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Grad ϕ Perturbations of Massless Gaussian Fields

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Abstract. We investigate weak perturbations of the continuum massless Gaussian measure by a class of approximately local analytic functionals and use our general results to give a new proof that the pressure of the dilute dipole gas is analytic in the activity.

1. Introduction, Notation, Results

The classical dipole gas at equilibrium is a difficult system. Many foundational results have been obtained [1–5] but for the purposes of this paper the work of Gawedzki-Kupiainen [4, 5] is most relevant. This was the first paper in which the renormalization group was explicitly used on this problem and this paper is an attempt to improve on their methods and results as a step on the way to other problems. For example we expect these methods to be effective in the analysis of self-repelling walk in four dimensions, screening or its absence in quantum statistical Coulomb systems and the φ_4^4 massless lattice quantum field theory.

Our results are designed to provide a framework for the renormalization group in the context of perturbations of the (continuum) massless Gaussian random field. We will use the dipole gas for motivation.

We will first describe the results and proof omitting some technical aspects and then return to give the definitions and state the results carefully.

The Dipole Gas

We consider N dipoles in a periodic box $\Lambda \subset \mathbb{R}^d$, $d \geqq 1$. Each dipole is described by a position coordinate x and a unit polarization vector $\hat{p}$. We unite these degrees of freedom into $\xi \equiv (x, p)$ and let $d\varrho(\xi) = dxdS(\hat{p})$, where dS is the normalized Lebesgue measure on the surface of the unit ball. The fundamental objects to study are all

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derived from the *partition function*,

$$Z(\Lambda, z, \beta) = \sum_N \frac{z^N}{N!} \int d^N\varrho \exp[-\beta V(\underset{\sim}{\xi})],$$

$$V(\underset{\sim}{\xi}) = \tfrac{1}{2} \sum_{i,j=1,N} v(\xi_i, \xi_j).$$

$v(\xi, \xi')$ is the potential energy of two dipoles in a periodic box,

$$v(\xi, \xi') = (\hat{p} \cdot \partial_x)(\hat{p}' \cdot \partial_{x'}) v(x - x'),$$

where $v(x-x') =$ "$|x-x'|^{2-d}$" is the Coulomb potential energy of two charges in a periodic box. We have put in the quotes because we will have to modify the Coulomb potential to resolve the ambiguities connected with periodic boundary conditions and stability.

An old question for this system is whether the pressure, which is defined by,

$$\beta P = \operatorname*{Lim}_{|\Lambda| \to \infty} P(\Lambda, z, \beta),$$

$$P(\Lambda, z, \beta) \equiv \frac{1}{|\Lambda|} \operatorname{Log} Z(\Lambda, z, \beta)$$

is analytic in the *activity* z, near the origin. If v (without periodic boundary conditions) had integrable decay as $|x-x'| \to \infty$, this would be a standard result in the theory of the Mayer expansion [7, 11]. However $|v(\xi, \xi')|$ behaves like $|x-x'|^{-d}$ when $|x-x'|$ is large. Nevertheless, as we shall prove in Corollary C, the pressure is analytic.

Results on the analyticity of the pressure for a dipole gas were first obtained in [4]. They considered a lattice dipole gas instead. We are studying the continuum problem. There are some advantages to the continuum: scaling is more convenient and estimates on covariances are immediate. A related continuum problem was studied in [6].

The first step is to rewrite the partition function as a functional integral using the Sine-Gordon transformation.

The Sine-Gordon Transformation

If $v(x, x')$ is positive semi-definite and sufficiently regular at the origin then there exists a Gaussian Borel measure $d\mu_v$ defined on a space of continuously differentiable functions $\phi(x)$ on the torus Λ such that

$$e^{-V(\underset{\sim}{\xi})} = \int d\mu_v(\phi) \exp\left[i \sum_j \phi(\xi_j)\right],$$

where $\phi(\xi) = \phi(x, \hat{p}) = (\hat{p} \cdot \partial)\phi(x)$. Upon substituting this into the partition function, we obtain

$$Z(\Lambda, z, \beta) = \int d\mu_v(\phi) Z(\Lambda, \phi),$$

$$Z(\Lambda, \phi) \equiv \exp[2z \int d\varrho(\xi) \cos[\beta^{1/2} \phi(\xi)]]. \tag{1.1}$$

Formulas of this type are called *Sine Gordon Transformations*. They were invented by Siegert [12] and Kac [13].

Wilson's Renormalization Group

The idea is to evaluate the functional integral (1.1) by a sequence of integrals (Fubini's Theorem), which are associated with increasing length scales, L^j, $j=1,2,\ldots$, and to examine the evolution of the perturbation under these integrations. To do this we split the covariance v into two pieces $v=v'+C'$, where

$$\hat{v}'(k)=L^2\hat{v}(Lk), \qquad \hat{C}'(k)=\hat{v}-\hat{v}'. \tag{1.2}$$

At $k=0$ the Fourier transform $\hat{v}(k)$ has a $1/k^2$ singularity. This singularity is cancelled in $\hat{C}'$ and therefore $C'(x)$ will have good decay properties characterized by the length L. Furthermore

$$Z(\Lambda,z,\beta)=\int d\mu_{v'}(\phi)\int d\mu_{C'}(f)Z(\Lambda,\phi+f)$$

because $\phi+f$ is a Gaussian random variable with covariance v and a gaussian measure is characterized by its covariance. This way of splitting the integration was used in [8]. Our objective is to describe the evolution of Z under the transformation

$$Z(\Lambda,\phi)\to\int d\mu_{C'}(f)Z(\Lambda,\phi+f)\equiv(\mu_{C'}*Z)(\Lambda,\phi). \tag{1.3}$$

After this transformation it remains to carry out an integral with respect to $d\mu_{v'}$, but $d\mu_{v'}$ is related to the original measure $d\mu_v$ by scale transformation. Thus it is customary to perform a rescaling $\mathscr{R}$ to change $d\mu_{v'}$ back into $d\mu_v$. A single renormalization group step consists of (1.3) followed by $\mathscr{R}$.

Wilson studied the transformation

$$V\to-\log(\mu_{C'}*e^{-V}) \tag{1.4}$$

of V instead of Z. In perturbation theory V changes in a simple way compared with Z.

The Large Field Problem

Gawedzki-Kupiainen managed to follow this route but they had to overcome a serious difficulty: they constructed the logarithm in (1.4) not by perturbation theory (which is an incomplete description because it diverges) but by convergent Mayer expansions. Unfortunately not even Mayer expansions can be expected to converge for all values of the field, so they had to resort to an ingenious hybrid representation in which the Mayer expansion was used to construct V in those regions of space where $\phi(x)$ is small, but in other regions of space where $\phi(x)$ is permitted to be large, the perturbation is left in the form $\exp(-V)$.

Our method is different: we replace the role of exponential and logarithm by two objects which we call $\mathscr{E}$xp and $\mathscr{L}$og respectively. They are indeed an exponential and a logarithm but they are defined by using a different algebraic structure on the class of allowed functionals. They resemble the standard exponential and logarithm sufficiently to allow us to use them in their place but they are defined by power series which terminate after finitely many terms so that convergence problems are absent: we do not have to resort to different procedures depending on the size of ϕ. The $\mathscr{E}$xp is not as new as we have made it sound, it is merely an efficient way of writing the "hard core polymer gas."

An Invariant Form for the Interaction

We will introduce the $\mathscr{E}$xp using the dipole gas example. Let us decompose the torus Λ into closed unit cubes Δ with disjoint interiors centered on an integral simple cubic lattice. We will call these cubes $\Lambda^{(0)}$ cubes. When we say a set *is in* $\Lambda^{(0)}$ we shall mean that it is a union of $\Lambda^{(0)}$ cubes. For each $\Lambda^{(0)}$ cube Δ we set

$$P(\Delta,\phi)=\exp\left[2z\int_{\Delta} d\varrho(\xi)\cos[\beta^{1/2}\phi(\xi)]\right]-1\,.$$

Then the initial interaction for the dipole gas is given by

$$Z(\Lambda,\phi)=\prod_{\Delta}[P(\Delta)+1]=\sum\frac{1}{N!}\sum_{\Delta_1,\ldots,\Delta_N}\prod_i P(\Delta_i)\,. \tag{1.5}$$

In the sum the Δ_i are required to have disjoint interiors. $P(\Delta,\phi)$ is a functional of ϕ that, intuitively speaking, depends on $\phi(x)$ for all x in an infinitesimal neighborhood of Δ. In view of this tendency of $P(\Delta)$ to spread out a little beyond Δ we glue together any two cubes Δ_i and Δ_j in the above sum whenever they are not strictly disjoint. This is done by defining

$$K(X)=\sum\frac{1}{N!}\sum_{\Delta_1,\ldots,\Delta_N\in\mathscr{C}(X)}\prod_i P(\Delta_i)\,, \tag{1.6}$$

where $\Delta_1,\ldots,\Delta_N\in\mathscr{C}(X)$ iff

1. $X=\cup\Delta_i$,
2. $\Delta_1,\ldots,\Delta_N$ are distinct,
3. X is a connected set.

With K defined this way, it follows that

$$Z(\Lambda,\phi)=\sum_{N=0}^{\infty}\frac{1}{N!}\sum_{X_1,\ldots,X_N\subset\Lambda}K(X_1,\phi)\ldots K(X_n,\phi)\,,$$

where the sets X_i are in $\Lambda^{(0)}$ and are disjoint. Since they are closed, disjointness implies that they are separated by a distance of at least one. This type of formula is often called a "polymer gas." We will now show that it is also an exponential.

We define[1] a commutative product, denoted $\circ$. on functions $F(X)$ of sets. For the moment let the domain of our functions be the null set and sets in $\Lambda^{(0)}$. The product is

$$(F_1\circ F_2)(X)=\sum_{Y,Z:Y\dot{\cup}Z=X}F_1(Y)F_2(Z)\,, \tag{1.7}$$

where $X=Y\dot{\cup}Z$ iff $X=Y\cup Z$ and $Y\cap Z=\Phi$. This product is suggested by an analogous product in [7]. The $\circ$ Identity $\mathscr{I}$ is

$$\mathscr{I}(X)=1 \quad \text{if} \quad X=\Phi$$
$$=0 \quad \text{otherwise}\,.$$

[1] We would like to thank Giovanni Gallavotti for suggesting that this product is the correct approach and Shelley Goldstein for suggesting we approach it correctly

We set $K(X)=0$ if $X=\Phi$. If we define the $\circ$ exponential $\mathscr{E}$xp by the power series $\mathscr{E}\text{xp}[K]=\mathscr{I}+K+K\circ K/2!+\ldots$, then $Z(\Lambda)=\sum_{X\subset\Lambda}\mathscr{E}\text{xp}[K](X)$. The sum over X is not very attractive and we can remove it by defining a "space filling" function $\square$. The space we wish to fill is $\Lambda - X$ but this is not in the domain of our functions so we have to enlarge the domain: define a *cell* to be an open block, face, edge, ..., a single point at the corner of a block. Let the domain of our functions be all unions of cells and the null set. We extend K by setting $K(X)=0$ whenever X is not a set in $\Lambda^{(0)}$. Define

$$\square(X)=1 \quad \text{if } X \text{ is a cell}$$
$$=0 \quad \text{otherwise}.$$

Now we have

$$Z(X,\phi)=\mathscr{E}\text{xp}[\square+K](X,\phi). \tag{1.8}$$

The right-hand side is a form which will remain invariant under renormalization group transformations. Notice that $\square$ is the "zero" perturbation because $\mathscr{E}\text{xp}[\square](X)=1$ for all X.

The Renormalization Group Step

A single renormalization group step has four parts, *changing the scale in* $\mathscr{E}$xp, *renormalization, rescaling, integrating out a fluctuation field.* They will be performed in this order but we will explain them in a different order.

Integrating out a fluctuation field. Given any function $F(X)$ with the property that $F(\Phi)=1$, we can define the logarithm by its power series:

$$(\mathscr{L}\text{og}F)(X)\equiv(\mathscr{L}\text{og}[\mathscr{I}+F-\mathscr{I}])(X)\equiv(F-\mathscr{I})-\tfrac{1}{2}(F-\mathscr{I})^2+\ldots.$$

Both the power series for $\mathscr{L}$og and $\mathscr{E}$xp terminate after finitely many terms. *Integrating out a fluctuation field* is the transformation

$$K\to K', \quad \text{where} \quad \square+K'\equiv\mathscr{L}\text{og}[\mu*\mathscr{E}\text{xp}[\square+K]], \tag{1.9}$$

where μ is a Gaussian measure, (such as $\mu_{C'}$).

Changing the scale in $\mathscr{E}$xp. The transformation (1.9) will have good properties when the decay length of the covariance is comparable with the side of the cubes used to define $\mathscr{E}$xp. This condition is imposed because the analysis of this map depends on fields in regions separated by more than the decay length being approximately independent. Thus we have to supplement (1.9) with another identity that changes the scale in $\mathscr{E}$xp because otherwise it will continue to be the unit scale while successive integrations are characterized by scales L^j with $j=1,2,\ldots$. To this end we write the torus as a union of closed cubes on the next scale. These are cubes with disjoint interiors and side L which are unions of the old $\Lambda^{(0)}$ cubes. We call these new cubes $\Lambda^{(1)}$ cubes and repeat our previous constructions on this scale.

We will produce a map which takes a functional K defined on sets in $\Lambda^{(0)}$ to a functional HK defined on sets in $\Lambda^{(1)}$. HK is a single symbol (not H times K). By construction HK is related to K by an identity of the form

$$\mathscr{E}\text{xp}[\square+K](\Lambda)=e^{-\delta E|\Lambda|-\frac{1}{2}\langle\phi,\delta\sigma\phi\rangle}\mathscr{E}\text{xp}[\square+HK](\Lambda). \tag{1.10}$$

This identity is *changing the scale in* $\mathscr{E}$xp. The type of the arguments K and HK indicate that the left-hand side contains an $\mathscr{E}$xp whose domain is the functionals on sets in $\Lambda^{(0)}$, while the right-hand side contains an $\mathscr{E}$xp whose domain is the functionals on sets in $\Lambda^{(1)}$. The other exponential on the right-hand side is the standard one. $\langle\phi, \delta\sigma\phi\rangle$ is a quadratic form in ϕ. The construction (Sect. 2) of the map $K \to HK$ is one of the principal contributions of this paper.

Renormalization. Before integrating out a fluctuation field the $\exp\{-\delta E|\Lambda| - \frac{1}{2}\langle\phi, \delta\sigma\phi\rangle\}$ in (1.10) will be removed by absorbing it into a shift in the covariance v' of the Gaussian measure $d\mu_{v'}$.

Rescaling is a trivial but confusing transformation where all lengths are scaled by L so that the $\Lambda^{(1)}$ cubes return to being $\Lambda^{(0)}$ cubes and the interaction is back in the invariant form.

The Class of Functionals K

We will need K to have derivatives with respect to ϕ. If we calculate the second variational derivatives with respect to $\phi(x)$ of the $K(X, \phi)$ given in (1.6) we obtain derivatives of delta functions. It is easier to regard K as a functional of the derivatives of ϕ, i.e. $\phi(\xi) \equiv \phi(x, p)$. We go a little further and allow K to be a functional of higher derivatives as well. If we take this view then any variational derivative of K at worst involves delta functions. Therefore we allow complex functionals whose variational derivatives exist as complex Borel measures. Our class of functionals $K = K(X, \phi)$ obey three conditions:

Smoothness. Variational derivatives with respect to the gradient of ϕ exist as Borel measures.

Locality. The variational derivatives of $K(X, \phi)$ have support within a small neighborhood of the set X. The neighborhoods are sufficiently small that neighborhoods of disjoint sets in $\Lambda^{(0)}$ do not overlap.

Analyticity and Bounds. $K(X, \phi)$ is bounded by a Gaussian functional of ϕ. If K were allowed to grow more rapidly than a Gaussian for ϕ large then we could not define the convolution by $\mu_{C'}$ in the transformation (1.9). We also require that $K(X, \phi)$ as a function of the set X becomes small when X is a large set. Finally we impose an analyticity condition which is roughly equivalent to demanding that, for any ϕ and any real continuous function $f(\xi)$ bounded uniformly by one, $K(X, \phi + \lambda f)$ is analytic in λ in a disc of radius $h > 0$ which is independent of f and ϕ.

On this class of functionals we will define a class of norms $\|K\|_{G,\Gamma,h}$ which measure these properties: G quantifies the rate of growth in ϕ, Γ quantifies the rate of growth in sets X and h is the radius of analyticity.

Outline of Results

We will prove two theorems, Theorems A and B, that together control a renormalization group step, if L is chosen sufficiently large and K is small enough.

Theorem A. *Given suitable G, Γ, h there exists $\tilde{G}$, $\bar{\Gamma}$, $\bar{h}$ such that $\|HK\|_{\tilde{G},\bar{\Gamma},\bar{h}} \leqq o(L) \|K\|_{G,\Gamma,h}$, where the $\|\cdot\|_{\tilde{G},\bar{\Gamma},\bar{h}}$ norm is stronger than $\|\cdot\|_{G,\Gamma,h}$ in Γ, weaker in G and h. The deterioration in G becomes arbitrarily small as $\|K\|_{G,\Gamma,h}$ tends to zero.*

Theorem B. *Given suitable G, Γ, h there exists G', Γ', h' such that $\|K'\|_{G',\Gamma',h'} \leqq \|K\|_{G,\Gamma,h}$, where the $\|\cdot\|_{G',\Gamma',h'}$ norm is weaker than $\|\cdot\|_{G,\Gamma,h}$. K' is defined in (1.9). The deterioration in Γ and h become arbitrarily small as $\|K\|_{G,\Gamma,h}$ tends to zero.*

The rescaling $\mathscr{R}$ reverses the deterioration in h and almost reverses the deterioration in G allowing these theorems to be combined to prove that a renormalization group step improves h and Γ. The deterioration in G slows down as the norm of K tends to zero which permits the renormalization group to be applied indefinitely.

Conclusion. We choose L sufficiently large and we choose a norm $\|\cdot\|_{G,\Gamma,h}$ such that (1) G prescribes a sufficiently small Gaussian growth in ϕ, (2) Γ prescribes a sufficiently rapid decay in X, (3) h prescribes a sufficiently large radius of analyticity. If $\|K\|_{G,\Gamma,h} \ll 1$ then each successive renormalization group step rewrites the partition function in terms of a slightly shifted Gaussian measure perturbed by an interaction whose norm is down by a factor $o(L^{-1})$ but the norm weakens its growth condition on ϕ slightly. Since each step also rescales Λ, $\Lambda \to \mathscr{R}(\Lambda) \equiv L^{-1}\Lambda$, we eventually rescale Λ into a unit block. From this we can deduce that the (L^N) finite volume pressures of the dipole gas are analytic uniformly in the volume and consequently any infinite volume limit point of these pressures is analytic in the activity.

Now we will fill in the missing details of the preceding outline.

The Coulomb Potential on the Torus

A dipole gas without additional short range forces or alternatively a modified Coulomb potential is not stable. Therefore we smooth the Coulomb potential at the origin. Let v be given by

$$v(x) = \frac{1}{(2\pi)^d |\Lambda|} \sum_{k \neq 0} \hat{v}(k) e^{ik \cdot x},$$

$$\hat{v}(k) = [k \cdot k P(k \cdot k) + k \cdot \sigma k]^{-1}, \tag{1.11}$$

where k is summed over the dual lattice. The omission of $k = 0$ removes an infra-red divergence without affecting the Dipole-Dipole interaction. For the moment we set σ, which will be a constant $d \times d$ *complex* matrix, to zero. $P(t)$ is a monotone positive $\mathscr{C}^\infty$ function with $P(t) = 1$ for t in a compact neighborhood of zero. It is chosen so that $\hat{v}(k) \geqq 0$, $\int dk |\hat{v}(k)|(1 + k^2)^s < \infty$ for some $s > d/2 + 2$, and v is a pseudo-differential operator. The inclusion of P smooths the singularity of v at $x = 0$ without changing the asymptotics of v as $|x|$ tends to infinity. Under the renormalization group σ will change but P is invariant.

The Massless Gaussian Measure

Let Λ be a d-dimensional torus, $d \geqq 1$. On the Sobolev space $H_s(\Lambda)/\{\text{constants}\}$, $s > d/2+2$, there exists a unique Gaussian Borel measure $d\mu_v$ defined by

$$\int d\mu_v(\phi) e^{i\int \phi(x) f(x) dx} = e^{-\langle f, vf\rangle/2}, \qquad f \in H_{-s},$$
$$\langle f, vf\rangle = \int dx dy f(x) v(x-y) f(y).$$

The existence of $d\mu_{v'}$ for σ small, is discussed in Appendix A.

We set $\phi_i(x) \equiv \phi(x,i) = \partial_i \phi(x)$, $\phi_{ij}(x) \equiv \phi(x,i,j) = \partial_i \partial_j \phi(x)$. We use ξ to denote (x,i) or (x,i,j); $\phi(\xi)$ is any one of these fields and $v(\xi,\xi')$ is the covariance for $\phi(\xi) \equiv (\phi_i(x), \phi_{ij}(x))$, $1 \leqq i, j \leqq d$. For example $v(x,i;y,j) \equiv \int d\mu_v \phi_i(x)\phi_j(y) = (\partial_i \partial_j v)(x-y)$.

Definition. A complex valued functional $K(\phi)$, defined for any continuous function $\phi(\xi)$, is said to be infinitely differentiable, if for each n there exists a regular complex Borel signed measure $K_n(\phi, d^n\xi)$ such that

$$\left.\frac{\partial K(\phi + \sum \lambda_i f_i)}{\partial \lambda_1 \dots \partial \lambda_n}\right|_{\lambda=0} = \int K_n(\phi, d^n\xi) f_1(\xi_1) \dots f_n(\xi_n)$$

for all real continuous functions, $f_1, \dots, f_n$. The functions f_i are defined on the index space (position space $\times$ discrete space for indices) and the integration includes summation over indices.

Distances in Λ will be measured in the norm

$$|x| \equiv |x|_\infty \equiv \sup_{i=1,\dots,d} |x_i|.$$

The distance $|X-Y|$ between two sets X and Y is the shortest distance between centers of blocks. We define for a set X in $\Lambda^{(i)}$, $i=0$ or 1.

$$\hat{X} \cong \{y : |x-y|/L^i \leqq \tfrac{1}{3}, \text{ for some } x \in X\}. \tag{1.12}$$

The $\cong$ means that the corners are to be rounded so that $\partial \hat{X}$ is smooth. The smoothing depends on the type of corner but not its location and is such that $\hat{X} \supset \{y : |x-y|/L^i \leqq \tfrac{1}{4}, \text{ for some } x \in X\}$.

Perturbations

Let the integrand be denoted by $Z(\Lambda, \phi)$. We shall require that there exist functionals $K(X,\phi)$, $X \subset \Lambda$, with $K(X=\Phi)=0$, such that

1. $Z(\Lambda,\phi) = \mathscr{E}\mathrm{xp}[\Box + K](\Lambda,\phi)$.
2. *Smoothness:* $K(X,\Psi) = K(X,(\Psi_i),(\Psi_{ij}))$ is an infinitely differentiable functional whose arguments Ψ_i, Ψ_{jk}, $1 \leqq i,j,k \leqq n$, are *arbitrary* real continuous functions. The derivatives of $K(X,\Psi)$ will be denoted $K_n(X,\Psi,d^n\xi)$, where $n=(n_1,n_2)$ denotes that n_1 of the derivatives are with respect to $\Psi_i(x)$ and n_2 are with respect to $\Psi_{jk}(x)$. We let $|n| = n_1 + n_2$. $K_n(X,\phi) = K_n(X,\Psi)$ with $\Psi_i = \phi_i \equiv \partial_i \phi$, $\Psi_{jk} = \phi_{ij} \equiv \partial_i \partial_j \phi$.
3. *Locality:* $K_n(X,\phi,d^n\xi)$ has support, as a measure, in $\hat{X}$, (more correctly $\hat{X}$ $\times$ discrete space for indices).
4. *Analyticity and Bounds:* We will specify a function $\Gamma(X)$ (called a *large set regulator*) by which the decay of $K(X,\phi)$ as a function of the set X is measured. We

will also specify a functional $G(X,\phi)$, (called a *large field regulator*) which will bound the growth of $K(X,\phi)$. Define the norms

$$\|K_n(X)\|_G = \sum_{\underline{\Delta}} \operatorname*{Sup}_{\phi} G^{-1}(X,\phi) \int_{\underline{\Delta}} |K_n(X,\phi,d^n\xi)|$$
$$\left(= \operatorname*{Sup}_{\phi} |K(X,\phi)G^{-1}(X,\phi)| \text{ if } n=0\right),$$

where for $n=(n_1,n_2)\,\underline{\Delta}=(\Delta_1,\ldots,\Delta_{|n|})$ is a set of $\Delta^{(0)}$ cubes which are the ranges of integration for the x components of $\xi_1,\ldots,\xi_{|n|}$ respectively,

$$\|K_n\|_{G,\Gamma} = \operatorname*{Sup}_{\Delta} \sum_{X\cap\Delta\neq\Phi} \Gamma(X)\,\|K_n(X)\|_G\,,$$
$$\|K\|_{G,\Gamma,h} = \sum_n \frac{h^n}{n!}\|K_n\|_{G,\Gamma}\,, \tag{1.13}$$
$$h^n = h_1^{n_1}h_2^{n_2} \quad \text{and} \quad n! = n_1!\,n_2!\,.$$

Condition (4) is that for some h, and G, Γ as specified below, $\|K\|_{G,\Gamma,h}<\infty$.

Definition. K is a *Local analytic functional* iff K satisfies conditions (2) to (4), for some choice of G,Γ.

Different locally analytic functionals $K(\Psi)$ can restrict to the same function $K(\phi)$ because ϕ_i and ϕ_{jk} are not independent variables. To avoid confusion we make the following

Definition. Let K_1 and K_2 be locally analytic functionals, $K_1 \doteq K_2$ means that $K_1(\Psi)=K_2(\Psi)$ when $\Psi_i=\partial_i\phi$, $\Psi_{ij}=\partial_i\partial_j\phi$. $K_1=K_2$ means they are equal as locally analytic functionals, i.e., for all Ψ.

The Large Field Regulator. Let $\|\phi\|_{s,\tilde{X}}$ be a Sobolev norm (of $\partial\phi$),

$$\|\phi\|^2_{s,\tilde{X}} = \sum_{\substack{\alpha:|\alpha|\leqq s\\ \alpha\neq 0}} \int_{\tilde{X}} dx\,|\phi_\alpha|^2\,. \tag{1.14}$$

α is a multi-index and ϕ_α is the distributional α derivative. $\tilde{X}$ is X together with a "collar" with smooth boundary to be specified later. Let κ be a (small) positive number and set

$$\tilde{G}(X,\phi) \equiv \tilde{G}_\kappa(X,\phi) = \exp\left[\tfrac{1}{2}\kappa\,\|\phi\|^2_{s,\tilde{X}}\right]. \tag{1.15}$$

We choose $s>d/2+2$ so that by the Sobolev lemma, e.g., p. 276 in [9], $\tilde{G}$ dominates $\phi(x,i)$ and $\phi(x,i,j)$ pointwise; there exists a constant c such that if $x\in\tilde{X}$, for all i,j,

(G0) $|\phi(x,i)|^2, |\phi(x,i,j)|^2 \leqq (c/\kappa)\tilde{G}(X,\phi)$.

$\tilde{G}$ also has the following properties,

(G1) $\tilde{G}(X,\phi=0)\geqq 1$, $\tilde{G}$ is μ_C integrable and measurable with respect to $\cap\{\sum(U): U\supset\tilde{X}, U \text{ open}\}$, where $\sum(U)=\sigma$ algebra generated by $\{\phi(x): x\in U\}$.

(G2) $\tilde{G}(X\mathring{\cup} Y)\geqq\tilde{G}(X)\tilde{G}(Y)$.

When $\tilde{X}=\hat{X}$, which will be the standard choice, we write G instead of $\tilde{G}$. In Appendix A we show that G is μ_C integrable.

Definition. If $g(X,\phi)$ satisfies (G0)–(G2) it is said to be a large field regulator. We shall use g for a general large field regulator. G and $\tilde{G}$ are the specific regulators mentioned above. (1.16)

Large Set Regulator. For A sufficiently large ($\geqq 2L^{d+1}$) and X a set in $\Lambda^{(i)}$, let

$$\Gamma(X)=A^{|X|}\operatorname*{Inf}_{T\text{ on }X}\prod_{b\in T}\theta(b).$$

$|X|$ is the number of $\Lambda^{(i)}$ cubes in X. The infimum over T runs over all tree graphs whose vertices are the centers of cubes in X. b runs over lines in T and $\theta(b)$ is a (sufficiently) increasing function of the length of b. We will see in Sect. 3 that a sufficient rate of increase is a power law dependent on L.

Relevant Parts and Small Sets. A set X in $\Lambda^{(0)}$ is *Small*, $X\in\mathscr{S}$, if $\operatorname{diam}(X)=\operatorname{Sup}\{|x-y|:x,y$ centers of $\Lambda^{(0)}$ cubes in $X\}\leqq 1$, (i.e., single cubes, nearest neighbor dimers, trimers, quadrimers in $d=2$).

Given $K(X)$ the *relevant part* $F(X)$ of $K(X)$ is given by

$$\begin{aligned}F(X,\phi)&=0 \quad\text{if}\quad X\notin\mathscr{S}\\ &=K(X,\phi=0)+1/2\sum_{i,j}\int_X dx\phi(x,i)Q_{ij}\phi(x,j),\end{aligned}\tag{1.17}$$

where Q_{ij} is a constant $d\times d$ matrix obtained from K differentiated with respect to ϕ_i and ϕ_j at $\phi=0$,

$$Q_{ij}=\frac{1}{|X|}\int K_{2,0}(X;dx,i;dy,j;\phi=0).\tag{1.18}$$

$\int$ does not include summation over i,j. In terms of F we define δE and $\delta\sigma$ by

$$\begin{aligned}\sum_{X\in\Lambda}F(X,\phi)&=-\delta E|\Lambda|-1/2\int dx\phi(x,i)\delta\sigma_{ij}\phi(x,j)\\ &=-\delta E|\Lambda|-1/2\langle\partial\phi,\delta\sigma\partial\phi\rangle.\end{aligned}\tag{1.19}$$

For $\gamma>0$ define

$$\begin{aligned}\bar{h}=\bar{h}(\gamma)&=((L/\gamma)^{-d/2}h_1,(L/\gamma)^{-d/2-1}h_2),\\ h_{\min}&=\operatorname{Min}(h_1,h_2).\end{aligned}\tag{1.20}$$

Theorem A. *Let L be sufficiently large. Then for any even local analytic functional K defined on sets in $\Lambda^{(0)}$, there exists an even local analytic functional $HK=HK(X,\phi)$ defined on sets X in $\Lambda^{(1)}$ such that*

$$\mathscr{E}\mathrm{xp}[\square+K](\Lambda)\doteq e^{-\delta E|\Lambda|-\frac{1}{2}\langle\phi,\delta\sigma\phi\rangle}\mathscr{E}\mathrm{xp}[\square+HK](\Lambda),$$

(with a formula (2.9) for HK) and:

1. *There exists $C>0$ such that*

$$|\delta E|\leqq C\|K\|_{G,\Gamma,h},\|\delta\sigma\|\leqq C\|K\|_{G,\Gamma,h}/h_1^2.$$

2. *Given $C_1>0$ there exist C_2, C_3, $C_4>0$ such that for all L, $A\geqq 2L^{d+1}$, all κ, h, γ, and K with*

$$\kappa\bar{h}_{\min}^2\geqq C_1,\ 1\leqq\gamma\leqq L/2,\ \|K\|_{G_\kappa,\Gamma,h}\leqq C_2L^{-d-1},\tag{1.21}$$

the following bound holds

$$\|HK\|_{\tilde{G}_{\kappa+\delta\kappa},\bar{\Gamma},\bar{h}} \leqq C_3 B \|K\|_{G_\kappa,\Gamma,h},$$

where $B=\gamma^{d+1}/L$ *if* $d\geqq 2$, $=\gamma^{3/2}/L^{1/2}$ *if* $d=1$, *and*

$$\delta\kappa = C_4 \|K\|_{G_\kappa,\Gamma,h}/h_1^2 .$$

$\tilde{G}$ *is defined by setting* $\tilde{X}=X\cup$*(Collar of width* 3*) and* $\bar{\Gamma}$ *is obtained by replacing* A *by* $\bar{A}=2^{2\delta}A$ *with* $\delta=2^{-d-3}$ *in* Γ.

The proof of Theorem A is mostly in Sect. 2. Sections 3–7 provide technical details.

Definition of $\mathcal{R}$. If x is a point $\mathcal{R}x\equiv x/L$. If X is a set $\mathcal{R}(X)\equiv L^{-1}X$. Collars rescale in the same way. If ϕ is a field $(\mathcal{R}\phi)(\mathcal{R}x)\equiv L^{d/2-1}\phi(x)$, $(\mathcal{R}\phi_i)(\mathcal{R}x)\equiv L^{d/2}\phi_i(x)$, $(\mathcal{R}\phi_{ij})(\mathcal{R}x)\equiv L^{d/2+1}\phi_{ij}(x)$. If W is a covariance $(\mathcal{R}W)(\mathcal{R}x,\mathcal{R}y)\equiv L^{d-2}W(x,y)$. The scaling for covariances and fields are designed so that

$$\mathcal{R}v'(x)=v(x);\ d\mu_{\mathcal{R}W}(\mathcal{R}\phi)=d\mu_W(\phi).$$

If $K(X,\phi)$ is a functional defined on sets in $\Lambda^{(1)}$ then $(\mathcal{R}K)(\mathcal{R}X,\mathcal{R}\phi)=K(X,\phi)$ defines the functional $\mathcal{R}K$ on sets in $\Lambda^{(0)}$. It follows that $(\mathcal{R}K)_n(\mathcal{R}X,\mathcal{R}\phi,d^n(\mathcal{R}\xi))=L^{-\dim(n)}K_n(X,\phi,d^n\xi)$, where $\dim(n)=n_1 d/2+n_2(d/2+1)$. If we require that $h=(h_1,h_2)$ scale according to $\mathcal{R}h=(L^{d/2}h_1,L^{d/2+1}h_2)$, then

$$\|\mathcal{R}K\|_{\mathcal{R}g,\Gamma,\mathcal{R}h}=\|K\|_{g,\Gamma,h}. \tag{1.22}$$

Γ is unchanged in form by rescaling and g scales like K.

Changing the scale in $\mathcal{E}$xp by Theorem A we obtain

$$\int d\mu_{v(\sigma)}\mathcal{E}\mathrm{xp}[\square+K](\Lambda)=\int d\mu_{v(\sigma)}e^{-\frac{1}{2}\langle\phi,\delta\sigma\phi\rangle-\delta E|\Lambda|}\mathcal{E}\mathrm{xp}[\square+HK](\Lambda).$$

Renormalization. The Gaussian factor from Theorem A is cancelled by a shift in σ in the covariance of ϕ. Set $N=\int d\mu_0\exp[-\delta E|\Lambda|-\langle\phi,\delta\sigma\phi\rangle/2]$, then we continue with

$$\int d\mu_{v(\sigma)}e^{-\frac{1}{2}\langle\phi,\delta\sigma\phi\rangle-\delta E|\Lambda|}\mathcal{E}\mathrm{xp}[\square+HK](\Lambda)=N\int d\mu_{v(\sigma+\delta\sigma)}\mathcal{E}\mathrm{xp}[\square+HK](\Lambda).$$

Rescaling. We split the integration over ϕ into two subintegrals, as described above, and rescale:

$$\begin{aligned}&=N\int d\mu_{v'(\sigma+\delta\sigma)}(\phi)\int d\mu_{C'(\sigma+\delta\sigma)}(f)\,\mathcal{E}\mathrm{xp}[\square+HK](\Lambda,\phi+f)\\&=N\int d\mu_{v(\sigma+\delta\sigma)}(\phi)\int d\mu_{C(\sigma+\delta\sigma)}(f)\,\mathcal{E}\mathrm{xp}[\square+\mathcal{R}(HK)](\mathcal{R}\Lambda,\phi+f),\end{aligned} \tag{1.23}$$

where $C\equiv\mathcal{R}C'$. By Theorem A, with $\gamma=16$, and (1.22),

$$\|\mathcal{R}HK\|_{\mathcal{R}\tilde{G}_{\kappa+\delta\kappa},\bar{\Gamma},\mathcal{R}\bar{h}}=\|HK\|_{\tilde{G}_{\kappa+\delta\kappa},\bar{\Gamma},\bar{h}}\leqq c(L)\|K\|_{G_\kappa,\Gamma,h}, \tag{1.24}$$

and $\mathcal{R}\bar{h}\geqq 4h$ and $c(L)\to 0$ as $L\to\infty$.

The integration over f is accomplished by Theorem B given below or Theorem B′ in Sect. 10. Theorem B requires the covariance to be real, which in turn means that the functional K must be real, because otherwise the shift $\delta\sigma$ in the covariance would be complex. We have separated out the real covariance case for this Introduction because Theorem B is simpler than Theorem B′. Theorem B is proved in Sect. 8. Theorem B′ is stated and proved in Sect. 10.

Define

$$\|C\| \equiv \sup_{\Delta} \sum_{\Delta'} C(\Delta \cdot \Delta')\theta(\mathrm{dist}[\Delta, \Delta']), \tag{1.25}$$

$$C(\Delta, \Delta') = \sup_{\xi \in \Delta, \xi' \in \Delta'} |C(\xi, \xi')|,$$

where Δ, Δ' are in $\Lambda^{(0)}$ and $\xi \in \Delta$ means that the position coordinate of $\xi \in \Delta$. If $C = \mathscr{R}C'$, C' as in (1.2), then $\|C\| < \infty$ by Lemma A5 in Appendix A.

Theorem B[2]. *Let $C: L_2 \to L_2$ be a real positive definite operator whose square root $C^{1/2}$ has a kernel $C^{1/2}(x, y)$ in $H_s(\Lambda \times \Lambda)$. Let g, g' be Large Field Regulators such that there exists a "homotopy" $g(u, X, \phi)$, $0 \leqq u \leqq 1$, of Large Field Regulators between $g = g(u=0)$ and $g' = g(u=1)$ with the property*

(G3) $\mu_{(t-u)C} * g(u) \leqq g(t)$ *for* $0 \leqq u \leqq t$.

Let $h' \equiv (h', h')$ and $h \equiv (h, h)$ with $0 < h' < h$ be given and let K be a local analytic functional small enough that,

$$\|K\|_{g, \Gamma, h} \leqq \frac{(h-h')^2}{16\|C\|}.$$

Then there is a local analytic functional K' such that

$$\mathscr{L}\mathrm{og}[\mu_C * \mathscr{E}\mathrm{xp}(\square + K)] = \square + K',$$

$$\|K'\|_{g', \Gamma, h'} \leqq \|K\|_{g, \Gamma, h}.$$

There is an explicit *finite* series for $K'(X) \equiv K(t=1, X)$ in terms of derivatives of $K(Y)$, $Y \subset X$, obtained by iterating

$$K(t) = \mu_{tC} * K + \tfrac{1}{2} \int_0^t ds\, \mu_{(t-s)C} * (K_\phi \circ K_\phi)(s).$$

(The notation is defined in Sect. 8.)

Integrating out a Fluctuation Field

By Theorem B we continue (1.23) with

$$= N \int d\mu_{v(\sigma + \delta\sigma)}(\phi)\, \mathscr{E}\mathrm{xp}[\square + [\mathscr{R}(HK)]'](\mathscr{R}\Lambda, \phi + f).$$

We apply Theorem B to $\mathscr{R}(HK)$, with $h' = h/2$, in such a way that we return to essentially the same norm we started with on K. To do this we need $g(u=0) = \mathscr{R}\tilde{G}_{\kappa + \delta\kappa}$ and $g(u=1) = G_{\kappa + \delta\kappa}$. Therefore let

$$g(u, X) = [\mathscr{R}\tilde{G}_{\kappa + \delta\kappa}(X)]^{1-u}[G_{\kappa + \delta\kappa}(X) 2^{\delta|X|}]^u. \tag{1.26}$$

[2] This theorem is related to the Glimm-Jaffe-Spencer Cluster Expansion in [10]

The $\gamma_\delta(X)=2^{\delta|X|}$ is allowed because we have a slightly stronger Large Set Regulator $\bar{\Gamma}(X)$ as a result of Theorem A, so we can absorb $\gamma(X)$ into $\bar{\Gamma}$ by,

$$\|[\mathscr{R}(HK)]'\|_{G_{\kappa+\delta\kappa}\gamma_\delta,\bar{\Gamma},h'}=\|[\mathscr{R}(HK)]'\|_{G_{\kappa+\delta\kappa},\bar{\Gamma}/\gamma_\delta,h'}$$
$$\geqq\|[\mathscr{R}(HK)]'\|_{G_{\kappa+\delta\kappa},\Gamma,h'}\,. \tag{1.27}$$

In Sect. 9 we prove (Proposition 9.1) that there exists $\kappa_{\max}>0$ such that for $\kappa+\delta\kappa\leqq\kappa_{\max}$ this choice of $g(u)$ satisfies the hypothesis G3 in Theorem B. In general $\kappa_{\max}$ depends on the covariance C but if C is the (rescaled) fluctuation covariance of (1.2) and $\|\delta\sigma\|\leqq 1/2$ then $\kappa_{\max}$ is a constant.

When we combine (1.27), the bound of Theorem B and (1.24), we find that if $\|K\|_{G_\kappa,\Gamma,h}$ is small then

$$\|[\mathscr{R}(HK)]'\|_{G_{\kappa+\delta\kappa},\Gamma,2h}\leqq c(L)\,\|K\|_{G_\kappa,\Gamma,h} \tag{1.28}$$

with $c(L)\to 0$ as $L\to\infty$.

Comment. The homotopy $g(u)$ is between two regulators with the same value of κ. It is surprising that this is possible: if $\exp(\frac{1}{2}\kappa x^2)$ is convolved by $\exp(-x^2/2)$ the result is bounded by $C\exp(\frac{1}{2}\kappa' x^2)$ but $\kappa'>\kappa$. It would be a disaster if κ had to increase in the Large Field Regulator when convolved by $\mu_{(t-s)C}$ because C is essentially the same for each scale, so κ would have to increase by the same increment each time we iterate. Instead of an increase in κ we get an increase in the width of the collar around X which we can afford because rescaling reduces the width back again. A similar issue arises in [4]. Both proofs are relying on more than the behaviour of the basic field $\phi(x,i)$ under scaling. The fact that $\phi(x,i)$ is a gradient is also needed.

Conclusion. Theorems A and B (B' if K complex), imply that L and A can be chosen large so that if σ, κ, $\|K\|_{G_\kappa,\Gamma,h}$ are small enough and $h=(h,h)$ is large enough, then

$$\int d\mu_{v(\sigma)}\,\mathscr{E}\mathrm{xp}[\square+K](\Lambda)=B\int d\mu_{v(\sigma+\delta\sigma)}\,\mathscr{E}\mathrm{xp}[\square+[\mathscr{R}(HK)]'](\mathscr{R}\Lambda),$$
$$B=e^{-\delta E|\Lambda|}[\int d\mu_\sigma\, e^{-\frac{1}{2}\langle\phi,\delta\sigma\phi\rangle}]\,, \tag{1.29}$$

$$\|[R(HK)]'\|_{G_{\kappa+\delta\kappa},\Gamma,2h}\leqq\tfrac{1}{2}\|K\|_{G_\kappa,\Gamma,h}\,, \tag{1.30}$$

$$\delta\sigma,\delta\kappa=O(\|K\|^2_{G_\kappa,\Gamma,h}/h_1^2)\,. \tag{1.31}$$

This means that we can continue iterating the two theorems until Λ is rescaled to a unit block, provided the torus Λ has all dimensions equal to L^N for some integer N. At the end, the interaction has decreased in norm by 2^{-N} and κ and σ have increased by less than $(\sum 2^{-3i})\,\|K\|_{G,\Gamma,h}$ with K and G being the initial K and Large Field Regulator. We can choose the norm of the initial K small enough so that at no point in the iteration are the smallness hypotheses on σ and κ violated, no matter how large Λ is. It is also possible to arrange that Γ becomes stronger in each iteration.

Corollary C. *Let L be a sufficiently large integer and let β be any real number. Then there exists a disc D, whose center is the origin, and a constant B such that, as a function of z in D, $P(\Lambda,z,\beta)$ is analytic and bounded by B, for any torus Λ whose dimensions are powers of L.*

Proof. We have shown in (1.6)–(1.8) that the initial perturbation

$$Z(\Lambda,\phi)=\exp\left[2z\int_{\Lambda} d\varrho(\xi)\cos(\varrho^{1/2}\phi(\xi)\right]$$

is of the form $\mathscr{E}\mathrm{xp}[\square+K](\Lambda)$. By Lemmas 4.2 and 5.1 we can prove that K is a local analytic functional. K is analytic in z for z in a (sufficiently small) disk $D=\{z:|z|<\varepsilon\}$ for all ϕ in H_s and all sets X. Also $\lim_{\varepsilon\to 0}\sup_{z\in D}\|K\|_{G,\Gamma,h}=0$, for any κ, h, and Λ.

The formulas (see (2.9) and Lemma 8.2) for HK and K' show that K' is analytic in z for $z\in D$ for X and ϕ fixed and $\sup_{z\in D}\|K'\|_{G_{\kappa+\delta\kappa},\Gamma,h}\leqq\gamma\sup_{z\in D}\|K\|_{G_\kappa,\Gamma,h}$ with $\gamma<1$, provided κ, h, and Γ are as described in the preceding summary. Furthermore $\delta\sigma$ defined in (1.19) is also analytic in $z\in D$ and bounded by $O(\|K\|/h^2)$. Thus analyticity is retained under iteration of Theorems A and B. After sufficiently many (N) iterations Λ becomes a unit cube Δ and then $\mathscr{E}\mathrm{xp}[\square+K](\Delta)=1+K(\Delta)$ with $|K(\Delta)|\leqq 2^{-N}G(\Delta)$. By the results on complex measures in Appendix A it is clear that $\int d\mu_{\sigma+\delta\sigma}(1+K(\Delta))$ has an analytic logarithm. Furthermore the logarithm of B in (1.29) is analytic in $z\in D$ (Appendix A).

End of Proof.

Open Problem. The pressure for the dipole gas is established to be analytic, but the proofs are lengthy and indirect. The discussion of the regulator after Proposition C suggests that the dipole gas is well behaved because of cancellations of the same type as are exploited in the theory of singular integral operators; the dipole – dipole potential is a singular integral operator, its fourier transform is $k_ik_j/k\cdot k$ near $k=0$. Is it possible to improve the standard estimates for the coefficients of the Mayer Expansion (the expansion of the pressure in z) to obtain a direct proof that this expansion is convergent? Progress in this direction has been made in [14].

2. A Formula for *HK*. The Proof of Theorem A

In this section we start the proof of Theorem A by deriving a formula (2.9) for HK. At the end of this section we will explain why one should expect to be able to prove Theorem A using this formula. The details appear in Sects. 3–7.

Recall that F, (the relevant parts) was defined in (1.17). Define

$$R(X)=\exp[F(X,\phi)]-1\,,$$
$$I\doteq K-R\,,$$
$$\Omega(X)=\sum_{Y\subset X}F(Y)\,. \tag{2.1}$$

$I=K-R$ leaves us with the freedom to choose I to be any local analytic functional which as a function of ϕ equals $K-R$. We postpone the choice of I to (4.5) and (4.6) in Sect. 4.

The underlying idea is that $\mathscr{E}\mathrm{xp}(\square+R)\cong\exp(\Omega)$. Observe that

$$e^{\Omega(X)}=\prod_{Y\subset X}(1+R(Y))=\sum\frac{1}{N!}\sum_{Y_1,\ldots,Y_N\subset X}R(Y_1)\ldots R(Y_N)\,. \tag{2.2}$$

If the sets Y_i were disjoint then this would equal $\mathscr{E}\mathrm{xp}(\square+R)$. Although the sets do not have to be disjoint, they are constrained by the condition that no two of them may be the same set. We obtain an $\mathscr{E}$xponential by rewriting this expression in terms of sets that do have to be disjoint by grouping Y_i's that intersect into new sets. Define a "grouping" relation $\mathscr{G}(Z)$ on $Y_1, \ldots, Y_N$ by $(Y_1, \ldots, Y_N) \in \mathscr{G}(Z)$ iff

1. $\cup Y_i = Z$.
2. Z is connected.
3. $Y_i \neq Y_j$ if $i \neq j$. (2.3)

Define

$$R^+(Z) = \sum_2^\infty \frac{1}{N!} \sum_{Y_1, \ldots, Y_N \in \mathscr{G}(Z)} R(Y_1) \ldots R(Y_N), \tag{2.4}$$

then

$$e^\Omega = \mathscr{E}\mathrm{xp}(\square + R + R^+). \tag{2.5}$$

Therefore

$$Z = \mathscr{E}\mathrm{xp}(\square + K) \doteq \mathscr{E}\mathrm{xp}(\square + R + I) = \mathscr{E}\mathrm{xp}(I - R^+) \circ e^\Omega.$$

Evaluate both sides on the set Λ,

$$\begin{aligned} Z(\Lambda) &\doteq \sum_{X, Y : X \dot{\cup} Y = 1} \mathscr{E}\mathrm{xp}(I - R^+)(X)\, e^{\Omega(Y)} \\ &= e^{\Omega(\Lambda)} \sum_{X \subset \Lambda} \mathscr{E}\mathrm{xp}(I - R^+)(X)\, H(X), \end{aligned} \tag{2.6}$$

where

$$H(X) = \exp\left[-\sum_{Y : Y \cap X \neq \Phi} F(Y)\right]. \tag{2.7}$$

Do not confuse $H(X)$ with HK which is a single symbol. Note that $H(X)$ is not a local analytic functional, because it does not depend only on fields $\phi(\xi)$ within $\hat{X}$. We shall see that this problem goes away if we pass to the $\Lambda^{(1)}$ scale. We set

$$\tilde{X} = \{y : |y - x| \leqq 3 \text{ for some } x \in X\},$$

and note that H has a factorisation property:

$$H(X) = H(Y) H(Z) \quad \text{if} \quad X = Y \cup Z \quad \text{with} \quad \tilde{Y} \cap \tilde{Z} = \Phi.$$

We will now rewrite Z as a local analytic functional on the next scale, $\Lambda^{(1)}$.

For any X in $\Lambda^{(0)}$ let

$$\bar{X} = \text{the smallest set in } \Lambda^{(1)} \text{ that contains } X.$$

Given U in $\Lambda^{(1)}$ define the reblocking relation $\mathscr{RB}$ by $X_1, \ldots, X_N \in \mathscr{RB}(U)$ iff

1. $U = \cup \bar{X}_i$.
2. The graph G, whose vertices are $1, \ldots, N$ and whose lines are those pairs ij such that $\bar{X}_i \cap \bar{X}_j \neq \Phi$, is connected.
3. $X_i \cap X_j = \Phi$. (2.8)

Define

$$HK(U) = \sum_{N=1}^\infty \frac{1}{N!} \sum_{\underset{\sim}{X} \in \mathscr{RB}(U)} H(\cup X_i) \prod_1^N (I - R^+)(X_i), \tag{2.9}$$

then, using the factorisation of H mentioned above, we have, provided L is large enough,

$$Z(\Lambda) \doteq e^{\Omega(\Lambda)} \mathscr{E}\mathrm{xp}(\square + HK)(\Lambda), \tag{2.10}$$

where HK is defined on sets in $\Lambda^{(1)}$ and $\mathscr{E}$xp is now defined using the $\circ$ product for sets in $\Lambda^{(1)}$. By definition $HK(U)$ depends on $\phi(x)$ with $x \in \tilde{U} = \{y : |y-x| \leqq 3$ for some x in $U\}$. We assume $L \geqq 9$ so that $\tilde{U} \subset \hat{U}$, where the $\hat{}$ is now being applied to a set in $\Lambda^{(1)}$, i.e., $\hat{U} = \{y : |y-x|/L \leqq 1/3$ for some $x \in U\}$. This gives a representation, which lives on $\Lambda^{(1)}$ and has properties in exact analogy to the starting representation on $\Lambda^{(0)}$.

Recall from the introduction the definitions of δE and $\delta\sigma$. Let $\|\delta\sigma\|$ be the operator norm of the matrix $\delta\sigma$ considered as an operator on a d – dimensional vector space. The proof of the following lemma is left to the reader.

Lemma 2.1. *There exists a constant C such that for any large field regulator g, any $h=(h_1,h_2)$, $A \geqq 1$,*

$$|\delta E| \leqq \|K\|_{g,\Gamma,h'}, \|\delta\sigma\| \leqq C\|K\|_{g,\Gamma,h}/h_1^2.$$

This lemma proves the claim in Theorem A part (1). The proof of part (2) will be Lemma 7.2.

Discussion. We shall omit all the subscripts G, Γ, h in this discussion. Let us review the construction of HK. First K is split into the "irrelevant part" I and the "relevant part" R. R is essentially the quadratic part of $K(X)$ with $X \in \mathscr{S}$ and I is the remainder of K. If $X \notin \mathscr{S}$ then $I(X) = K(X)$. We then define R^+ from R by the grouping relation (2.3). Finally HK is defined from $I - R^+$ and H (2.8, 2.9) with $H \cong \exp[O(\|K\| h_1^{-2})(\partial\phi)^2]$.

Roughly speaking Theorem A says that, after rescaling, HK is smaller than K in $\|\cdot\|$ norm, provided we allow a small deterioration $\kappa \to \kappa + O(\|K\|/h_1^2)$ in the large field behaviour measured by G. The key to proving Theorem A is to show that I and R^+ have smaller norm than K. The deterioration in large field behaviour is due to the $H(X)$ factor in (2.9) which is bounded by $\exp(\|\delta\sigma\| \, \|\phi\|_{s,X}^2/2)$ and by Lemma 2.1 $\|\delta\sigma\| \leqq C\|K\|/h_1^2$.

The factor R^+. From its definition we see that R^+ is a sum of products of $K(X)$'s where the sets X must intersect. Since at least two K's with overlapping sets must be present we expect R^+ to be bounded by $O(\|K\|^2)$ which is much smaller than $\|K\|$. By (3) of the relation $\mathscr{G}(Z)$ in (2.3), Z is a union of small sets X_i of which no two are the same as sets. Therefore the number of factors $R(X)$ that can overlap at a given block $\Delta \in \Lambda^{(0)}$ is less than some number τ depending only on the dimension. Each $R(X)$ has an $\exp(C\|K\| \, \|\phi\|_{s,X}^2/h_1^2)$ large field behaviour, so the worst growth that can arise in any given block is less than $\exp(C\tau\|K\| \, \|\phi\|_{s,X}^2/h_1^2) \leqq G_\kappa(\Delta, \phi)$, if K is such that $\|K\|/h_1^2 \ll \kappa$.

The factor I: R has a large field behavior $\exp[O(\|K\|/h_1^2) \, \|\phi\|_{s,X}^2)$, so for $\|K\|$ small enough $I \equiv K - R$ has large field behavior no worse than that of K and hence controllable by G. I has a small set part and a large set part. The large set part becomes smaller under $\mathscr{RB}$ because if A is large ($> L^{d+1}$) then a large set X has the property that $\Gamma(\bar{X}) < L^{-d-1}\Gamma(X)$. See Fig. 1 below. The small set part of I increases by a factor of L^d under $\mathscr{RB}$ because up to L^d different small sets X have the same

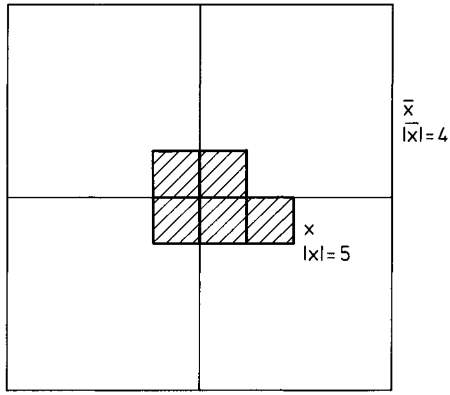

Fig. 1. $|x|=5$ and $|\bar{x}|=4 \Rightarrow \Gamma(\bar{x}) \leqq \frac{1}{A}\Gamma(x)$

image in $\Lambda^{(1)}$ under the map $X \to \bar{X}$. However these parts of I are $O(\partial\phi^3)$ or $O(\partial\phi\,\partial\partial\phi)$. To see this note that $I = K - R \cong K - F$. F has been chosen to cancel the quadratic part of the Taylor series for K so that the leading terms are $O(\partial\phi^3)$ or $O(\partial\phi\,\partial\partial\phi)$. Upon rescaling each factor of $\partial\phi$ contributes a small factor of $L^{-d/2}$ and each factor of $\partial\partial\phi$ contributes $L^{-d/2-1}$. These factors more than compensate for the L^d increase.

Summary. The mechanisms which make I smaller are: (1) large sets are irrelevant due to our choice of regulator, (2) for small sets the subtraction of R ensures that we only have higher order terms which scale down in size faster than L^{-d} under rescaling. In the physics literature, the second mechanism has been emphasized since Wilson and the first one is implicit in the calculations that assume only the local terms are important. In fact, terms corresponding to large sets and/or large fields are always present but the goal of this paper is to show that neglecting them gives qualitatively correct answers.

3. Large Sets

In this section we show that the important parts of the perturbation are local. We assess the decay of $K(X)$ as a function of X by multiplying it with the large set regulator $\Gamma(X)$. If by reblocking we pass to the next scale so that X becomes $\bar{X}$, which is the smallest set in $\Lambda^{(1)}$ containing X, then the appropriate regulator is $\Gamma(\bar{X})$. We remind the reader that if the regulator Γ is evaluated on a set in $\Lambda^{(1)}$, then distance and volume are measured in units appropriate to $\Lambda^{(1)}$. For example $|\bar{X}|$ is the number of $\Lambda^{(1)}$ blocks in $\bar{X}$. We will show that if X is *not a small set*, $X \notin \mathcal{S}$, then $\Gamma(\bar{X})$ is much smaller than $\Gamma(X)$. This change in the value of Γ will cause the large set part of K to look smaller when K is transferred from $\Lambda^{(0)}$ to $\Lambda^{(1)}$.

The large set regulator is

$$\Gamma(X) = A^{|X|} \inf_{T \text{ on } X} \prod_{b \in T} \theta(|b|), \qquad A \geqq 2L^{d+1}. \tag{3.1}$$

θ is called the *Line Regulator*. $|b|$ is the length of line b connecting centers of blocks in X. θ is chosen to be an increasing function with the property,

$$(\theta)\ \ \theta(1)=1\,,\ \theta(\{s/L\})\leqq\tfrac{1}{2}L^{-d-1}\theta(s)\quad\text{for}\quad s\geqq 2\,,$$

where $\{r\}$ = smallest integer $\geqq r$. Functions satisfying (θ) with equality grow as a power of s for s large.

For J a local annalytic functional on $\Lambda^{(0)}$, define

$$\|J\|^{(1)}_{G,\Gamma,h}=\sum_n\frac{h^n}{n!}\sup_{\Delta\in\Lambda^{(1)}}\sum_{X:\bar{X}\cap\Delta\neq\Phi}\|J_N(X)\|_G\Gamma(\bar{X})\,. \tag{3.2}$$

Notice that Δ is a cube in $\Lambda^{(1)}$ and the large set regulator is being evaluated on a $\Lambda^{(1)}$ set. This norm is technically useful in the passage from $\Lambda^{(0)}$ to $\Lambda^{(1)}$ because it estimates a $\Lambda^{(0)}$ functional as if it were a $\Lambda^{(1)}$ functional. We summarise the main properties in Lemma 3.1 below.

Definition. $\gamma(X)$ is a Large Set Regulator iff

$$(\Gamma 0)\ \ \gamma(X)\geqq 1\ \text{all}\ X\,,$$
$$(\Gamma 1)\ \ \gamma(X\cup Y)\leqq\gamma(X)\gamma(Y)\theta(\operatorname{dist}(X,Y))\,. \tag{3.3}$$

Lemma 3.1. *Let Γ be defined as described above and let Γ' be obtained from Γ by replacing A by A', where*

$$2L^{d+1}\leqq A\leqq A'\leqq 2^{(2-d)}A\,.$$

If $J(X,\phi)=0$ when $X\in\mathscr{S}$ then for any large field regulator g

$$\|J\|^{(1)}_{g,\Gamma',h}\leqq 3^d\frac{2}{L}\|J\|_{g,\Gamma,h}\,.$$

If $J(X,\phi)\neq 0$ for $X\in\mathscr{S}$ then

$$\|J\|^{(1)}_{g,\Gamma',h}\leqq 2(3L)^d\|J\|_{g,\Gamma,h}\,.$$

Proof. Immediate consequence of Lemma 3.2 below. Note that a factor of $(3L)^d$ comes from bounding $\sum_{X:\bar{X}\cap\Delta\neq\Phi}$ with $\Delta\in\Lambda^{(1)}$ by $(3L)^d\sum_{X:X\cap\Delta\neq\Phi}$ with $\Delta\in\Lambda^{(0)}$.

Lemma 3.2. *Suppose $L\geqq 2$. Let A, A', Γ, Γ' be defined as in Lemma 3.1, then*

$$(\Gamma 2')\ \ \Gamma'(\bar{X})\leqq 2L^{-d-1}\Gamma(X)\quad \text{if } X\notin\mathscr{S}$$
$$\leqq 2\Gamma(X)\qquad \text{for } X\in\mathscr{S}\,.$$

Proof. $\Gamma 2'$ is clear for $X\in\mathscr{S}$, because $\Gamma'(\bar{X})=A'^{|\bar{X}|}\leqq A'^{|X|}\leqq A^{|X|}2^{(2-d)|X|}=\Gamma(X)2^{(2-d)|X|}$ and $\operatorname{Max}\{|X|:X\in\mathscr{S}\}\leqq 2^d$.

Now assume $\Gamma 2'$ holds for sets $X\in\mathscr{S}$, which are connected. These, by definition, are sets X such that the minimal tree $T(X)$ on X only has lines of length one.

Reduction to Connected Case. Let X be a set not in $\mathscr{S}$ which has at least one line of length $\geqq 2$ in its minimal tree graph, $T(X)$. By erasing all lines of length $\geqq 2$ in $T(X)$

we split $T(X)$ into connected subtrees $T_1, \ldots, T_N$ with T_i consisting of lines of length one which link a connected component X_i of X. Since $\theta(1)=1$,

$$\Gamma(X)=A^{|X|} \inf_{T \text{ on } \{X_1, \ldots, X_N\}} \prod_{ij \in T} \theta(|ij|),$$

where $|ij|=\operatorname{dist}(X_i, X_j) \geqq 2$ by construction. By relabelling, if necessary we may suppose that X_1 has coordination number one in a minimising T. By induction on N we may assume $\Gamma 2'$ for X_1 and $Y=X-X_1$. Therefore $\Gamma'(\bar{X}_1) \leqq 2\Gamma(X_1)$ and $\Gamma'(\bar{Y}) \leqq 2\Gamma(Y)$. By $\Gamma 1$ and (θ) in (3.1),

$$\begin{aligned} \Gamma'(\bar{X}) &\leqq \Gamma'(\bar{X}_1)\Gamma'(\bar{Y})\,\theta(\operatorname{dist}(\bar{X}_1, \bar{Y})) \\ &\leqq 4\Gamma(X_1)\Gamma(Y)\tfrac{1}{2}L^{-d-1}\theta(\operatorname{dist}(X_1, Y)) \leqq 2L^{-d-1}\Gamma(X). \end{aligned}$$

This reduces $\Gamma 2'$ to the connected case.

Case $X \notin \mathcal{S}$ and X Connected. Since X is connected $\Gamma(X)=A^{|X|}$ and $\Gamma'(\bar{X})=A'^{|\bar{X}|}$. We proceed by induction on $|X|$. X not in $\mathcal{S}$ and connected implies that there exists a block Δ in $\Lambda^{(1)}$ such that $|X \cap \Delta| \geqq 2$, where $X \cap \Delta$ is a set in $\Lambda^{(0)}$. Therefore we can choose two $\Lambda^{(0)}$ blocks δ_1 and δ_2 in $X \cap \Delta$ and divide X into two connected subsets, X_1 containing δ_1 and $X_2=X-X_1$ containing δ_2. Then

$$\Gamma'(\bar{X})=A'^{|\bar{X}|} \leqq \Gamma(\bar{X}_1)\Gamma(\bar{X}_2)/A'$$

because Δ is counted both in $\bar{X}_1$ and $\bar{X}_2$. The inductive assumption together with $A' \geqq 2L^{d+1}$ allows us to continue with

$$\leqq 4\Gamma(X_1)\Gamma(X_2)/A' \leqq 2L^{-d-1}\Gamma(X).$$

End of Proof of Lemma.

4. A Bound on I

The main result in this section is Lemma 4.4, a bound on the factor I which appears in our formula (2.9) for HK.

Lemma 4.1. *Let G_ε be the large field regulator in* (1.16) *and let*

$$\begin{aligned} m(X) &= \int |K_2(X, \phi=0, d^2\xi)|, \\ e(X) &= K(X, \phi=0). \end{aligned}$$

Then there exists $C>0$ such that

$$\begin{aligned} 1.\quad \|F_N(X,\phi)\|_{G_\varepsilon} &\leqq C\left[e(X)+\frac{1}{\varepsilon}m(X)\right] && \text{if} \quad N=0 \\ &\leqq \left(\frac{C}{\varepsilon}\right)^{1/2} m(X) && N=1 \\ &\leqq m(X) && N=2 \\ &\leqq 0 && N \geqq 3. \end{aligned}$$

2. *There exists $C>0$ such that for any large field regulator g,*

$$\|F\|_{G_\varepsilon,\Gamma,h} \leqq C\left(1+\frac{1}{\varepsilon h_1^2}\right)\|K\|_{g,\Gamma,h}.$$

The proof is left to the reader. (1) is an immediate consequence of G0 in (1.16). (2) follows from (1) and the bounds

$$\sum_{X\cap\Delta\neq\Phi} e(X)\Gamma(X) \leqq \|K\|_{g,\Gamma,h}; \qquad \sum_{X\cap\Delta\neq\Phi} m(X)\Gamma(X) \leqq \|K\|_{g,\Gamma,h}/h_1^2.$$

Lemma 4.2. *There exist $C_1, C_2, C_3>0$ such that for any large field regulator g, for all $\varepsilon>0$, $h_1>0$ and K such that*

$$\|K\|_{g,\Gamma,h} \leqq C_1\varepsilon h_1^2/[1+\varepsilon h_1^2],$$

the following bounds hold

$$\|e^{\pm F}-1\|_{G_\varepsilon,\Gamma,h} \leqq C_3[1+1/(\varepsilon h_1^2)]\,\|K\|_{g,\Gamma,h'}$$
$$\|e^F-1-F\|_{G_\varepsilon,\Gamma,h} \leqq [C_3[1+1/(\varepsilon h_1^2)]\,\|K\|_{g,\Gamma,h}]^2,$$

Proof. Let $G=G_\varepsilon$. It is not difficult to check that for $n=1,2,\ldots$,

$$\|F^n\|_{G,\Gamma,h} \leqq \|F\|^n_{G^{1/n},\Gamma,h}. \tag{4.1}$$

Note that $1/n$ appears in the exponent of G. From the definitions

$$\|e^{\pm F}-1\|_{G,\Gamma,h} \leqq \sum_{n=1}^{\infty}\frac{1}{n!}\|F^n\|_{G,\Gamma,h} \leqq \sum_{n=1}^{\infty}\frac{1}{n!}\|F\|^n_{G^{1/n},\Gamma,h}.$$

By Lemma 4.1, we continue with

$$\leqq \sum_{n=1}^{\infty}\frac{1}{n!}C^n\left(1+\frac{n}{\varepsilon h_1^2}\right)^n\|K\|^n_{g,\Gamma,h} \leqq \sum_{n=1}^{\infty}\text{Const}^n\left(1+\frac{1}{\varepsilon h_1^2}\right)^n\|K\|^n_{G,\Gamma,h}.$$

Now we sum this geometric series to get the result. The proof of the second claim follows by exactly the same reasoning with $n=1$ replaced by $n=2$ as the lower limit of the sums. □

Let $p\geqq 0$ be an integer or half integer. For $n=(n_1,n_2)$, define

$$\dim(n)=\frac{d}{2}n_1+\left(\frac{d}{2}+1\right)n_2,$$

$$\|J\|_{g,\Gamma,h,\dim\geqq p}=\sum_{n:\dim(n)\geqq p}\frac{h^n}{n!}\|J_n\|_{g,\Gamma}.$$

Note that $\phi_i^{n_1}\phi_{jk}^{n_2}$ has rescaling dimension $\dim(n)$.

Lemma 4.3. *Let p be non-negative. Suppose J is an analytic functional with $J_n(X,\phi=0)=0$ if $\dim(n)<p$, then there exists C_p depending on p such that for all $\varepsilon>0$, h,*

$$\|J\|_{G_\varepsilon,\Gamma,h} \leqq C_p[1+1/(\varepsilon h_{\min}^2)]^p\,\|J\|_{G_\varepsilon,\Gamma,h,\dim\geqq p},$$

where $h_{\min}=\min(h_1,h_2)$.

Proof. For n with $\dim(n) < p$,

$$J_n(X, \phi, d^n\xi) = \int_0^1 dt \frac{d}{dt} J_n(X, t\phi, d^n\xi)$$
$$= \sum_\alpha \int_0^1 dt \int J_{n+\alpha}(X, t\phi, d^{n+\alpha}\xi)\phi(\xi_{n+\alpha}),$$

where $\alpha = (1,0)$ or $(0,1)$. We rewrite the sum over α as a sum over m with the conditions $|m| = |n| + 1$ and $m \geqq n$, where $m \geqq n$ means $m_i \geqq n_i$ for $i = 1, 2$. Let $G = G_\varepsilon$. Then

$$\frac{h^n}{n!} \|J_n(X)\|_G \leqq \frac{h^n}{n!} \sum_m \int_0^1 dt \sum_{\Delta_1, \ldots, \Delta_m} \operatorname{Sup}_\phi \operatorname{Var}[J_m(X, t\phi, d^m\xi) 1_{\underset{\sim}{\Delta}}]$$
$$\times G^{-1}(X, t\phi) \operatorname{Sup}_{\xi \in \Delta_{n+1}} |\phi(\xi)| \exp\left[-\frac{\varepsilon}{2}(1-t^2) \|\phi\|^2_{s, \hat{X}}\right]$$
$$\leqq \operatorname{Const}(|n|+1)\varepsilon^{-1/2} h_{\min}^{-1} \sum_m \|J_m(X)\|_G h^m/m! \tag{4.2}$$

We have used G0 in (1.16) to bound $\phi(\xi)\exp\left[-\frac{\varepsilon}{2}(1-t^2) \quad \|\phi\|^2_{s, \hat{X}}\right]$ by $\operatorname{Const} \varepsilon^{-1/2}(1-t^2)^{-1/2}$. Now iterate this inequality until there is no m with $\dim(m) \leqq p$. Note that we can also require that $\dim(m) \leqq p + d/2 + 1$. Since $d \geqq 1$ each iteration increases the $\dim(m)$ by at least 1/2. Therefore we need at most $2p$ iterations. We multiply the resulting inequality by $\Gamma(X)$ and sum over X,

$$\frac{h^n}{n!} \|J_n\|_{G, \Gamma} \leqq \operatorname{Const}(p) \cdot \operatorname{Max}\{(eh^2_{\min})^{-j/2} | j = 1, \ldots, 2p\}$$
$$\times \sum_{m : p \leqq \dim(m) \leqq p + d/2 + 1} \|J_m\|_{G, \Gamma} \frac{h^m}{m!}. \tag{4.3}$$

Clearly the maximum is bounded by $\operatorname{Const}(1 + 1/(\varepsilon h^2_{\min}))^p$. Also the upper bound on the summation can be removed to obtain a further bound. Hence the right-hand side of (4.3) can be bounded by $\operatorname{Const}(p) \cdot (1 + 1/(\varepsilon h^2_{\min}))^p \ \|J\|_{G, \Gamma, h, \dim > p}$. Lemma 4.3 then follows from the inequality by summing over n with $\dim(n) \leqq p$. □

In (2.1) I was defined to be any local analytic functional such that $I \doteq K - R$. We will now describe the choice for I. The objective is to choose I to be a local analytic functional such that $I_n(\phi = 0) \cong 0$ if $\dim(n) \leqq d$. Let $Q_{ij}(X, dx, dy) = K_{2,0}(X; \phi = 0; dx, i; dy, j)$ and

$$K^{(2)}(X, \phi) = K(X, \phi = 0) + \tfrac{1}{2} \sum_{i,j} \int \phi_i(x) Q_{ij}(X, dx, dy) \phi_j(y), \tag{4.4}$$

i.e., $K^{(2)}$ is the Taylor series about $\phi = 0$ of K up to terms of dimension d in ϕ. Next we observe that

$$(K^{(2)} - F)(X, \phi) = \tfrac{1}{2} \sum_{i,j} \frac{1}{|X|} \int_X dz \int Q_{ij}(X, dx, dy) W_{ij}(x, y, z, \phi),$$

where F is defined in (1.17) and $W_{ij}(x,y,z,\phi)=\phi_i(x)\phi_j(y)-\phi_i(z)\phi_j(z)$. Given arbitrary continuous functions Ψ_i, Ψ_{jk}, $i,j,k=1,\dots,d$, let

$$\text{“}W\text{”}_{ij}(x,y,z\,\Psi)=\int_0^1 ds\sum_k[(x-z)_k\Psi_{ik}(z+s(x-z))\Psi_j(z+s(y-z))$$
$$+\Psi_i(z+s(x-z))(y-z)_k\Psi_{jk}(z+s(y-z)],$$

where $(x-z)_k$ = the k^{th} coordinate of $x-z$. By the fundamental theorem of calculus "W"$_{ij}(x,y,z,\Psi=\phi)=W_{ij}(x,y,z,\phi)$, i.e., "$W$" $\doteq W$. We define I as a local analytic functional by

$$I(X)=(K-K^{(2)})(X)+\text{“}(K^{(2)}-F)\text{”}(X)+(F-R)(X)\quad\text{if}\quad X\in\mathscr{S}\tag{4.5}$$

with

$$\text{“}(K^{(2)}-F)\text{”}(X,\Psi)=\tfrac{1}{2}\sum_{i,j}\frac{1}{|X|}\int_X dz\int Q_{ij}(X,dx,dy)\cdot\text{“}W\text{”}_{ij}(x,y,z,\Psi),\tag{4.6}$$

and set $I(X)=K(X)$ if $X\notin\mathscr{S}$. By construction $I\doteq K-R$ and if $X\in\mathscr{S}$, then by construction and Lemma 4.2 $I_n(X)=O(\|K\|^2_{G,\Gamma,h})$ for $\dim(n)\leqq d$.

Recall the definition of the $\|\cdot\|^{(1)}_{g,\Gamma,h}$ norm from Sect. 3. Let γ and $\bar{h}(\gamma)$ be as in Theorem A in the Introduction.

Lemma 4.4. *Let L be sufficiently large and let $C_1>0$ be given, then there exist C_2, C_3 such that for all L, all $A\geqq 2L^{d+1}$, $\kappa>0$, h, γ with $1\leqq\gamma\leqq L/2$, and K such that*

$$\kappa\bar{h}_{\min}(\gamma)^2\geqq C_1,\ \|K\|_{G_\kappa,\Gamma,h}\leqq C_2L^{-d-1},$$

the following bound holds

$$\|I\|^{(1)}_{G_\kappa,\Gamma',\bar{h}}\leqq C_3B\|K\|_{G_\kappa,\Gamma,h},$$

where $B=\gamma^{d+1}/L$ for $d\geqq 2$, $=\gamma^{3/2}/L^{1/2}$ for $d=1$ and Γ' is defined by replacing A by $A'=2^{(2-d)}A$ in Γ.

Proof. Let $G=G_\kappa$. First

$$\|I1_{X\notin\mathscr{S}}\|^{(1)}_{G,\Gamma',\bar{h}}\leqq 3^d\frac{2}{L}\|K\|_{G,\Gamma,h},$$

where we have used Lemma 3.1 and $\bar{h}\leqq h$. This bound reduces the proof to considering $\tilde{I}=I1_{X\in\mathscr{S}}$. By (4.5) $\tilde{I}$ has three terms but the last term contains $F-R$ and, by Lemmas 3.1 and 4.2, is already small enough to satisfy the conclusion of Lemma 4.4. Thus we will drop this term from I in the rest of this proof.

By construction (with $F-R$ dropped) $\tilde{I}_n(\phi=0)=0$ if $\dim(n)\leqq d$, therefore by Lemma 3.1 and Lemma 4.3,

$$\|\tilde{I}\|^{(1)}_{G,\Gamma',\bar{h}}\leqq 2(3L)^d\|\tilde{I}\|_{G,\Gamma,\bar{h}}\leqq\text{Const}\cdot L^d\cdot\|\tilde{I}\|_{G,\Gamma,\bar{h},\dim>d}.$$

Recall the definition of $\bar{h}$ (1.20); by changing $\bar{h}$ to h in the dim $>d$ norm we can extract a factor of $(\gamma/L)^q$, where q is the lowest dimension with dim $>d$. If $d=1$ then $q=d+1/2$, corresponding to $(\partial\phi)^3$. If $d\geqq 2$ then the lowest dimension is $d+1$, corresponding to $(\partial\phi)(\partial\partial\phi)$. By the definition of B we continue the inequality with

$$\leqq\text{Const}\cdot B\|\tilde{I}\|_{G,\Gamma,h,\dim>d}.$$

By the triangle inequality and definition (4.5) of I it is sufficient to bound $\|(K-K^{(2)})1_{X\in\mathscr{S}}\|_{G,\Gamma,h,\dim>d}$ and $\|``(K^{(2)}-F)"1_{X\in\mathscr{S}}\|_{G,\Gamma,h,\dim>d}$. The first term is equal to $\|K1_{X\in\mathscr{S}}\|_{G,\Gamma,h,\dim>d}$ and then bounded by $\|K\|_{G,\Gamma,h}$. By a short calculation which resembles Lemma 4.1 the second term is also bounded by Const $\cdot\,\|K\|_{G,\Gamma,h}$. □

5. Bound on R^+

The main result of this section is Lemma 5.2, which states that the $\|\ \ \|^{(1)}$ norm of R^+ is smaller than $\|K\|$ by a factor of L if $\|K\|$ is sufficiently small. The definition of R^+ suggests that $\|R^+\|$ is of order $\|K\|^2$ so this is not surprising. The proof in this section is just an exercise in techniques that are quite standard in the practice of cluster expansions. See, for example, [15].

Let

$$R^+(X)=\sum_{N=2}^{\infty}\frac{1}{N!}\sum_{\underset{\sim}{Y}\in\mathscr{G}(X,N)}R(Y_1)\ldots R(Y_N),\tag{5.1}$$

where $\underset{\sim}{Y}\equiv(Y_1,\ldots,Y_N)\in\mathscr{G}(X,N)$ implies

($\mathscr{G}$1) $X=\bigcup_{i=1,N}Y_i$.

($\mathscr{G}$2) X is connected.

($\mathscr{G}$3) For any box Δ there are at most τ sets $Y_1,\ldots,Y_N$ such that $\Delta\cap\hat{Y}_i\neq\Phi$. (5.2)

Clearly the condition (3) of (2.3) implies ($\mathscr{G}$3).

Let

$$\Gamma_{-\delta}(X)=\Gamma(X)2^{-\delta|X|}.\tag{5.3}$$

Lemma 5.1. *There exists $C>0$ such that for any large field regulator g, and any $\delta>0$,*

$$\|R^+\|_{g^\tau,\Gamma_{-\delta},h}\leqq\sum_{N=2}^{\infty}(C/\delta)^{N-1}\|R\|^N_{g,\Gamma,h},$$

where $g^\tau(X,\phi)=(g(X,\phi))^\tau$.

Proof.

$$R^+_M(X)=\sum_{N=2}^{\infty}\frac{1}{N!}\sum_{M_1,\ldots,M_N:\Sigma M_i=M}\frac{M!}{M_1!\ldots M_N!}\sum_{\underset{\sim}{Y}\in\mathscr{G}(X,N)}\prod_{i=1}^{N}R_{M_i}(Y_i).$$

X is connected, therefore, if G is the graph on vertices $1,\ldots,N$ whose lines ij are those pairs such that $Y_i\cap Y_j\neq\Phi$, then G is connected. G contains a maximal connected tree graph, so the contribution of lines in G to the line regulator part of $\Gamma_{-\delta}$ equals one. Property ($\mathscr{G}$1) then implies that

$$\Gamma_{-\delta}(X)\leqq\Gamma_{-\delta}(Y_1)\ldots\Gamma_{-\delta}(Y_N)=2^{-\delta|X|}\prod\Gamma(Y_i).$$

Property ($\mathscr{G}$3) implies $g(X)^\tau\geqq\prod g(Y_i)$, therefore

$$\|R^+_M\|_{g^\tau,\Gamma_{-\delta}}\leqq\sum_{N=2}^{\infty}\frac{1}{N!}\sum_{\underset{\sim}{M}}\frac{M!}{M_1!\ldots M_N!}I_{N,M},$$

where

$$I_{N,M}=\operatorname*{Sup}_{\Delta}\sum_{X\cap\Delta\neq\Phi}\sum_{\underset{\sim}{Y}\in\mathscr{G}(X,N)}\prod_{i=1}^{N}\|R_{M_i}(Y_i)\|_g\Gamma_{-\delta}(Y_i).$$

Insert the following consequence of ($\mathscr{G}$2), which says that if X is connected then there is a tree graph as described in the remark above,

$$1(\underset{\sim}{Y}\in\mathscr{G})\leqq\sum_{T\text{ on }1,\dots,N}\prod_{ij\in T}\sum_{\Delta_i,\Delta_j:\operatorname{dist}(\Delta_i,\Delta_j)\leqq 1}\times 1(Y_i\cap\Delta_i\neq\Phi)\,1(Y_j\cap\Delta_j\neq\Phi),$$

followed by $1(X\cap\Delta\neq\Phi)\leqq\sum_j 1(Y_j\cap\Delta\neq\Phi)$. For each T we estimate the sums over X, Y's, and Δ's. The result is

$$I(N,\underset{\sim}{M})\leqq\text{Const}^{N-1}\sum_{T,j}\prod_i\left[\operatorname*{Sup}_{\Delta}\sum_Y\|R_{M_i}(Y)\|_g\Gamma_{-\delta}(Y)\times|Y|^{\eta(i,T)}1(Y\cap\Delta\neq\Phi)\right],$$

where $\eta(i,T)=d(i,T)\ -1$ for $i=1,\dots,\hat{j},\dots,N$ and $\eta(j,T)=d(j,T)$. $d(i,T)=$co-ordination of T at vertex i. This estimate is not difficult and is left to the reader. Details can be found on p. 180 of [15]. By Cayley's Theorem on the number of tree graphs,

$$\sum_T(\cdot)\leqq\sum_{d_1,\dots,d_N}\frac{(N-2)!}{(d_1-1)!\dots(d_N-1)!}\operatorname*{Sup}_{T:\underset{\sim}{d}(T)=\underset{\sim}{d}}(\cdot),$$

and $|Y|^\eta\leqq\frac{\eta!}{\delta^\eta}2^{\delta|Y|}(\log 2)^{-\eta}$ followed by $\eta(j,T)\leqq(N-1)\,[d(1,T)-1]!$, we arrive at

$$I(N,\underset{\sim}{M})\leqq N!\,\text{Const}^{N-1}\delta^{-1}\sum_{\underset{\sim}{d}}\prod_i\left[\delta^{1-d_i}\operatorname*{Sup}_{\Delta}\sum_Y\times\|R_{M_i}(Y)\|_g\Gamma(Y)\,1(Y\cap\Delta\neq\Phi)\right].$$

For a tree graph on $1,\dots,N$ there are $N-1$ lines and $\sum(d_i-1)=N-2$. Thus $\prod\delta^{1-d_i}=\delta^{2-N}$ and the bracket inside the summantions $\underset{\sim}{d}$ is independent of $\underset{\sim}{d}$. Use $\sum_{\underset{\sim}{d}}1=2^{N-2}\sum_{\underset{\sim}{d}}2^{-\Sigma(d_i-1)}\leqq 2^{N-2}2^N$,

$$I(N,\underset{\sim}{M})\leqq N!\,\text{Const}^{N-1}\delta^{1-N}\prod_i\left[\operatorname*{Sup}_{\Delta}\sum_Y\|R_{M_i}(Y)\|_g\Gamma(Y)\,1(Y\cap\Delta\neq\Phi)\right].$$

Thus

$$\|R_M^+\|_{g^\tau,\Gamma_{-\delta}}\leqq\sum_{N=2}^{\infty}\frac{1}{N!}\sum_{\underset{\sim}{M}}\frac{M!}{M_1!\dots M_N!}I(N,\underset{\sim}{M})$$

$$\leqq\sum_{N=2}^{\infty}\sum_{\underset{\sim}{M}}\frac{M!}{M_1!\dots M_N!}(\text{Const}/\delta)^{N-1}\prod_i\left[\operatorname*{Sup}_{\Delta}\sum_Y\|R_{M_i}(Y)\|_g\Gamma(Y)\,1(Y\cap\Delta\neq\Phi)\right].$$

Finally we multiply by $h^M/M!$ and sum over M

$$\|R\|_{g^\tau, \Gamma_{-\delta}, h} \leqq \sum_{N=2}^{\infty} (\mathrm{Const}/\delta)^{N-1} \left[\sum_M \frac{h^M}{M!} \operatorname{Sup}_{\Delta} \sum_Y \|R_M(Y)\|_g \right.$$
$$\left. \times \Gamma(Y) 1(Y \cap \Delta \neq \Phi) \right]^N \equiv \sum_{N=2}^{\infty} (\mathrm{Const}/\delta)^{N-1} \|R\|_{g,\Gamma,h}^N,$$

which is the conclusion of the lemma. □

Lemma 5.2. *There exist C_1, C_2 such that for any large field regulator g, $A \geqq 2L^{d+1}$, $\varepsilon > 0$, h, and K with*

$$\|K\|_{g,\Gamma,h} \leqq \frac{C_1}{L^{d+1}} \left[\frac{\varepsilon h_1^2}{1+\varepsilon h_1^2} \right]^2,$$

the following bound holds:

$$\|R^+\|_{G_\varepsilon, \Gamma', h}^{(1)} \leqq \frac{C_2}{L} \|K\|_{g,\Gamma,h},$$

where Γ' is defined by replacing A by $A' = 2^{(2-d)}A$ in Γ as in Lemma 4.4.

Proof. By Lemma 3.1, followed by Lemma 5.1,

$$\begin{aligned}
\|R^+\|_{G_\varepsilon, \Gamma', h}^{(1)} &\leqq 2(3L)^d \|R^+\|_{G_\varepsilon, \Gamma, h} \\
&\leqq \mathrm{Const}\, L^d \sum_{N=2}^{\infty} (\mathrm{Const}/\delta)^{N-1} \|R\|_{G_{\varepsilon/\tau}, \Gamma_\delta, h}^N \\
&\leqq \mathrm{Const}\, L^d \sum_{N=2}^{\infty} (\mathrm{Const}/\delta)^{N-1} [2^{(2^d\delta)} \|R\|_{G_{\varepsilon/\tau}, \Gamma, h}]^N,
\end{aligned}$$

where we have used $\Gamma_\delta(X) \leqq 2^{(2^d\delta)}\Gamma(X)$ because R vanishes off sets not in $\mathscr{S}$. Choose $\delta = 2^{-d}$. By Lemma 4.2 we continue with

$$\leqq \mathrm{Const}\, L^d \sum_{N=2}^{\infty} (\mathrm{Const})^{N-1} \|K\|_{g,\Gamma,h}^N \leqq \mathrm{Const}\, L^{-1} \|K\|_{g,\Gamma,h}. \quad \square$$

6. Lemmas on H

For K a local analytic functional on $\mathcal{A}^{(0)}$ define

$$|K|_{G,\Gamma,h}^{(1)} = \sum_0^{\infty} \frac{h^N}{N!} \operatorname{Sup}_Y \Gamma(\bar{Y}) \|K_N(Y)\|_G. \tag{6.1}$$

For any large field regulators g and g' and any large set regulators (3.3) γ, γ', it is not difficult to check that

$$\|JJ'\|_{gg', \gamma\gamma', h}^{(1)} \leqq |J|_{g,\gamma,h}^{(1)} \|J'\|_{g',\gamma',h}^{(1)}. \tag{6.2}$$

Let

$$\tilde{X} = \{y : |y-x| \leqq 3 \text{ for some } x \in X\}, \tag{6.3}$$

$$\gamma_\delta(X) = 2^{\delta|X|}. \tag{6.4}$$

Recall from the introduction that $\tilde{G}_\varepsilon$ is defined by using $\tilde{X}$ in the place of $\hat{X}$ in the definition of G_ε. For the following Lemma $\bar{h}$ is defined as in Theorem A in the Introduction.

Lemma 6.1. *There exists C_1 such that for $\varepsilon>0$, $1\geqq\delta>0$, h, any large field regulator g, and K such that*

$$\|K\|_{g,\Gamma,h}\leqq\frac{C_1}{L^d}\delta\frac{\varepsilon h_1^2}{1+\varepsilon h_1^2},$$

the following bound holds:

$$|H|^{(1)}_{\tilde{G}_\varepsilon,\gamma_{-\delta},\bar{h}}\leqq 4.$$

Proof. Let $\alpha_i=\bar{h}_i/h_i$, $n=(n_1,n_2)$,

$$\frac{\alpha^n h^n}{n!}\|H(Y)_n\|_{\tilde{G}_\varepsilon}\leqq\alpha^n\sum_N\frac{h^N}{N!}\left\|\prod_{X:X\cap Y\neq\Phi}[e^{-F(X)}]_{N_X}\right\|_{\tilde{G}_\varepsilon},$$

where $N=(N_X)$ with each $N_X=(N_{1,X},N_{2,X})$, a multi-index. $N!=\prod N_X!$ and $h^N=\prod h^{N_X}$,

$$\leqq\alpha^n\sum_N\frac{h^N}{N!}\prod_{X:X\cap Y\neq\Phi}\|[e^{-F(X)}]_{N_X}\|_{\tilde{G}_{\varepsilon/\tau}},$$

because at most τ sets in $\mathscr{S}$ can overlap, i.e.,

$$\begin{aligned}\tau=\operatorname*{Sup}_x|\{X\in\mathscr{S}:x\subset\hat{X}\}|&=\alpha^n\prod_{X:X\cap Y\neq\Phi}\sum_m\frac{h^m}{m!}\|[e^{-F(X)}]_m\|_{\tilde{G}_{\varepsilon/\tau}},\\&=\alpha^n\prod_{X:X\cap Y\neq\Phi}\left[1+\sum_m\frac{h^m}{m!}\|[e^{-F(X)}-1]_m\|_{\tilde{G}_{\varepsilon/\tau}}\right],\\&\leqq\alpha^n\prod_{X:X\cap Y\neq\Phi}[1+\|e^{-F}-1\|_{G_{\varepsilon/\tau},\Gamma,h}],\\&\leqq\alpha^n\exp[\tau|Y|\,\|e^{-F}-1\|_{G_{\varepsilon/\tau},\Gamma,h}],\end{aligned}$$

because the number of terms in the product is at most $\tau|Y|$. Now use Lemma 4.2 and choose C_1 so that,

$$\leqq\alpha^n\exp[\delta|\bar{Y}|].$$

The conclusion of the Lemma follows by multiplying through by $\gamma_{-\delta}(Y)=2^{-\delta|Y|}$, taking the supremum over Y and summing over n. □

7. Conclusion of Proof of Theorem A

We have proved in Lemmas 4.4 and 5.2 that the norms of I and R^+ are smaller than $\|K\|/L$, (roughly speaking). Also, the norm of H is bounded in Lemma 6.1. We now put these results together to prove Theorem A.

Let J be an analytic functional defined for sets in $\Lambda^{(0)}$. Define $\bar{J}$, an analytic functional defined on the next scale, i.e., on sets in $\Lambda^{(1)}$, by

$$\bar{J}(X)=\sum\frac{1}{N!}\sum_{\underset{\sim}{X}\in\mathscr{RB}(X,N)}J(X_1)\dots J(X_N),\tag{7.1}$$

where $\underset{\sim}{X} \equiv (X_1, \ldots, X_N) \in \mathscr{RB}(X, N)$ iff

1. $X = \cup X_i$.
2. The graph G, whose vertices are $1, \ldots, N$ and whose lines ij are those pairs such that $\bar{X}_i \cap \bar{X}_j \neq \Phi$, is connected.
3. $X_i \cap X_j = \Phi$ for all i, j. (7.2)

Note that we have defined $\mathscr{RB}$ on sets in $\Lambda^{(0)}$, whereas in Sect. 2 it was defined only for sets in $\Lambda^{(1)}$. Conditions (2) and (3) of (7.2) and (2.8) are the same. Condition (1) is different.

Recall the definitions of $\|\cdot\|^{(1)}_{g,\Gamma,h}$ in (3.2), γ_δ in (6.4) and $\Gamma_\delta = \Gamma\gamma_\delta$ in (5.3).

Lemma 7.1. *There exists $C_1 > 0$ such that for any large field regulator g and $\delta > 0$,*

$$\|\bar{J}\|^{(1)}_{g,\Gamma,h} \leqq \sum_{N=1}^{\infty} (C_1/\delta)^{N-1} (\|J\|^{(1)}_{g,\Gamma_\delta,h})^N .$$

Proof. Exactly the same as the proof of Lemma 5.1.

Let $\delta = 2^{-d-3}$ and recall the definitions of $\tilde{G}_\kappa$, $\bar{\Gamma}$, and $\bar{h}(\gamma)$ from Theorem A in the Introduction. Recall the definition of Γ' in Lemmas 4.4 and 5.2. $\bar{\Gamma}$ and Γ' are related by $\bar{\Gamma}_{6\delta} = \Gamma'$.

Proof of Theorem A. Set $J(X) = I - R^+$. Let U be in $\Lambda^{(1)}$, then

$$HK(U) = \sum_{X : \bar{X} = U} H(X)\bar{J}(X), \tag{7.3}$$

where $\bar{J}$ was defined above Lemma 7.1,

$$\begin{aligned}
\|HK\|_{\tilde{G}_{\kappa+\delta\kappa},\bar{\Gamma},\bar{h}} &\equiv \sum_n \frac{\bar{h}^n}{n!} \operatorname*{Sup}_{\Delta \in \Lambda^{(1)}} \sum_{U : U \cap \Delta \neq \Phi} \|(HK)_n(U)\|_{\tilde{G}_{\kappa+\delta\kappa}} \bar{\Gamma}(U) \\
&\leqq \sum_n \frac{\bar{h}^n}{n!} \operatorname*{Sup}_{\Delta \in \Lambda^{(1)}} \sum_{U : U \cap \Delta \neq \Phi} \sum_{X : \bar{X} = U} \cdot \|[H(X)\bar{J}(X)]_n\|_{\tilde{G}_{\kappa+\delta\kappa}} \bar{\Gamma}(U) \\
&= \sum_n \frac{\bar{h}^n}{n!} \operatorname*{Sup}_{\Delta \in \Lambda^{(1)}} \sum_{X : \bar{X} \cap \Delta \neq \Phi} \|[H(X)\bar{J}(X)]_n\|_{\tilde{G}_{\kappa+\delta\kappa}} \bar{\Gamma}(X) \\
&\equiv \|H\bar{J}\|^{(1)}_{\tilde{G}_{\kappa+\delta\kappa},\bar{\Gamma},\bar{h}} \leqq |H|^{(1)}_{\tilde{G}_{\delta\kappa},\gamma-\delta,\bar{h}} \, \|\bar{J}\|^{(1)}_{\tilde{G}_\kappa,\bar{\Gamma}_\delta,\bar{h}} ,
\end{aligned} \tag{7.4}$$

where we have used $\sum_{U : U \cap \Delta \neq \Phi} \sum_{X : \bar{X} = U} = \sum_{X : \bar{X} \cap \Delta \neq \Phi}$ in the second equality and (6.2) in the last inequality. We make it bigger by replacing $\bar{G}$ by G and use Lemmas 6.1 and 7.1,

$$\leqq \text{Const} \sum_{N=1}^{\infty} (\text{Const}/\delta)^{N-1} (\|J\|^{(1)}_{G_\kappa,\bar{\Gamma}_{2\delta},\bar{h}})^N .$$

Note that $\bar{\Gamma}_{2\delta} \leqq \Gamma'$ which appears in Lemma 4.4. By Lemma 4.4 we estimate the norm of J,

$$\leqq \text{Const} \sum_{N=1}^{\infty} \text{Const}^{N-1} (B \|K\|_{G_\kappa,\Gamma,h})^N .$$

We sum the series using the small K hypothesis and obtain the result. $\square$

8. Integration with Real Covariances. The Proof of Theorem B

The strategy of the proof is as follows: If we set $Z(t)=\mu_{tC}*Z_0$, then Z solves a functional heat equation $\frac{\partial Z}{\partial t}=\frac{1}{2}Z_{\phi\phi}$, where $Z_{\phi\phi}$ is a functional Laplacian of Z. Since $Z(\Phi,\phi)=1$, Z is in the domain of $\mathscr{L}$og, which is defined by the (finite) power series

$$\mathscr{L}\text{og}(Z)=(Z-\mathscr{I})-\tfrac{1}{2}(Z-\mathscr{I})\circ(Z-\mathscr{I})+\dots. \tag{8.1}$$

Since $\square+K=\mathscr{L}\text{og}(Z)$ the heat equation for Z is equivalent to

$$\frac{\partial K}{\partial t}=\tfrac{1}{2}(K_{\phi\phi}+K_\phi\circ K_\phi);\qquad \square+K(t=0)=\mathscr{L}\text{og}\,Z_0. \tag{8.2}$$

K_ϕ is a functional gradient of K. This equation is equivalent to the integral equation

$$K(t)=\mu_{tC}*K(0)+\tfrac{1}{2}\int_0^t ds\,\mu_{(t-s)C}*(K_\phi\circ K_\phi).$$

By taking the $\|\cdot\|_{g,\gamma,h}$ norm of this equation we learn that it has an iterative solution such that $\|K(t)\|_{g(t),\gamma,h}$ is dominated by a function $k(t,h)$ which solves $\frac{\partial k}{\partial t}=\|C\|\left(\frac{\partial k}{\partial h}\right)^2$, i.e., a one dimensional version of the original flow equation without the Laplacian. The bounds in the theorem are obtained by studying the explicit solution to this hamilton-Jacobi equation. This approach is a method in [11], except that products have been replaced by $\circ$ products. It is a simple method for generating and studying a cluster expansion originally due to Glimm, Jaffe, and Spencer [10].

The Functional Laplacian

Let μ_t be the gaussian measure on $H_s(\Lambda)$ with fluctuation covariance tC, where C is positive definite. Let $F(\Psi)=F(\Lambda,\phi)$ be a functional whose domain is all $\Psi(\xi)=(\Psi_i(x),\Psi_{jk}(x))$, $i,j,k=1,\dots,d$, where Ψ_i, Ψ_{jk} are arbitrary continuous functions on Λ. We assume that F is infinitely differentiable and for some $h>0$ and $\varepsilon>0$,

$$\|F\|_{G,h}\equiv\sum\frac{h^n}{n!}\|F_n\|_{G_\varepsilon}<\infty. \tag{8.3}$$

If F has these properties we shall say that F is analytic in ϕ.

We define the functional Laplacian $F_{\phi\phi}$ of an analytic functional F by

$$\tfrac{1}{2}F_{\phi\phi}=\lim_{t\to 0}\frac{1}{t}(\mu_t*F-F). \tag{8.4}$$

The limit exists and is given by

$$F_{\phi\phi}=\int F_2(d^2\xi)C(\xi,\xi'). \tag{8.5}$$

F_2 denotes a second order derivative, specified by the indices inside ξ and ξ'. We omit the easy proof which consists of expanding the F in $\mu*F$ in a Taylor series to second order in ϕ and using a Chebycheff inequality to show that the remainder gives no contribution as $t \to 0$. The bound on the remainder uses G0 of (1.16) and (8.3). The Chebycheff inequality is based on the μ_t integrability of $G_{\delta/t}$ for δ sufficiently small.

Lemma 8.1. *If $Z(t,\phi) = (\mu_t * Z_0)(\phi)$, where $Z_0(\Psi)$ is analytic, then,*

$$\frac{\partial Z}{\partial t} = \tfrac{1}{2} Z_{\phi\phi}.$$

Moreover, if $\tilde{Z}(t,\phi)$ is any other solution to this heat equation and the same initial data, which is continuously differentiable in t as a map from t to the space of analytic functionals, then $Z = \tilde{Z}$.

Proof. It is clear from the definition of the Laplacian that $Z(t)$ satisfies the equation. To prove uniqueness: let $\mu_{ts} \equiv \mu_{(t-s)C}$ and consider

$$\tilde{Z}(t) - \mu_t * \tilde{Z}(0) = \int_0^t ds \frac{\partial}{\partial s} \mu_{ts} * \tilde{Z}(s),$$

(where the integral is a Riemann integral).

$$= \int_0^t ds \left\{ -\mu_{ts} * \tfrac{1}{2} \tilde{Z}_{\phi\phi}(s) + \mu_{ts} * \frac{\partial \tilde{Z}(s)}{\partial s} \right\} = 0. \quad \square$$

Lemma 8.2. *If $Z(t) = \mu_t * \mathscr{E}\mathrm{xp}(\square + K_0)$, where $K = K_0(X,\phi)$ is analytic, then $Z(t) = \mathscr{E}\mathrm{xp}(\square + K(t))$, where $K(t,X,\phi)$ is the finite series obtained by iterating*

$$K(t) = \mu_t * K_0 + \tfrac{1}{2} \int_0^t ds \mu_{ts} * (K_\phi \circ K_\phi)(s,\phi),$$

where

$$K_\phi \circ K_\phi(\phi) = \int C(\xi,\eta) K_1(\phi, d\xi) \circ K_1(\phi, d\eta).$$

Proof. The iterative solution to the integral equation is a finite series because eventually one builds up terms containing $K_\phi \circ \ldots \circ K_\phi$ with more factors than there are blocks in Λ and such terms are zero. It is easy to check that the iterative solution is a continuously differentiable map from t into analytic functions of ϕ which solves (8.2). As remarked earlier (8.2) is equivalent to the heat equation for $\mathscr{E}\mathrm{xp}(\square + K(t))$, so Lemma 8.1 implies $Z(t) = \mathscr{E}\mathrm{xp}(\square + K(t))$. $\square$

Proposition 8.3. *Let g and γ obey conditions* G0–G3 *of Theorem B and $\Gamma 0$, $\Gamma 1$ of (3.3). Let $h = (h,h)$ and set*

$$k(t,h) = \|K(t)\|_{g(t),\gamma,h},$$

then $k(t,h)$, as a power series in h, is term by term majorised by the solution of

$$k^*(t,h) = k(0,h) + \|C\| \int_0^t ds \left(\frac{\partial k^*(s,h)}{\partial h} \right)^2 \tag{8.6}$$

or equivalently,

$$\frac{\partial k^*}{\partial t} = \|C\| \left[\frac{\partial k^*}{\partial h} \right]^2; \quad k^*(t=0) = k(t=0).$$

Proof.

$$K_n(t,X)=\mu_t * K_n(0,X)+\tfrac{1}{2}\int_0^t ds\,\mu_{ts} * (K_\varphi \circ K_\varphi)_n(s,X),$$

where,

$$(K_\varphi \circ K_\varphi)_n(s,X)$$

$$=\sum_{a,b:a+b=n}\frac{n!}{a!\,b!}\sum_{Y,Z:Y\cup Z=X}\int K_{a+1}(s,Y,d\xi)C(\xi,\xi')K_{b+1}(s,Z,d\xi').$$

$K_{a+1}(d\xi)$ denotes a derivative of K_a with respect to the variable specified by $d\xi$. Take the $\|\ \|_g$ norm. Condition (G3) in Theorem B implies $\|\mu_{ts} * A\|_{g(t)} \leqq \|A\|_{g(s)}$, where $A=A(X,\phi)$ or $A=A_n(X,\phi,d^n\xi)$. Therefore

$$\|K_n(t,X)\|_g \leqq \|K_n(0,X)\|_g$$

$$+\tfrac{1}{2}\int_0^t ds\sum_{a,b}\frac{n!}{a!\,b!}\sum_{\Delta_1,\Delta_2}C(\Delta_1,\Delta_2)\sum_{Y,Z}\|K_{a+1}(s,Y,\xi\in\Delta_1)\|_g\|K_{b+1}(s,Z,\xi'\in\Delta_2)\|_g,$$

where

$$C(\Delta_1,\Delta_2)=\operatorname*{Sup}_{\xi\in\Delta_1,\xi'\in\Delta_2}|C(\xi,\xi')|$$

$$\times\|K_{a+1}(s,Y,\xi\in\Delta_1)\|_g=\sum_{\Delta_2,\ldots,\Delta_{a+1}}\|\operatorname{Var}K_{a+1}(Y)1_{\Delta_1,\ldots,\Delta_{a+1}}\|_g.$$

Multiply both sides by $\gamma(X)$ and sum over X such that $X\cap\Delta \neq \Phi$,

$$\|K_n(t)\|_{g,\gamma} \leqq \|K_n(0)\|_{g,\gamma}$$

$$+\int_0^t ds\sum_{a,b}\frac{n!}{a!\,b!}\operatorname*{Sup}_{\Delta}\sum_{\Delta_1,\Delta_2}C_\theta(\Delta_1,\Delta_2)\sum_{Y\cap\Delta,\Delta_1\neq\Phi}\|K_{a+1}(s,Y,\xi\in\Delta_1)\|_g\gamma(Y)$$

$$\times\sum_{Z\cap\Delta_2\neq\Phi}\|K_{b+1}(s,Z,\xi'\in\Delta_2)\|_g\gamma(Z).$$

We have used $\Gamma 1$ of (3.3) and have set

$$C_\theta(\Delta_1,\Delta_2)=C(\Delta_1,\Delta_2)\,\theta(\operatorname{dist}(\Delta_1,\Delta_2)).$$

Now we use $\sum_{Z\cap\Delta_2\neq\Phi}\|K_{b+1}(s,Z,\xi'\in\Delta_2)\|\gamma(Z)\leqq\|K_{b+1}(s)\|_{g,\gamma}$, followed by $\sum_{\Delta_2}C_\theta(\Delta_1,\Delta_2)\leqq\|C\|$, followed by $\sum_{Y\cap\Delta,\Delta_1\neq\Phi}\|K_{a+1}(s,Y,y\in\Delta_1)\|_g\gamma(Y)=\|K_{a+1}\|_{g,\gamma}$. The result is

$$\|K_n(t)\|_{g,\gamma}\leqq\|K_n(0)\|_{g,\gamma}$$

$$+\int_0^t ds\sum_{a,b}\frac{n!}{a!\,b!}\|C\|\,\|K_{a+1}(s)\|_{g,\gamma}\cdot\|K_{b+1}(s)\|_{g,\gamma}.$$

Multiply both sides by $h^n/n!$ and sum over n.

$$\|K(t)\|_{g,\gamma,h}\leqq\|K(0)\|_{g,\gamma,h}$$

$$+\int_0^t ds\,\|C\|\,\frac{\partial}{\partial h}\|K(s)\|_{g,\gamma,h}\,\frac{\partial}{\partial h}\|K(s)\|_{g,\gamma,h}.$$

This inequality and the one before show that when equation (8.6) is iterated, a power series in h is generated which term by term majorises $\|K(s)\|_{g,\gamma,h}$. □

Proof of Theorem B. The proof is to combine Proposition 8.3 with the following lemma.

Lemma 8.4. *Suppose $k_0(h)$ is a power series with positive coefficients in a single variable h and it is convergent for $|h| \leqq h_0$. Then there is a unique function $k(t,h)$, analytic in t and h near the origin, which solves*

$$\frac{\partial k}{\partial t} = \|C\| \left[\frac{\partial k}{\partial h}\right]^2; \qquad k(t=0,h) = k_0(h), \tag{8.7}$$

and if $k_0(h_0) \leqq \frac{1}{16\|C\|}(h_0 - h_1)^2$ *for some* $h_1, 0 < h_1 < h_0$, *then*

$$k(1,h_1) \leqq k(0,h_0).$$

Proof of Lemma 8.4. The solution of the Hamilton Jacobi equation in the lemma is unique within the class of functions of t and h which are analytic near the origin and have the same initial data because the differential equation determines a formal power series solution. Such a solution is easily verified to be given by

$$k(t,h) = k(0,h_{cl}) - \tfrac{1}{4}(t\|C\|)^{-1}(h - h_{cl})^2, \tag{8.8}$$

provided $h_{cl} = h_{cl}(t,h)$ is analytic in t and h near the origin and is a stationary point for the right-hand side, i.e., satisfies

$$f_h(h_{cl} - h) = 0; \qquad f_h(z) \equiv -2t\|C\| k'(0, z+h) - z, \tag{8.9}$$

$$\lim_{t \to 0} h_{cl} = h,$$

where $k' = \partial k/\partial h$. We will now show that h_{cl} exists and is unique. For $|h| \leqq h_1$ let $B = \{z \,|\, |z| \leqq (h_0 - h)/2)$. By Cauchy's representation of analytic functions and the assumption on k we have, for $z \in \partial B$, $|k'(0,z+h)| \leqq 2k(0,h_0)/(h_0 - h_1)$ $\leqq (h_0 - h_1)/(8\|C\|) \leqq |z|/(4\|C\|)$. Hence $f_h(z) \neq 0$ for $z \in \partial B$ for $0 \leqq t \leqq 1$. Since $f_h(z) = 0$ has a unique solution ($z = 0$) in B for $t = 0$, it follows by Rouché's theorem that $f_h(z) = 0$ has a unique solution in B for $0 \leqq t \leqq 1$. Furthermore since k_0 has real coefficients the intermediate value theorem implies h_{cl} is real if h is real. Hence $k(t,h_1) \leqq k(0,h_{cl}) \leqq k_0(h_0)$ by (9.2) followed by the maximum principle. □

9. The Large Field Regulator

We consider complex Gaussian measure with covariances $C: L_2 \to L_2$ that can be written in the form $C = C_1^{1/2}(1 + iV)C_1^{1/2}$, where C_1 is a positive definite operator on L_2 whose square root $C_1^{1/2}$ has a kernel $C_1^{1/2}(x,y)$ which belongs to $H_s(\Lambda \times \Lambda)$. V is a real symmetric trace class operator on L_2 with operator norm less than one half. We assume that there exists a constant A such that

$$\begin{gathered} \|C_1\|_{s, X \times \Lambda} \leqq A|X|, \qquad X \subset \Lambda, \\ C_1 : H_{-s} \to H_s \text{ has norm less than } A, \\ \|V\|^2_{0, X \times X} \leqq A|X|, \qquad X \subset \Lambda, \end{gathered} \tag{9.1}$$

$\|V\|_{0, X \times X}$ is the L_2 norm of the kernel of V as a function on $X \times X$.

Given a set X, let $X(\tau)$ be X with a (smooth boundary) collar of width τ. Given τ_0 and τ_1 satisfying $0 \leqq \tau_0 < \tau_1 < 1/3$ and $\tau_1 - \tau_0 \geqq 1/12$ we let

$$\begin{aligned} X_0 &= X(\tau_0), \quad X_1 = X(\tau_1), \\ g_0(X,\phi) &= \exp\left[\frac{\kappa}{2} \|\phi\|^2_{s,X_0}\right], \\ g_1(X,\phi) &= \exp\left[\frac{\kappa}{2} \|\phi\|^2_{s,X_1} + \frac{\kappa}{2} \|\phi\|'^2_{s,X_1} + \delta|X|\right], \\ g_u &= g_0^{1-u} g_1^u, \quad 0 \leqq u \leqq 1, \end{aligned} \tag{9.2}$$

$\|\phi\|'^2_{s,X(\tau)}$ is defined in the same way as $\|\phi\|^2_{s,X(\tau)}$ but with the $\sum\int |\phi_i|^2$ term omitted. Let $\delta = 2^{-d-3}$.

Proposition 9.1. *Let g_u be as defined above. Under the above assumptions on C there exists $\kappa_{\max} > 0$ such that*

1. *If $V=0$ then for all $\kappa \leqq \kappa_{\max}$ and $0 \leqq u \leqq t \leqq 1$,*

$$\mu_{(t-u)C} * g_u(X) \leqq g_t(X).$$

2. *If $V \neq 0$ and $C^{1/2}(x,y) = 0$ if $|x-y| \geqq 1/12$, then for all $\kappa \leqq \kappa_{\max}$, $0 \leqq u \leqq t \leqq 1$, $t-u \geqq 1/4$,*

$$|\mu_{(t-u)C}| * g_u(X) \leqq g_t(X).$$

In (2) $|\mu_C|$ is the absolute value of μ_C restricted to the Borel σ-algebra of $H_s(X(1/3))$.

Comments. We have summarised some useful facts about complex Gaussian measures in Appendix A. In particular, in part (2) it is necessary to restrict μ_C to a small σ algebra before taking the absolute value, otherwise $|\mu_C| \cong O(\exp[\text{Const}|\Lambda|])$. By definition, μ_C restricted to $H_s(X(1/3))$ is the Gaussian measure μ_W on $H_s(X(1/3))$ with covariance $W(x,y) = 1(x)C(x,y)1(y)$, where 1 is the indicator function of the set $X(1/3)$. Let $\tilde{C}$ be the covariance obtained from C by setting $V(x,y)=0$ if either x or y lie outside $X(1)$. By the short range hypothesis on $C_1^{1/2}$, $W(x,y) = 1(x)\tilde{C}(x,y)1(y)$ so that the restrictions of μ_C and $\mu_{\tilde{C}}$ coincide and therefore $|\mu_W| \leqq |\mu_{\tilde{C}}|$. We will obtain part (2) by estimating $|\mu_{\tilde{C}}|$.

Proof. We will suppress the s indices on the norms. Repeated indices are to be summed over. Let $D = X_1 - X_0$, $\mu_u = \mu_{uC}$.

Part (1), Case $V=0$. g_u can be rewritten in the form

$$g_u(X) = \exp\frac{\kappa}{2}\left[\int_{X_0} \phi_i^2 + u\int_D \phi_i^2 + [1+u]\,\|\phi\|'^2_{X_0} + 2u\,\|\phi\|'^2_D + u\delta|X|\right]. \tag{9.3}$$

Let $F_u(\phi) = \exp\left[\frac{\kappa}{2}[1+u]\,\|\phi\|'^2_{X_0}\right]$, then for any $p \geqq 1$,

$$\mu_{t-u} * F_u^p = F_u^p \int d\mu_{t-u}(f) F_u^p(f) \exp[p\kappa[1+u]\langle f, T\phi\rangle],$$

where $T: H_s \to H_{-s}$ is defined by $\langle g, Tg\rangle = \|g\|'^2_{X_0}$ for all $g \in H_s$. By (9.1), Corollary A4 and $\|T\phi\|^2_{-s} \leqq \|\phi\|'^2$, we find that for κ sufficiently small, depending on p,

$$\mu_{t-u} * F_u^p \leqq F_u^p e^{\delta(t-u)|X|} \exp[\tfrac{1}{2}O(p\kappa[t-u])^2\,\|\phi\|'^2_{X_0}].$$

We may summarise our progress so far by saying that given $p \geqq 1$ if we choose κ sufficiently small then for all $u \leqq t$, $u, t \in [0,1]$,

$$\mu_{t-u} * F_u^p \leqq F_{\tilde{t}}^p e^{\delta(t-u)|X|},$$

where $\tilde{t} = [u+t]/2$. The same argument can be applied to other terms in the exponent of g_u in (9.3) (all but the first) and then Holder's inequality proves that

$$\begin{aligned} &\mu_{t-u} * \exp\left[p\frac{\kappa}{2}\left[u \int_D \phi_i^2 + [1+u]\, \|\phi\|_{X_0}^{\prime 2} + 2u \|\phi\|_D^{\prime 2}\right]\right] \\ &\leqq e^{\delta(t-u)|X|} \exp\left[p\frac{\kappa}{2}\left[\tilde{t} \int_D \phi_i^2 + [1+\tilde{t}]\, \|\phi\|_{X_0}^{\prime 2} + 2\tilde{t} \|\phi\|_D^{\prime 2}\right]\right] \end{aligned} \tag{9.4}$$

for any $p \geqq 1$ and for κ small depending on p. Thus the first term in the exponent of g_u is the only dangerous one and we will now show how to bound it.

For $p \geqq 1$ we let $E(\phi) = \exp\left[\frac{\kappa}{2} \int_{X_0} \phi_i^2\right]$ so that

$$\mu_{t-u} * E^p = E^p \int d\mu_{t-u}(f) E^p(f) \exp\left[p\kappa \int_{X_0} \phi_i f_i\right].$$

By integration by parts we rewrite the exponent,

$$\begin{aligned} \int_{X_0} \phi_i f_i &\equiv -\int_{X_0} \phi_{ii} f + \int_{\partial X_0} \phi_i f \\ &= -\int_{X_0} \phi_{ii} f - \int_D \partial_i(\phi_i f \chi) \\ &= -\int_{X_0} \phi_{ii} f - \int_D \phi_{ii} \chi f - \int_D \phi_i \chi_i f - \int_D \phi_i \chi f_i, \end{aligned}$$

where χ is smooth and $\chi = 1$ on X_0, 0 off X_1, $|\chi| \leqq 1$, $|\partial_i \chi| \leqq O(l_1 - l_0)^{-1}$. We can now use Corollary A4 and the same arguments we have just used to obtain (9.3) to show that

$$\mu_{t-u} * E^p \leqq E^p e^{\delta(t-u)|X|} \cdot \exp\left[O([t-u]p^2\kappa^2)\left\{\int_D \phi_i^2 + \|\phi\|_{X_0}^{\prime 2} + \|\phi\|_D^{\prime 2}\right\}\right]. \tag{9.5}$$

We estimate g_u by the Holder inequality using (9.5) and (9.4). We obtain the result claimed in part (1) of the Proposition.

Part (2), $V \neq 0$. We repeat the argument above with $V \neq 0$ using Corollary A4. The $\operatorname{Tr} V^2$ term in Corollary A4 is $O(|X|)$ as opposed to $O(|\Lambda|)$ because, as discussed in the comment just before this proof, we may set $V(x,y) = 0$ if either x or y is not in $X(1)$. The hypothesis $t - u \geqq 1/4$ is needed because the $\operatorname{Tr} V^2$ term does not tend to zero as $t \to u$. □

10. Integration with Complex Covariances

In this section the main result is Theorem B′ which replaces Theorem B in the Introduction when the covariance is complex. The method of Sect. 8 requires estimates on convolution by a Gaussian measure $\mu_{(t-s)C}$. If C is complex the absolute value of $\mu_{(t-s)C}$ is not well normalized (cf. Lemma A2) and the method of Sect. 8 fails. Nevertheless results analogous to the basic lemmas of Sect. 8 can be found. We begin with the analogue to Proposition 8.3.

Proposition 8.3′. *Let $C = C_1^{1/2}(I+iV)C_1^{1/2}$ satisfy the assumptions set out at the beginning of Sect. 9 and suppose $C_1^{1/2}(x,y)=0$ if $|x-y| \geqq 1/12$. Let $g(u)=g_u$ be the large field regulator of (9.2). Let γ be a large set regulator satisfying $\Gamma 0$, $\Gamma 1$ of (3.3). Let $h=(h,h)$ and set*

$$k(t,h) = \|K(t)\|_{g(t),\gamma,h},$$

where $K(t)$ is defined by $\mu_{tC} * \mathscr{E}\mathrm{xp}(K) = \mathscr{E}\mathrm{xp}(K(t))$. Then for $t \geqq 1/4$, $k(t,h)$ as a power series in h is term by term majorised by the solution $k^*(t,h)$ of

$$\frac{\partial k^*}{\partial t} = \|C\| \left[\frac{\partial k^*}{\partial h}\right]^2; \qquad k^*(t=0) = k(t=0).$$

The assumption $t \geqq 1/4$ is made for the convenience of the proof. We can always scale $t \to at$ by changing $C \to C/a$.

Theorem B′. *Let $C = C_R + iC_I$ be a convolution operator on L_2 whose Fourier transform satisfies*

$$A_1(1+k^2)^{-m} \leqq \hat{C}_R(k) \leqq A_2(1+k^2)^{-m},$$
$$\int dk (1+k^2)^m |\hat{C}_I(k)| \leqq A_3,$$

where A_1, A_2, and A_3 are strictly positive. A_3 is sufficiently small depending on A_1 and A_2 and $m > d+2s$. Let $g = g_0$ and $g' = g_1$ be the large field regulators of (9.2) and let $h=(h,h)$, $h'=(h',h')$ with $0<h'<h$. Then if K is any local analytic functional with

$$\|K\|_{g,\Gamma,h} < \frac{1}{64\|C\|}(h-h')^2,$$

then

$$\|K'\|_{g',\Gamma,h'} \leqq \|K\|_{g,\Gamma,h}.$$

Proof of Theorem B′. (Assuming Proposition 8.3′). We write the covariance in the form $C = C_s + C_1$, where C_s obeys the hypotheses of Proposition 8.3′ and C_1 is real. We shall prove that such a decomposition is possible later. By using the identity $\mu_C * \mathscr{E}\mathrm{xp}(K) = \mu_{C_1} * \mu_{C_s} * \mathscr{E}\mathrm{xp}(K)$ and applying Proposition 8.3′ (with C replaced by $2C_s$ and $t \in [0,1/2]$) followed by Proposition 8.3 (with C replaced by $2C_1$ and $t \in [1/2,1]$) we find that $\|K'\|_{g',\Gamma,h'} \leqq k(t=1,h')$, where $k(t,h)$ solves $\partial k/\partial t = 2\,\mathrm{Max}(\|C_s\|, \|C_1\|)(\partial k/\partial h)^2$ with $k(t=0,h) = \|K\|_{g,\Gamma,h}$. Lemma 8.4 completes the proof.

Choice of C_s: let χ be a smooth positive definite function such that $\chi(x)=0$ if $|x| \geqq 1/12$, $\chi(0)=1$, and $1 \geqq \chi \geqq 0$. Let $R(x-y)$ denote the kernel of $C_R^{1/2}$ and let D be the operator on L_2 with kernel $R(x-y)\chi(x-y)$. The Fourier transform of D is a multiplication operator with $\hat{D}(k) = (\hat{\chi} * \hat{R})(k)$. The rapid decay and positivity of $\hat{\chi}$ imply that $\hat{D}^2$ satisfies the same bounds as $\hat{C}_R$ with different constants replacing A_1 and A_2. Therefore we can choose $\varepsilon > 0$ small so that if we let $C_s = \varepsilon D^2 + iC_2$, then $C_1 = C - C_s$ is a positive operator on L_2 and $\hat{C}_1(k)$ decays like $(1+k^2)^{-m}$. □

The proof of Proposition 8.3′ is closely related to the method in Sect. 8. The integral equation of Proposition 8.3 continues to hold for a complex covariance by the same argument as in Sect. 8. It is

$$K(t)=\mu_t * K(0)+\tfrac{1}{2}\int_0^t ds\, \mu_{t-s} * (K_\phi \circ K_\phi)(s,\phi). \tag{10.1}$$

By iterating (10.1) we obtain a finite series for $K'=K(t)$. We estimate the series term by term. In the proof of Proposition 8.3 we were carrying out this program using the bound

$$\|\mu_{t-s} * F(X)\|_{g_s} \leqq \|F(X)\|_{g_t}$$

which is implied by part (1) of Proposition 9.1. In part (2) of Proposition 9.1 there is a similar bound but it does not hold when t is arbitrarily close to s and the proof of Proposition 8.3 fails.

We illustrate the main idea in the proof of Proposition 8.3′ by considering as an example one of the terms that result when the integral equation is iterated. The series contains the term

$$I(X,\phi)=\tfrac{1}{2}\sum_{Y,Z:\, Y\cup Z=X}\int_0^t ds\, \mu_{t-s} * [(\mu_s * K_\phi^0(Y))\circ(\mu_s * K_\phi^0(Z))], \tag{10.2}$$

where $K^0=K(t=0)$. Consider a single term in the Y, Z sum and set $F(\phi)=K_\phi^0(Y)K_\phi^0(Z)$. Given any set U, let $\hat{U}=\hat{U}$ be U with a smooth collar of width 1/3. Given any function $f(x,y)$, in particular $C(x,y)$, define

$$\begin{aligned} f_1(x,y)&=f(x,y) \text{ if } x,y \text{ in } (Y\cup Z)' \\ &=0 \quad \text{otherwise}, \\ f_2(x,y)&=f(x,y) \text{ if } x,y \text{ both in } \hat{Y} \text{ or both in } \hat{Z} \\ &=0 \quad \text{otherwise}, \\ f(t,s,x,y)&=(t-s)f_1(x,y)+sf_2(x,y). \end{aligned} \tag{10.3}$$

(We shall say that f_1 and f_2 are obtained from f by *compression*. More generally, given $X_1,\ldots,X_N$ disjoint we define the compression f_N by: $f_N(x,y)=0$ unless both x and y belong to the same X_i, $=f(x,y)$ otherwise.) $C(t,s)=C(t,s,x,y)$ has been defined so that

$$\mu_{t-s} * [(\mu_s * K_\phi^0(Y))\circ(\mu_s * K_\phi^0(Z))]=\mu_{C(t,s)} * F(\phi). \tag{10.4}$$

By hypothesis $C=C_1^{1/2}(I+iV)C_1^{1/2}$ and $C_1^{1/2}$ has range less than 1/12. This together with $\operatorname{dist}(Y,Z)\geqq 1$ implies that $C(t,s,x,y)$ equals the kernel of $tC_1^{1/2}\left(I+\frac{i}{t}V(t,s)\right)C_1^{1/2}$ if $x,y\in(Y\cup Z)'$. By Proposition 9.1, for $t\geqq 1/4$ and $s\in[0,t]$,

$$\|\mu_{C(t,s)} * F(\phi)\|_{g(t)} \leqq \|F\|_{g(0)}. \tag{10.5}$$

To check the hypotheses of Proposition 9.1 note that the operator norm of $\frac{1}{t}V(t,s)$ as an operator on L_2 is less than 1/2 because $\frac{1}{t}V(t,s)$ is a convex combination of compressions V_1 and V_2, and compressions inherit this property from V. The hypothesis on the L_2 norm of $V(t,s)$ is valid for the same reason.

From (10.5) and (10.4) we conclude that

$$\|\mu_{t-s} * [(\mu_s * K^0_\phi(Y)) \circ (\mu_s * K^0_\phi(Z))]\|_{g(t)} \leqq \|K^0_\phi(Y)\|_{g(0)} \|K^0_\phi(Z)\|_{g(0)}$$

which exemplifies the main idea in the proof of Proposition 8.3′.

The proof of Proposition 8.3′ requires a formula for the general term in the series. This is Lemma 10.1 below.

Let T denote a tree graph on vertices $1, \ldots, N$ and let $b=(i,j)$ denote a typical bond in T. With each bond b is associated a variable s_b and $\underset{\sim}{s}=(s_b)_{b\in T}$, $d^T s = \prod ds_b$. Define

$$D_b \equiv -\int_{X_i} d\xi \int_{X_j} d\zeta\, C(\xi,\zeta) \frac{\partial}{\partial\phi(\xi)} \frac{\partial}{\partial\phi(\zeta)},$$

$$D^T = \prod_{b\in T} D_b,$$

$$\begin{aligned} s(x,y) &= t \text{ if either } x \text{ or } y \text{ not in } \cup \hat{X}_i, \\ &= \max\{s_b : b \text{ in path in } T \text{ joining } j \text{ to } k \text{ if } x\in \hat{X}_j, y \in \hat{X}_k\} \\ &= 0 \quad \text{if} \quad j=k. \end{aligned}$$

Since T is a tree graph there is a unique path in T connecting any vertices l and k. Let $C(t,\underset{\sim}{s},x,y)$ be *any* covariance satisfying

$$C(t,\underset{\sim}{s},x,y) \equiv C(x,y)[t-s(x,y)] \quad \text{if} \quad x,y\in \hat{X}. \tag{10.6}$$

Lemma 10.1. *The results of iterating the integral equation is*

$$K(X,t) = \mu_{tC} * K^0(X) + \sum_\pi \sum_T \int_0^t d^T s\, \mu_{C(t,\underset{\sim}{s})} * D^T \prod_1^N K^0(X_i),$$

where π is summed over all partitions of X into subsets $X_1, \ldots, X_N$ with $X = \mathring{\cup} X_i$ and summation over N is included.

Proof. The proof is to check that this expression for $K(X,t)$ satisfies the differential equation (8.2). We will omit this verification since a similar argument is given on p. 30 of [11]. See also the simpler Lemma 10.2 below. Note that the integrand is measurable relative to ϕ in $H_s(\hat{X})$ and so the formula depends on $C(t,s,x,y)$ only through its restriction to $x,y\in\hat{X}\times\hat{X}$ as claimed.

The following lemma will be required for combinatoric reasons. $d_i = d_i(T)$ is the number of lines in T meeting vertex i.

Lemma 10.2. *Suppose $k^0(h)$ is a power series in h with positive coefficients, convergent for $|h|$ small. Then*

$$k(t,h) = \sum_{N=1}^\infty \frac{(t\|C\|)^{N-1}}{N!} \sum_{T \text{ on } \{1,\ldots,N\}} \prod_i^N \left[\left(\frac{\partial}{\partial h}\right)^{d_i} k^0(h)\right]$$

is the unique solution, which is analytic in t and h near the origin, to

$$\frac{\partial k}{\partial t} = \tfrac{1}{2}\|C\| \left[\frac{\partial k}{\partial h}\right]^2; \quad k(t=0) = k^0.$$

Proof. We omit the details. The idea is that when the series is differentiated with respect to t, the t derivative can be visualised as selecting a line in a tree graph T and erasing it, i.e., the corresponding t factor is differentiated. The tree graph without the line splits into two sub-tree graphs which contribute to the two $\partial k/\partial h$ factors on the right-hand side.

Proof of Proposition 8.3'. In the formula in Lemma 10.1 we are at liberty to take

$$C(t,\underset{\sim}{s})=C_1^{1/2}(I+iV(\underset{\sim}{s},t))C_1^{1/2}, \tag{10.7}$$

where $V(\underset{\sim}{s},t,x,y)=[t-s(\chi,y)]\,V(x,y)$. As in the example above $\frac{1}{t}\,V(\underset{\sim}{s},t)$ is a convex combination of V and kernels obtained from V by compression, so the operator norm and L_2 norms of $V(\underset{\sim}{s},t)$ are bounded for all $\underset{\sim}{s}$ in $[0,t]$ and $t\in[0,1]$ by the same norms of V. Let $\underset{\sim}{M}=(M_1,\ldots,M_N)$ be a multi-index with $M=\sum M_i$. By differentiating the tree graph representation of Lemma 10.1,

$$K_M(X,t)=\mu_{tC}*K_M^0(X)+\sum_\pi\sum_T\sum_{\underset{\sim}{M}}\frac{M!}{\underset{\sim}{M}!}\int_0^t d^Ts\,\mu_{C(t,\underset{\sim}{s})}*D^T\prod_1^N K_{M_i}^0(X_i).$$

(We remind the reader that each M_i is itself a multi-index specifying the number of ϕ_i, ϕ_{ij} derivatives.)

To each bond $b=(i,j)$, we associate Δ_b^i, Δ_b^j, unit blocks that intersect $\hat{X}_i$ and $\hat{X}_j$ respectively. The number of bonds in T that meet at i is $d_i=d_i(T)$ so there are d_i unit blocks $\{\Delta_i^b : b\in T\}$ intersecting $\hat{X}_i$. We take the $\|\cdot\|_{g(t)}$ norm of each side of the Tree Graph Formula using Proposition 9.1,

$$\|K_M(t,X)\|_{g(t)}\leqq\|K_M^0(X)\|_g$$
$$+\sum_\pi\sum_T\sum_{\underset{\sim}{M}}\frac{M!}{\underset{\sim}{M}!}\sum_{\underset{\sim}{\Delta}}\prod_{b=(i,j)\in T}tC(\Delta_i^b,\Delta_j^b)\prod_{i=1}^N\|K_{M_i+d_i}^0(X_i,\underset{\sim}{\Delta}_i)\|_{g(0)},$$

where $\underset{\sim}{\Delta}_i$ represents the d_i blocks associated to X_i and

$$\|K_{M+d}(t,X,\underset{\sim}{\Delta})\|_{g(t)}=\sum_{\Delta_j,\,j=1,M}\|\mathrm{Var}\,K_{M+d}(X)\,1_{\Delta_1,\ldots,\Delta_{M+d}}\|_{g(0)}$$

$$C(\Delta,\Delta')=\sup_{\xi\in\Delta,\,\zeta\in\Delta'}|C(\xi,\zeta)|.$$

We multiply both sides by $\Gamma(X)$, sum over $X\cap\Delta\neq\Phi$, Δ fixed, set $C_\theta(\Delta,\Delta')=C(\Delta,\Delta')\theta(\mathrm{dist}(\Delta,\Delta'))$ and use property $\Gamma1$ in (3.3) of $\Gamma(X)$,

$$\|K_M(t)\|_{g(t),\Gamma}\leqq\|K_M^0\|_{g(0),\Gamma}$$
$$+\sum_N\frac{1}{N!}\sum_T\sum_{\underset{\sim}{M}}\frac{M!}{\underset{\sim}{M}!}\cdot\sum_{X_1,\ldots,X_N:\,\cup X_i\supset\Delta}\cdot\sum_{\underset{\sim}{\Delta}}\prod_{b=(i,j)\in T}tC_\theta(\Delta_i^b,\Delta_j^b)$$
$$\times\prod_{i=1}^N\|K_{M_i+d_i}^0(X_i,\underset{\sim}{\Delta}_i)\|_{g(0)}\Gamma(X_i).$$

We sum over the sets X_i, $i=1,\dots,N$, and the blocks $\underset{\sim}{\Delta}$, working inwards from the leaves of the tree. The argument is not difficult, some details may be found in [13, p. 180]

$$\leqq \|K^0_{\underset{\sim}{M}}\|_{g(0),\Gamma} + \sum_N \frac{N}{N!}\cdot\sum_T\sum_{\underset{\sim}{M}}\frac{M!}{\underset{\sim}{M}!}t^{N-1}\|C\|^{N-1}\prod_{i=1}^N \|K^0_{M_i+d_i}\|_{g(0),\Gamma}\,.$$

We multiply by $h^M/M!$ and sum over M,

$$\leqq \|K^0\|_{g(0),\Gamma,h} + \sum_N \frac{N}{N!}\cdot\sum_T t^{N-1}\|C\|^{N-1}\prod_{i=1}^N\left[\sum_M \frac{h^M}{M!}\cdot\|K^0_{M_i+d_i}\|_{g(0),\Gamma}\right]$$

$$\leqq \|K^0\|_{g(0),\Gamma,h} + \sum_N \frac{N}{N!}\cdot\sum_T t^{N-1}\|C\|^{N-1}\cdot\prod_{i=1}^N\left[\left(\frac{d}{dh}\right)^{d_i}\|K^{(0)}\|_{g(0),\Gamma,h}\right].$$

Use $N\leqq 2^{N-1}$ and note that by Lemma 10.2 this series solves $\partial k/\partial t=\|C\|\,(\partial k/\partial h)^2$ with $k(t=0)=\|K^{(0)}\|_{g(0),\Gamma,h}$. Therefore $\|K(t)\|_{g(0),\Gamma,h}\leqq k(t,h)$. □

Appendix A (Covariances and Complex Gaussian Measures)

Gaussian Processes [16, p. 16]. Let $H_-,(\cdot,\cdot)_-$ be a real Hilbert space. According to general theory there exists an abstract measure space $(\Omega, d\mu)$ and a linear map, $f\to\langle\phi,f\rangle$, from H_- into random variables (functions on Ω), such that

$$\int d\mu\, e^{i\alpha\langle\phi,f\rangle}=e^{-1/2\alpha^2(f,f)_-}\equiv e^{-1/2\alpha^2\langle f,Cf\rangle},\qquad \alpha\in\mathbb{C}, \tag{A.1}$$

where $C:H_-\to H_+$ is the canonical isomorphism between H_- and the dual Hilbert space H_+. We shall refer to C as the covariance of $d\mu$ and write $d\mu_C$ when necessary. This collection of random variables is called a Gaussian Process indexed by H_-. If the σ algebra of measurable subsets of Ω is the smallest such that each $\langle\phi,f\rangle$ is measurable, $(\Omega,d\mu)$ is unique up to measure preserving isomorphisms.

Let $S:H_+\to H_-$ be symmetric (this means $S=S'$, where S' is the operator dual to S) and suppose $SC:H_-\to H_-$ is a trace class self adjoint operator. It follows that it has a complete set of eigenvectors f_n and eigenvalues λ_n such that $Sg=\sum\lambda_n f_n\langle f_n,g\rangle$ (e.g., $g=Ch$ with $h\in H_-$). Define

$$\langle\phi,S\phi\rangle=\sum\lambda_n\langle\phi,f_n\rangle^2, \tag{A.2}$$

where the sum exists as a limit in $L_2(\Omega,d\mu)$.

Lemma A1. *Let $S:H_+\to H_-$ be symmetric with $SC:H_-\to H_-$ trace class. Then*

$$\int d\mu\, e^{-\alpha/2\langle\phi,S\phi\rangle}=\det{}^{-1/2}(I+\alpha SC),$$

$$\int d\nu\, e^{i\beta\langle\phi,f\rangle}=e^{-\beta^2/2(f,[I+\alpha SC]^{-1}f)_-},\qquad f\in H_-,$$

where

$$d\nu=d\mu\, e^{-\alpha/2\langle\phi,S\phi\rangle}/\int d\mu\, e^{-\alpha/2\langle\phi,S\phi\rangle}$$

for all complex β and all complex α such that $I+zSC$ is invertible for $|z|\leqq|\alpha|$.

Proof. Pages 29 and 30 of [16], combined with analytic continuation in α and β. A good reference for determinants is [19], p. 44.

Note that $(f,[I+\alpha SC]^{-1}f)_- = \langle f,[C^{-1}+\alpha S]^{-1}f\rangle$ so $d\nu$ is a (complex) Gaussian measure with covariance $[C^{-1}+\alpha S]^{-1}$.

Lemma A2. *Suppose H_- and H_+ are dual Hilbert spaces with canonical isomorphism $C_1: H_- \to H_+$. Let $C_2: H_- \to H_+$ be symmetric with $T = -C_1^{-1}C_2: H_- \to H_-$ trace class and $\|T\| < 1$. Let $C = C_1 + iC_2$, $\bar{C} = C_1 - iC_2$ and define the complex Gaussian measure $d\mu_C$ by*

$$d\mu_C(\phi) = \det{}^{1/2}(I+iT)\, d\mu_{C_R}(\phi)\, e^{-i/2\langle\phi, S\phi\rangle},$$

where $C_R = \bar{C}C_1^{-1}C$, $S = -C^{-1}C_2\bar{C}^{-1}$. Then

1. $\int d\mu_C(\phi)\, e^{i\langle\phi,f\rangle} = e^{-1/2\langle f, Cf\rangle}$ *for all $f \in H_-$.*
2. $d|\mu_C|(\phi) \leqq e^{1/2\,\mathrm{Tr}\, T^2} d\mu_{C_R}(\phi)$.

Proof. For (1) we apply Lemma A1. The result follows from $C = (C_R^{-1} + iS)^{-1}$ and

$$\begin{aligned}\det{}_{H_-}(I + iSC_R) &= \det{}_{H_-}(I - iC^{-1}C_2C_1^{-1}C)\\ &= \det{}_{H_+}(I - iCC^{-1}C_2C_1^{-1}CC^{-1})\\ &= \det{}_{H^+}(I - iC_2C_1^{-1}) = \det{}_{H^-}(I - iC_1^{-1}C_2).\end{aligned}$$

We should be using H_- with the inner product $\langle f, C_R g\rangle$, but this is equivalent to the H_- inner product and the determinant is a similarity invariant.

For (2) we use $|\det(I+iT)|^2 = \det(I+iT)(I-iT)$ and the estimate $|\det(I+X)| \leqq$ Trace norm of X. See [19, p. 47]. □

Models for $d\mu$ and Regularity of Sample Points.

The pairing $f, \phi \to \langle\phi, f\rangle$ makes it tempting to search for a model in which ϕ is a random element of H_+, but this is not possible. Prohorov's Theorem, [17] p. 29 or [18] p. 67, describes how much H_+ must be enlarged to serve as a model for the measure space in which ϕ lives. In this context it implies the following result.

Let $H_s = H_s/\{\text{constants}\}$, where H_s is the Sobolev space of index s of functions on the torus Λ and s is an integer that will be specified below. From now on we will omit to write the "/{constants}".

Proposition A3. *Let $C: L_2 \to L_2$ be a positive definite operator such that $C^{1/2}$ has a kernel $C^{1/2}(x,y)$ in $H_s(\Lambda\times\Lambda)$. Let H_- be the completion of L_2 with respect to the inner product $f, g \to \langle f, Cg\rangle$. Then there exists a unique Borel measure $d\mu_C$ on H_s such that the measure space $(H_s, d\mu_C)$ is a model for the Gaussian process (H_-, ϕ). The random variable $\langle\phi, f\rangle$ is given by the canonical pairing between H_s and H_{-s} whenever $f \in H_{-s}$. When $f \in H_-$ we set $\langle\phi, f\rangle = \mathrm{Lim}\langle\phi, f_n\rangle$, where $f_n \in H_{-s}$, $f_n \to f$ in H_-. The limit exists in $L_2(H_s, d\mu)$.*

Suppose $S: H_s \to H_{-s}$, then we can define $\langle\phi, S\phi\rangle$ either by using the model of Proposition A3, so that $\phi \in H_s$, or using the definition in Eq. (A2). (The hypothesis on S implies that SC is a trace class operator on H_{-s}.) The two definitions coincide.

Corollary A4. *Let s be a non-negative integer and suppose $C = C_1^{1/2}(I+iV)C_1^{1/2}$, where $C_1: L_2 \to L_2$ is a positive definite operator such that $C_1^{1/2}$ has a kernel that is a* *function in $H_s(\Lambda\times\Lambda)$. $V: L_2\to L_2$ is symmetric and trace class with $\|V\|<1$. Then for any subset $B\subset\Lambda$ with smooth boundary and any $f\in H_{-s}$,*

$$\int d|\mu_C|(\phi)\exp[\kappa/2\|\phi\|_{s,B}^2+\langle\phi,f\rangle],$$

$$\leqq\exp\left[\operatorname{Tr}V^2+\frac{\kappa}{1-A\kappa}\|C_1^{1/2}\|_{s,B\times\Lambda}^2+\frac{1}{2}\frac{A}{1-\kappa A}\|f\|_{-s}^2\right],$$

where $A=2\|C_1\|_{H_{-s}\to H_{+s}}$.

Proof. Let $Df=(\partial^\alpha f)$, where α runs over all multi-indices such that $|\alpha|\leqq s$ and let χ be the characteristic function of B, so that $f,g\to\langle Df,\chi Dg\rangle\equiv\langle f,D'\chi Dg\rangle$ is the inner product on $H_s(B)$. Let $W=C_R^{1/2}C_1^{-1/2}$. W is a bounded operator on L_2 and $\|W^*W\|\leqq 2$ by the hypothesis on V and $C_R=C_1^{1/2}(1+V^2)C_1^{1/2}$. The Corollary is an application of Lemma A2 followed by Lemma A1 together (A.3) and (A.4), below. To obtain (A.3):

$$\begin{aligned}\det{}_{H_-}^{-1}(I-\kappa D'\chi DC_R)&=\det{}_{H_-}([I-\kappa D'\chi DC_R]^{-1})\\&=\det{}_{L_2}([I-\kappa C_R^{1/2}D'\chi DC_R^{1/2}]^{-1}).\end{aligned}$$

Let $X=C_R^{1/2}D'\chi DC_R^{1/2}$. Then $\det([I-X]^{-1})=\det(I+X+X^2+\dots)$ $\leqq\exp(\operatorname{Tr}(X+X^2+\dots))$ and we continue with

$$\leqq\exp(\operatorname{Tr}(X)/(1-\|X\|)).$$

Since $X=WC_1^{1/2}D'\chi DC_1^{1/2}W^*$ and $\|W\|\leqq\sqrt{2}$, $\operatorname{Tr}(X)\leqq 2\|C_1^{1/2}\|_{s,B\times\Lambda}^2$ and $\|X\|\leqq 2\|C_1^{1/2}D'\chi DC_1^{1/2}\|=2\|DC_1^{1/2}C_1^{1/2}D'\|\leqq 2A$ we obtain

$$\det{}_{H_-}^{-1}(I-\kappa D'\chi DC_R)\leqq\exp\left(\frac{2\kappa}{1-A\kappa}\right). \tag{A.3}$$

The other estimate is

$$\langle f,C_R(I-\kappa D'\chi DC_R)^{-1}f\rangle\leqq\frac{A\kappa}{1-A\kappa}(f,f)_{-s}, \tag{A.4}$$

which holds because by $\|W\|\leqq\sqrt{2}$, the same $C_R: H_{-s}\to H_s$ has norm less or equal to A, $D: H_s\to L_2$ and $D': L_2\to H_{-s}$ have norm one. End of Proof.

Decay of the Fluctuation Covariance

In (1.2) we defined the fluctuation covariance C'. Let $C=\mathscr{R}C'$ be the rescaled fluctuation covariance. It is defined on the rescaled torus $L^{-1}\Lambda$, but we will replace $L^{-1}\Lambda$ by Λ because we are proving estimates which are uniform in Λ provided the periods are greater than one. The Fourier transform of C is

$$\begin{aligned}\hat{C}(k)&=\frac{1}{k^2P(k^2/L^2)+k\cdot\sigma k}-\frac{1}{k^2P(k^2)+k\cdot\sigma k}\\&=0\quad\text{if}\quad k=0,\end{aligned}$$

k takes values in $\hat{\Lambda}$, the lattice dual to the torus Λ.

Lemma A5. *Let $P(t)$ be a positive non-decreasing smooth function on $\mathbb{R}$ which equals one near $t=0$ and is such that $P(t) \geqq \text{Const}\, t^\tau$, $\tau > s + \frac{d}{2}$ and for $m=0,1,\ldots$, $|P^{(m)}(t)| \leqq C_m(1+|t|)^{\tau-m}$. Then for each $n=0,1,\ldots$, there exists A_n such that for all multiindices α, $|\alpha| \leqq 2s$, all $d \times d$ complex matrices σ with $\|\sigma\| \leqq 1/2$, all $L \geqq 1$ and any torus Λ,*

$$|\partial^\alpha C(x)| \leqq A_n L^{d+|\alpha|}(1+|x|)^{-n},$$

where $|x|$ denotes distance on the torus Λ.

Proof. Define $D(x) = 2\varepsilon^{-2} \sum_j (1-\cos(\varepsilon x_j))$, where $x_j, j=1,\ldots,d$, are the coordinates of $x \in \Lambda$ and ε is the dual lattice spacing, i.e., $\hat{\Lambda} = (\varepsilon Z)^d$. There exists C such that for all x in Λ and all Λ, $|x|^2 \leqq CD(x)$. By this remark and summation by parts it follows that for any multi-index α and $n=0,1,\ldots$,

$$|x|^{2n} |\partial_x^\alpha C(x)| \leqq \text{Const} \left| \varepsilon^d \sum_k (-\Delta)^n [(ik)^\alpha \hat{C}(k)] e^{ikx} \right|$$

$$\leqq \text{Const}\, \varepsilon^d \sum_k \left| (-\Delta)^n \frac{k^\alpha}{k^2 P(k^2/L^2) + k \cdot \sigma k} - (-\Delta)^n \frac{k^\alpha}{k^2 P(k^2) + k \cdot \sigma k} \right|,$$

where Const is independent of Λ and L and Δ is the finite difference Laplacian on $\hat{\Lambda}$. The integrand is vanishing near $k=0$ because $P=1$ near $k=0$. It decays as $k \to \infty$ better than integrably. By increasing Const we can replace the Riemann sum over k by integration and the finite difference Laplacian by the continuum Laplacian. If $n=0$ then we split the range of the integral into $|k| \geqq 1$ and $|k| \leqq 1$. For $|k| \leqq 1$ the integrand is bounded uniformly in L. For $|k| \geqq 1$ and L large, the integral is dominated by

$$\int_{|k| \geqq 1} dk \frac{|k^\alpha|}{P(k^2/L^2)} \leqq L^{d+|\alpha|} \int dk \frac{|k^\alpha|}{P(k^2)} \leqq \text{Const}\, L^{d+|\alpha|}.$$

Similar arguments hold for $n > 0$ by using $|P^{(m)}(t)| \leqq C_m(1+|t|)^{\tau-m}$. □

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