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GRADED INFINITE ORDER JET MANIFOLDS

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The relevant material on differential calculus on graded infinite order jet manifolds and its cohomology is summarized. This mathematics provides the adequate formulation of Lagrangian theories of even and odd variables on smooth manifolds in terms of the Grassmann-graded variational bicomplex.

Keywords: Jet manifolds; graded manifolds; graded differential forms; variational bicomplex.

1. Introduction

Let $Y \to X$ be a smooth fiber bundle and $J^\infty Y$ the Fréchet manifold of infinite order jets of its sections. The differential calculus on $J^\infty Y$ and its cohomology provide the adequate mathematical description of Lagrangian theories on $Y \to X$ in terms of the variational bicomplex [1, 13, 28]. This description has been extended to Lagrangian theories on graded manifolds in terms of the Grassmann-graded variational bicomplex of differential forms on a graded infinite order jet manifold [3, 5, 6, 14].

Different geometric models of odd variables are phrased in terms of both graded manifolds and supermanifolds. Note that graded manifolds are characterized by sheaves on smooth manifolds, while supermanifolds are defined by gluing of sheaves on supervector spaces [4, 15]. Treating odd variables on smooth manifolds, we follow the Serre–Swan theorem for graded manifolds (Theorem 14). It states that a graded commutative $C^\infty(X)$-ring is isomorphic to an algebra of graded functions on a graded manifold with a body X iff it is the exterior algebra of some projective $C^\infty(X)$-module of finite rank. By virtues of the Batchelor theorem [4], any graded manifold $(Z, \mathfrak{A})$ with a body Z and a structure sheaf $\mathfrak{A}$ of graded functions is isomorphic to a graded manifold $(Z, \mathfrak{A}_Q)$ modeled over some vector bundle $Q \to Z$, i.e. its structure sheaf $\mathfrak{A}_Q$ is the sheaf of sections of the exterior bundle $\wedge Q^*$, where Q^* is the dual of $Q \to Z$. Our goal is the following differential bigraded algebra (henceforth DBGA) $\mathcal{S}^*_\infty[F; Y]$ and its relevant cohomology.

Let $F \to Y \to X$ be a composite bundle where $F \to Y$ is a vector bundle. Jet manifolds J^rF of $F \to X$ are also vector bundles over J^rY. Let $(J^rY, \mathfrak{A}_r)$ be a graded manifold modeled over $J^rF \to J^rY$, and let $\mathcal{S}^*_r[F;Y]$ be the DBGA of Grassmann-graded differential forms on the graded manifold $(J^rY, \mathfrak{A}_r)$. There is the inverse system of jet manifolds

$$Y \xleftarrow{\pi} J^1Y \longleftarrow \cdots J^{r-1}Y \xleftarrow{\pi^r_{r-1}} J^rY \longleftarrow \cdots . \tag{1}$$

Its projective limit $J^\infty Y$ is a paracompact Fréchet manifold, called the infinite order jet manifold. This inverse system yields the direct system of DBGAs

$$\mathcal{S}^*[F;Y] \xrightarrow{\pi^*} \mathcal{S}^*_1[F;Y] \longrightarrow \cdots \mathcal{S}^*_{r-1}[F;Y] \xrightarrow{\pi^{r*}_{r-1}} \mathcal{S}^*_r[F;Y] \longrightarrow \cdots , \tag{2}$$

where π^{r*}_{r-1} is the pull-back monomorphisms. Its direct limit is the above-mentioned DBGA $\mathcal{S}^*_\infty[F;Y]$ of all Grassmann-graded differential forms on graded manifolds $(J^rY, \mathfrak{A}_r)$ modulo the pull-back identification. One can think of elements of $\mathcal{S}^*_\infty[F;Y]$ as being Grassmann-graded differential forms on a graded manifold $(J^\infty Y, \mathfrak{A}_\infty)$, called the graded infinite order jet manifold, whose body is $J^\infty Y$ and the structure sheaf $\mathfrak{A}_\infty$ is the sheaf of germs of elements of $\mathcal{S}^*_\infty[F;Y]$.

The DBGA $\mathcal{S}^*_\infty[F;Y]$ is split into the above-mentioned Grassmann-graded variational bicomplex, describing Lagrangian theories of even and odd variables on a smooth manifold X. Grassmann-graded Lagrangians and their Euler–Lagrange operators are elements of this bicomplex. Its cohomology results in the global first variational formula, the first Noether theorem and defines a class of variationally trivial Lagrangians.

It should be emphasized that this description of Grassmann-graded Lagrangian systems differs from that phrased in terms of fibered graded manifolds [19, 24], but reproduces the heuristic formulation of Lagrangian BRST theory [3, 7]. Namely, $(J^\infty Y, \mathfrak{A}_\infty)$ is a graded manifold of jets of smooth fiber bundles, but not jets of fibered graded manifolds.

2. Technical Preliminary

Throughout the paper, smooth manifolds are real and finite-dimensional. They are Hausdorff and second-countable topological spaces (i.e. have a countable base for topology). Consequently, they are paracompact, separable (i.e. have a countable dense subset) and locally compact topological spaces, which are countable at infinity. Unless otherwise stated, smooth manifolds are assumed to be connected and, consequently, arcwise connected. It is essential for our consideration that a paracompact smooth manifold admits the partition of unity by smooth functions. Real-analytic manifolds are also considered as smooth ones because they need not possess the partition of unity by real-analytic functions.

Only proper covers $\mathfrak{U} = \{U_\iota\}$ of smooth manifolds are considered, i.e. $U_\iota \neq U_{\iota'}$ if $\iota \neq \iota'$. A cover $\mathfrak{U}'$ is said to be a refinement of a cover $\mathfrak{U}$ if, for each $U' \in \mathfrak{U}'$, there exists $U \in \mathfrak{U}$ such that $U' \subset U$. For any cover $\mathfrak{U}$ of an n-dimensional smooth manifold X, there exists a countable atlas $\{(U'_\iota, \varphi_\iota)\}$ of X such that: (i) the cover $\{U'_\iota\}$ refines $\mathfrak{U}$, (ii) $\varphi_\iota(U'_\iota) = \mathbb{R}^n$ and (iii) the closure $\overline{U}'_\iota$ of any U_ι is compact [18].

Let $\pi : Y \to X$ be a smooth fiber bundle. There exist the following particular covers of X which one can choose for its bundle atlas [18].

(i) There is a bundle atlas of Y over a countable cover $\mathfrak{U}$ of X where each member U_ι of $\mathfrak{U}$ is a domain (i.e. a contractible open subset) and its closure $\overline{U}_\iota$ is compact.

(ii) There exists a bundle atlas of Y over a finite cover of X. Indeed, let Ψ be a bundle atlas of $Y \to X$ over a cover $\mathfrak{U}$ of X. For any cover $\mathfrak{U}$ of a manifold X, there exists its refinement $\{U_{ij}\}$, where $j \in \mathbb{N}$ and i runs through a finite set such that $U_{ij} \cap U_{ik} = \emptyset$, $j \neq k$. Let $\{(U_{ij}, \psi_{ij})\}$ be the corresponding bundle atlas of a fiber bundle $Y \to X$. Then Y has the finite bundle atlas $U_i = \cup_j U_{ij}$, $\psi_i(x) = \psi_{ij}(x)$, $x \in U_{ij} \subset U_i$, whose members U_i, however, need not be contractible and connected.

Without a loss of generality, we further assume that a cover $\mathfrak{U}$ for a bundle atlas of $Y \to X$ is also a cover for a manifold atlas of its base X. Given such an atlas, a fiber bundle Y is provided with the associated bundle coordinates (x^λ, y^i), where (x^λ) are coordinates on X.

Given a manifold X, its tangent and cotangent bundles TX and T^*X are endowed with the bundle coordinates $(x^\lambda, \dot{x}^\lambda)$ and $(x^\lambda, \dot{x}_\lambda)$ with respect to holonomic frames $\{\partial_\lambda\}$ and $\{dx^\lambda\}$, respectively. Given a smooth bundle $Y \to X$, its vertical tangent and cotangent bundles VY and V^*Y are provided with the bundle coordinates $(x^\lambda, y^i, \dot{y}^i)$ and $(x^\lambda, y^i, \overline{y}_i)$, respectively.

By $\Lambda = (\lambda_1 \ldots \lambda_k)$, $|\Lambda| = k$, $\lambda + \Lambda = (\lambda\lambda_1 \ldots \lambda_k)$ are denoted symmetric multi-indices. Summation over a multi-index Λ means separate summation over each of its index λ_i. The notation

$$d_\lambda = \partial_\lambda + \sum_{0 \leq |\Lambda|} y^i_{\lambda+\Lambda} \partial_i^\Lambda, \quad d_\Lambda = d_{\lambda_r} \circ \cdots \circ d_{\lambda_1}, \tag{3}$$

stands for the total derivatives.

3. Finite Order Jet Manifolds

Given a smooth fiber bundle $Y \to X$, its r-order jet $j^r_x s$ is defined as the equivalence class of sections s of Y identified by their $r+1$ terms of their Taylor series at a point $x \in X$. The disjoint union $J^r Y = \bigcup_{x \in X} j^r_x s$ of these jets is a smooth manifold provided with the adapted coordinates

$$(x^\lambda, y^i, y^i_\Lambda)_{|\Lambda| \leq r}, \quad (x^\lambda, y^i_\Lambda) \circ s = (x^\lambda, \partial_\Lambda s^i(x)), \quad {y'}^i_{\lambda+\Lambda} = \frac{\partial x^\mu}{\partial x'^\lambda} d_\mu y'^i_\Lambda.$$

For the sake of brevity, the index $r = 0$ further stands for Y. The jet manifolds of $Y \to X$ form the inverse system (1), where π^r_{r-1}, $r > 0$ are affine bundles.

Given fiber bundles Y and Y' over X, every bundle morphism $\Phi : Y \to Y'$ over a diffeomorphism f of X admits the r-order jet prolongation to the morphism of the r-order jet manifolds

$$J^r\Phi : J^rY \ni j^r_x s \mapsto j^r_{f(x)}(\Phi \circ s \circ f^{-1}) \in J^rY'.$$

If Φ is an injection or surjection, so is $J^r\Phi$. It preserves an algebraic structure. If $Y \to X$ is a vector bundle, $J^rY \to X$ is also a vector bundle. If $Y \to X$ is an affine bundle modeled over a vector bundle $\overline{Y} \to X$, then $J^rY \to X$ is an affine bundle modeled over the vector bundle $J^r\overline{Y} \to X$.

Every section s of a fiber bundle $Y \to X$ admits the r-order jet prolongation to the section $(J^rs)(x) = j^r_x s$ of the jet bundle $J^rY \to X$.

Every exterior form ϕ on the jet manifold J^kY gives rise to the pull-back form $\pi^{k+i}_k{}^*\phi$ on the jet manifold $J^{k+i}Y$. Let $\mathcal{O}^*_k$ be the differential graded algebra (henceforth DGA) of exterior forms on the jet manifold J^kY. We have the direct system of DGAs

$$\mathcal{O}^*X \xrightarrow{\pi^*} \mathcal{O}^*Y \xrightarrow{\pi^1_0{}^*} \mathcal{O}^*_1 \to \cdots \mathcal{O}^*_{r-1} \xrightarrow{\pi^r_{r-1}{}^*} \mathcal{O}^*_r \to \cdots. \tag{4}$$

Every projectable vector field $u = u^\mu\partial_\mu + u^i\partial_i$ on a fiber bundle $Y \to X$ has the k-order jet prolongation onto J^kY to the vector field

$$j^ku = u^\lambda\partial_\lambda + u^i\partial_i + \sum_{0<|\Lambda|\leq k} [d_\Lambda(u^i - y^i_\mu u^\mu) + y^i_{\mu+\Lambda}u^\mu]. \tag{5}$$

Jet manifold provides the conventional language of theory of nonlinear differential equations and differential operators on fiber bundles [9, 22]. A k-order differential equation on a fiber bundle $Y \to X$ is defined as a closed subbundle $\mathfrak{E}$ of the jet bundle $J^kY \to X$. Its classical solution is a (local) section s of $Y \to X$ whose k-order jet prolongation J^ks lives in $\mathfrak{E}$.

Differential equations can come from differential operators. Let $E \to X$ be a vector bundle coordinated by (x^λ, v^A), $A = 1, \ldots, m$. A bundle morphism $\mathcal{E} : J^kY \to E$ over X is called a k-order differential operator on a fiber bundle $Y \to X$. It sends each section s of $Y \to X$ onto the section $(\mathcal{E} \circ J^ks)^A(x)$ of the vector bundle $E \to X$. Let us suppose that the canonical zero section $\widehat{0}(X)$ of the vector bundle $E \to X$ belongs to $\mathcal{E}(J^kY)$. Then the kernel of a differential operator $\mathcal{E}$ is defined as $\operatorname{Ker}\mathcal{E} = \mathcal{E}^{-1}(\widehat{0}(X)) \subset J^kY$. If $\operatorname{Ker}\mathcal{E}$ is a closed subbundle of the jet bundle $J^kY \to X$, it is a k-order differential equation, associated to the differential operator $\mathcal{E}$. For instance, the kernel of an Euler–Lagrange operator need not be a closed subbundle. Therefore, it may happen that associated Euler–Lagrange equations are not a differential equation in a strict sense.

4. Infinite Order Jet Manifold

Given the inverse system (1) of jet manifolds, its projective limit $J^\infty Y$ is defined as a minimal set such that there exist surjections

$$\pi^\infty : J^\infty Y \to X, \quad \pi^\infty_0 : J^\infty Y \to Y, \quad \pi^\infty_k : J^\infty Y \to J^kY, \tag{6}$$

obeying the commutative diagrams $\pi^\infty_r = \pi^k_r \circ \pi^\infty_k$ for any admissible k and $r < k$. A projective limit of the inverse system (1) always exists. It consists of those elements

$(\ldots, z_r, \ldots, z_k, \ldots)$, $z_r \in J^rY$, $z_k \in J^kY$, of the Cartesian product $\prod_k J^kY$ which obey the relations $z_r = \pi^k_r(z_k)$ for all $k > r$. One can think of elements of $J^\infty Y$ as being infinite order jets of sections of $Y \to X$ identified by their Taylor series at points of X.

The set $J^\infty Y$ is provided with the projective limit topology. This is the coarsest topology such that the surjections π^∞_r (6) are continuous. Its base consists of inverse images of open subsets of J^rY, $r = 0, \ldots,$ under the mappings π^∞_r. With this topology, $J^\infty Y$ is a paracompact Fréchet (complete metrizable, but not Banach) manifold modeled on a locally convex vector space of formal number series $\{a^\lambda, a^i, a^i_\lambda, \ldots\}$ [28]. Moreover, the surjections π^∞_r are open maps, i.e. $J^\infty Y \to J^rY$ are topological bundles. A bundle coordinate atlas $\{U_Y, (x^\lambda, y^i)\}$ of $Y \to X$ provides $J^\infty Y$ with the manifold coordinate atlas

$$\{(\pi^\infty_0)^{-1}(U_Y), (x^\lambda, y^i_\Lambda)\}_{0\leq|\Lambda|}, \quad {y'}^i_{\lambda+\Lambda} = \frac{\partial x^\mu}{\partial x'^\lambda} d_\mu y'^i_\Lambda. \tag{7}$$

It is essential for our consideration that Y is a strong deformation retract of $J^\infty Y$ [1, 13] (see Appendix A). This result follows from the fact that a base of any affine bundle is a strong deformation retract of its total space. Consequently, a fiber bundle Y is a strong deformation retract of any finite order jet manifold J^rY. Therefore by virtue of the Vietoris–Begle theorem [8], there are isomorphisms

$$H^*(J^\infty Y, \mathbb{R}) = H^*(J^rY, \mathbb{R}) = H^*(Y, \mathbb{R}) \tag{8}$$

of cohomology groups of $J^\infty Y$, J^rY, $0 < r$ and Y with coefficients in the constant sheaf $\mathbb{R}$.

Though $J^\infty Y$ fails to be a smooth manifold, one can introduce the differential calculus on $J^\infty Y$ as follows. Let us consider the direct system (4) of DGAs. Its direct limit $\mathcal{O}^*_\infty$ exists, and consists of all exterior forms on finite order jet manifolds modulo the pull-back identification. It is a DGA, inheriting the DGA operations of $\mathcal{O}^*_r$ [23].

Theorem 1. *The cohomology $H^*(\mathcal{O}^*_\infty)$ of the de Rham complex*

$$0 \to \mathbb{R} \to \mathcal{O}^0_\infty \xrightarrow{d} \mathcal{O}^1_\infty \xrightarrow{d} \cdots \tag{9}$$

of the DGA $\mathcal{O}^*_\infty$ equals the de Rham cohomology of a fiber bundle Y [1].

Proof. By virtue of the well-known theorem, the operation of taking homology groups of cochain complexes commutes with the passage to a direct limit [23]. Since the DGA $\mathcal{O}^*_\infty$ is a direct limit of DGAs $\mathcal{O}^*_r$, its cohomology is isomorphic to the direct limit of the direct system

$$H^*_{\rm DR}(Y) \to H^*_{\rm DR}(J^1Y) \to \cdots H^*_{\rm DR}(J^{r-1}Y) \to H^*_{\rm DR}(J^rY) \to \cdots \tag{10}$$

of the de Rham cohomology groups $H^*_{\rm DR}(J^rY) = H^*(\mathcal{O}^*_r)$ of finite order jet manifolds J^rY. By virtue of the de Rham theorem [20], the de Rham cohomology

$H^*_{\rm DR}(J^rY)$ of J^rY equals its cohomology $H^*(J^rY,\mathbb{R})$ with coefficients in the constant sheaf $\mathbb{R}$. Since Y is a strong deformations retract of J^rY, this cohomology coincides with the cohomology $H^*(Y,\mathbb{R})$ of Y. Consequently, the direct limit of the direct system (10) is the de Rham cohomology $H^*(Y,\mathbb{R}) = H^*_{\rm DR}(Y)$ of Y. □

Corollary 2. *Any closed form $\phi \in \mathcal{O}^*_\infty$ is decomposed into the sum $\phi = \sigma + d\xi$, where σ is a closed form on Y.*

One can think of elements of $\mathcal{O}^*_\infty$ as being differential forms on the infinite order jet manifold $J^\infty Y$ as follows. Let $\mathfrak{O}^*_r$ be the sheaf of germs of exterior forms on J^rY and $\overline{\mathfrak{O}}^*_r$ the canonical presheaf of local sections of $\mathfrak{O}^*_r$ (we follow the terminology of Ref. 20). Since π^r_{r-1} are open maps, there is the direct system of presheaves

$$\overline{\mathfrak{O}}^*_0 \xrightarrow{\pi^{1\,*}_0} \overline{\mathfrak{O}}^*_1 \cdots \xrightarrow{\pi^{r\;*}_{r-1}} \overline{\mathfrak{O}}^*_r \longrightarrow \cdots .$$

Its direct limit $\overline{\mathfrak{O}}^*_\infty$ is a presheaf of DGAs on $J^\infty Y$. Let $\mathfrak{T}^*_\infty$ be the sheaf of DGAs of germs of $\overline{\mathfrak{O}}^*_\infty$ on $J^\infty Y$. The structure module $\mathcal{Q}^*_\infty = \Gamma(\mathfrak{T}^*_\infty)$ of global sections of $\mathfrak{T}^*_\infty$ is a DGA such that, given an element $\phi \in \mathcal{Q}^*_\infty$ and a point $z \in J^\infty Y$, there exist an open neighborhood U of z and an exterior form $\phi^{(k)}$ on some finite order jet manifold J^kY so that $\phi|_U = \pi^{\infty *}_k \phi^{(k)}|_U$. Therefore, there is the DGA monomorphism $\mathcal{O}^*_\infty \to \mathcal{Q}^*_\infty$. It should be emphasized that the paracompact space $J^\infty Y$ admits a partition of unity by elements of the ring $\mathcal{Q}^0_\infty$, but not $\mathcal{O}^0_\infty$.

Since elements of the DGA $\mathcal{Q}^*_\infty$ are locally exterior forms on finite order jet manifolds, the following Poincaré lemma holds.

Lemma 3. *For closed element $\phi \in \mathcal{Q}^*_\infty$, there exists a neighborhood U of each point $z \in J^\infty Y$ such that $\phi|_U$ is exact.*

Theorem 4. *The cohomology $H^*(\mathcal{Q}^*_\infty)$ of the de Rham complex*

$$0 \to \mathbb{R} \to \mathcal{Q}^0_\infty \xrightarrow{d} \mathcal{Q}^1_\infty \xrightarrow{d} \cdots . \qquad (11)$$

of the DGA $\mathcal{Q}^*_\infty$ equals the de Rham cohomology of a fiber bundle Y [28].

Proof. Let us consider the de Rham complex of sheaves

$$0 \to \mathbb{R} \to \mathfrak{T}^0_\infty \xrightarrow{d} \mathfrak{T}^1_\infty \xrightarrow{d} \cdots \qquad (12)$$

on $J^\infty Y$. By virtue of Lemma 3, it is exact at all terms, except $\mathbb{R}$. Being the sheaves of $\mathcal{Q}^0_\infty$-modules, the sheaves $\mathfrak{T}^r_\infty$ are fine and, consequently acyclic because the paracompact space $J^\infty Y$ admits the partition of unity by elements of the ring $\mathcal{Q}^0_\infty$ [20]. Thus, the complex (12) is a resolution of the constant sheaf $\mathbb{R}$ on $J^\infty Y$. In accordance with the abstract de Rham theorem (see Appendix B), cohomology $H^*(\mathcal{Q}^*_\infty)$ of the complex (11) equals the cohomology $H^*(J^\infty Y,\mathbb{R})$ of $J^\infty Y$ with coefficients in the constant sheaf $\mathbb{R}$. Since Y is a strong deformation retract of $J^\infty Y$, we obtain

$$H^*(\mathcal{Q}^*_\infty) = H^*(J^\infty Y,\mathbb{R}) = H^*(Y,\mathbb{R}) = H^*_{\rm DR}(Y). \qquad \square$$

Due to the monomorphism $\mathcal{O}^*_\infty \to \mathcal{Q}^*_\infty$, one can restrict $\mathcal{O}^*_\infty$ to the coordinate chart (7) where horizontal forms dx^λ and contact one-forms $\theta^i_\Lambda = dy^i_\Lambda - y^i_{\lambda+\Lambda}dx^\lambda$ make up a local basis for the $\mathcal{O}^0_\infty$-algebra $\mathcal{O}^*_\infty$. Though $J^\infty Y$ is not a smooth manifold, elements of $\mathcal{O}^*_\infty$ are exterior forms on finite order jet manifolds and, therefore, their coordinate transformations are smooth. Moreover, there is the canonical decomposition $\mathcal{O}^*_\infty = \oplus \mathcal{O}^{k,m}_\infty$ of $\mathcal{O}^*_\infty$ into $\mathcal{O}^0_\infty$-modules $\mathcal{O}^{k,m}_\infty$ of k-contact and m-horizontal forms together with the corresponding projectors

$$h_k : \mathcal{O}^*_\infty \to \mathcal{O}^{k,*}_\infty, \quad h^m : \mathcal{O}^*_\infty \to \mathcal{O}^{*,m}_\infty.$$

Accordingly, the exterior differential on $\mathcal{O}^*_\infty$ is split into the sum $d = d_{\rm H} + d_{\rm V}$ of the total and vertical differentials

$$d_{\rm H} \circ h_k = h_k \circ d \circ h_k, \quad d_{\rm H} \circ h_0 = h_0 \circ d, \quad d_{\rm H}(\phi) = dx^\lambda \wedge d_\lambda(\phi),$$

$$d_{\rm V} \circ h^m = h^m \circ d \circ h^m, \quad d_{\rm V}(\phi) = \theta^i_\Lambda \wedge \partial^\Lambda_i \phi, \quad \phi \in \mathcal{O}^*_\infty,$$

such that $d_{\rm H} \circ d_{\rm H} = 0$, $d_{\rm V} \circ d_{\rm V} = 0$, $d_{\rm H} \circ d_{\rm V} + d_{\rm V} \circ d_{\rm H} = 0$. These differentials make $\mathcal{O}^{*,*}_\infty$ into a bicomplex.

Let $\vartheta \in \mathfrak{d}\mathcal{O}^0_\infty$ be the $\mathcal{O}^0_\infty$-module of derivations of the $\mathbb{R}$-ring $\mathcal{O}^0_\infty$.

Proposition 5. *The derivation module $\mathfrak{d}\mathcal{O}^0_\infty$ is isomorphic to the $\mathcal{O}^0_\infty$-dual $(\mathcal{O}^1_\infty)^*$ of the module of one-forms $\mathcal{O}^1_\infty$.*

Proof. At first, let us show that $\mathcal{O}^*_\infty$ is generated by elements df, $f \in \mathcal{O}^0_\infty$. It suffices to justify that any element of $\mathcal{O}^1_\infty$ is a finite $\mathcal{O}^0_\infty$-linear combination of elements df, $f \in \mathcal{O}^0_\infty$. Indeed, every $\phi \in \mathcal{O}^1_\infty$ is an exterior form on some finite order jet manifold J^rY. By virtue of the Serre–Swan theorem extended to non-compact manifolds [15, 26], the $C^\infty(J^rY)$-module $\mathcal{O}^1_r$ of one-forms on J^rY is a projective module of finite rank, i.e. ϕ is represented by a finite $C^\infty(J^rY)$-linear combination of elements df, $f \in C^\infty(J^rY) \subset \mathcal{O}^0_\infty$. Any element $\Phi \in (\mathcal{O}^1_\infty)^*$ yields a derivation $\vartheta_\Phi(f) = \Phi(df)$ of the $\mathbb{R}$-ring $\mathcal{O}^0_\infty$. Since the module $\mathcal{O}^1_\infty$ is generated by elements df, $f \in \mathcal{O}^0_\infty$, different elements of $(\mathcal{O}^1_\infty)^*$ provide different derivations of $\mathcal{O}^0_\infty$, i.e. there is a monomorphism $(\mathcal{O}^1_\infty)^* \to \mathfrak{d}\mathcal{O}^0_\infty$. By the same formula, any derivation $\vartheta \in \mathfrak{d}\mathcal{O}^0_\infty$ sends $df \mapsto \vartheta(f)$ and, since $\mathcal{O}^0_\infty$ is generated by elements df, it defines a morphism $\Phi_\vartheta : \mathcal{O}^1_\infty \to \mathcal{O}^0_\infty$. Moreover, different derivations ϑ provide different morphisms Φ_ϑ. Thus, we have a monomorphism and, consequently, an isomorphism $\mathfrak{d}\mathcal{O}^0_\infty \to (\mathcal{O}^1_\infty)^*$. □

The proof of Proposition 5 gives something more. The DGA $\mathcal{O}^*_\infty$ is a minimal Chevalley–Eilenberg differential calculus over the $\mathbb{R}$-ring $\mathcal{O}^0_\infty$ of smooth real functions on finite order jet manifolds of $Y \to X$.

Remark 1. Let $\mathcal{K}$ be a commutative ring and $\mathcal{A}$ a commutative $\mathcal{K}$-ring. The module $\mathfrak{d}\mathcal{A}$ of derivations of $\mathcal{A}$ is a Lie $\mathcal{K}$-algebra. The Chevalley–Eilenberg complex of the Lie algebra $\mathfrak{d}\mathcal{A}$ with coefficients in the ring $\mathcal{A}$ contains a subcomplex of

$\mathcal{A}$-multilinear skew-symmetric maps [15]. It is called the Chevalley–Eilenberg differential calculus over a $\mathcal{K}$-ring $\mathcal{A}$. The minimal Chevalley–Eilenberg calculus is generated by monomials $a_0 da_1 \wedge \cdots \wedge da_k$, $a_i \in \mathcal{A}$. For instance, the DGA of exterior forms on a smooth manifold Z is the minimal Chevalley–Eilenberg differential calculus over the $\mathbb{R}$-ring $C^\infty(Z)$.

Restricted to a coordinate chart (7), $\mathcal{O}^1_\infty$ is a free $\mathcal{O}^0_\infty$-module generated by the exterior forms dx^λ, θ^i_Λ. Since $\mathfrak{d}\mathcal{O}^0_\infty = (\mathcal{O}^1_\infty)^*$, any derivation of the $\mathbb{R}$-ring $\mathcal{O}^0_\infty$ takes the coordinate form

$$\vartheta = \vartheta^\lambda \partial_\lambda + \vartheta^i \partial_i + \sum_{0<|\Lambda|} \vartheta^i_\Lambda \partial^\Lambda_i, \tag{13}$$

where $\partial^\Lambda_i(s^j_\Sigma) = \partial^\Lambda_i \rfloor ds^j_\Sigma = \delta^j_i \delta^\Lambda_\Sigma$ up to permutations of multi-indices Λ and Σ. Its coefficients ϑ^λ, ϑ^i, ϑ^i_Λ are local smooth functions of finite jet order possessing the transformation law

$$\vartheta'^\lambda = \frac{\partial x'^\lambda}{\partial x^\mu}\vartheta^\mu, \quad \vartheta'^i = \frac{\partial y'^i}{\partial y^j}\vartheta^j + \frac{\partial y'^i}{\partial x^\mu}\vartheta^\mu, \quad \vartheta'^i_\Lambda = \sum_{|\Sigma|\leq|\Lambda|} \frac{\partial y'^i_\Lambda}{\partial y^j_\Sigma}\vartheta^j_\Sigma + \frac{\partial y'^i_\Lambda}{\partial x^\mu}\vartheta^\mu.$$

Extended to the DGA $\mathcal{O}^*_\infty$, the interior product obeys the rule

$$\vartheta\rfloor(\phi\wedge\sigma) = (\vartheta\rfloor\phi)\wedge\sigma + (-1)^{|\phi|}\phi\wedge(\vartheta\rfloor\sigma).$$

Any derivation ϑ (13) of the ring $\mathcal{O}^0_\infty$ yields a derivation (a Lie derivative $\mathbf{L}_\vartheta$) of the DGA $\mathcal{O}^*_\infty$ given by the relations

$$\mathbf{L}_\vartheta\phi = \vartheta\rfloor d\phi + d(\vartheta\rfloor\phi), \quad \mathbf{L}_\vartheta(\phi\wedge\phi') = \mathbf{L}_\vartheta(\phi)\wedge\phi' + \phi\wedge\mathbf{L}_\vartheta(\phi').$$

In particular, the total derivatives (3) are defined as the local derivations of $\mathcal{O}^0_\infty$ and the corresponding Lie derivatives $d_\lambda\phi = \mathbf{L}_{d_\lambda}\phi$ of $\mathcal{O}^*_\infty$.

A derivation ϑ (13) is called contact if the Lie derivative $\mathbf{L}_\upsilon$ preserves the contact ideal of the DGA $\mathcal{O}^*_\infty$, i.e. the Lie derivative $\mathbf{L}_\upsilon$ of a contact form is a contact form.

Proposition 6. *A derivation* ϑ (13) *is contact iff it takes the form*

$$\vartheta = \vartheta^\lambda \partial_\lambda + \vartheta^i \partial_i + \sum_{|\Lambda|>0} [d_\Lambda(\vartheta^i - y^i_\mu \vartheta^\mu) + y^i_{\mu+\Lambda}\vartheta^\mu]. \tag{14}$$

Proof. The expression (14) results from a direct computation similar to that of the first part of Bäcklund's theorem [21]. □

A glance at the expression (5) enables one to regard a contact derivation (14) as an infinite order jet prolongation of its restriction

$$\upsilon = \vartheta^\lambda \partial_\lambda + \vartheta^i \partial_i \tag{15}$$

to the ring $C^\infty(Y)$. Since coefficients ϑ^λ and ϑ^i depend on jet coordinates y^i_Λ, $0 < |\Lambda|$, in general, one calls υ (15) a generalized vector field. Generalized symmetries of differential equations and Lagrangians have been intensively studied [2, 21, 22, 25].

Any contact derivation admits the horizontal splitting

$$\vartheta = \upsilon_{\rm H} + \upsilon_{\rm V} = \vartheta^\lambda d_\lambda + \left[\vartheta^i\partial_i + \sum_{0<|\Lambda|} d_\Lambda(\vartheta^i - y^i_\mu\vartheta^\mu)\partial_i^\Lambda\right] \tag{16}$$

relative to the canonical connection $\nabla = dx^\lambda \otimes d_\lambda$ on the $C^\infty(X)$-ring $\mathcal{O}^0_\infty$. One can show [14] that a vertical contact derivation

$$\upsilon = \upsilon^i\partial_i + \sum_{0<|\Lambda|} d_\Lambda \upsilon^i\partial_i^\Lambda$$

obeys the relations

$$\upsilon\rfloor d_{\rm H}\phi = -d_{\rm H}(\upsilon\rfloor\phi), \quad \mathbf{L}_\upsilon(d_{\rm H}\phi) = d_{\rm H}(\mathbf{L}_\upsilon\phi), \;\; \phi\in\mathcal{O}^*_\infty. \tag{17}$$

They follow from the equalities

$$\begin{gathered} \upsilon\rfloor\theta^i_\Lambda = \upsilon^i_\Lambda, \quad d_{\rm H}(\upsilon^i_\Lambda) = \upsilon^i_{\lambda+\Lambda}dx^\lambda, \quad d_{\rm H}\theta^i_\lambda = dx^\lambda\wedge\theta^i_{\lambda+\Lambda}, \\ d_\lambda\circ\upsilon^i_\Lambda\partial_i^\Lambda = \upsilon^i_\Lambda\partial_i^\Lambda\circ d_\lambda. \end{gathered} \tag{18}$$

5. Variational Bicomplex on Fiber Bundles

In order to transform the bicomplex $\mathcal{O}^{*,*}_\infty$ into the variational bicomplex, one introduces the $\mathbb{R}$-module projector

$$\varrho = \sum_{k>0}\frac{1}{k}\overline{\varrho}\circ h_k\circ h^n, \qquad \overline{\varrho}(\phi) = \sum_{|\Lambda|\geq 0}(-1)^{|\Lambda|}\theta^i\wedge[d_\Lambda(\partial_i^\Lambda\rfloor\phi)], \;\; \phi\in\mathcal{O}^{>0,n}_\infty, \tag{19}$$

such that $\varrho\circ d_{\rm H} = 0$ and the nilpotent variational operator $\delta = \varrho\circ d$ on $\mathcal{O}^{*,n}_\infty$ which obeys the relation

$$\delta\circ\varrho - \varrho\circ d = 0. \tag{20}$$

Let us denote $\mathbf{E}_k = \varrho(\mathcal{O}^{k,n}_\infty)$. Then the DGA $\mathcal{O}^*_\infty$ is split into the variational bicomplex

$$\begin{array}{ccccccccc}
 & \vdots & & \vdots & & \vdots & & \vdots & & \vdots \\
 & {\scriptstyle d_{\rm V}}\uparrow & & {\scriptstyle d_{\rm V}}\uparrow & & {\scriptstyle d_{\rm V}}\uparrow & & {\scriptstyle d_{\rm V}}\uparrow & & {\scriptstyle -\delta}\uparrow \\
0\to & \mathcal{O}^{1,0}_\infty & \overset{d_{\rm H}}{\to} & \mathcal{O}^{1,1}_\infty & \overset{d_{\rm H}}{\to}\cdots & \mathcal{O}^{1,m}_\infty & \overset{d_{\rm H}}{\to}\cdots & \mathcal{O}^{1,n}_\infty & \overset{\varrho}{\to} & \mathbf{E}_1\to 0 \\
 & {\scriptstyle d_{\rm V}}\uparrow & & {\scriptstyle d_{\rm V}}\uparrow & & {\scriptstyle d_{\rm V}}\uparrow & & {\scriptstyle d_{\rm V}}\uparrow & & {\scriptstyle -\delta}\uparrow \\
0\to\mathbb{R}\to & \mathcal{O}^0_\infty & \overset{d_{\rm H}}{\to} & \mathcal{O}^{0,1}_\infty & \overset{d_{\rm H}}{\to}\cdots & \mathcal{O}^{0,m}_\infty & \overset{d_{\rm H}}{\to}\cdots & \mathcal{O}^{0,n}_\infty & \equiv & \mathcal{O}^{0,n}_\infty \\
 & {\scriptstyle \pi^{\infty*}}\uparrow & & {\scriptstyle \pi^{\infty*}}\uparrow & & {\scriptstyle \pi^{\infty*}}\uparrow & & {\scriptstyle \pi^{\infty*}}\uparrow & & \\
0\to\mathbb{R}\to & \mathcal{O}^0X & \overset{d}{\to} & \mathcal{O}^1X & \overset{d}{\to}\cdots & \mathcal{O}^mX & \overset{d}{\to}\cdots & \mathcal{O}^n(X) & \overset{d}{\to} & 0 \\
 & \uparrow & & \uparrow & & \uparrow & & \uparrow & & \\
 & 0 & & 0 & & 0 & & 0 & &
\end{array} \tag{21}$$

Its relevant cohomology has been obtained as follows [13, 27]. One starts from the algebraic Poincaré lemma [25, 29].

Lemma 7. *If Y is a contractible bundle $\mathbb{R}^{n+p} \to \mathbb{R}^n$, the variational bicomplex* (21) *is exact at all terms, except* $\mathbb{R}$.

Proof. The homotopy operators for d_{V}, d_{H}, δ and ϱ are given by the formulas (5.72), (5.109), (5.84) in [25] and (4.5) in [29], respectively. □

Theorem 8. (i) *The second row from the bottom and the last column of this bicomplex make up the variational complex*

$$0 \to \mathbb{R} \to \mathcal{O}^0_\infty \xrightarrow{d_{\mathrm{H}}} \mathcal{O}^{0,1}_\infty \cdots \xrightarrow{d_{\mathrm{H}}} \mathcal{O}^{0,n}_\infty \xrightarrow{\delta} \mathbf{E}_1 \xrightarrow{\delta} \mathbf{E}_2 \to \cdots . \tag{22}$$

Its cohomology is isomorphic to the de Rham cohomology of the fiber bundle Y, namely,

$$H^{k<n}(d_{\mathrm{H}}; \mathcal{Q}^*_\infty) = H^{k<n}(Y), \quad H^{k-n}(\delta; \mathcal{Q}^*_\infty) = H^{k\geq n}(Y). \tag{23}$$

(ii) *The rows of contact forms of the bicomplex* (21) *are exact sequences.*

Proof. Let $\mathfrak{T}^*_\infty$ be the sheaf of germs of differential forms on $J^\infty Y$. It is split into the variational bicomplex $\mathfrak{T}^{*,*}_\infty$. Let $\mathcal{Q}^*_\infty$ be the DGA of global sections of $\mathfrak{T}^*_\infty$. It is also decomposed into the variational bicomplex $\mathcal{Q}^{*,*}_\infty$. Since the paracompact space $J^\infty Y$ admits a partition of unity by elements of the ring $\mathcal{Q}^0_\infty$, the d_{H}- and δ-cohomology of $\mathcal{Q}^{*,*}_\infty$ can be obtained as follows [1, 13, 27, 28]. Let us consider the variational subcomplex of $\mathfrak{T}^{*,*}_\infty$ and the subcomplexes of sheaves of contact forms

$$0 \to \mathbb{R} \to \mathfrak{T}^0_\infty \xrightarrow{d_{\mathrm{H}}} \mathfrak{T}^{0,1}_\infty \cdots \xrightarrow{d_{\mathrm{H}}} \mathfrak{T}^{0,n}_\infty \xrightarrow{\delta} \mathfrak{E}_1 \xrightarrow{\delta} \mathfrak{E}_2 \to \cdots , \quad \mathfrak{E}_k = \varrho(\mathfrak{T}^{k,n}_\infty), \tag{24}$$

$$0 \to \mathfrak{T}^{k,0}_\infty \xrightarrow{d_{\mathrm{H}}} \mathfrak{T}^{k,1}_\infty \cdots \xrightarrow{d_{\mathrm{H}}} \mathfrak{T}^{k,n}_\infty \xrightarrow{\varrho} \mathfrak{E}_k \to 0. \tag{25}$$

By virtue of Lemma 7, these complexes are exact at all terms, except $\mathbb{R}$. Since $\mathfrak{T}^{m,k}_\infty$ are sheaves of $\mathcal{Q}^0_\infty$-modules, they are fine. The sheaves $\mathfrak{E}_k$ are also proved to be fine (see Appendix C). Consequently, all sheaves, except $\mathbb{R}$, in the complexes (24) and (25) are acyclic. Therefore, these complexes are resolutions of the constant sheaf $\mathbb{R}$ and the zero sheaf over $J^\infty Y$, respectively. Let us consider the corresponding subcomplexes

$$0 \to \mathbb{R} \to \mathcal{Q}^0_\infty \xrightarrow{d_{\mathrm{H}}} \mathcal{Q}^{0,1}_\infty \cdots \xrightarrow{d_{\mathrm{H}}} \mathcal{Q}^{0,n}_\infty \xrightarrow{\delta} \Gamma(\mathfrak{E}_1) \xrightarrow{\delta} \Gamma(\mathfrak{E}_2) \to \cdots , \tag{26}$$

$$0 \to \mathcal{Q}^{k,0}_\infty \xrightarrow{d_{\mathrm{H}}} \mathcal{Q}^{k,1}_\infty \cdots \xrightarrow{d_{\mathrm{H}}} \mathcal{Q}^{k,n}_\infty \xrightarrow{\varrho} \Gamma(\mathfrak{E}_k) \to 0 \tag{27}$$

of the DGA $\mathcal{Q}^*_\infty$. In accordance with the abstract de Rham theorem (see Appendix B), cohomology of the complex (26) equals the cohomology of $J^\infty Y$ with coefficients in the constant sheaf $\mathbb{R}$, while the complex (27) is exact. Since Y is a strong deformation retract of $J^\infty Y$, cohomology of the complex (26) equals the de Rham cohomology of Y by virtue of the isomorphisms (8). Note that, in order to prove the exactness of the complex (27), the acyclicity of the sheaves $\mathfrak{E}_k$ need not be justified. Finally, the subalgebra $\mathcal{O}^*_\infty \subset \mathcal{Q}^*_\infty$ is proved to have the same

d_H- and δ-cohomology as $\mathcal{Q}^*_\infty$ [12, 27] (see Appendix D). Similarly, one can show that, restricted to $\mathcal{O}^{k,n}_\infty$, the operator ϱ remains exact. $\square$

Note that the cohomology isomorphism (23) gives something more. The relation (20) for ϱ and the relation $h_0 d = d_H h_0$ for h_0 define a cochain morphism of the de Rham complex (1) of the DGA $\mathcal{O}^*_\infty$ to its variational complex (22). The corresponding homomorphism of their cohomology groups is an isomorphism by virtue of Theorem 1 and item (i) of Theorem 8. Then the splitting of a closed form $\phi \in \mathcal{O}^*_\infty$ in Corollary 2 leads to the following decompositions:

Proposition 9. *Any d_H-closed form $\phi \in \mathcal{O}^{0,m}$, $m < n$, is represented by a sum*

$$\phi = h_0\sigma + d_H\xi, \quad \xi \in \mathcal{O}^{m-1}_\infty, \tag{28}$$

where σ is a closed m-form on Y. Any δ-closed form $\phi \in \mathcal{O}^{k,n}$ is split into

$$\phi = h_0\sigma + d_H\xi, \quad k = 0, \ \xi \in \mathcal{O}^{0,n-1}_\infty, \tag{29}$$

$$\phi = \varrho(\sigma) + \delta(\xi), \quad k = 1, \ \xi \in \mathcal{O}^{0,n}_\infty, \tag{30}$$

$$\phi = \varrho(\sigma) + \delta(\xi), \quad k > 1, \ \xi \in \mathbf{E}_{k-1}, \tag{31}$$

where σ is a closed $(n+k)$-form on Y.

One can think of the elements

$$L = \mathcal{L}\omega \in \mathcal{O}^{0,n}_\infty, \quad \delta L = \sum_{|\Lambda|\geq 0} (-1)^{|\Lambda|} d_\Lambda(\partial^\Lambda_i \mathcal{L})\theta^i \wedge \omega \in \mathbf{E}_1, \quad \omega = dx^1 \wedge \cdots \wedge dx^n,$$

of the variational complex (22) as being a finite order Lagrangian and its Euler–Lagrange operator, respectively. Then the following are corollaries of Theorem 8:

Corollary 10. (i) *A finite order Lagrangian $L \in \mathcal{O}^{0,n}_\infty$ is variationally trivial, i.e. $\delta(L) = 0$ iff*

$$L = h_0\sigma + d_H\xi, \quad \xi \in \mathcal{O}^{0,n-1}_\infty, \tag{32}$$

where σ is a closed n-form on Y.

(ii) *A finite order Euler–Lagrange-type operator $\mathcal{E} \in \mathbf{E}_1$ satisfies the Helmholtz condition $\delta(\mathcal{E}) = 0$ iff*

$$\mathcal{E} = \delta(L) + \varrho(\sigma), \quad L \in \mathcal{O}^{0,n}_\infty,$$

where σ is a closed $(n+1)$-form on Y.

Corollary 11. *The exactness of the row of one-contact forms of the variational bicomplex (21) at the term $\mathcal{O}^{1,n}_\infty$ relative to the projector ϱ provides the $\mathbb{R}$-module decomposition*

$$\mathcal{O}^{1,n}_\infty = \mathbf{E}_1 \oplus d_H(\mathcal{O}^{1,n-1}_\infty).$$

Given a Lagrangian $L \in \mathcal{O}^{0,n}_\infty$, we have the corresponding decomposition

$$dL = \delta L - d_H\Xi. \tag{33}$$

The form Ξ in the decomposition (33) is not uniquely defined. It reads

$$\Xi = \sum_{s=0} F_i^{\lambda\nu_s\ldots\nu_1}\theta^i_{\nu_s\ldots\nu_1}\wedge\omega_\lambda, \quad F_i^{\nu_k\ldots\nu_1} = \partial_i^{\nu_k\ldots\nu_1}\mathcal{L} - d_\lambda F_i^{\lambda\nu_k\ldots\nu_1} + h_i^{\nu_k\ldots\nu_1},$$

$$\omega_\lambda = \partial_\lambda\rfloor\omega,$$

where local functions $h\in\mathcal{O}^0_\infty$ obey the relations $h_i^\nu = 0$, $h_i^{(\nu_k\nu_{k-1})\ldots\nu_1} = 0$. It follows that $\Xi_{\rm L} = \Xi + L$ is a Lepagean equivalent of a finite order Lagrangian L [17].

The decomposition (33) leads to the global first variational formula and the first Noether theorem as follows:

Theorem 12. *Given a Lagrangian* $L = \mathcal{L}\omega\in\mathcal{O}^{0,n}_\infty$, *its Lie derivative* $\mathbf{L}_\upsilon L$ *along a contact derivation* υ (16) *fulfils the first variational formula*

$$\mathbf{L}_\vartheta L = \upsilon_{\rm V}\rfloor\delta L + d_{\rm H}(h_0(\vartheta\rfloor\Xi_{\rm L})) + \mathcal{L}d_{\rm V}(\upsilon_{\rm H}\rfloor\omega), \tag{34}$$

where $\Xi_{\rm L}$ *is a Lepagean equivalent.*

Proof. The formula (34) comes from the splitting of (33) and the relation (17) as follows:

$$\begin{aligned}\mathbf{L}_\vartheta L &= \vartheta\rfloor dL + d(\vartheta\rfloor L) = [\upsilon_{\rm V}\rfloor dL - d_{\rm V}\mathcal{L}\wedge\upsilon_{\rm H}\rfloor\omega] + [d_{\rm H}(\upsilon_{\rm H}\rfloor L) + d_{\rm V}(\mathcal{L}\upsilon_{\rm H}\rfloor\omega)]\\ &= \upsilon_{\rm V}\rfloor dL + d_{\rm H}(\upsilon_{\rm H}\rfloor L) + \mathcal{L}d_{\rm V}(\upsilon_{\rm H}\rfloor\omega) = \upsilon_{\rm V}\rfloor\delta L - \upsilon_{\rm V}\rfloor d_{\rm H}\Xi + d_{\rm H}(\upsilon_{\rm H}\rfloor L)\\ &\quad + \mathcal{L}d_{\rm V}(\upsilon_{\rm H}\rfloor\omega)\\ &= \upsilon_{\rm V}\rfloor\delta L + d_{\rm H}(\upsilon_{\rm V}\rfloor\Xi + \upsilon_{\rm H}\rfloor L) + \mathcal{L}d_{\rm V}(\upsilon_{\rm H}\rfloor\omega),\end{aligned}$$

where $\upsilon_{\rm V}\rfloor\Xi = h_0(\vartheta\rfloor\Xi)$ since Ξ is a one-contact form, $\upsilon_{\rm H}\rfloor L = h_0(\upsilon\rfloor L)$, and $\Xi_{\rm L} = \Xi + L$. $\square$

A contact derivation ϑ (14) is called a variational symmetry of a Lagrangian L if the Lie derivative $\mathbf{L}_\vartheta L = d_{\rm H}\xi$ is $d_{\rm H}$-exact. A glance at the expression (34) shows that: (i) a contact derivation ϑ is a variational symmetry only if it is projected onto X (i.e. its components ϑ^λ depend only on coordinates of X), (ii) ϑ is a variational symmetry iff its vertical part $\upsilon_{\rm V}$ is well, (iii) it is a variational symmetry iff the density $\upsilon_{\rm V}\rfloor\delta L$ is $d_{\rm H}$-exact.

Theorem 13. *If a contact derivation* ϑ (14) *is a variational symmetry of a Lagrangian* L, *the first variational formula* (34) *restricted to* $\operatorname{Ker}\delta L$ *leads to the weak conservation law*

$$0\approx d_{\rm H}(h_0(\vartheta\rfloor\Xi_{\rm L}) - \xi).$$

Remark 2. Let a contact derivation ϑ (14) be the jet prolongation of a vector field $\vartheta^\lambda\partial_\lambda + \vartheta^i\partial_i$ on Y. If ϑ is a variational symmetry of a Lagrangian L, then it is also a symmetry of the Euler–Lagrange operator δL of L, i.e. $\mathbf{L}_\vartheta\delta L = 0$ by virtue of the equality $\mathbf{L}_\vartheta\delta L = \delta(\mathbf{L}_\vartheta L)$. However, this equality fails to be true in the case of generalized symmetries [25].

6. Polynomial Variational Bicomplex

Let $Y \to X$ be an affine bundle. Since X is a strong deformation retract of Y, the de Rham cohomology of Y and, consequently, $J^\infty Y$ equals that of X. An immediate consequence of this fact is the following cohomology isomorphisms:

$$H^{<n}(d_{\rm H}; \mathcal{O}^*_\infty) = H^{<n}(X), \qquad H^0(\delta; \mathcal{O}^*_\infty) = H^n(X), \qquad H^k(\delta; \mathcal{O}^*_\infty) = 0.$$

It follows that every $d_{\rm H}$-closed form $\phi \in \mathcal{O}^{0,m<n}_\infty$ is represented by the sum

$$\phi = \sigma + d_{\rm H}\xi, \quad \xi \in \mathcal{O}^{0,m-1}_\infty, \tag{35}$$

where σ is a closed form on X. Similarly, any variationally trivial Lagrangian takes the form

$$L = \sigma + d_{\rm H}\xi, \quad \xi \in \mathcal{O}^{0,n-1}_\infty,$$

where σ is a closed n-form on X.

Let us restrict our consideration to the short variational complex

$$0 \to \mathbb{R} \to \mathcal{O}^0_\infty \xrightarrow{d_{\rm H}} \mathcal{O}^{0,1}_\infty \cdots \xrightarrow{d_{\rm H}} \mathcal{O}^{0,n}_\infty \xrightarrow{\delta} \mathbf{E}_1 \tag{36}$$

and the similar complex of sheaves

$$0 \to \mathbb{R} \to \mathfrak{T}^0_\infty \xrightarrow{d_{\rm H}} \mathfrak{T}^{0,1}_\infty \cdots \xrightarrow{d_{\rm H}} \mathfrak{T}^{0,n}_\infty \xrightarrow{\delta} \mathfrak{E}_1. \tag{37}$$

In the case of an affine bundle $Y \to X$, we can lower this complex onto the base X as follows.

Let us consider the open surjection $\pi^\infty : J^\infty Y \to X$ and the direct image $\mathfrak{X}^*_\infty = \pi^\infty_* \mathfrak{T}^*_\infty$ on X of the sheaf $\mathfrak{T}^*_\infty$. Its stalk over a point $x \in X$ consists of the equivalence classes of sections of the sheaf $\mathfrak{T}^*_\infty$ which coincide on the inverse images $(\pi^\infty)^{-1}(U_x)$ of neighborhoods U_x of x. Since $\pi^\infty_*\mathbb{R} = \mathbb{R}$, we have the following complex of sheaves on X:

$$0 \to \mathbb{R} \to \mathfrak{X}^0_\infty \xrightarrow{d_{\rm H}} \mathfrak{X}^{0,1}_\infty \xrightarrow{d_{\rm H}} \cdots \xrightarrow{d_{\rm H}} \mathfrak{X}^{0,n}_\infty \xrightarrow{\delta} \pi^\infty_* \mathfrak{E}_1. \tag{38}$$

Every point $x \in X$ has a base of open contractible neighborhoods $\{U_x\}$ such that the sheaves $\mathfrak{T}^{0,*}_\infty$ of $\mathcal{Q}^*_\infty$-modules are acyclic on the inverse images $(\pi^\infty)^{-1}(U_x)$ of these neighborhoods. Then, in accordance with the Leray theorem [16], cohomology of $J^\infty Y$ with coefficients in the sheaves $\mathfrak{T}^{0,*}_\infty$ are isomorphic to that of X with coefficients in their direct images $\mathfrak{X}^{0,*}_\infty$, i.e. the sheaves $\mathfrak{X}^{0,*}_\infty$ on X are acyclic. Furthermore, Lemma 7 also shows that the complexes of sections of sheaves $\mathfrak{T}^{0,*}_\infty$ over $(\pi^\infty_0)^{-1}(U_x)$ are exact. It follows that the complex (38) on X is exact at all terms, except $\mathbb{R}$, and it is a resolution of the constant sheaf $\mathbb{R}$ on X. Due to the $\mathbb{R}$-algebra isomorphism $\mathcal{Q}^*_\infty = \Gamma(\mathfrak{X}^*_\infty)$, one can think of the short variational subcomplex of the complex (24) as being the complex of the structure algebras of the sheaves in the complex (38) on X.

Given the sheaf $\mathfrak{X}^*_\infty$ on X, let us consider its subsheaf $\mathfrak{P}^*_\infty$ of germs of exterior forms which are polynomials in the fiber coordinates y^i_Λ, $|\Lambda| \geq 0$, of the topological fiber bundle $J^\infty Y \to X$. This property is coordinate-independent due to the transition functions (7). The sheaf $\mathfrak{P}^*_\infty$ is a sheaf of $C^\infty(X)$-modules. The DGA P^*_∞ of its global sections is a $C^\infty(X)$-subalgebra of $\mathcal{Q}^*_\infty$. We have the subcomplex

$$0 \to \mathbb{R} \to \mathfrak{P}^0_\infty \xrightarrow{d_{\rm H}} \mathfrak{P}^{0,1}_\infty \xrightarrow{d_{\rm H}} \cdots \xrightarrow{d_{\rm H}} \mathfrak{P}^{0,n}_\infty \xrightarrow{\delta} \pi^\infty_* \mathfrak{E}_1 \tag{39}$$

of the complex (38) on X. As a particular variant of the algebraic Poincaré lemma, the exactness of the complex (39) at all terms, except $\mathbb{R}$, follows from the form of the homotopy operator for $d_{\rm H}$ or can be proved in a straightforward way [3]. Since the sheaves $\mathfrak{P}^{0,*}_\infty$ of $C^\infty(X)$-modules on X are acyclic, the complex (39) is a resolution of the constant sheaf $\mathbb{R}$ on X. Hence, cohomology of the complex

$$0 \to \mathbb{R} \to P^0_\infty \xrightarrow{d_{\rm H}} P^{0,1}_\infty \xrightarrow{d_{\rm H}} \cdots \xrightarrow{d_{\rm H}} P^{0,n}_\infty \xrightarrow{\delta} \Gamma(\mathfrak{E}_1) \tag{40}$$

of the DGAs $P^{0,<n}_\infty$ equals the de Rham cohomology of X. It follows that every $d_{\rm H}$-closed polynomial form $\phi \in P^{0,m<n}_\infty$ is decomposed into the sum

$$\phi = \sigma + d_{\rm H}\xi, \quad \xi \in \mathcal{P}^{0,m-1}_\infty, \tag{41}$$

where σ is a closed form on X.

Let $\mathcal{P}^*_\infty$ be $C^\infty(X)$-subalgebra of the polynomial algebra P^*_∞ which consists of exterior forms which are polynomials in the fiber coordinates y^i_Λ. Obviously, $\mathcal{P}^*_\infty$ is a subalgebra of $\mathcal{O}^*_\infty$. Finally, one can show that $\mathcal{P}^*_\infty$ have the same cohomology as P^*_∞, i.e. if ϕ in the decomposition (41) is an element of $\mathcal{P}^{0,*}_\infty$ then ξ is so. The proof of this fact follows the proof in Appendix D, but differential forms on X (not $J^\infty Y$) are also considered.

7. Differential Calculus on Graded Manifolds

We restrict our consideration to graded manifolds $(Z, \mathfrak{A})$ with structure sheaves $\mathfrak{A}$ of Grassmann algebras of finite rank [4, 15]. By a Grassmann algebra over a ring $\mathcal{K}$ is meant a $\mathbb{Z}_2$-graded exterior algebra of some $\mathcal{K}$-module. The symbol $[\,\cdot\,]$ stands for the Grassmann parity.

Treating Lagrangian systems of odd variables on a smooth manifold, we are based on the following variant of the Serre–Swan theorem [6].

Theorem 14. *Let Z be a smooth manifold. A graded commutative $C^\infty(Z)$-algebra $\mathcal{A}$ is isomorphic to the algebra of graded functions on a graded manifold with a body Z iff it is the exterior algebra of some projective $C^\infty(Z)$-module of finite rank.*

Proof. The proof follows at once from the Batchelor theorem [4] and the classical Serre–Swan theorem generalized to an arbitrary smooth manifold [15, 26]. By virtue of the first one, any graded manifold $(Z, \mathfrak{A})$ with a body Z is isomorphic to the one

$(Z,\mathfrak{A}_Q)$, modeled over some vector bundle $Q\to Z$, whose structure sheaf $\mathfrak{A}_Q$ is the sheaf of germs of sections of the exterior bundle

$$\wedge Q^* = \mathbb{R} \underset{Z}{\oplus} Q^* \underset{Z}{\oplus} \overset{2}{\wedge} Q^* \underset{Z}{\oplus} \cdots , \tag{42}$$

where Q^* is the dual of $Q\to Z$. The structure ring $\mathcal{A}_Q$ of graded functions (sections of $\mathfrak{A}_Q$) on a graded manifold $(Z,\mathfrak{A}_Q)$ consists of sections of the exterior bundle (42). The classical Serre–Swan theorem states that a $C^\infty(Z)$-module is isomorphic to the module of sections of a smooth vector bundle over Z iff it is a projective module of finite rank. $\square$

Assuming that Batchelor's isomorphism is fixed from the beginning, we associate to $(Z,\mathfrak{A}_Q)$ the following DBGA $\mathcal{S}^*[Q;Z]$ [4,15]. Let us consider the sheaf $\mathfrak{d}\mathfrak{A}_Q$ of graded derivations of $\mathfrak{A}_Q$. One can show that its sections over an open subset $U\subset Z$ exhaust all graded derivations of the graded commutative $\mathbb{R}$-ring $\mathcal{A}_U$ of graded functions on U [4]. Global sections of $\mathfrak{d}\mathfrak{A}_Q$ make up the real Lie superalgebra $\mathfrak{d}\mathcal{A}_Q$ of graded derivations of the $\mathbb{R}$-ring $\mathcal{A}_Q$, i.e.

$$u(ff') = u(f)f' + (-1)^{[u][f]} f u(f'), \quad f,f'\in\mathcal{A}_Q,\;\; u\in\mathcal{A}_Q.$$

Then one can construct the Chevalley–Eilenberg complex of $\mathfrak{d}\mathcal{A}_Q$ with coefficients in $\mathcal{A}_Q$ [11]. Its subcomplex $\mathcal{S}^*[Q;Z]$ of $\mathcal{A}_Q$-linear morphism is the Grassmann-graded Chevalley–Eilenberg differential calculus

$$0\to\mathbb{R}\to\mathcal{A}_Q \xrightarrow{d} \mathcal{S}^1[Q;Z] \xrightarrow{d} \cdots \mathcal{S}^k[Q;Z] \xrightarrow{d} \cdots \tag{43}$$

over a graded commutative $\mathbb{R}$-ring $\mathcal{A}_Q$ [15]. The Chevalley–Eilenberg coboundary operator d and the graded exterior product $\wedge$ make $\mathcal{S}^*[Q;Z]$ into a DBGA whose elements obey the relations

$$\phi\wedge\phi' = (-1)^{|\phi||\phi'|+[\phi][\phi']}\phi'\wedge\phi, \qquad d(\phi\wedge\phi') = d\phi\wedge\phi' + (-1)^{|\phi|}\phi\wedge d\phi'. \tag{44}$$

Given the DGA $\mathcal{O}^*Z$ of exterior forms on Z, there are the canonical monomorphism $\mathcal{O}^*Z\to\mathcal{S}^*[Q;Z]$ and the body epimorphism $\mathcal{S}^*[Q;Z]\to\mathcal{O}^*Z$ which are cochain morphisms.

Lemma 15. *The DBGA $\mathcal{S}^*[Q;Z]$ is a minimal differential calculus over $\mathcal{A}_Q$, i.e. it is generated by elements df, $f\in\mathcal{A}_Q$.*

Proof. One can show that elements of $\mathfrak{d}\mathcal{A}_Q$ are represented by sections of some vector bundle over Z, i.e. $\mathfrak{d}\mathcal{A}_Q$ is a projective $C^\infty(Z)$- and $\mathcal{A}_Q$-module of finite rank, and so is its $\mathcal{A}_Q$-dual $\mathcal{S}^1[Q;Z]$ [14,15]. Hence, $\mathfrak{d}\mathcal{A}_Q$ is the $\mathcal{A}_Q$-dual of $\mathcal{S}^1[Q;Z]$ and, consequently, $\mathcal{S}^1[Q;Z]$ is generated by elements df, $f\in\mathcal{A}_Q$ [15]. $\square$

This fact is essential for our consideration because of the following [15]:

Lemma 16. *Given a ring R, let $\mathcal{K}$, $\mathcal{K}'$ be R-rings and $\mathcal{A}$, $\mathcal{A}'$ the Grassmann algebras over $\mathcal{K}$ and $\mathcal{K}'$, respectively. Then any homomorphism $\rho:\mathcal{A}\to\mathcal{A}'$ yields* *the homomorphism of the minimal Chevalley–Eilenberg differential calculus over a $\mathbb{Z}_2$-graded R-ring $\mathcal{A}$ to that over $\mathcal{A}'$ given by the map $da \mapsto d(\rho(a))$, $a \in \mathcal{A}$. This map provides a monomorphism if ρ is a monomorphism of R-algebras.*

One can think of elements of the DBGA $\mathcal{S}^*[Q;Z]$ as being Grassmann-graded or, simply, graded) differential forms on Z as follows. Given an open subset $U \subset Z$, let $\mathcal{A}_U$ be the Grassmann algebra of sections of the sheaf $\mathfrak{A}_Q$ over U, and let $\mathcal{S}^*[Q;U]$ be the corresponding Chevalley–Eilenberg differential calculus over $\mathcal{A}_U$. Given an open set $U' \subset U$, the restriction morphisms $\mathcal{A}_U \to \mathcal{A}_{U'}$ yield the restriction morphism of the DBGAs $\mathcal{S}^*[Q;U] \to \mathcal{S}^*[Q;U']$. Thus, we obtain the presheaf $\{U, \mathcal{S}^*[Q;U]\}$ of DBGAs on a manifold Z and the sheaf $\mathfrak{S}^*[Q;Z]$ of DBGAs of germs of this presheaf. Since $\{U, \mathcal{A}_U\}$ is the canonical presheaf of the sheaf $\mathfrak{A}_Q$, the canonical presheaf of $\mathfrak{S}^*[Q;Z]$ is $\{U, \mathcal{S}^*[Q;U]\}$. In particular, $\mathcal{S}^*[Q;Z]$ is the DBGA of global sections of the sheaf $\mathfrak{S}^*[Q;Z]$, and there is the restriction morphism $\mathcal{S}^*[Q;Z] \to \mathcal{S}^*[Q;U]$ for any open $U \subset Z$.

Due to this restriction morphism, elements of the DBGA $\mathcal{S}^*[Q;Z]$ can be written in the following local form. Given bundle coordinates (z^A, q^a) on Q and the corresponding fiber basis $\{c^a\}$ for $Q^* \to X$, the tuple (z^A, c^a) is called a local basis for the graded manifold $(Z, \mathfrak{A}_Q)$ [4]. With respect to this basis, the graded functions read

$$f = \sum_{k=0} \frac{1}{k!} f_{a_1 \ldots a_k} c^{a_1} \cdots c^{a_k}, \tag{45}$$

where $f_{a_1 \cdots a_k}$ are smooth real functions on Z, and we omit the symbol of the exterior product of elements c^a. Due to the canonical splitting $VQ = Q \times Q$, the fiber basis $\{\partial_a\}$ for vertical tangent bundle $VQ \to Q$ of $Q \to Z$ is the dual of $\{c^a\}$. Then graded derivations take the local form $u = u^A \partial_A + u^a \partial_a$, where u^A and u^a are local graded functions. They act on graded functions (45) by the rule

$$u(f_{a \ldots b} c^a \cdots c^b) = u^A \partial_A (f_{a \ldots b}) c^a \cdots c^b + u^d f_{a \ldots b} \partial_d \rfloor (c^a \cdots c^b). \tag{46}$$

Relative to the dual local bases $\{dz^A\}$ for T^*Z and $\{dc^b\}$ for Q^*, graded one-forms read $\phi = \phi_A dz^A + \phi_a dc^a$. The duality morphism is given by the interior product

$$u \rfloor \phi = u^A \phi_A + (-1)^{[\phi_a]} u^a \phi_a, \quad u \in \mathfrak{d}\mathcal{A}_Q, \ \phi \in \mathcal{S}^1[Q;Z].$$

The Chevalley–Eilenberg coboundary operator d, called the graded exterior differential, reads

$$d\phi = dz^A \wedge \partial_A \phi + dc^a \wedge \partial_a \phi,$$

where the derivations ∂_A and ∂_a act on coefficients of graded differential forms by the formula (46), and they are graded commutative with the graded differential forms dz^A and dc^a.

Since $\mathcal{S}^*[Q;Z]$ is a DBGA of graded differential forms on Z, one can obtain its de Rham cohomology by means of the abstract de Rham theorem as follows:

Theorem 17. *The cohomology of the de Rham complex* (43) *of the DBGA* $\mathcal{S}^*[Q;Z]$ *equals the de Rham cohomology of the body* Z.

Proof. We have the complex

$$0 \to \mathbb{R} \to \mathfrak{S}^0[Q;Z] \xrightarrow{d} \mathfrak{S}^1[Q;Z] \xrightarrow{d} \cdots \mathfrak{S}^k[Q;Z] \xrightarrow{d} \cdots \tag{47}$$

of sheafs of germs of graded differential forms on Z. Its members $\mathfrak{S}^k[Q;Z]$ are sheaves of $C^\infty(Z)$-modules on Z and, consequently, are fine and acyclic. Furthermore, the Poincaré lemma for graded differential forms holds [4]. It follows that the complex (47) is a fine resolution of the constant sheaf $\mathbb{R}$ on the manifold Z. Then, by virtue of Theorem 26, there is an isomorphism

$$H^*(\mathcal{S}^*[Q;Z]) = H^*(Z;\mathbb{R}) = H^*_{\mathrm{DR}}(Z) \tag{48}$$

of the cohomology of the complex (43) to the de Rham cohomology of Z. Moreover, the cohomology isomorphism (48) accompanies the cochain monomorphism of the de Rham complex of $\mathcal{O}^*Z$ to the complex (43). Hence, any closed graded differential form is split into a sum $\phi = \sigma + d\xi$ of a closed exterior form σ on Z and an exact graded differential form. □

8. Graded Infinite Order Jet Manifold

As was mentioned above, we consider graded manifolds of jets of smooth fiber bundles, but not jets of fibered graded manifolds. To motivate this construction, let us return to the case of even variables when $Y \to X$ is a vector bundle. The jet bundles $J^kY \to X$ are also vector bundles. Let $\mathcal{P}^*_\infty \subset \mathcal{O}^*_\infty$ be a subalgebra of exterior forms on these bundles whose coefficients are polynomial in fiber coordinates. In particular, $\mathcal{P}^0_\infty$ is the ring of polynomials of these coordinates with coefficients in the ring $C^\infty(X)$. One can associate to such a polynomial of degree m, a section of the symmetric product $\vee^m(J^kY)^*$ of the dual to some jet bundle $J^kY \to X$, and *vice versa*. Moreover, any element of $\mathcal{P}^*_\infty$ is an element of the Chevalley–Eilenberg differential calculus over $\mathcal{P}^0_\infty$. Following this example, let $F \to X$ be a vector bundle, and let us consider graded manifolds $(X, \mathcal{A}_{J^rF})$ modeled over the vector bundles $J^rF \to X$. There is a direct system of the corresponding DBGAs

$$\mathcal{S}^*[F;X] \to \mathcal{S}^*[J^1F;X] \to \cdots \mathcal{S}^*[J^rF;X] \to \cdots,$$

whose direct limit $\mathcal{S}^*_\infty[F;X]$ is the Grassmann-graded counterpart of an even DGA $\mathcal{P}^*_\infty$.

In a general setting, let us consider a composite bundle $F \to Y \to X$ where $F \to Y$ is a vector bundle provided with bundle coordinates (x^λ, y^i, q^a). Jet manifolds J^rF of $F \to X$ are vector bundles $J^rF \to J^rY$ coordinated by $(x^\lambda, y^i_\Lambda, q^a_\Lambda)$, $0 \leq |\Lambda| \leq r$. Let $(J^rY, \mathfrak{A}_r)$ be a graded manifold modeled over this vector bundle. Its local basis is $(x^\lambda, y^i_\Lambda, c^a_\Lambda)$, $0 \leq |\Lambda| \leq r$. Let $\mathcal{S}^*_r[F;Y]$ be the DBGA of graded differential forms on the graded manifold $(J^rY, \mathfrak{A}_r)$.

There is an epimorphism of graded manifolds $(J^{r+1}Y, \mathfrak{A}_{r+1}) \to (J^rY, \mathfrak{A}_r)$, seen as local-ringed spaces. It consists of the surjection π^{r+1}_r and the sheaf monomorphism $\pi^{r+1*}_r\mathfrak{A}_r \to \mathfrak{A}_{r+1}$, where $\pi^{r+1*}_r\mathfrak{A}_r$ is the pull-back onto $J^{r+1}Y$ of the topological fiber bundle $\mathfrak{A}_r \to J^rY$. This sheaf monomorphism induces the monomorphism of the canonical presheaves $\overline{\mathfrak{A}}_r \to \overline{\mathfrak{A}}_{r+1}$, which associates to each open subset $U \subset J^{r+1}Y$ the ring of sections of $\mathfrak{A}_r$ over $\pi^{r+1}_r(U)$. Accordingly, there is the monomorphism of graded commutative rings $\mathcal{A}_r \to \mathcal{A}_{r+1}$. By virtue of Lemmas 15 and 16, this monomorphism yields the monomorphism of DBGAs

$$\mathcal{S}^*_r[F;Y] \to \mathcal{S}^*_{r+1}[F;Y]. \tag{49}$$

As a consequence, we have the direct system (2) of DBGAs. Its direct limit $\mathcal{S}^*_\infty[F;Y]$ is a DBGA of all graded differential forms $\phi \in \mathcal{S}^*[F_r; J^rY]$ on graded manifolds $(J^rY, \mathfrak{A}_r)$ modulo monomorphisms (49). Its elements obey the relations (44).

The monomorphisms $\mathcal{O}^*_r \to \mathcal{S}^*_r[F;Y]$ provide a monomorphism of the direct system (4) to the direct system (2) and, consequently, the monomorphism

$$\mathcal{O}^*_\infty Y \to \mathcal{S}^*_\infty[F;Y] \tag{50}$$

of their direct limits. In particular, $\mathcal{S}^*_\infty[F;Y]$ is an $\mathcal{O}^0_\infty Y$-algebra. Accordingly, the body epimorphisms $\mathcal{S}^*_r[F;Y] \to \mathcal{O}^*_r$ yield the epimorphism of $\mathcal{O}^0_\infty$-algebras

$$\mathcal{S}^*_\infty[F;Y] \to \mathcal{O}^*_\infty. \tag{51}$$

The morphisms (50) and (51) are cochain morphisms between the de Rham complex (9) of the DGA $\mathcal{O}^*_\infty$ and the de Rham complex

$$0 \to \mathbb{R} \to \mathcal{S}^0_\infty[F;Y] \xrightarrow{d} \mathcal{S}^1_\infty[F;Y] \cdots \xrightarrow{d} \mathcal{S}^k_\infty[F;Y] \to \cdots \tag{52}$$

of the DBGA $\mathcal{S}^0_\infty[F;Y]$. Moreover, the corresponding homomorphisms of cohomology groups of these complexes are isomorphisms as follows:

Theorem 18. *There is an isomorphism*

$$H^*(\mathcal{S}^*_\infty[F;Y]) = H^*(Y) \tag{53}$$

of cohomology $H^*(\mathcal{S}^*_\infty[F;Y])$ *of the de Rham complex* (52) *to the de Rham cohomology* $H^*_{\mathrm{DR}}(Y)$ *of* Y.

Proof. The complex (52) is the direct limit of the de Rham complexes of the DBGAs $\mathcal{S}^*_r[F;Y]$. Therefore, the direct limit of cohomology groups of these complexes is the cohomology of the de Rham complex (52). By virtue of Theorem 17, cohomology of the de Rham complex of $\mathcal{S}^*_r[F;Y]$ for any r equals the de Rham cohomology of J^rY and, consequently, that of Y, which is the strong deformation retract of any J^rY. Hence, the isomorphism (53) holds. □

It follows that any closed graded differential form $\phi \in \mathcal{S}^*_\infty[F;Y]$ is split into the sum $\phi = d\sigma + d\xi$ of a closed exterior form σ on Y and an exact graded differential form.

One can think of elements of $\mathcal{S}^*_\infty[F;Y]$ as being graded differential forms on the infinite order jet manifold $J^\infty Y$. Indeed, let $\mathfrak{S}^*_r[F;Y]$ be the sheaf of DBGAs on J^rY and $\overline{\mathfrak{S}}^*_r[F;Y]$ its canonical presheaf. Then the above-mentioned presheaf monomorphisms $\overline{\mathfrak{A}}_r \to \overline{\mathfrak{A}}_{r+1}$, yield the direct system of presheaves

$$\overline{\mathfrak{S}}^*[F;Y] \to \overline{\mathfrak{S}}^*_1[F;Y] \to \cdots \overline{\mathfrak{S}}^*_r[F;Y] \to \cdots, \tag{54}$$

whose direct limit $\overline{\mathfrak{S}}^*_\infty[F;Y]$ is a presheaf of DBGAs on the infinite order jet manifold $J^\infty Y$. Let $\mathfrak{T}^*_\infty[F;Y]$ be the sheaf of DBGAs of germs of the presheaf $\overline{\mathfrak{S}}^*_\infty[F;Y]$. One can think of the pair $(J^\infty Y, \mathfrak{T}^0_\infty[F;Y])$ as being a graded manifold, whose body is the infinite order jet manifold $J^\infty Y$ and the structure sheaf $\mathfrak{T}^0_\infty[F;Y]$ is the sheaf of germs of graded functions on graded manifolds $(J^rY, \mathfrak{A}_r)$. We agree to call it the graded infinite order jet manifold. The structure module $\mathcal{Q}^*_\infty[F;Y]$ of sections of $\mathfrak{T}^*_\infty[F;Y]$ is a DBGA such that, given an element $\phi \in \mathcal{Q}^*_\infty[F;Y]$ and a point $z \in J^\infty Y$, there exist an open neighborhood U of z and a graded exterior form $\phi^{(k)}$ on some finite order jet manifold J^kY so that $\phi|_U = \pi^{\infty *}_k \phi^{(k)}|_U$. In particular, there is the monomorphism $\mathcal{S}^*_\infty[F;Y] \to \mathcal{Q}^*_\infty[F;Y]$.

Due to this monomorphism, one can restrict $\mathcal{S}^*_\infty[F;Y]$ to the coordinate chart (7) and say that $\mathcal{S}^*_\infty[F;Y]$ as an $\mathcal{O}^0_\infty Y$-algebra is locally generated by the elements

$$(1, c^a_\Lambda, dx^\lambda, \theta^a_\Lambda = dc^a_\Lambda - c^a_{\lambda+\Lambda}dx^\lambda, \theta^i_\Lambda = dy^i_\Lambda - y^i_{\lambda+\Lambda}dx^\lambda), \quad 0 \le |\Lambda|,$$

where c^a_Λ, θ^a_Λ are odd and dx^λ, θ^i_Λ are even. We agree to call (y^i, c^a) the local basis for $\mathcal{S}^*_\infty[F;Y]$. Let the collective symbol s^A stand for its elements. Accordingly, the notation s^A_Λ and $\theta^A_\Lambda = ds^A_\Lambda - s^A_{\lambda+\Lambda}dx^\lambda$ is introduced. For the sake of simplicity, we further denote $[A] = [s^A]$.

Similarly to $\mathcal{O}^*_\infty$, the DBGA $\mathcal{S}^*_\infty[F;Y]$ is decomposed into $\mathcal{S}^0_\infty[F;Y]$-modules $\mathcal{S}^{k,r}_\infty[F;Y]$ of k-contact and r-horizontal graded forms. Accordingly, the graded exterior differential d on $\mathcal{S}^*_\infty[F;Y]$ falls into the sum $d = d_{\rm H} + d_{\rm V}$ of the total and vertical differentials, where

$$d_{\rm H}(\phi) = dx^\lambda \wedge d_\lambda(\phi), \qquad d_\lambda = \partial_\lambda + \sum_{0\le|\Lambda|} s^A_{\lambda+\Lambda}\partial^\Lambda_A.$$

Let $\mathfrak{d}\mathcal{S}^0_\infty[F;Y]$ be a $\mathcal{S}^0_\infty[F;Y]$-module of graded derivation of the $\mathbb{R}$-ring $\mathcal{S}^0_\infty[F;Y]$. It is a real Lie superalgebra. Similarly to Proposition 5, one can show that the DBGA $\mathcal{S}^*_\infty[F;Y]$ is minimal differential calculus over the graded commutative $\mathbb{R}$-ring $\mathcal{S}^0_\infty[F;Y]$. The interior product $\vartheta\rfloor\phi$ and the Lie derivative $\mathbf{L}_\vartheta\phi$, $\phi \in \mathcal{S}^*_\infty[F;Y]$, $\vartheta \in \mathfrak{d}\mathcal{S}^0_\infty[F;Y]$, obey the relations

$$\vartheta\rfloor(\phi\wedge\sigma) = (\vartheta\rfloor\phi)\wedge\sigma + (-1)^{|\phi|+[\phi][\vartheta]}\phi\wedge(\vartheta\rfloor\sigma), \quad \phi,\sigma \in \mathcal{S}^*_\infty[F;Y]$$
$$\mathbf{L}_\vartheta\phi = \vartheta\rfloor d\phi + d(\vartheta\rfloor\phi), \quad \mathbf{L}_\vartheta(\phi\wedge\sigma) = \mathbf{L}_\vartheta(\phi)\wedge\sigma + (-1)^{[\vartheta][\phi]}\phi\wedge\mathbf{L}_\vartheta(\sigma).$$

A graded derivation $\vartheta \in \mathfrak{d}\mathcal{S}^0_\infty[F;Y]$ is called contact if the Lie derivative $\mathbf{L}_\vartheta$ preserves the ideal of contact graded forms of the DBGA $\mathcal{S}^*_\infty[F;Y]$. With respect to the local basis $(x^\lambda, s^A_\Lambda, dx^\lambda, \theta^A_\Lambda)$ for the DBGA $\mathcal{S}^*_\infty[F;Y]$, any contact graded derivation takes the form

$$\vartheta = \upsilon_{\rm H} + \upsilon_{\rm V} = \vartheta^\lambda d_\lambda + \left[\vartheta^A\partial_A + \sum_{|\Lambda|>0} d_\Lambda(\vartheta^A - s^A_\mu\vartheta^\mu)\partial^\Lambda_A\right], \tag{55}$$

where $\upsilon_{\rm H}$ and $\upsilon_{\rm V}$ denotes its horizontal and vertical parts. Furthermore, one can justify that any vertical contact graded derivation

$$\vartheta = \vartheta^A\partial_A + \sum_{|\Lambda|>0} d_\Lambda\vartheta^A\partial^\Lambda_A \tag{56}$$

satisfies the relations

$$\vartheta\rfloor d_{\rm H}\phi = -d_{\rm H}(\vartheta\rfloor\phi), \quad \mathbf{L}_\vartheta(d_{\rm H}\phi) = d_{\rm H}(\mathbf{L}_\vartheta\phi), \ \ \phi\in\mathcal{S}^*_\infty[F;Y]. \tag{57}$$

9. Grassmann-Graded Variational Bicomplex

Similarly to the DGA $\mathcal{O}^*_\infty$, the DBGA $\mathcal{S}^*_\infty[F;Y]$ is provided with the graded projection endomorphism

$$\varrho = \sum_{k>0}\frac{1}{k}\overline{\varrho}\circ h_k\circ h^n, \qquad \overline{\varrho}(\phi) = \sum_{0\leq|\Lambda|}(-1)^{|\Lambda|}\theta^A\wedge[d_\Lambda(\partial^\Lambda_A\rfloor\phi)], \quad \phi\in\mathcal{S}^{>0,n}_\infty[F;Y],$$

such that $\varrho\circ d_{\rm H} = 0$ and the nilpotent graded variational operator $\delta = \varrho\circ d$. With these operators the bicomplex BGDA $\mathcal{S}^*_\infty[F;Y]$ is completed to the Grassmann-graded variational bicomplex. We restrict our consideration to its short variational subcomplex

$$\begin{gathered} 0\to\mathbb{R}\to\mathcal{S}^0_\infty[F;Y]\xrightarrow{d_{\rm H}}\mathcal{S}^{0,1}_\infty[F;Y]\cdots\xrightarrow{d_{\rm H}}\mathcal{S}^{0,n}_\infty[F;Y]\xrightarrow{\delta}\mathbf{E}_1, \\ \mathbf{E}_1 = \varrho(\mathcal{S}^{1,n}_\infty[F;Y]), \end{gathered} \tag{58}$$

and its subcomplex of one-contact graded forms

$$0\to\mathcal{S}^{1,0}_\infty[F;Y]\xrightarrow{d_{\rm H}}\mathcal{S}^{1,1}_\infty[F;Y]\cdots\xrightarrow{d_{\rm H}}\mathcal{S}^{1,n}_\infty[F;Y]\xrightarrow{\varrho}\mathbf{E}_1\to 0. \tag{59}$$

One can think of its even elements

$$L = \mathcal{L}\omega\in\mathcal{S}^{0,n}_\infty[F;Y], \ \ \delta L = \theta^A\wedge\mathcal{E}_A\omega = \sum_{0\leq|\Lambda|}(-1)^{|\Lambda|}\theta^A\wedge d_\Lambda(\partial^\Lambda_A L)\omega\in\mathbf{E}_1 \tag{60}$$

as being a Grassmann-graded Lagrangian and its Euler–Lagrange operator, respectively.

Theorem 19. *Cohomology of the complex* (58) *equals the de Rham cohomology* $H^*_{\rm DR}(Y)$ *of* Y. *The complex* (59) *is exact.*

The proof of Theorem 19 follows the scheme of the proof of Theorem 8. It falls into three steps.

(i) We start with showing that the complexes (58) and (59) are locally exact.

Lemma 20. *If* $Y = \mathbb{R}^{n+k} \to \mathbb{R}^n$, *the complex* (58) *at all terms, except* $\mathbb{R}$, *is exact.*

Proof. Referring to [3, 10] for the proof, we summarize a few formulas. Any horizontal graded form $\phi \in \mathcal{S}^{0,*}_\infty$ admits the decomposition

$$\phi = \phi_0 + \widetilde{\phi}, \quad \widetilde{\phi} = \int_0^1 \frac{d\lambda}{\lambda} \sum_{0\leq|\Lambda|} s^A_\Lambda \partial^\Lambda_A \phi, \tag{61}$$

where ϕ_0 is an exterior form on $\mathbb{R}^{n+k}$. Let $\phi \in \mathcal{S}^{0,m<n}_\infty$ be d_{H}-closed. Then its component ϕ_0 (61) is an exact exterior form on $\mathbb{R}^{n+k}$ and $\widetilde{\phi} = d_{\mathrm{H}}\xi$, where ξ is given by the following expressions. Let us introduce the operator

$$D^{+\nu}\widetilde{\phi} = \int_0^1 \frac{d\lambda}{\lambda} \sum_{0\leq k} k\delta^\nu_{(\mu_1} \delta^{\alpha_1}_{\mu_2} \cdots \delta^{\alpha_{k-1}}_{\mu_k)} \lambda s^A_{(\alpha_1\ldots\alpha_{k-1})} \partial^{\mu_1\ldots\mu_k}_A \widetilde{\phi}(x^\mu, \lambda s^A_\Lambda, dx^\mu). \tag{62}$$

The relation $[D^{+\nu}, d_\mu]\widetilde{\phi} = \delta^\nu_\mu \widetilde{\phi}$ holds, and leads to the desired expression

$$\xi = \sum_{k=0} \frac{(n-m-1)!}{(n-m+k)!} D^{+\nu} P_k \partial_\nu \rfloor \widetilde{\phi}, \quad P_0 = 1, \;\; P_k = d_{\nu_1} \cdots d_{\nu_k} D^{+\nu_1} \cdots D^{+\nu_k}. \tag{63}$$

Now, let $\phi \in \mathcal{S}^{0,n}_\infty$ be a graded density such that $\delta\phi = 0$. Then its component ϕ_0 (61) is an exact n-form on $\mathbb{R}^{n+k}$ and $\widetilde{\phi} = d_{\mathrm{H}}\xi$, where ξ is given by the expression

$$\xi = \sum_{|\Lambda|\geq 0} \sum_{\Sigma+\Xi=\Lambda} (-1)^{|\Sigma|} s^A_\Xi d_\Sigma \partial^{\mu+\Lambda}_A \widetilde{\phi} \omega_\mu. \tag{64}$$

Since elements of $\mathcal{S}^*_\infty$ are polynomials in s^A_Λ, the sum in the expression (63) is finite. However, the expression (63) contains a d_{H}-exact summand which prevents its extension to $\mathcal{O}^*_\infty$. In this respect, we also quote the homotopy operator (5.107) in [25] which leads to the expression

$$\xi = \int_0^1 I(\phi)(x^\mu, \lambda s^A_\Lambda, dx^\mu) \frac{d\lambda}{\lambda}, \tag{65}$$

$$I(\phi) = \sum_{0\leq|\Lambda|} \sum_\mu \frac{\Lambda_\mu + 1}{n-m+|\Lambda|+1} d_\Lambda \left[\sum_{0\leq|\Xi|} (-1)^\Xi \frac{(\mu+\Lambda+\Xi)!}{(\mu+\Lambda)!\Xi!} s^A d_\Xi \partial^{\mu+\Lambda+\Xi}_A (\partial_\mu \rfloor \phi) \right],$$

where $\Lambda! = \Lambda_{\mu_1}! \cdots \Lambda_{\mu_n}!$ and Λ_μ denotes the number of occurrences of the index μ in Λ [25]. The graded forms (64) and (65) differ in a d_{H}-exact graded form. □

Lemma 21. *If* $Y = \mathbb{R}^{n+k} \to \mathbb{R}^n$, *the complex* (59) *is exact.*

Proof. The fact that a $d_{\rm H}$-closed graded $(1,m)$-form $\phi\in\mathcal{S}^{1,m<n}_\infty$ is $d_{\rm H}$-exact is derived from Lemma 20 as follows. We write

$$\phi=\sum\phi^\Lambda_A\wedge\theta^A_\Lambda, \tag{66}$$

where $\phi^\Lambda_A\in\mathcal{S}^{0,m}_\infty$ are horizontal graded m-forms. Let us introduce additional variables $\overline{s}^A_\Lambda$ of the same Grassmann parity as s^A_Λ. Then one can associate to each graded $(1,m)$-form ϕ (66) a unique horizontal graded m-form

$$\overline{\phi}=\sum\phi^\Lambda_A\overline{s}^A_\Lambda, \tag{67}$$

whose coefficients are linear in the variables $\overline{s}^A_\Lambda$, and *vice versa*. Let us consider the modified total differential

$$\overline{d}_{\rm H}=d_{\rm H}+dx^\lambda\wedge\sum_{0<|\Lambda|}\overline{s}^A_{\lambda+\Lambda}\overline{\partial}^\Lambda_A,$$

acting on graded forms (67), where $\overline{\partial}^\Lambda_A$ is the dual of $d\overline{s}^A_\Lambda$. Comparing the equality $\overline{d}_{\rm H}\overline{s}^A_\Lambda=dx^\lambda s^A_{\lambda+\Lambda}$ and the last equality (18), one can easily justify that $\overline{d_{\rm H}\phi}=\overline{d}_{\rm H}\overline{\phi}$. Let a graded $(1,m)$-form ϕ (66) be $d_{\rm H}$-closed. Then the associated horizontal graded m-form $\overline{\phi}$ (67) is $\overline{d}_{\rm H}$-closed and, by virtue of Lemma 20, it is $\overline{d}_{\rm H}$-exact, i.e. $\overline{\phi}=\overline{d}_{\rm H}\overline{\xi}$, where $\overline{\xi}$ is a horizontal graded $(m-1)$-form given by the expression (63) depending on additional variables $\overline{s}^A_\Lambda$. A glance at this expression shows that, since $\overline{\phi}$ is linear in the variables $\overline{s}^A_\Lambda$, so is $\overline{\xi}=\sum\xi^\Lambda_A\overline{s}^A_\Lambda$. It follows that $\phi=d_{\rm H}\xi$ where $\xi=\sum\xi^\Lambda_A\wedge\theta^A_\Lambda$. It remains to prove the exactness of the complex (59) at the last term $\mathbf{E}_1$. If

$$\varrho(\sigma)=\sum_{0\leq|\Lambda|}(-1)^{|\Lambda|}\theta^A\wedge[d_\Lambda(\partial^\Lambda_A\rfloor\sigma)]=\sum_{0\leq|\Lambda|}(-1)^{|\Lambda|}\theta^A\wedge[d_\Lambda\sigma^\Lambda_A]\omega=0,\quad \sigma\in\mathcal{S}^{1,n}_\infty,$$

a direct computation gives

$$\sigma=d_{\rm H}\xi,\quad \xi=-\sum_{0\leq|\Lambda|}\sum_{\Sigma+\Xi=\Lambda}(-1)^{|\Sigma|}\theta^A_\Xi\wedge d_\Sigma\sigma^{\mu+\Lambda}_A\omega_\mu. \tag{68}$$

□

Remark 3. The proof of Lemma 21 fails to be extended to complexes of higher contact forms because the products $\theta^A_\Lambda\wedge\theta^B_\Sigma$ and $s^A_\Lambda s^B_\Sigma$ obey different commutation rules.

(i) Let us now prove Theorem 19 for the DBGA $\mathcal{Q}^*_\infty[F;Y]$. Similarly to $\mathcal{S}^*_\infty[F;Y]$, the sheaf $\mathfrak{T}^*_\infty[F;Y]$ and the DBGA $\mathcal{Q}^*_\infty[F;Y]$ are split into the Grassmann-graded variational bicomplexes. We consider their subcomplexes

$$0\to\mathbb{R}\to\mathfrak{T}^0_\infty[F;Y]\xrightarrow{d_{\rm H}}\mathfrak{T}^{0,1}_\infty[F;Y]\cdots\xrightarrow{d_{\rm H}}\mathfrak{T}^{0,n}_\infty[F;Y]\xrightarrow{\delta}\mathfrak{E}_1, \tag{69}$$

$$0\to\mathfrak{T}^{1,0}_\infty[F;Y]\xrightarrow{d_{\rm H}}\mathfrak{T}^{1,1}_\infty[F;Y]\cdots\xrightarrow{d_{\rm H}}\mathfrak{T}^{1,n}_\infty[F;Y]\xrightarrow{\varrho}\mathfrak{E}_1\to 0, \tag{70}$$

$$0\to\mathbb{R}\to\mathcal{Q}^0_\infty[F;Y]\xrightarrow{d_{\rm H}}\mathcal{Q}^{0,1}_\infty[F;Y]\cdots\xrightarrow{d_{\rm H}}\mathcal{Q}^{0,n}_\infty[F;Y]\xrightarrow{\delta}\Gamma(\mathfrak{E}_1), \tag{71}$$

$$0\to\mathcal{Q}^{1,0}_\infty[F;Y]\xrightarrow{d_{\rm H}}\mathcal{Q}^{1,1}_\infty[F;Y]\cdots\xrightarrow{d_{\rm H}}\mathcal{Q}^{1,n}_\infty[F;Y]\xrightarrow{\varrho}\Gamma(\mathfrak{E}_1)\to 0, \tag{72}$$

where $\mathfrak{E}_1 = \varrho(\mathfrak{T}^{1,n}_\infty[F;Y])$. By virtue of Lemmas 20 and 21, the complexes (69) and (70) at all terms, except $\mathbb{R}$, are exact. The terms $\mathfrak{T}^{*,*}_\infty[F;Y]$ of the complexes (69) and (70) are sheaves of $\mathcal{Q}^0_\infty$-modules. Since $J^\infty Y$ admits the partition of unity just by elements of $\mathcal{Q}^0_\infty$, these sheaves are fine and, consequently, acyclic. By virtue of the abstract de Rham theorem (see Appendix B), cohomology of the complex (71) equals the cohomology of $J^\infty Y$ with coefficients in the constant sheaf $\mathbb{R}$ and, consequently, the de Rham cohomology of Y in accordance with isomorphisms (8). Similarly, the complex (72) is proved to be exact.

(ii) It remains to prove that cohomology of the complexes (58) and (59) equals that of the complexes (71) and (72). The proof of this fact straightforwardly follows the proof of Theorem 8, and it is a slight modification of the proof of [14], Theorem 4.1, where graded exterior forms on the infinite order jet manifold $J^\infty Y$ of an affine bundle are treated as those on X.

Proposition 22. *Every $d_{\rm H}$-closed graded form $\phi \in \mathcal{S}^{0,m<n}_\infty[F;Y]$ falls into the sum*

$$\phi = h_0\sigma + d_{\rm H}\xi, \quad \xi \in \mathcal{S}^{0,m-1}_\infty[F;Y], \tag{73}$$

where σ is a closed m-form on Y. Any δ-closed graded density (e.g. a variationally trivial Grassmann-graded Lagrangian) $L \in \mathcal{S}^{0,n}_\infty[F;Y]$ is the sum

$$L = h_0\sigma + d_{\rm H}\xi, \quad \xi \in \mathcal{S}^{0,n-1}_\infty[F;Y], \tag{74}$$

where σ is a closed n-form on Y. In particular, an odd δ-closed graded density is always $d_{\rm H}$-exact.

Proof. The complex (58) possesses the same cohomology as the short variational complex

$$0 \to \mathbb{R} \to \mathcal{O}^0_\infty \xrightarrow{d_{\rm H}} \mathcal{O}^{0,1}_\infty \cdots \xrightarrow{d_{\rm H}} \mathcal{O}^{0,n}_\infty \xrightarrow{\delta} \mathbf{E}_1 \tag{75}$$

of the DGA $\mathcal{O}^*_\infty$. The monomorphism (50) and the body epimorphism (51) yield the corresponding cochain morphisms of the complexes (58) and (75). Therefore, cohomology of the complex (58) is the image of the cohomology of $\mathcal{O}^*_\infty$. □

The global exactness of the complex (59) at the term $\mathcal{S}^{1,n}_\infty[F;Y]$ results in the following [14].

Proposition 23. *Given a Grassmann-graded Lagrangian $L = \mathcal{L}\omega$, there is the decomposition*

$$dL = \delta L - d_{\rm H}\Xi, \quad \Xi \in \mathcal{S}^{1,n-1}_\infty[F;Y], \tag{76}$$

$$\Xi = \sum_{s=0} \theta^A_{\nu_s\ldots\nu_1} \wedge F^{\lambda\nu_s\ldots\nu_1}_A \omega_\lambda, \quad F^{\nu_k\ldots\nu_1}_A = \partial^{\nu_k\ldots\nu_1}_A \mathcal{L} - d_\lambda F^{\lambda\nu_k\ldots\nu_1}_A + h^{\nu_k\ldots\nu_1}_A, \tag{77}$$

where local graded functions h obey the relations $h^\nu_a = 0$, $h^{(\nu_k\nu_{k-1})\ldots\nu_1}_a = 0$.

Note that, locally, one can always choose Ξ (77) where all functions h vanish.

The decomposition (76) leads to the global first variational formula for Grassmann-graded Lagrangians as follows [5, 14].

Proposition 24. *Let $\vartheta \in \mathfrak{d}\mathcal{S}^0_\infty[F;Y]$ be a contact graded derivation* (55) *of the $\mathbb{R}$-ring $\mathcal{S}^0_\infty[F;Y]$. Then the Lie derivative $\mathbf{L}_\vartheta L$ of a Lagrangian L fulfills the first variational formula*

$$\mathbf{L}_\vartheta L = \vartheta_{\mathrm{V}} \rfloor \delta L + d_{\mathrm{H}}(h_0(\vartheta\rfloor \Xi_{\mathrm{L}})) + d_{\mathrm{V}}(\vartheta_{\mathrm{H}}\rfloor\omega)\mathcal{L}, \tag{78}$$

where $\Xi_{\mathrm{L}} = \Xi + L$ is a Lepagean equivalent of L given by the coordinate expression (77).

Proof. The proof follows from the splitting (76) similarly to the proof of Proposition 12. □

A contact graded derivation ϑ (55) is called a variational symmetry of a Lagrangian L if the Lie derivative $\mathbf{L}_\vartheta L = d_{\mathrm{H}}\xi$ is d_{H}-exact. A glance at the expression (78) shows that: (i) a contact graded derivation ϑ is a variational symmetry only if it is projected onto X, (ii) ϑ is a variational symmetry iff its vertical part υ_{V} is well, (iii) it is a variational symmetry iff the density $\upsilon_{\mathrm{V}}\rfloor\delta L$ is d_{H}-exact.

Theorem 25. *If a contact graded derivation ϑ* (55) *is a variational symmetry of a Lagrangian L, the first variational formula* (34) *restricted to* $\mathrm{Ker}\,\delta L$ *leads to the weak conservation law*

$$0 \approx d_{\mathrm{H}}(h_0(\vartheta\rfloor\Xi_{\mathrm{L}}) - \xi).$$

Remark 4. If $Y \to X$ is an affine bundle, one can consider the subalgebra $\mathcal{P}[F;Y] \subset \mathcal{S}[F;Y]$ of graded differential forms whose coefficients are polynomials in fiber coordinates of $Y \to X$ and their jets. This subalgebra is also split into the Grassmann-graded variational bicomplex. One can show that, the cohomology of its short variational subcomplex as like as that of the complex (40) equals the de Rham cohomology of X.

10. Appendixes

10.1. *Appendix A*

To show that Y is a strong deformation retract of $J^\infty Y$, let us construct a homotopy from $J^\infty Y$ to Y in an explicit form. Let $\gamma_{(k)}$, $k \leq 1$, be a global sections of the affine jet bundles $J^kY \to J^{k-1}Y$. Then, we have a global section

$$\gamma : Y \ni (x^\lambda, y^i) \to (x^\lambda, y^i, y^i_\Lambda = \gamma_{(|\Lambda|)}{}^i_\Lambda \circ \gamma_{(|\Lambda|-1)} \circ \cdots \circ \gamma_{(1)}) \in J^\infty Y \tag{79}$$

of the open surjection $\pi^\infty_0 : J^\infty Y \to Y$. Let us consider the map

$$\begin{aligned} &[0,1]\times J^\infty Y \ni (t;x^\lambda,y^i,y^i_\Lambda) \to (x^\lambda,y^i,y'^i_\Lambda)\in J^\infty Y, \quad 0<|\Lambda|, \\ &y'^i_\Lambda = f_k(t)y^i_\Lambda + (1-f_k(t))\gamma_{(k)}{}^i_\Lambda(x^\lambda,y^i,y^i_\Sigma), \quad |\Sigma|<k=|\Lambda|, \end{aligned} \tag{80}$$

where $f_k(t)$ is a continuous monotone real function on $[0,1]$ such that

$$f_k(t) = \begin{cases} 0 & t\leq 1-2^{-k}, \\ 1 & t\geq 1-2^{-(k+1)}. \end{cases} \tag{81}$$

A glance at the transition functions (7) shows that, although written in a coordinate form, this map is globally defined. It is continuous because, given an open subset $U_k \subset J^kY$, the inverse image of the open set $(\pi^\infty_k)^{-1}(U_k)\subset J^\infty Y$, is the open subset

$$\begin{aligned} &(t_k,1]\times(\pi^\infty_k)^{-1}(U_k)\cup(t_{k-1},1]\times(\pi^\infty_{k-1})^{-1}(\pi^k_{k-1}[U_k\cap\gamma_{(k)}(J^{k-1}Y)])\cup\cdots \\ &\quad \cup[0,1]\times(\pi^\infty_0)^{-1}(\pi^k_0[U_k\cap\gamma_{(k)}\circ\cdots\circ\gamma_{(1)}(Y)]) \end{aligned}$$

of $[0,1]\times J^\infty Y$, where $[t_r,1]=\operatorname{supp} f_r$. Then, the map (80) is a desired homotopy from $J^\infty Y$ to Y which is identified with its image under the global section (79).

10.2. *Appendix B*

We quote the following minor generalization of the abstract de Rham theorem ([20], Theorem 2.12.1) [13, 28]. Let

$$0\to S \xrightarrow{h} S_0 \xrightarrow{h^0} S_1 \xrightarrow{h^1}\cdots\xrightarrow{h^{p-1}} S_p \xrightarrow{h^p} S_{p+1}, \quad p>1,$$

be an exact sequence of sheaves of Abelian groups over a paracompact topological space Z, where the sheaves S_q, $0\leq q<p$, are acyclic, and let

$$0\to\Gamma(Z,S)\xrightarrow{h_*}\Gamma(Z,S_0)\xrightarrow{h^0_*}\Gamma(Z,S_1)\xrightarrow{h^1_*}\cdots\xrightarrow{h^{p-1}_*}\Gamma(Z,S_p)\xrightarrow{h^p_*}\Gamma(Z,S_{p+1}) \tag{82}$$

be the corresponding cochain complex of sections of these sheaves.

Theorem 26. *The q-cohomology groups of the cochain complex* (82) *for* $0\leq q\leq p$ *are isomorphic to the cohomology groups* $H^q(Z,S)$ *of* Z *with coefficients in the sheaf* S.

10.3. *Appendix C*

The sheaves $\mathfrak{E}_k$ in proof of Theorem 8 are fine as follows [13]. Though the $\mathbb{R}$-modules $\Gamma(\mathfrak{E}_{k>1})$ fail to be $\mathcal{Q}^0_\infty$-modules [29], one can use the fact that the sheaves $\mathfrak{E}_{k>0}$ are projections $\varrho(\mathfrak{T}^{k,n}_\infty)$ of sheaves of $\mathcal{Q}^0_\infty$-modules. Let $\{U_i\}_{i\in I}$ be a locally finite open covering of $J^\infty Y$ and $\{f_i\in\mathcal{Q}^0_\infty\}$ the associated partition of unity. For any open subset $U \subset J^\infty Y$ and any section φ of the sheaf $\mathfrak{T}^{k,n}_\infty$ over U, let us put $h_i(\varphi) = f_i\varphi$. The endomorphisms h_i of $\mathfrak{T}^{k,n}_\infty$ yield the $\mathbb{R}$-module endomorphisms

$$\overline{h}_i = \varrho \circ h_i : \mathfrak{E}_k \xrightarrow{\text{in}} \mathfrak{T}^{k,n}_\infty \xrightarrow{h_i} \mathfrak{T}^{k,n}_\infty \xrightarrow{\varrho} \mathfrak{E}_k$$

of the sheaves $\mathfrak{E}_k$. They possess the properties required for $\mathfrak{E}_k$ to be a fine sheaf. Indeed, for each $i \in I$, supp $f_i \subset U_i$ provides a closed set such that $\overline{h}_i$ is zero outside this set, while the sum $\sum_{i\in I} \overline{h}_i$ is the identity morphism.

10.4. *Appendix D*

Let the common symbol D stand for $d_{\rm H}$ and δ. Bearing in mind decompositions (28)–(31), it suffices to show that if an element $\phi \in \mathcal{O}^*_\infty$ is D-exact in the algebra $\mathcal{Q}^*_\infty$, then it is so in the algebra $\mathcal{O}^*_\infty$. Lemma 7 states that, if Y is a contractible bundle and a D-exact form ϕ on $J^\infty Y$ is of finite jet order $[\phi]$ (i.e. $\phi \in \mathcal{O}^*_\infty$), there exists a differential form $\varphi \in \mathcal{O}^*_\infty$ on $J^\infty Y$ such that $\phi = D\varphi$. Moreover, a glance at the homotopy operators for $d_{\rm H}$ and δ shows that the jet order $[\varphi]$ of φ is bounded by an integer $N([\phi])$, depending only on the jet order of ϕ. Let us call this fact the finite exactness of the operator D. Given an arbitrary bundle Y, the finite exactness takes place on $J^\infty Y|_U$ over any domain $U \subset Y$. Let us prove the following.

(i) Given a family $\{U_\alpha\}$ of disjoint open subsets of Y, let us suppose that the finite exactness takes place on $J^\infty Y|_{U_\alpha}$ over every subset U_α from this family. Then, it is true on $J^\infty Y$ over the union $\underset{\alpha}{\cup} U_\alpha$ of these subsets.

(ii) Suppose that the finite exactness of the operator D takes place on $J^\infty Y$ over open subsets U, V of Y and their non-empty overlap $U \cap V$. Then, it is also true on $J^\infty Y|_{U\cup V}$.

Proof of (i). Let $\phi \in \mathcal{O}^*_\infty$ be a D-exact form on $J^\infty Y$. The finite exactness on $(\pi^\infty_0)^{-1}(\cup U_\alpha)$ holds since $\phi = D\varphi_\alpha$ on every $(\pi^\infty_0)^{-1}(U_\alpha)$ and $[\varphi_\alpha] < N([\phi])$.

Proof of (ii). Let $\phi = D\varphi \in \mathcal{O}^*_\infty$ be a D-exact form on $J^\infty Y$. By assumption, it can be brought into the form $D\varphi_U$ on $(\pi^\infty_0)^{-1}(U)$ and $D\varphi_{\rm V}$ on $(\pi^\infty_0)^{-1}(V)$, where φ_U and $\varphi_{\rm V}$ are differential forms of bounded jet order. Let us consider their difference $\varphi_U - \varphi_{\rm V}$ on $(\pi^\infty_0)^{-1}(U \cap V)$. It is a D-exact form of bounded jet order $[\varphi_U - \varphi_{\rm V}] < N([\phi])$ which, by assumption, can be written as $\varphi_U - \varphi_{\rm V} = D\sigma$, where σ is also of bounded jet order $[\sigma] < N(N([\phi]))$. Lemma 27 below shows that $\sigma = \sigma_U + \sigma_{\rm V}$, where σ_U and $\sigma_{\rm V}$ are differential forms of bounded jet order on $(\pi^\infty_0)^{-1}(U)$ and $(\pi^\infty_0)^{-1}(V)$, respectively. Then, putting

$$\varphi'|_U = \varphi_U - D\sigma_U, \quad \varphi'|_{\rm V} = \varphi_{\rm V} + D\sigma_{\rm V},$$

we have the form ϕ, equal to $D\varphi'_U$ on $(\pi^\infty_0)^{-1}(U)$ and $D\varphi'_{\rm V}$ on $(\pi^\infty_0)^{-1}(V)$, respectively. Since the difference $\varphi'_U - \varphi'_{\rm V}$ on $(\pi^\infty_0)^{-1}(U \cap V)$ vanishes, we obtain $\phi = D\varphi'$

on $(\pi_0^\infty)^{-1}(U \cup V)$ where

$$\varphi' \stackrel{\text{def}}{=} \begin{cases} \varphi'|_U = \varphi'_U, \\ \varphi'|_{\mathrm{V}} = \varphi'_{\mathrm{V}} \end{cases}$$

is of bounded jet order $[\varphi'] < N(N([\phi]))$.

To prove the finite exactness of D on $J^\infty Y$, it remains to choose an appropriate cover of Y. A smooth manifold Y admits a countable cover $\{U_\xi\}$ by domains U_ξ, $\xi \in \mathbf{N}$, and its refinement $\{U_{ij}\}$, where $j \in \mathbf{N}$ and i runs through a finite set, such that $U_{ij} \cap U_{ik} = \emptyset$, $j \neq k$ [18]. Then Y has a finite cover $\{U_i = \cup_j U_{ij}\}$. Since the finite exactness of the operator D takes place over any domain U_ξ, it also holds over any member U_{ij} of the refinement $\{U_{ij}\}$ of $\{U_\xi\}$ and, in accordance with item (i) above, over any member of the finite cover $\{U_i\}$ of Y. Then by virtue of item (ii) above, the finite exactness of D takes place over Y.

Lemma 27. *Let U and V be open subsets of a bundle Y and $\sigma \in \mathfrak{O}^*_\infty$ a differential form of bounded jet order on $(\pi_0^\infty)^{-1}(U \cap V) \subset J^\infty Y$. Then, σ is split into a sum $\sigma_U + \sigma_{\mathrm{V}}$ of differential forms σ_U and σ_{V} of bounded jet order on $(\pi_0^\infty)^{-1}(U)$ and $(\pi_0^\infty)^{-1}(V)$, respectively.*

Proof. By taking a smooth partition of unity on $U \cup V$ subordinate to the cover $\{U, V\}$ and passing to the function with support in V, one gets a smooth real function f on $U \cup V$ which is 0 on a neighborhood of $U - V$ and 1 on a neighborhood of $V - U$ in $U \cup V$. Let $(\pi_0^\infty)^* f$ be the pull-back of f onto $(\pi_0^\infty)^{-1}(U \cup V)$. The differential form $((\pi_0^\infty)^* f)\sigma$ is 0 on a neighborhood of $(\pi_0^\infty)^{-1}(U)$ and, therefore, can be extended by 0 to $(\pi_0^\infty)^{-1}(U)$. Let us denote it σ_U. Accordingly, the differential form $(1 - (\pi_0^\infty)^* f)\sigma$ has an extension σ_{V} by 0 to $(\pi_0^\infty)^{-1}(V)$. Then, $\sigma = \sigma_U + \sigma_{\mathrm{V}}$ is a desired decomposition because σ_U and σ_{V} are of the jet order which does not exceed that of σ. □

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