  $\lim_{t \rightarrow \infty} s_i(t) = R_i(\theta_i) s_i^i(t) + p_i = \frac{1}{n} \sum_{i=1}^n p_i$ .

Regarding the convergence time of the proposed algorithm, the general analysis for gossip algorithms based on randomized and pairwise averaging proposed in [8] can be applied. In particular, by following this analysis let us assume each agent  $i$  to have a clock which ticks at the times of a rate 1 Poisson process. At each tick, this agent  $i$  performs a state update. Furthermore, let us define the  $\epsilon$ -averaging time of a gossip algorithm as:

$$T_{conv}(\epsilon) = \sup_{s(0)} \inf \left\{ t : Pr \left( \frac{\|s(t) - \bar{s}\|}{\|s(0)\|} \geq \epsilon \right) \leq \epsilon \right\},$$

where  $\bar{s}$  is the point where the algorithm converges, i.e., the average of the initial conditions. In particular, the following upper-bound for  $T_{conv}(\epsilon)$  has been given in [8]:

$$T_{conv}(\epsilon) \leq \frac{3 \log \epsilon^{-1}}{\log \lambda_2 (E[W])^{-1}}, \quad (10)$$

where the term  $\lambda_2(E[W])$  represent the second smallest eigenvalue of the expectation of matrix  $W(k)$  which is chosen according to a stochastic process.

## V. DECENTRALIZED ESTIMATION OF A COMMON REFERENCE FRAME

In this section, a technique to build a common reference frame in a decentralized fashion by exploiting the algorithm introduced in Section IV is described. The key idea is to let the network of agents estimate two common reference points in local coordinates and use them to compute a reference frame common to every agent. More specifically, the algorithm works as follows:

- 1) Estimate two common reference points in local coordinates  $\mathcal{F}_i = \{f_{1,i}, f_{2,i}\}$
- 2) Use the first point  $f_{1,i}$  to identify the origin of the common frame  $O_r = f_{1,i}$  and the second point  $f_{2,i}$  to compute a common heading vector  $x$  with unitary norm.

The proposed approach is based on distance and line of sight measurements affected by noise, bias or errors. Therefore, agreement on these two points (one being the barycenter of

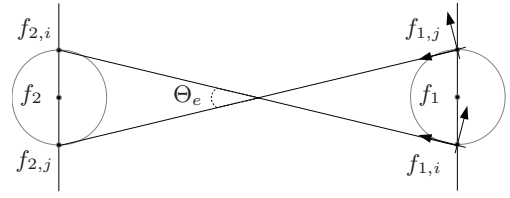


Fig. 1. Maximum orientation error between two estimated common frames

the network and another being a random point in space) is affected by a maximum error characterized in Theorem 3, that is, each agent estimates a point inside a circle centered at the barycenter of the initial condition of the estimates with radius  $d(\mathcal{G})\bar{\delta}\sqrt{2}$ . Therefore, the common reference frame between any pair agents might have an origin that differs at most by  $2d(\mathcal{G})\bar{\delta}\sqrt{2}$ . The error in orientation depends also on the estimation of the second common point. As shown in Figure 1, to compute the maximum error we draw two parallel lines passing through the average of the estimated points  $f_1 = \frac{1}{n} \sum_{i=1}^n f_{1,i}$  and  $f_2 = \frac{1}{n} \sum_{i=1}^n f_{2,i}$  and then connect the intersection of the circles with the parallel lines to find the worst case scenario that constructs two reference frames with maximum orientation error. Let  $r = d(\mathcal{G})\bar{\delta}\sqrt{2}$  and  $d = \|f_2 - f_1\|_2$ . The maximum orientation error  $\Theta_e$  is:

$$\Theta_e = 2 \arcsin \left( \frac{2r}{\sqrt{4r^2 + d^2}} \right).$$

Notably, the orientation error depends both on the maximum agreement error and on the distance between the two estimated points. In particular, to reduce this error is sufficient to increase numerically the amplitude of the random initial conditions used to estimate  $f_2$ .

## VI. SIMULATION RESULTS

A first campaign of simulations was carried out to evaluate the tightness of the upper bound given in eq. (10) by assuming  $\epsilon = 1\%$ . In particular, we considered a varying number of agents ranging from 5 to 25 and ideal (noise-free) measurements. Each simulation was performed on a random network topology and the average over 100 trials was computed. Each agent executed a state update with a random neighbor according to a stochastic process with Poisson probability distribution. Table I reports the obtained results with time units in seconds. It can be noticed that the actual averaging time  $T_{conv}(\epsilon)$  is significantly lower than its upper bound.

TABLE I  
CONVERGENCE TIME ANALYSIS FOR  $\epsilon = 1\%$  WITHOUT NOISE AND VARYING NODES

Number of Agents	5	10	15	20	25
Average Time	5.02	13.78	14.74	11.04	9.81
Upper Bound	42.45	73.57	130.65	186.68	219.09

A second campaign of simulations was carried out to evaluate the effectiveness of the algorithm for noisy measurements of the relative distance and of the angle of the line of sight between a pair of agents. An additive noise bounded by a percentage  $\bar{\delta} \leq \frac{\epsilon}{d_{ij}} \cdot 100$  of the distance between the given pair of agents was considered on distance measurements.



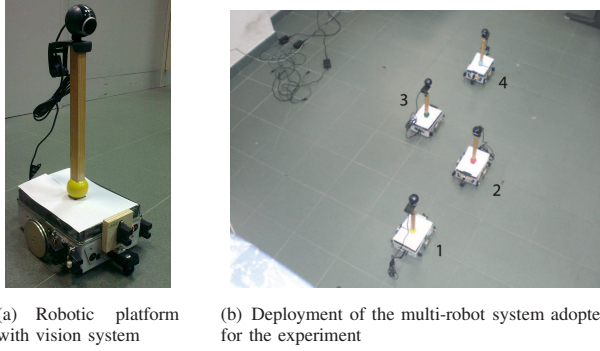


Fig. 2.

Furthermore, an additive bounded noise  $\phi \in [-a, a]$  was also considered on the measurements of the angle of the line of sight between a pair of agents. The compact notation  $(p, a)$  to specify noise parameters is used in Table II. Different network topologies with a fixed number of agents (10) and an increasing magnitude of noise were considered. For each simulation results refer to the average over 100 trials. A randomized uniform edge selection process was used. Table II reports the obtained results in terms of relative errors with respect to the true distance between the generic agent and the average of the initial conditions. Initial conditions were chosen to be zero in local coordinates for every agent, the common heading vector was computed with random initial condition with magnitude 10. It can be noticed that the estimation error of the location of the agreement point increases with the increase of noise magnitude.

TABLE II  
RANDOM NETWORK TOPOLOGY (10 NODES) WITH DISTANCE AND ANGULAR ERRORS ( $\pm[\% \text{DISTANCE}]$ ,  $\pm[\text{deg}]$ )

Noise	(10, 4)	(15, 8)	(20, 12)	(25, 16)	(30, 20)
Max % Err.	0.019%	0.036%	0.058%	0.081%	0.132%
Mean % Err.	0.010%	0.018%	0.034%	0.051%	0.080%
Max Head. Err.	0.225°	0.424°	0.733°	0.912°	1.306°

A third campaign of simulations was carried out to evaluate the performance of the algorithm on ring topologies. We considered a maximum error on the distance measurements bounded by 10% of the true distance between pairs of nodes and angular errors bounded by  $\pm 4$  deg degrees. We considered a varying number of agents ranging from 5 to 25 with constant edge length. A randomized uniform edge selection process was used. The common heading vector was computed with random initial conditions with magnitude equal to 10. Table III reports the obtained results in terms of relative errors with respect to the true distance between the generic agent and the average of the initial conditions. It can be seen that even if significant angular errors are considered the algorithm still achieves small relative errors with respect to the distance between the agents and the agreement point. The absolute size of the errors increases as the diameter of the network increases. Notably, the case with 10 agents provides similar results to the ones obtained in Table II showing that the existence of cycles does not affect the algorithm performance.

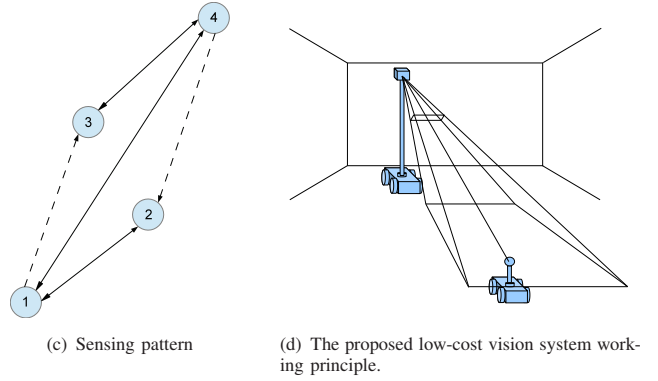


TABLE III  
RING NETWORK TOPOLOGY WITH BOUNDED NOISE ( $\pm 10\%$ ,  $\pm 4$  deg) AND VARYING NODES

Number of Agents	5	10	15	20	25
Max % Error	0.023%	0.030%	0.029%	0.064%	0.165%
Mean % Error	0.016%	0.017%	0.015%	0.046%	0.147%
Max Heading Err.	0.384°	0.177°	0.147°	0.238°	0.461°

## VII. EXPERIMENTAL RESULTS

Experiments have been carried out by exploiting 4 units of the mobile robotic platform SAETTA which is a low-cost robotic platform developed at the Rome Tre University, see [27] for details. In particular, each unit was equipped with the low-cost vision system shown in Figure 2(a), which allows to compute the relative distance among robots. Figure 2(b) depicts the deployment of the multi-robot system adopted for the experiments.

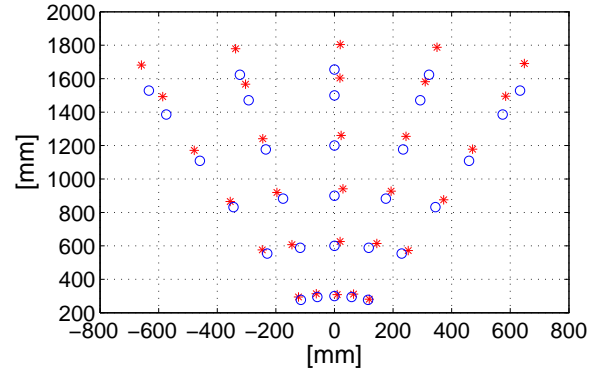


Fig. 3. Vision System Calibration Results. Crosses (red) represent the real measured location while circles (blue) the measured location after calibration.

In Section IV, a theoretical characterization of the proposed algorithm in the presence of noisy measurements has been carried out under the assumption of accurate angular measurements. The validity of this assumption for the low-cost vision system used for the experiments is now investigated.

Figure 2(d) depicts the low-cost vision system working principle: assuming landmarks (colored spheres) to be located at a certain height  $h_m$  and a webcam to be mounted on top of a pole at a given height  $h_p$  with a given inclination  $\alpha$ , it is possible to obtain a simple geometrical relationship between the landmark location and the pixels of the CCD of the camera.

Figure 3 describes the accuracy achievable by the adopted

vision system. Crosses (red) represent the real locations (colored spheres) detected by the webcam, while circles (blue) describe the estimated locations computed by the vision system after calibration. Due to the bandwidth constraints imposed by the hardware communication, i.e., USB 2.0 full-speed port, frames were collected at very low resolution,  $176 \times 144$  pixel. It should be noticed that while the error grows with the distance on the radial coordinate, it can be considered negligible on the angular coordinate. In particular, the radial error is proportional to the measured distance with an upper bound after the calibration less than 10% of the measured distance, while the angular error can be described with a bounded uniform distribution  $[-4 \text{ deg}, +4 \text{ deg}]$ . As a matter of fact, the angular error is sufficiently small for the adopted low-cost vision system to justify the approximation made in in eq. (5).

Figure 2(c) describes the sensing patterns where continuous lines represent bi-directional sensing links over which an update between a pair of robots can be carried out, while dashed lines represent unidirectional sensing links which cannot be used to perform an update. The maximum distance among robots 1 and 4 is roughly 1500 [mm]. Robots 1 and 4 are able to sense all the other robots, robots 2 and 3 are able to sense only robots 1 and 4, respectively. Regarding the

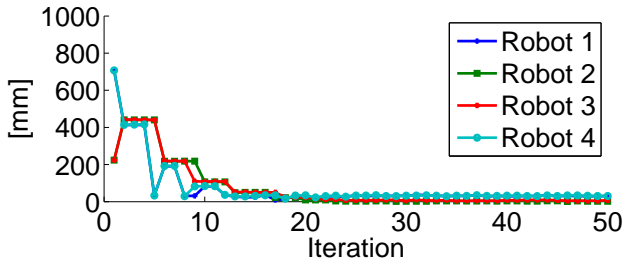


Fig. 4. Robots Network Centroid Estimation Error.

algorithm implementation, collaboration among pairs of robots was achieved by means of a gossip communication scheme: i) each robot woke up randomly every 5 seconds and interacted with one of its neighbors at random, ii) interacting robots performed the state update rule given in eq. (3).

Figure 4 depicts the network centroid estimation error for the four robots running the proposed algorithm in the presence of noisy measurements due to the adopted low-cost vision system. According to the obtained results, each robot was able to achieve an estimate of the network centroid with a bounded error. Note that, all the estimation errors  $\{32.84, 3.50, 6.60, 32.95\}$  [mm] are significantly below the theoretical upper bound given in Theorem 3 which is fixed to  $\simeq 300$  [mm], being  $d(\mathcal{G}) = 3$ , and  $\bar{\delta} = 70.7$  [mm]. Furthermore, a second point was computed to build a common reference frame. To this end, the agents' initial conditions were chosen uniformly at random such that  $\|s_i^z(0)\| \in [0, 10000]$  [mm] resulting in an agreement point at coordinates  $f_2 = [4140, 1880]$  [mm]. The estimation errors are  $\{29.95, 8.29, 10.75, 35.78\}$  [mm]. The distance between the two agreement points is  $d = 2562$  [mm]. The maximum angular error in the estimated common reference frame is  $\Theta_e = 0.0052^\circ$ , corroborating the fact that by choosing initial

conditions sufficiently large for the second point ( $f_2$ ) results in a negligible angular error on the computation of the common reference frame.

## VIII. CONCLUSIONS

In this work we presented a decentralized algorithm to estimate the centroid and common reference frame in a network of agents. We considered noisy relative positions measurements with respect to local reference frames and characterized analytically the worst case performance. We provided simulations and experiments to validate the effectiveness of the proposed algorithm. Future work will focus on extending the method to a dynamic setting.

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## APPENDIX

### A. Proof of Lemma 1

Assume an update for a pair of agents  $i$  and  $j$  is carried out at time  $t$ . According to eq (1), the update of the agent  $i$  with respect to a common frame  $\Sigma$  can be expressed as:

$$\begin{aligned} s_i(t+1) &= \frac{(R(\theta_i) s_i^i(t) + p_i) + (R(\theta_j) s_j^j(t) + p_j)}{2} \\ &= \frac{s_i(t) + s_j(t) + \bar{\delta}_{ij} c_{ij}}{2}. \end{aligned}$$

Equivalently, it is possible to write the update of the agent  $j$  with respect to a common frame  $\Sigma$  as given in eq. (1) where it should be noticed that  $\bar{\delta}_{ji} = \bar{\delta}_{ij}$  since it is the computed average of the errors introduced by the two agents noisy measurements.

At this point, by recalling the definition of  $c_{ij}$  and by considering that the update rule is decoupled along the  $x$ -axis and  $y$ -axis, the thesis follows. ■

### B. Proof of Lemma 2

Consider eq. (8) along the  $x$ -axis. Since  $W(t)$  is a symmetric doubly stochastic matrix, it holds:

$$\begin{aligned} \mathbf{1}_n^T x(t+1) &= \mathbf{1}_n^T \left[ W(t)x(t) + \frac{\bar{\delta}_{ij}}{d_{ij}} \left( \frac{(e_i - e_j)(e_i - e_j)^T}{2} p_x \right) \right] \\ &= \mathbf{1}_n^T W(t)x(t) + \frac{\bar{\delta}_{ij}}{d_{ij}} \mathbf{1}_n^T \left( \frac{(e_i - e_j)(e_i - e_j)^T}{2} p_x \right) \\ &= \mathbf{1}_n^T x(t) \end{aligned}$$

where the fact  $\mathbf{1}_n^T \left( \frac{(e_i - e_j)(e_i - e_j)^T}{2} \right) = \mathbf{0}_n^T$  has been exploited. It follows:

$$\mathbf{1}_n^T x(t) = \mathbf{1}_n^T x(0), \quad \forall t \in \mathbb{R}^+.$$

Furthermore, since the same holds also along the  $y$ -axis, then  $\bar{s}(t) = \bar{s}$ ,  $\forall t \geq 0$ , thus proving the statement. ■

### C. Proof of Theorem 3

Consider the following Lyapunov function:

$$V(t) = V_x(t) + V_y(t) = x(t)^T P x(t) + y(t)^T P y(t),$$

with  $P = I - \frac{\mathbf{1}\mathbf{1}^T}{n}$  such that  $P \geq 0 \quad \forall z_x(t) \notin \text{span}(\mathbf{1})$ . Thus,  $P$  is positive semi-definite and its null space represents a

consensus between the agents estimates. In particular,  $V_x(t+1)$  can be written as:

$$\begin{aligned} V_x(t+1) &= x(t)^T W^T P W x(t) \\ &\quad + \frac{\bar{\delta}_{ij}^2}{d_{ij}^2} \left( \frac{(e_i - e_j)(e_i - e_j)^T}{2} p_x \right)^T P \left( \frac{(e_i - e_j)(e_i - e_j)^T}{2} p_x \right) \\ &= x(t)^T \left( W - \frac{\mathbf{1}\mathbf{1}^T}{n} \right) x(t) + \frac{\bar{\delta}_{ij}^2}{d_{ij}^2} \frac{(p_{i,x} - p_{j,x})^2}{2} \end{aligned}$$

where  $W^T P W = W - \frac{\mathbf{1}\mathbf{1}^T}{n}$  follows from the fact that  $W^2 = W$  and  $W \frac{\mathbf{1}\mathbf{1}^T}{n} = \frac{\mathbf{1}\mathbf{1}^T}{n}$ . Let us now consider the difference  $V_x(t+1) - V_x(t)$  as follows:

$$V_x(t+1) - V_x(t) = -\frac{(x_i(t) - x_j(t))^2}{2} + \frac{\bar{\delta}_{ij}^2}{d_{ij}^2} \frac{(p_{i,x} - p_{j,x})^2}{2}.$$

By noticing that  $(p_{i,x} - p_{j,x})^2 \leq d_{ij}^2$ , it holds:

$$\frac{\bar{\delta}_{ij}^2}{d_{ij}^2} \frac{(p_{i,x} - p_{j,x})^2}{2} \leq \frac{\bar{\delta}_{ij}^2}{2} \leq \frac{\bar{\delta}^2}{2},$$

with  $\bar{\delta} = \max_{\{(i,j) \in E\}} \bar{\delta}_{ij}$ . This implies that the difference  $V_x(t+1) - V_x(t)$  can be bounded by:

$$V_x(t+1) - V_x(t) \leq -\frac{(x_i(t) - x_j(t))^2}{2} + \frac{\bar{\delta}^2}{2}. \quad (11)$$

Therefore the component  $V_x(t)$  of  $V(t)$  is decreasing if  $\bar{\delta}^2 \leq (x_i(t) - x_j(t))^2$  when agents  $i$  and  $j$  perform a local update. The same reasoning with an equivalent result can be applied to the term  $V_y(t+1) - V_y(t)$ . Therefore:

$$V(t+1) - V(t) \leq -\frac{(x_i(t) - x_j(t))^2}{2} - \frac{(y_i(t) - y_j(t))^2}{2} + \bar{\delta}^2.$$

Then, the following inequality is obtained:

$$2\bar{\delta}^2 \leq (x_i(t) - x_j(t))^2 + (y_i(t) - y_j(t))^2, \quad \forall (i, j) \in E.$$

Thus,  $V(t)$  is decreasing when agents  $i$  and  $j$  perform a local update if  $\bar{\delta} \sqrt{2} \leq \|s_i - s_j\|$ . After a sufficient number of iterations, it holds:

$$\max_{i,j \in V} \|s_i(t) - s_j(t)\| \leq d(\mathcal{G}) \bar{\delta} \sqrt{2}, \quad (12)$$

where  $d(\mathcal{G})$  is the diameter of the graph  $\mathcal{G}$ . This is due to the fact that for any pair of agents  $i, j \in V$  linked by a path  $p_{ij} = \{i, k, m, \dots, h, j\}$  with  $|p_{ij}| = p$ , it holds:

$$\begin{aligned} \|s_i(t) - s_j(t)\| &= \|s_i - s_k + s_k - \dots - s_h + s_h - s_j\| \\ &\leq \|s_i(t) - s_k(t)\| + \|s_k(t) - s_m(t)\| + \dots \\ &\quad + \|s_h(t) - s_j(t)\| \leq p \bar{\delta} \sqrt{2}, \end{aligned}$$

Therefore, the inequality given in eq. (12) follows from the fact that  $d(\mathcal{G})$  is the longest among the shortest paths between any pair of agents. Furthermore, from eq. (12) it holds:

$$\begin{aligned} \left\| s_i(t) - \sum_{j=1}^n \frac{s_j(t)}{n} \right\| &= \frac{1}{n} \left\| n s_i(t) - \sum_{j=1}^n s_j(t) \right\| \\ &\leq \frac{1}{n} \sum_{j=1}^n \|s_i(t) - s_j(t)\| \\ &\leq d(\mathcal{G}) \bar{\delta} \sqrt{2}, \quad \forall i \in V. \end{aligned}$$

Finally, let us recall that according to Lemma 2 the average of the agents estimates is preserved over time. Thus, it holds  $\|s_i(t) - \bar{s}\| \leq d(\mathcal{G}) \bar{\delta} \sqrt{2}$ ,  $\forall i \in V$ , proving the statement. ■