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GRADED LAGRANGIAN FORMALISM

G. SARDANASHVILY
Department of Theoretical Physics
Moscow State University
117234 Moscow, Russia

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Graded Lagrangian formalism in terms of a Grassmann-graded variational bicomplex on graded manifolds is developed in a very general setting. This formalism provides the comprehensive description of reducible degenerate Lagrangian systems, characterized by hierarchies of non-trivial higher-order Noether identities and gauge symmetries. This is a general case of classical field theory and Lagrangian non-relativistic mechanics.

Keywords: Lagrangian formalism; variational bicomplex; graded manifold; Noether identities; gauge symmetries.

1. Introduction

Conventional Lagrangian formalism on fiber bundles $Y \to X$ over a smooth manifold X is formulated in algebraic terms of a variational bicomplex of exterior forms on jet manifolds of sections of $Y \to X$ [2, 9, 16, 17, 19, 30, 36, 37]. The cohomology of this bicomplex provides the global first variational formula for Lagrangians and Euler–Lagrange operators, without appealing to the calculus of variations. For instance, this is the case of classical field theory if $\dim X > 1$ and non-autonomous mechanics if $X = \mathbb{R}$ [19, 20, 35].

However, this formalism is not sufficient in order to describe reducible degenerate Lagrangian systems whose degeneracy is characterized by a hierarchy of higher-order Noether identities. They constitute the Koszul–Tate chain complex whose cycles are Grassmann-graded elements of certain graded manifolds [7, 8, 19]. Moreover, many field models also deal with Grassmann-graded fields, e.g. fermion fields, antifields and ghosts [19, 21, 35].

These facts motivate us to develop graded Lagrangian formalism of even and odd variables [8, 17, 19, 34].

Different geometric models of odd variables are described either on graded manifolds or supermanifolds. Both graded manifolds and supermanifolds are phrased in terms of sheaves of graded commutative algebras [5, 19]. However, graded manifolds are characterized by sheaves on smooth manifolds, while supermanifolds are constructed by gluing of sheaves on supervector spaces. Treating odd variables on a smooth manifold X, we follow the Serre–Swan theorem generalized to graded manifolds (Theorem 7). It states that, if a graded commutative $C^\infty(X)$-ring is generated by a projective $C^\infty(X)$-module of finite rank, it is isomorphic to a ring of graded functions on a graded manifold whose body is X. In accordance with this theorem, we describe odd variables in terms of graded manifolds [8, 17, 19, 34].

We consider a generic Lagrangian theory of even and odd variables on an n-dimensional smooth real manifold X. It is phrased in terms of the Grassmann-graded variational bicomplex (28) [4, 7, 8, 17, 19, 34]. Graded Lagrangians L and Euler–Lagrange operators δL are defined as elements of terms $\mathcal{S}^{0,n}_\infty[F;Y]$ and $\varrho(\mathcal{S}^{1,n}_\infty[F;Y])$ of this bicomplex, respectively. Cohomology of the Grassmann-graded variational bicomplex (28) (Theorems 13 and 14) defines a class of variationally trivial graded Lagrangians (Theorem 15) and results in the global decomposition (33) of dL (Theorem 16), the first variational formula (37) and the first Noether Theorem 20.

A problem is that any Euler–Lagrange operator satisfies Noether identities, which therefore must be separated into the trivial and non-trivial ones. These Noether identities obey first-stage Noether identities, which in turn are subject to the second-stage ones, and so on. Thus, there is a hierarchy of higher-stage Noether identities. In accordance with general analysis of Noether identities of differential operators [33], if certain conditions hold, one can associate to a graded Lagrangian system the exact antifield Koszul–Tate complex (62) possessing the boundary operator (60) whose nilpotentness is equivalent to all non-trivial Noether and higher-stage Noether identities [7, 8, 18].

It should be noted that the notion of higher-stage Noether identities has come from that of reducible constraints. The Koszul–Tate complex of Noether identities has been invented similarly to that of constraints under the condition that Noether identities are locally separated into independent and dependent ones [4, 13]. This condition is relevant for constraints, defined by a finite set of functions which the inverse mapping theorem is applied to. However, Noether identities unlike constraints are differential equations. They are given by an infinite set of functions on a Fréchet manifold of infinite-order jets where the inverse mapping theorem fails to be valid. Therefore, the regularity condition for the Koszul–Tate complex of constraints is replaced with homology regularity, Condition 27, in order to construct the Koszul–Tate complex (62) of Noether identities.

The second Noether theorems (Theorems 32–34) is formulated in homology terms, and it associates to this Koszul–Tate complex the cochain sequence of ghosts (71) with the ascent operator (72) whose components are non-trivial gauge and higher-stage gauge symmetries of Lagrangian theory.

2. Variational Bicomplex on Fiber Bundles

Given a smooth fiber bundle $Y \to X$, the jet manifolds J^rY of its sections provide the conventional language of theory of differential equations and differential operators on $Y \to X$ [12, 26]. Though we restrict our consideration to finite-order Lagrangian formalism, it is conveniently formulated on an infinite-order jet manifold $J^\infty Y$ of Y in terms of the above-mentioned variational bicomplex of differential forms on $J^\infty Y$. However, different variants of a variational sequence of finite jet order on jet manifolds J^rY are also considered [1, 27, 38].

Remark 1. Smooth manifolds throughout are assumed to be Hausdorff, second-countable and, consequently, paracompact and locally compact, countable at infinity. It is essential that a paracompact smooth manifold admits the partition of unity by smooth functions. Given a manifold X, its tangent and cotangent bundles TX and T^*X are endowed with bundle coordinates $(x^\lambda, \dot{x}^\lambda)$ and $(x^\lambda, \dot{x}_\lambda)$ with respect to holonomic frames $\{\partial_\lambda\}$ and $\{dx^\lambda\}$, respectively. By $\Lambda = (\lambda_1 \cdots \lambda_k)$, $|\Lambda| = k$, $\lambda + \Lambda = (\lambda\lambda_1 \cdots \lambda_k)$, are denoted symmetric multi-indices. Summation over a multi-index Λ means separate summation over each index λ_i.

Let $Y \to X$ be a fiber bundle provided with bundle coordinates (x^λ, y^i). An r-order jet manifold J^rY of its sections is provided with the adapted coordinates $(x^\lambda, y^i, y^i_\Lambda)_{|\Lambda| \leq r}$. These jet manifolds form an inverse system

$$Y \xleftarrow{\pi} J^1Y \leftarrow \cdots J^{r-1}Y \xleftarrow{\pi^r_{r-1}} J^rY \leftarrow \cdots, \tag{1}$$

where π^r_{r-1}, $r > 0$, are affine bundles. Its projective limit $J^\infty Y$ is defined as a minimal set such that there exist surjections

$$\pi^\infty : J^\infty Y \to X, \quad \pi^\infty_0 : J^\infty Y \to Y, \quad \pi^\infty_k : J^\infty Y \to J^kY, \tag{2}$$

obeying the relations $\pi^\infty_r = \pi^k_r \circ \pi^\infty_k$ for all admissible k and $r < k$. One can think of elements of $J^\infty Y$ as being infinite-order jets of sections of $Y \to X$.

A set $J^\infty Y$ is provided with the coarsest topology such that the surjections π^∞_r (2) are continuous. Its base consists of inverse images of open subsets of J^rY, $r = 0, \ldots$, under the maps π^∞_r. With this topology, $J^\infty Y$ is a paracompact Fréchet (complete metrizable) manifold [17, 19, 37]. It is called the infinite-order jet manifold. One can show that surjections π^∞_r are open maps admitting local sections, i.e. $J^\infty Y \to J^rY$ are continuous bundles. A bundle coordinate atlas $\{U_Y, (x^\lambda, y^i)\}$ of $Y \to X$ provides $J^\infty Y$ with a manifold coordinate atlas

$$\{(\pi^\infty_0)^{-1}(U_Y), (x^\lambda, y^i_\Lambda)\}_{0\leq|\Lambda|}, \quad {y'}^i_{\lambda+\Lambda} = \frac{\partial x^\mu}{\partial x'^\lambda} d_\mu y'^i_\Lambda, \quad d_\lambda = \partial_\lambda + \sum_{0\leq|\Lambda|} y^i_{\lambda+\Lambda}\partial^\Lambda_i. \tag{3}$$

Theorem 2. *A fiber bundle Y is a strong deformation retract of an infinite-order jet manifold $J^\infty Y$ [2, 16, 19].*

Corollary 3. *By virtue of the well-known Vietoris–Begle theorem* [11], *there is an isomorphism*

$$H^*(J^\infty Y;\mathbb{R}) = H^*(Y;\mathbb{R}) \tag{4}$$

between the cohomology of $J^\infty Y$ *with coefficients in the constant sheaf* $\mathbb{R}$ *and that of* Y.

The inverse sequence (1) of jet manifolds yields a direct sequence

$$\mathcal{O}^*(X) \xrightarrow{\pi^*} \mathcal{O}^*(Y)p \xrightarrow{\pi_0^{1*}} \mathcal{O}_1^* \to \cdots \mathcal{O}_{r-1}^* \xrightarrow{\pi^r_{r-1}{}^*} \mathcal{O}_r^* \to \cdots \tag{5}$$

of differential graded algebras (henceforth DGAs) $\mathcal{O}^*(X)$, $\mathcal{O}^*(Y)$, $\mathcal{O}_r^* = \mathcal{O}^*(J^rY)$ of exterior forms on X, Y and jet manifolds J^rY, where $\pi^r_{r-1}{}^*$ are the pull-back monomorphisms. Its direct limit $\mathcal{O}_\infty^*$ consists of all exterior forms on finite-order jet manifolds modulo the pull-back identification. It is a DGA which inherits operations of an exterior differential d and an exterior product $\wedge$ of DGAs $\mathcal{O}_r^*$.

Theorem 4. *The cohomology* $H^*(\mathcal{O}_\infty^*)$ *of the de Rham complex*

$$0 \to \mathbb{R} \to \mathcal{O}_\infty^0 \xrightarrow{d} \mathcal{O}_\infty^1 \xrightarrow{d} \cdots \tag{6}$$

of a DGA $\mathcal{O}_\infty^*$ *equals the de Rham cohomology* $H^*_{\mathrm{DR}}(Y)$ *of a fiber bundle* Y [1, 9, 19].

One can think of elements of $\mathcal{O}_\infty^*$ as being differential forms on an infinite-order jet manifold $J^\infty Y$ as follows. Let $\mathfrak{G}_r^*$ be a sheaf of germs of exterior forms on J^rY and $\overline{\mathfrak{G}}_r^*$ the canonical presheaf of local sections of $\mathfrak{G}_r^*$. Since π^r_{r-1} are open maps, there is a direct sequence of presheaves

$$\overline{\mathfrak{G}}_0^* \xrightarrow{\pi_0^{1*}} \overline{\mathfrak{G}}_1^* \cdots \xrightarrow{\pi^r_{r-1}{}^*} \overline{\mathfrak{G}}_r^* \longrightarrow \cdots .$$

Its direct limit $\overline{\mathfrak{G}}_\infty^*$ is a presheaf of DGAs on $J^\infty Y$. Let $\mathfrak{Q}_\infty^*$ be a sheaf of DGAs of germs of $\overline{\mathfrak{G}}_\infty^*$ on $J^\infty Y$. The structure module $\mathcal{Q}_\infty^* = \Gamma(\mathfrak{Q}_\infty^*)$ of global sections of $\mathfrak{Q}_\infty^*$ is a DGA such that, given an element $\phi \in \mathcal{Q}_\infty^*$ and a point $z \in J^\infty Y$, there exist an open neighborhood U of z and an exterior form $\phi^{(k)}$ on some finite-order jet manifold J^kY so that $\phi|_U = \pi_k^{\infty*}\phi^{(k)}|_U$. Therefore, one can think of $\mathcal{Q}_\infty^*$ as being an algebra of locally exterior forms on finite-order jet manifolds. In particular, there is a monomorphism $\mathcal{O}_\infty^* \to \mathcal{Q}_\infty^*$.

A DGA $\mathcal{O}_\infty^*$ is split into a variational bicomplex [15–17, 19]. If $Y \to X$ is a contractible bundle $\mathbb{R}^{n+p} \to \mathbb{R}^n$, a variational bicomplex is exact [30, 36]. A problem is to determine cohomology of this bicomplex in a general case. One also considers a variational bicomplex of a DGA $\mathcal{Q}_\infty^*$ [2, 37]. It is essential that a paracompact space $J^\infty Y$ admits a partition of unity by elements of a ring $\mathcal{Q}_\infty^0$ [37]. This fact enabled one to apply the abstract de Rham theorem (Theorem A.1) in order to obtain cohomology of a variational bicomplex $\mathcal{Q}_\infty^*$ [2, 37]. Then we have proved that cohomology of a variational bicomplex $\mathcal{O}^*_\infty$ equals that of a variational bicomplex $\mathcal{Q}^*_\infty$ [15–17, 19, 32].

Remark 5. Let $Y \to X$ be a vector bundle. Its global section constitute a projective $C^\infty(X)$-module of finite rank. The converse is also true by virtue of the well-known Serre–Swan theorem, extended to an arbitrary manifold X [19, 31]. In this case, a DGA $\mathcal{O}^*_0$ of exterior forms on Y is isomorphic to the minimal Chevalley–Eilenberg differential calculus over a real commutative ring $C^\infty(Y)$ of smooth real functions on Y. Jet bundles $J^rY \to X$ also are vector bundles. Then one can consider a differential graded subalgebra $\mathcal{P}^*_r \subset \mathcal{O}^*_r$ of differential forms whose coefficients are polynomials in jet coordinates y^i_Λ, $0 \leq |\Lambda| \leq r$, on $J^rY \to X$. In particular, $\mathcal{P}^0_r$ is a $C^\infty(X)$-ring of polynomials of coordinates y^i_Λ. One can associate to such a polynomial of degree m a section of a symmetric tensor product $\overset{m}{\vee}(J^kY)^*$ of the dual of a jet bundle $J^kY \to X$, and vice versa. A DGA $\mathcal{P}^*_r$ is isomorphic to the minimal Chevalley–Eilenberg differential calculus over a real ring $\mathcal{P}^0_r$. Accordingly, there exists a differential graded subalgebra $\mathcal{P}^*_\infty \subset \mathcal{O}^*_\infty$ of differential forms whose coefficients are polynomials in jet coordinates y^i_Λ, $0 \leq |\Lambda|$, of the continuous bundle $J^\infty Y \to X$. This property is coordinate-independent due to the linear transition functions (3). In particular, $\mathcal{P}^0_\infty$ is a ring of polynomials of coordinates y^i_Λ, $0 \leq |\Lambda|$, with coefficients in a ring $C^\infty(X)$. A DGA $\mathcal{P}^*_\infty$ is the direct system of the above-mentioned DGAs $\mathcal{P}^0_r$. It is split into a variational bicomplex. Its cohomology can be obtained [15, 17, 19, 34].

We follow this example in order to construct a Grassmann-graded variational bicomplex.

3. Differential Calculus Over a Graded Commutative Ring

Let us start with the differential calculus over a graded commutative ring (henceforth GCR) as a generalization of that over a commutative ring.

By a Grassmann gradation (or, simply, a gradation if there is no danger of confusion) throughout is meant a $\mathbb{Z}_2$-gradation. Hereafter, the symbol $[\cdot]$ stands for a Grassmann parity.

An additive group $\mathcal{A}$ is said to be graded if it is a product $\mathcal{A} = \mathcal{A}_0 \oplus \mathcal{A}_1$ of two additive subgroups $\mathcal{A}_0$ and $\mathcal{A}_1$ whose elements are called even and odd, respectively.

An algebra $\mathcal{A}$ is called graded if it is a graded additive group so that

$$[aa'] = ([a] + [a'])\,\mathrm{mod}\,2, \quad a \in \mathcal{A}_{[a]},\ a' \in \mathcal{A}_{[a']}.$$

Its even part $\mathcal{A}_0$ is a subalgebra of $\mathcal{A}$, while the odd one $\mathcal{A}_1$ is an $\mathcal{A}$-module. If $\mathcal{A}$ is a graded ring, then $[1] = 0$. A graded ring $\mathcal{A}$ is called graded commutative if $aa' = (-1)^{[a][a']}a'a$.

Given a graded algebra $\mathcal{A}$, an $\mathcal{A}$-module Q is called graded if it is a graded additive group such that

$$[aq] = [qa] = ([a] + [q])\,\mathrm{mod}\,2, \quad a \in \mathcal{A},\ q \in Q.$$

If $\mathcal{A}$ is a GCR, a graded $\mathcal{A}$-module Q is usually assumed to obey the condition $qa = (-1)^{[a][q]}aq$.

In particular, a graded $\mathbb{R}$-module $B = B_0 \oplus B_1$ is called the graded vector space. It is said to be (n, m)-dimensional if $B_0 = \mathbb{R}^n$, $B_1 = \mathbb{R}^m$.

Let $\mathcal{K}$ be a commutative ring. A graded algebra $\mathcal{A}$ is said to be a $\mathcal{K}$-algebra if it is a $\mathcal{K}$-module. For instance, it is called a real graded algebra if $\mathcal{K} = \mathbb{R}$.

Let $\mathcal{A}$ be a GCR. The following are standard constructions of new graded $\mathcal{A}$-modules from the old ones.

- A direct sum of graded modules is defined just as that of modules over a commutative ring.
- A tensor product $P \otimes Q$ of graded $\mathcal{A}$-modules P and Q is an additive group generated by elements $p \otimes q$, $p \in P$, $q \in Q$, obeying relations

$$(p + p') \otimes q = p \otimes q + p' \otimes q, \quad p \otimes (q + q') = p \otimes q + p \otimes q',$$
$$ap \otimes q = (-1)^{[p][a]}pa \otimes q = (-1)^{[p][a]}p \otimes aq, \quad a \in \mathcal{A}.$$

  In particular, a tensor algebra $\otimes P$ of a graded $\mathcal{A}$-module P is defined as that of a module over a commutative algebra. Its quotient $\wedge P$ with respect to an ideal generated by elements

$$p \otimes p' + (-1)^{[p][p']}p' \otimes p, \quad p, p' \in P,$$

  is a bigraded exterior algebra of a graded module P with respect to a graded exterior product

$$p \wedge p' = -(-1)^{[p][p']}p' \wedge p.$$

- A morphism $\Phi : P \to Q$ of graded $\mathcal{A}$-modules seen as additive groups is said to be an even (respectively, odd) morphism if Φ preserves (respectively, change) the Grassmann parity of all graded-homogeneous elements of P and if it obeys the relations

$$\Phi(ap) = (-1)^{[\Phi][a]}a\Phi(p), \quad p \in P, \ a \in \mathcal{A}.$$

  A morphism $\Phi : P \to Q$ of graded $\mathcal{A}$-modules as additive groups is called a graded $\mathcal{A}$-module morphism if it is represented by a sum of even and odd morphisms. A set $\mathrm{Hom}_{\mathcal{A}}(P, Q)$ of graded morphisms of P to Q is naturally a graded $\mathcal{A}$-module. A graded $\mathcal{A}$-module $P^* = \mathrm{Hom}_{\mathcal{A}}(P, \mathcal{A})$ is called the dual of a graded $\mathcal{A}$-module P.

A real graded algebra $\mathfrak{g}$ is called a Lie superalgebra if its product $[\cdot, \cdot]$, called the Lie superbracket, obeys relations

$$[\varepsilon, \varepsilon'] = -(-1)^{[\varepsilon][\varepsilon']}[\varepsilon', \varepsilon],$$
$$(-1)^{[\varepsilon][\varepsilon'']}[\varepsilon, [\varepsilon', \varepsilon'']] + (-1)^{[\varepsilon'][\varepsilon]}[\varepsilon', [\varepsilon'', \varepsilon]] + (-1)^{[\varepsilon''][\varepsilon']}[\varepsilon'', [\varepsilon, \varepsilon']] = 0.$$

Obviously, an even part $\mathfrak{g}_0$ of a Lie superalgebra $\mathfrak{g}$ is a Lie algebra. A graded vector space P is called a $\mathfrak{g}$-module if it is provided with an $\mathbb{R}$-bilinear map

$$\mathfrak{g} \times P \ni (\varepsilon, p) \to \varepsilon p \in P, \quad [\varepsilon p] = ([\varepsilon] + [p]) \bmod 2,$$
$$[\varepsilon, \varepsilon']p = (\varepsilon \circ \varepsilon' - (-1)^{[\varepsilon][\varepsilon']}\varepsilon' \circ \varepsilon)p.$$

Let $\mathcal{A}$ be a real GCR. Let P and Q be graded $\mathcal{A}$-modules. The real graded module $\mathrm{Hom}_{\mathbb{R}}(P, Q)$ of $\mathbb{R}$-linear graded homomorphisms $\Phi : P \to Q$ can be endowed with the two graded $\mathcal{A}$-module structures

$$(a\Phi)(p) = a\Phi(p), \quad (\Phi \bullet a)(p) = \Phi(ap), \quad a \in \mathcal{A},\ p \in P,$$

called $\mathcal{A}$- and $\mathcal{A}^\bullet$-module structures, respectively. Let us put

$$\delta_a \Phi = a\Phi - (-1)^{[a][\Phi]}\Phi \bullet a, \quad a \in \mathcal{A}.$$

An element $\Delta \in \mathrm{Hom}_{\mathbb{R}}(P, Q)$ is said to be a Q-valued graded differential operator of order s on P if $\delta_{a_0} \circ \cdots \circ \delta_{a_s} \Delta = 0$ for any tuple of $s+1$ elements $a_0, \ldots, a_s$ of $\mathcal{A}$.

In particular, zero-order graded differential operators coincide with graded $\mathcal{A}$-module morphisms $P \to Q$. A first-order graded differential operator Δ satisfies the relation

$$\delta_a \circ \delta_b \, \Delta(p) = ab\Delta(p) - (-1)^{([b]+[\Delta])[a]} b\Delta(ap) - (-1)^{[b][\Delta]} a\Delta(bp)$$
$$+ (-1)^{[b][\Delta]+([\Delta]+[b])[a]} = 0, \quad a, b \in \mathcal{A},\ p \in P.$$

For instance, let $P = \mathcal{A}$. A first-order Q-valued graded differential operator Δ on $\mathcal{A}$ fulfills the condition

$$\Delta(ab) = \Delta(a)b + (-1)^{[a][\Delta]} a\Delta(b) - (-1)^{([b]+[a])[\Delta]} ab\Delta(\mathbf{1}), \quad a, b \in \mathcal{A}.$$

It is called a Q-valued graded derivation of $\mathcal{A}$ if $\Delta(\mathbf{1}) = 0$, i.e. the graded Leibniz rule

$$\Delta(ab) = \Delta(a)b + (-1)^{[a][\Delta]} a\Delta(b), \quad a, b \in \mathcal{A},$$

holds. If ∂ is a graded derivation of $\mathcal{A}$, then $a\partial$ is so for any $a \in \mathcal{A}$. Hence, graded derivations of $\mathcal{A}$ constitute a graded $\mathcal{A}$-module $\mathfrak{d}(\mathcal{A}, Q)$, called the graded derivation module.

If $Q = \mathcal{A}$, a graded derivation module $\mathfrak{d}\mathcal{A}$ is also a real Lie superalgebra with respect to a superbracket

$$[u, u'] = u \circ u' - (-1)^{[u][u']} u' \circ u, \quad u, u' \in \mathcal{A}. \tag{7}$$

Then one can consider the Chevalley–Eilenberg complex $C^*[\mathfrak{d}\mathcal{A};\mathcal{A}]$ where a real GCR $\mathcal{A}$ is regarded as an $\mathfrak{d}\mathcal{A}$-module [14, 19]. It reads

$$0 \to \mathbb{R} \xrightarrow{\text{in}} \mathcal{A} \xrightarrow{d} C^1[\mathfrak{d}\mathcal{A};\mathcal{A}] \xrightarrow{d} \cdots C^k[\mathfrak{d}\mathcal{A};\mathcal{A}] \xrightarrow{d} \cdots,$$
$$C^k[\mathfrak{d}\mathcal{A};\mathcal{A}] = \operatorname{Hom}_{\mathbb{R}}\left(\overset{k}{\bigwedge}\mathfrak{d}\mathcal{A},\mathcal{A}\right). \tag{8}$$

Let us bring homogeneous elements of $\overset{k}{\wedge}\mathfrak{d}\mathcal{A}$ into the form

$$\varepsilon_1\wedge\cdots\varepsilon_r\wedge\epsilon_{r+1}\wedge\cdots\wedge\epsilon_k, \quad \varepsilon_i\in(\mathfrak{d}\mathcal{A})_0, \quad \epsilon_j\in(\mathfrak{d}\mathcal{A})_1.$$

Then an even coboundary operator d of the complex (8) is given by the expression

$$\begin{aligned} & dc(\varepsilon_1\wedge\cdots\wedge\varepsilon_r\wedge\epsilon_1\wedge\cdots\wedge\epsilon_s) \\ &= \sum_{i=1}^{r}(-1)^{i-1}\varepsilon_i c(\varepsilon_1\wedge\cdots\widehat{\varepsilon}_i\cdots\wedge\varepsilon_r\wedge\epsilon_1\wedge\cdots\epsilon_s) \\ &\quad + \sum_{j=1}^{s}(-1)^{r}\varepsilon_i c(\varepsilon_1\wedge\cdots\wedge\varepsilon_r\wedge\epsilon_1\wedge\cdots\widehat{\epsilon}_j\cdots\wedge\epsilon_s) \\ &\quad + \sum_{1\le i<j\le r}(-1)^{i+j}c([\varepsilon_i,\varepsilon_j]\wedge\varepsilon_1\wedge\cdots\widehat{\varepsilon}_i\cdots\widehat{\varepsilon}_j\cdots\wedge\varepsilon_r\wedge\epsilon_1\wedge\cdots\wedge\epsilon_s) \\ &\quad + \sum_{1\le i<j\le s}c([\epsilon_i,\epsilon_j]\wedge\varepsilon_1\wedge\cdots\wedge\varepsilon_r\wedge\epsilon_1\wedge\cdots\widehat{\epsilon}_i\cdots\widehat{\epsilon}_j\cdots\wedge\epsilon_s) \\ &\quad + \sum_{1\le i<r,1\le j\le s}(-1)^{i+r+1}c([\varepsilon_i,\epsilon_j]\wedge\varepsilon_1\wedge\cdots\widehat{\varepsilon}_i\cdots\wedge\varepsilon_r\wedge\epsilon_1\wedge\cdots\widehat{\epsilon}_j\cdots\wedge\epsilon_s), \end{aligned} \tag{9}$$

where the caret $\widehat{}$ denotes omission.

It is easily justified that the complex (8) contains a subcomplex $\mathcal{O}^*[\mathfrak{d}\mathcal{A}]$ of $\mathcal{A}$-linear graded morphisms. It is provided with a structure of a bigraded $\mathcal{A}$-algebra with respect to a graded exterior product

$$\begin{aligned} & \phi\wedge\phi'(u_1,\ldots,u_{r+s}) \\ &\quad = \sum_{i_1<\cdots<i_r;j_1<\cdots<j_s}\operatorname{Sgn}^{i_1\cdots i_r j_1\cdots j_s}_{1\cdots r+s}\phi(u_{i_1},\ldots,u_{i_r})\phi'(u_{j_1},\ldots,u_{j_s}), \end{aligned} \tag{10}$$

where $u_1,\ldots,u_{r+s}$ are graded-homogeneous elements of $\mathfrak{d}\mathcal{A}$ and

$$u_1\wedge\cdots\wedge u_{r+s} = \operatorname{Sgn}^{i_1\cdots i_r j_1\cdots j_s}_{1\cdots r+s}u_{i_1}\wedge\cdots\wedge u_{i_r}\wedge u_{j_1}\wedge\cdots\wedge u_{j_s}.$$

The coboundary operator d (9) and the graded exterior product $\wedge$ (10) bring $\mathcal{O}^*[\mathfrak{d}\mathcal{A}]$ into a differential bigraded algebra (henceforth DBGA) whose elements obey relations

$$\phi\wedge\phi' = (-1)^{|\phi||\phi'|+[\phi][\phi']}\phi'\wedge\phi, \quad d(\phi\wedge\phi') = d\phi\wedge\phi' + (-1)^{|\phi|}\phi\wedge d\phi'.$$

It is called the graded Chevalley–Eilenberg differential calculus over a real GCR $\mathcal{A}$.

In particular, we have

$$\mathcal{O}^1[\mathfrak{d}\mathcal{A}] = \mathrm{Hom}_{\mathcal{A}}(\mathfrak{d}\mathcal{A}, \mathcal{A}) = \mathfrak{d}\mathcal{A}^*. \tag{11}$$

One can extend this duality relation to the graded interior product of $u \in \mathfrak{d}\mathcal{A}$ with any element $\phi \in \mathcal{O}^*[\mathfrak{d}\mathcal{A}]$ by the rules

$$u\rfloor(bda) = (-1)^{[u][b]} bu(a), \quad a, b \in \mathcal{A},$$
$$u\rfloor(\phi \wedge \phi') = (u\rfloor\phi) \wedge \phi' + (-1)^{|\phi|+[\phi][u]} \phi \wedge (u\rfloor\phi').$$

As a consequence, any graded derivation $u \in \mathfrak{d}\mathcal{A}$ of $\mathcal{A}$ yields a derivation

$$\mathbf{L}_u\phi = u\rfloor d\phi + d(u\rfloor\phi), \quad \phi \in \mathcal{O}^*[\mathfrak{d}\mathcal{A}], \quad u \in \mathfrak{d}\mathcal{A},$$
$$\mathbf{L}_u(\phi \wedge \phi') = \mathbf{L}_u(\phi) \wedge \phi' + (-1)^{[u][\phi]} \phi \wedge \mathbf{L}_u(\phi'),$$

called the graded Lie derivative of a DBGA $\mathcal{O}^*[\mathfrak{d}\mathcal{A}]$.

Note that, if $\mathcal{A}$ is a commutative ring, the graded Chevalley–Eilenberg differential calculus comes to the familiar one.

The minimal graded Chevalley–Eilenberg differential calculus $\mathcal{O}^*\mathcal{A} \subset \mathcal{O}^*[\mathfrak{d}\mathcal{A}]$ over a GCR $\mathcal{A}$ consists of monomials $a_0 da_1 \wedge \cdots \wedge da_k$, $a_i \in \mathcal{A}$. The corresponding complex

$$0 \to \mathbb{R} \to \mathcal{A} \xrightarrow{d} \mathcal{O}^1\mathcal{A} \xrightarrow{d} \cdots \mathcal{O}^k\mathcal{A} \xrightarrow{d} \cdots \tag{12}$$

is called the bigraded de Rham complex of a real GCR $\mathcal{A}$.

4. Differential Calculus on Graded Manifolds

As was mentioned above, we follow Serre–Swan Theorem 7 below and consider a real GCR $\mathcal{A}$ of graded functions on a graded manifold. Then the minimal graded Chevalley–Eilenberg differential calculus $\mathcal{O}^*\mathcal{A}$ over $\mathcal{A}$ is a DBGA of graded exterior forms on this graded manifold [17, 19, 34].

A real GCR Λ is called the Grassmann algebra if it is a free ring such that

$$\Lambda = \Lambda_0 \oplus \Lambda_1 = (\mathbb{R} \oplus (\Lambda_1)^2) \oplus \Lambda_1,$$

i.e. a Grassmann algebra is generated by the unit element $\mathbf{1}$ and its odd elements. Note that there is a different definition of a Grassmann algebra [25].

Hereafter, we restrict our consideration to Grassmann algebras which are finite-dimensional vector spaces. In this case, there exists a real vector space V such that $\Lambda = \wedge V$ is its exterior algebra endowed with the Grassmann gradation

$$\Lambda_0 = \mathbb{R} \bigoplus_{k=1} \overset{2k}{\wedge} V, \quad \Lambda_1 = \bigoplus_{k=1} \overset{2k-1}{\wedge} V. \tag{13}$$

One calls $\dim V$ the rank of a Grassmann algebra Λ. Given a basis $\{c^i\}$ for a vector space V, elements of the Grassmann algebra Λ (13) take the form

$$a = \sum_{k=0,1,\ldots} \sum_{(i_1\cdots i_k)} a_{i_1\cdots i_k} c^{i_1}\cdots c^{i_k},$$

where the second sum runs through all the tuples $(i_1 \cdots i_k)$ such that no two of them are permutations of each other.

A graded manifold of dimension (n, m) is defined as a local-ringed space $(Z, \mathfrak{A})$ whose body Z is an n-dimensional smooth manifold and whose structure sheaf $\mathfrak{A} = \mathfrak{A}_0 \oplus \mathfrak{A}_1$ is a sheaf of Grassmann algebras of rank m such that [5, 19]:

- there is an exact sequence of sheaves
$$0 \to \mathcal{R} \to \mathfrak{A} \xrightarrow{\sigma} C^\infty_Z \to 0, \quad \mathcal{R} = \mathfrak{A}_1 + (\mathfrak{A}_1)^2,$$
where C^∞_Z is a sheaf of smooth real functions on Z;
- $\mathcal{R}/\mathcal{R}^2$ is a locally free sheaf of C^∞_Z-modules of finite rank (with respect to pointwise operations), and a sheaf $\mathfrak{A}$ is locally isomorphic to an exterior product $\underset{C^\infty_Z}{\wedge}(\mathcal{R}/\mathcal{R}^2)$.

Sections of a sheaf $\mathfrak{A}$ are called graded functions on a graded manifold $(Z, \mathfrak{A})$. They make up a real GCR $\mathfrak{A}(Z)$ which is a $C^\infty(Z)$-ring, called the structure ring of $(Z, \mathfrak{A})$. Let us recall the well-known Batchelor theorem [5, 19].

Theorem 6. *Let $(Z, \mathfrak{A})$ be a graded manifold. There exists a vector bundle $E \to Z$ with an m-dimensional typical fiber V such that the structure sheaf $\mathfrak{A}$ of $(Z, \mathfrak{A})$ is isomorphic to the structure sheaf $\mathfrak{A}_E = S_{\wedge E^*}$ of germs of sections of an exterior bundle $\wedge E^*$, whose typical fiber is a Grassmann algebra $\Lambda = \wedge V^*$.*

Though Batchelor's isomorphism in Theorem 6 fails to be canonical, we restrict our consideration to graded manifolds $(Z, \mathfrak{A}_E)$, called simple graded manifolds modeled over a vector bundle $E \to Z$. Accordingly, the structure ring $\mathcal{A}_E$ of a simple graded manifold $(Z, \mathfrak{A}_E)$ is a module $\mathcal{A}_E = \wedge E^*(Z)$ of sections of an exterior bundle $\wedge E^*$.

The above-mentioned Serre–Swan theorem and Theorem 6 lead to the Serre–Swan theorem for graded manifolds [7, 19].

Theorem 7. *Let Z be a smooth manifold. A $C^\infty(Z)$-GCR $\mathcal{A}$ is generated by some projective $C^\infty(Z)$-module of finite rank if and only if it is isomorphic to the structure ring $\mathfrak{A}(Z)$ of some graded manifold $(Z, \mathfrak{A})$ with a body Z.*

Given a simple graded manifold $(Z, \mathfrak{A}_E)$, a trivialization chart $(U; z^A, y^a)$ of a vector bundle $E \to Z$ yields its splitting domain $(U; z^A, c^a)$. Graded functions on it are Λ-valued functions

$$f = \sum_{k=0}^{m} \frac{1}{k!} f_{a_1\cdots a_k}(z) c^{a_1} \cdots c^{a_k}, \tag{14}$$

where $f_{a_1\cdots a_k}(z)$ are smooth functions on U and $\{c^a\}$ is a fiber basis for E^*. One calls $\{z^A, c^a\}$ the local basis for a graded manifold $(Z,\mathfrak{A}_E)$ [5, 19]. Transition functions $y'^a = \rho^a_b(z^A)y^b$ of bundle coordinates on $E\to Z$ yield the corresponding transformation $c'^a = \rho^a_b(z^A)c^b$ of the associated local basis for a graded manifold $(Z,\mathfrak{A}_E)$ and the according coordinate transformation law of graded functions (14).

Given a graded manifold $(Z,\mathfrak{A})$, let $\mathfrak{d}\mathfrak{A}(Z)$ be a graded derivation module of its real structure ring $\mathfrak{A}(Z)$. Its elements are called graded vector fields on a graded manifold $(Z,\mathfrak{A})$. A key point is that graded vector fields $u\in\mathfrak{d}\mathcal{A}_E$ on a simple graded manifold $(Z,\mathfrak{A}_E)$ can be represented by sections of some vector bundle as follows [17, 19]. Due to a canonical splitting $VE = E\times E$, the vertical tangent bundle VE of $E\to Z$ can be provided with fiber bases $\{\partial_a\}$, which are the duals of bases $\{c^a\}$. Then graded vector fields on a splitting domain $(U; z^A, c^a)$ of $(Z,\mathfrak{A}_E)$ read

$$u = u^A\partial_A + u^a\partial_a, \tag{15}$$

where u^A, u^a are local Λ-valued functions on U. In particular,

$$\partial_a\circ\partial_b = -\partial_b\circ\partial_a, \quad \partial_A\circ\partial_a = \partial_a\circ\partial_A.$$

The graded derivations (15) act on graded functions $f\in\mathfrak{A}_E(U)$ (14) by the rule

$$u(f_{a\cdots b}c^a\cdots c^b) = u^A\partial_A(f_{a\cdots b})c^a\cdots c^b + u^k f_{a\cdots b}\partial_k\rfloor(c^a\cdots c^b). \tag{16}$$

This rule implies the corresponding coordinate transformation law

$$u'^A = u^A, \quad u'^a = \rho^a_j u^j + u^A\partial_A(\rho^a_j)c^j$$

of graded vector fields. It follows that graded vector fields (15) can be represented by sections of a vector bundle $\mathcal{V}_E$ which is locally isomorphic to a vector bundle $\wedge E^*\otimes_Z(E\oplus_Z TZ)$.

Given a real GCR $\mathcal{A}_E$ of graded functions on a graded manifold $(Z,\mathfrak{A}_E)$ and a real Lie superalgebra $\mathfrak{d}\mathcal{A}_E$ of its graded derivations, let us consider the graded Chevalley–Eilenberg differential calculus

$$\mathcal{S}^*[E;Z] = \mathcal{O}^*[\mathfrak{d}\mathcal{A}_E] \tag{17}$$

over $\mathcal{A}_E$. Since a graded derivation module $\mathfrak{d}\mathcal{A}_E$ is isomorphic to a module of sections of a vector bundle $\mathcal{V}_E\to Z$, elements of $\mathcal{S}^*[E;Z]$ are represented by sections of an exterior bundle $\wedge\overline{\mathcal{V}}_E$ of the $\wedge E^*$-dual $\overline{\mathcal{V}}_E\to Z$ of $\mathcal{V}_E$ which is locally isomorphic to a vector bundle $\wedge E^*\otimes_Z(E^*\oplus_Z T^*Z)$. With respect to the dual fiber bases $\{dz^A\}$ for T^*Z and $\{dc^b\}$ for E^*, sections of $\overline{\mathcal{V}}_E$ take the coordinate form

$$\phi = \phi_A dz^A + \phi_a dc^a, \quad \phi'_a = \rho^{-1b}_a\phi_b, \quad \phi'_A = \phi_A + \rho^{-1b}_a\partial_A(\rho^a_j)\phi_b c^j,$$

ϕ_A, f_a are local Λ-valued functions on U. The duality isomorphism $\mathcal{S}^1[E;Z] = \mathfrak{d}\mathcal{A}^*_E$ (11) is given by a graded interior product

$$u\rfloor\phi = u^A\phi_A + (-1)^{[\phi_a]}u^a\phi_a.$$

Elements of $\mathcal{S}^*[E;Z]$ are called graded exterior forms on a graded manifold $(Z,\mathfrak{A}_E)$.

Seen as an $\mathcal{A}_E$-algebra, the DBGA $\mathcal{S}^*[E;Z]$ (17) on a splitting domain $(U; z^A, c^a)$ is locally generated by graded one-forms dz^A, dc^i such that

$$dz^A \wedge dc^i = -dc^i \wedge dz^A, \quad dc^i \wedge dc^j = dc^j \wedge dc^i.$$

Accordingly, the coboundary operator d (9), called the graded exterior differential, reads

$$d\phi = dz^A \wedge \partial_A \phi + dc^a \wedge \partial_a \phi,$$

where derivatives ∂_A, ∂_a act on coefficients of graded exterior forms by the formula (16), and they are graded commutative with graded forms dz^A, dc^a.

Lemma 8. *The* DBGA $\mathcal{S}^*[E;Z]$ (17) *is a minimal differential calculus over* $\mathcal{A}_E$ [19].

The bigraded de Rham complex (12) of the minimal graded Chevalley–Eilenberg differential calculus $\mathcal{S}^*[E;Z]$ reads

$$0 \to \mathbb{R} \to \mathcal{A}_E \xrightarrow{d} \mathcal{S}^1[E;Z] \xrightarrow{d} \cdots \mathcal{S}^k[E;Z] \xrightarrow{d} \cdots . \tag{18}$$

Its cohomology $H^*(\mathcal{A}_E)$ is called the de Rham cohomology of a graded manifold $(Z, \mathfrak{A}_E)$. In particular, given a DGA $\mathcal{O}^*(Z)$ of exterior forms on Z, there exists a canonical monomorphism

$$\mathcal{O}^*(Z) \to \mathcal{S}^*[E;Z] \tag{19}$$

and a body epimorphism $\mathcal{S}^*[E;Z] \to \mathcal{O}^*(Z)$ which are cochain morphisms of the de Rham complex (18) and the de Rham complex of $\mathcal{O}^*(Z)$. Then one can show the following [19, 34].

Theorem 9. *The de Rham cohomology of a graded manifold* $(Z, \mathfrak{A}_E)$ *equals the de Rham cohomology of its body* Z.

Corollary 10. *Any closed graded exterior form is decomposed into a sum* $\phi = \sigma + d\xi$ *where* σ *is a closed exterior form on* Z.

5. Grassmann-Graded Variational Bicomplex

Let X be an n-dimensional smooth manifold and $Y \to X$ a vector bundle over X. In Remark 5, we mention a polynomial variational bicomplex of a DGA $\mathcal{P}^*_\infty$. The latter is the direct limit of DGAs $\mathcal{P}^*_r$ where $\mathcal{P}^*_r$ is the minimal Chevalley–Eilenberg differential calculus over a ring $\mathcal{P}^0_r$ of sections of symmetric tensor products of a vector jet bundle $J^rY \to X$.

Let $(X, \mathfrak{A}_E)$ be a simple graded manifold modeled over a vector bundle $E \to X$. Any jet bundle $J^rE \to X$ is also a vector bundle. Then let $(X, \mathcal{A}_{J^rE})$ denote a simple graded manifold modeled over a vector bundle $J^rE \to X$. Its structure module $\mathcal{A}_{J^rE}$ is a real GCR of sections of an exterior bundle $\wedge(J^rE)^*$ where $(J^rE)^*$ denotes the dual of $J^rE \to X$. Let $S^*[J^rE, X]$ be the minimal Chevalley–Eilenberg differential calculus over a real GCR $\mathcal{A}_{J^rE}$. It is a BGDA of graded exterior forms on a simple graded manifold $(X, \mathcal{A}_{J^rE})$. There is a direct system

$$\mathcal{S}^*[E;X] \to \mathcal{S}^*[J^1E;X] \to \cdots \mathcal{S}^*[J^rE;X] \to \cdots$$

of BGDAs $S^*[J^rE,X]$. Its direct limit $\mathcal{S}^*_\infty[E;X]$ is the Grassmann-graded counterpart of the above-mentioned DGA $\mathcal{P}^*_\infty$. A BDGA $\mathcal{S}^*_\infty[E;X]$ is split into a Grassmann-graded variational bicomplex which leads to graded Lagrangian formalism of odd variables represented by generating elements of the structure ring $\mathcal{A}_E$ of a graded manifold $(X, \mathfrak{A}_E)$ [17, 19, 34].

Note that the definition of jets of these odd variables as elements of structure rings of graded manifolds $\mathcal{A}_{J^rE}$ differs from that of jets of fibered-graded manifolds [23, 29], but it reproduces the heuristic notion of jets of odd variables in Lagrangian field theory [4, 10].

In order to formulate graded Lagrangian theory both of even and odd variables, let us consider a composite bundle $F \to Y \to X$ where $F \to Y$ is a vector bundle provided with bundle coordinates (x^λ, y^i, q^a). Jet manifolds J^rF of $F \to X$ are also vector bundles $J^rF \to J^rY$ coordinated by $(x^\lambda, y^i_\Lambda, q^a_\Lambda)$, $0 \leq |\Lambda| \leq r$. Let $(J^rY, \mathfrak{A}_r)$ (where $J^0Y = Y$, $\mathfrak{A}_0 = \mathfrak{A}_F$) be a simple-graded manifold modeled over such a vector bundle. Its local basis is $(x^\lambda, y^i_\Lambda, c^a_\Lambda)$, $0 \leq |\Lambda| \leq r$. Let $\mathcal{S}^*_r[F;Y] = \mathcal{S}^*_r[J^rF;J^rY]$ denote a DBGA of graded exterior forms on a graded manifold $(J^rY, \mathfrak{A}_r)$. In particular, there is the cochain monomorphism (19):

$$\mathcal{O}^*_r = \mathcal{O}^*(J^rY) \to \mathcal{S}^*_r[F;Y]. \tag{20}$$

A surjection $\pi^{r+1}_r : J^{r+1}Y \to J^rY$ yields an epimorphism of graded manifolds

$$(\pi^{r+1}_r, \widehat{\pi}^{r+1}_r) : (J^{r+1}Y, \mathfrak{A}_{r+1}) \to (J^rY, \mathfrak{A}_r),$$

including a sheaf monomorphism $\widehat{\pi}^{r+1}_r : \pi^{r+1*}_r\mathfrak{A}_r \to \mathfrak{A}_{r+1}$, where $\pi^{r+1*}_r\mathfrak{A}_r$ is the pull-back onto $J^{r+1}Y$ of a continuous fiber bundle $\mathfrak{A}_r \to J^rY$. This sheaf monomorphism induces a monomorphism of canonical presheaves $\overline{\mathfrak{A}}_r \to \overline{\mathfrak{A}}_{r+1}$, which associates to each open subset $U \subset J^{r+1}Y$ a ring of sections of $\mathfrak{A}_r$ over $\pi^{r+1}_r(U)$. Accordingly, there is a monomorphism

$$\pi^{r+1*}_r : \mathcal{S}^0_r[F;Y] \to \mathcal{S}^0_{r+1}[F;Y] \tag{21}$$

of structure rings of graded functions on graded manifolds $(J^rY, \mathfrak{A}_r)$ and $(J^{r+1}Y, \mathfrak{A}_{r+1})$. By virtue of Lemma 8, the differential calculus $\mathcal{S}^*_r[F;Y]$ and $\mathcal{S}^*_{r+1}[F;Y]$ are minimal. Therefore, the monomorphism (21) yields a monomorphism of DBGAs

$$\pi^{r+1*}_r : \mathcal{S}^*_r[F;Y] \to \mathcal{S}^*_{r+1}[F;Y]. \tag{22}$$

As a consequence, we have a direct system of DBGAs

$$\mathcal{S}^*[F;Y] \xrightarrow{\pi^*} \mathcal{S}^*_1[F;Y] \to \cdots \mathcal{S}^*_{r-1}[F;Y] \xrightarrow{\pi^{r*}_{r-1}} \mathcal{S}^*_r[F;Y] \to \cdots . \tag{23}$$

Its direct limit $\mathcal{S}^*_\infty[F;Y]$ consists of all graded exterior forms $\phi \in \mathcal{S}^*[F_r;J^rY]$ on graded manifolds $(J^rY, \mathfrak{A}_r)$ modulo the monomorphisms (22).

The cochain monomorphisms $\mathcal{O}_r^* \to \mathcal{S}_r^*[F;Y]$ (20) provide a monomorphism of the direct system (5) to the direct system (23) and, consequently, a monomorphism

$$\mathcal{O}_\infty^* \to \mathcal{S}_\infty^*[F;Y] \tag{24}$$

of their direct limits. In particular, $\mathcal{S}_\infty^*[F;Y]$ is an $\mathcal{O}_\infty^0$-algebra. Accordingly, the body epimorphisms $\mathcal{S}_r^*[F;Y] \to \mathcal{O}_r^*$ yield an epimorphism of $\mathcal{O}_\infty^0$-algebras

$$\mathcal{S}_\infty^*[F;Y] \to \mathcal{O}_\infty^*. \tag{25}$$

It is readily observed that the morphisms (24) and (25) are cochain morphisms between the de Rham complex (6) of a DGA $\mathcal{O}_\infty^*$ and the de Rham complex

$$0 \to \mathbb{R} \to \mathcal{S}_\infty^0[F;Y] \xrightarrow{d} \mathcal{S}_\infty^1[F;Y] \cdots \xrightarrow{d} \mathcal{S}_\infty^k[F;Y] \to \cdots \tag{26}$$

of a DBGA $\mathcal{S}_\infty^*[F;Y]$. Moreover, the corresponding homomorphisms of cohomology groups of these complexes are isomorphisms as follows.

Theorem 11. *There is an isomorphism*

$$H^*(\mathcal{S}_\infty^*[F;Y]) = H_{DR}^*(Y) \tag{27}$$

of the cohomology of the de Rham complex (26) *to the de Rham cohomology of* Y*.*

Proof. The complex (26) is the direct limit of the de Rham complexes of DBGAs $\mathcal{S}_r^*[F;Y]$. In accordance with the well-known theorem [19, 28], the direct limit of cohomology groups of these complexes is the cohomology of the de Rham complex (26). By virtue of Theorem 9, cohomology of the de Rham complex of $\mathcal{S}_r^*[F;Y]$ equals the de Rham cohomology of J^rY and, consequently, that of Y, which is the strong deformation retract of any jet manifold J^rY because $J^kY \to J^{k-1}Y$ are affine bundles. Hence, the isomorphism (27) holds. □

Corollary 12. *Any closed graded form* $\phi \in \mathcal{S}_\infty^*[F;Y]$ *is decomposed into the sum* $\phi = \sigma + d\xi$ *where* σ *is a closed exterior form on* Y*.*

One can think of elements of $\mathcal{S}_\infty^*[F;Y]$ as being graded differential forms on an infinite-order jet manifold $J^\infty Y$. Indeed, let $\mathfrak{S}_r^*[F;Y]$ be a sheaf of DBGAs on J^rY and $\overline{\mathfrak{S}}_r^*[F;Y]$ its canonical presheaf. Then the above-mentioned presheaf monomorphisms $\overline{\mathfrak{A}}_r \to \overline{\mathfrak{A}}_{r+1}$ yield a direct system of presheaves

$$\overline{\mathfrak{S}}^*[F;Y] \to \overline{\mathfrak{S}}_1^*[F;Y] \to \cdots \overline{\mathfrak{S}}_r^*[F;Y] \to \cdots,$$

whose direct limit $\overline{\mathfrak{S}}_\infty^*[F;Y]$ is a presheaf of DBGAs on an infinite-order jet manifold $J^\infty Y$. Let $\mathfrak{Q}_\infty^*[F;Y]$ be a sheaf of DBGAs of germs of a presheaf $\overline{\mathfrak{S}}_\infty^*[F;Y]$. One can think of a pair $(J^\infty Y, \mathfrak{Q}_\infty^0[F;Y])$ as being a graded-Fréchet manifold, whose body is an infinite-order jet manifold $J^\infty Y$ and a structure sheaf $\mathfrak{Q}_\infty^0[F;Y]$ is a sheaf of germs of graded functions on graded manifolds $(J^rY, \mathfrak{A}_r)$. The structure module $\mathcal{Q}_\infty^*[F;Y] = \Gamma(\mathfrak{Q}_\infty^*[F;Y])$ of sections of $\mathfrak{Q}_\infty^*[F;Y]$ is a DBGA such that, given an element $\phi \in \mathcal{Q}_\infty^*[F;Y]$ and a point $z \in J^\infty Y$, there exist an open neighborhood U

of z and a graded exterior form $\phi^{(k)}$ on some finite-order jet manifold J^kY so that $\phi|_U = \pi_k^{\infty *}\phi^{(k)}|_U$.

In particular, there is a monomorphism $\mathcal{S}^*_\infty[F;Y] \to \mathcal{Q}^*_\infty[F;Y]$. Due to this monomorphism, one can restrict $\mathcal{S}^*_\infty[F;Y]$ to the coordinate chart (3) of $J^\infty Y$ and can say that $\mathcal{S}^*_\infty[F;Y]$ as an $\mathcal{O}^0_\infty$-algebra is locally generated by elements

$$(c^a_\Lambda, dx^\lambda, \theta^a_\Lambda = dc^a_\Lambda - c^a_{\lambda+\Lambda}dx^\lambda, \theta^i_\Lambda = dy^i_\Lambda - y^i_{\lambda+\Lambda}dx^\lambda), \quad 0 \leq |\Lambda|,$$

where c^a_Λ, θ^a_Λ are odd and dx^λ, θ^i_Λ are even. We agree to call (y^i, c^a) the local generating basis for $\mathcal{S}^*_\infty[F;Y]$. Let the collective symbol s^A stand for its elements. Accordingly, the notations s^A_Λ of their jets and $\theta^A_\Lambda = ds^A_\Lambda - s^A_{\lambda+\Lambda}dx^\lambda$ of contact forms are introduced. For the sake of simplicity, we further denote $[A] = [s^A]$.

A DBGA $\mathcal{S}^*_\infty[F;Y]$ is decomposed into $\mathcal{S}^0_\infty[F;Y]$-modules $\mathcal{S}^{k,r}_\infty[F;Y]$ of k-contact and r-horizontal graded forms together with the corresponding projections

$$h_k : \mathcal{S}^*_\infty[F;Y] \to \mathcal{S}^{k,*}_\infty[F;Y], \quad h^m : \mathcal{S}^*_\infty[F;Y] \to \mathcal{S}^{*,m}_\infty[F;Y].$$

Accordingly, a graded exterior differential d on $\mathcal{S}^*_\infty[F;Y]$ falls into the sum $d = d_V + d_H$ of a vertical graded differential

$$d_V \circ h^m = h^m \circ d \circ h^m, \quad d_V(\phi) = \theta^A_\Lambda \wedge \partial^\Lambda_A \phi, \quad \phi \in \mathcal{S}^*_\infty[F;Y],$$

and a total graded differential

$$d_H \circ h_k = h_k \circ d \circ h_k, \quad d_H \circ h_0 = h_0 \circ d, \quad d_H(\phi) = dx^\lambda \wedge d_\lambda(\phi),$$
$$d_\lambda = \partial_\lambda + \sum_{0\leq|\Lambda|} s^A_{\lambda+\Lambda}\partial^\Lambda_A.$$

These differentials obey the nilpotent relations

$$d_H^2 = 0, \quad d_V^2 = 0, \quad d_H d_V + d_V d_H = 0.$$

A DBGA $\mathcal{S}^*_\infty[F;Y]$ is also provided with a graded projection endomorphism

$$\varrho = \sum_{k>0} \frac{1}{k}\overline{\varrho} \circ h_k \circ h^n : \mathcal{S}^{*>0,n}_\infty[F;Y] \to \mathcal{S}^{*>0,n}_\infty[F;Y],$$
$$\overline{\varrho}(\phi) = \sum_{0\leq|\Lambda|} (-1)^{|\Lambda|}\theta^A \wedge [d_\Lambda(\partial^\Lambda_A \rfloor \phi)], \quad \phi \in \mathcal{S}^{>0,n}_\infty[F;Y],$$

such that $\varrho \circ d_H = 0$, and with a nilpotent graded variational operator

$$\delta = \varrho \circ d : \mathcal{S}^{*,n}_\infty[F;Y] \to \mathcal{S}^{*+1,n}_\infty[F;Y].$$

These operators split a DBGA $\mathcal{S}^{*}_\infty[F;Y]$ into a Grassmann-graded variational bicomplex

$$\begin{array}{ccccccccc}
 & & \vdots & & \vdots & & \vdots & & \vdots \\
 & & {\scriptstyle d_V}\uparrow & & {\scriptstyle d_V}\uparrow & & {\scriptstyle d_V}\uparrow & & {\scriptstyle -\delta}\uparrow \\
 & 0 \to & \mathcal{S}^{1,0}_\infty & \xrightarrow{d_H} & \mathcal{S}^{1,1}_\infty & \xrightarrow{d_H}\cdots & \mathcal{S}^{1,n}_\infty & \xrightarrow{\varrho} & \varrho(\mathcal{S}^{1,n}_\infty) \to 0 \\
 & & {\scriptstyle d_V}\uparrow & & {\scriptstyle d_V}\uparrow & & {\scriptstyle d_V}\uparrow & & {\scriptstyle -\delta}\uparrow \\
0 \to \mathbb{R} \to & & \mathcal{S}^{0}_\infty & \xrightarrow{d_H} & \mathcal{S}^{0,1}_\infty & \xrightarrow{d_H}\cdots & \mathcal{S}^{0,n}_\infty & \equiv & \mathcal{S}^{0,n}_\infty \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 \to \mathbb{R} \to & & \mathcal{O}^0(X) & \xrightarrow{d} & \mathcal{O}^1(X) & \xrightarrow{d}\cdots & \mathcal{O}^n(X) & \xrightarrow{d} & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array} \tag{28}$$

We restrict our consideration to its short variational subcomplex

$$0 \to \mathbb{R} \to \mathcal{S}^0_\infty[F;Y] \xrightarrow{d_H} \mathcal{S}^{0,1}_\infty[F;Y] \cdots \xrightarrow{d_H} \mathcal{S}^{0,n}_\infty[F;Y] \xrightarrow{\delta} \varrho(\mathcal{S}^{1,n}_\infty[F;Y]) \tag{29}$$

and a subcomplex of one-contact graded forms

$$0 \to \mathcal{S}^{1,0}_\infty[F;Y] \xrightarrow{d_H} \mathcal{S}^{1,1}_\infty[F;Y] \cdots \xrightarrow{d_H} \mathcal{S}^{1,n}_\infty[F;Y] \xrightarrow{\varrho} \varrho(\mathcal{S}^{1,n}_\infty[F;Y]) \to 0. \tag{30}$$

Theorem 13. *Cohomology of the complex* (29) *equals the de Rham cohomology of Y.*

Theorem 14. *The complex* (30) *is exact.*

These theorems are proved in Appendix B.

6. Graded Lagrangian Formalism

Decomposed into the variational bicomplex, a DBGA $\mathcal{S}^*_\infty[F;Y]$ describes graded Lagrangian theory on a graded manifold $(Y, \mathfrak{A}_F)$. Its graded Lagrangian is defined as an element

$$L = \mathcal{L}\omega \in \mathcal{S}^{0,n}_\infty[F;Y], \quad \omega = dx^1 \wedge \cdots \wedge dx^n,$$

of the graded variational complex (29). Accordingly, a graded exterior form

$$\delta L = \theta^A \wedge \mathcal{E}_A\omega = \sum_{0\leq|\Lambda|} (-1)^{|\Lambda|}\theta^A \wedge d_\Lambda(\partial^\Lambda_A\mathcal{L})\omega \in \varrho(\mathcal{S}^{1,n}_\infty[F;Y]) \tag{31}$$

is said to be its graded Euler–Lagrange operator. We agree to call a pair $(\mathcal{S}^{0,n}_\infty[F;Y], L)$ the graded Lagrangian system.

The following is a corollary of Theorems 13 and B.3 [17, 19].

Theorem 15. *Every d_H-closed graded form $\phi \in \mathcal{S}_\infty^{0,m<n}[F;Y]$ falls into the sum*

$$\phi = h_0\sigma + d_H\xi, \quad \xi \in \mathcal{S}_\infty^{0,m-1}[F;Y],$$

where σ is a closed m-form on Y. Any δ-closed (i.e. variationally trivial) graded Lagrangian $L \in \mathcal{S}_\infty^{0,n}[F;Y]$ is the sum

$$L = h_0\sigma + d_H\xi, \quad \xi \in \mathcal{S}_\infty^{0,n-1}[F;Y],$$

where σ is a closed n-form on Y.

The exactness of the complex (30) at a term $\mathcal{S}_\infty^{1,n}[F;Y]$ results in the following [17, 19].

Theorem 16. *Given a graded Lagrangian L, there is the decomposition*

$$dL = \delta L - d_H\Xi_L, \quad \Xi \in \mathcal{S}_\infty^{n-1}[F;Y], \tag{32}$$

$$\Xi_L = L + \sum_{s=0} \theta^A_{\nu_s\cdots\nu_1} \wedge F_A^{\lambda\nu_s\cdots\nu_1}\omega_\lambda, \quad \omega_\lambda = \partial_\lambda\rfloor\omega, \tag{33}$$

$$F_A^{\nu_k\cdots\nu_1} = \partial_A^{\nu_k\cdots\nu_1}\mathcal{L} - d_\lambda F_A^{\lambda\nu_k\cdots\nu_1} + \sigma_A^{\nu_k\cdots\nu_1}, \quad k = 1,2,\ldots,$$

where local graded functions σ obey the relations $\sigma_A^\nu = 0, \sigma_A^{(\nu_k\nu_{k-1})\cdots\nu_1} = 0$.

Proof. The decomposition (33) is a straightforward consequence of the exactness of the complex (30) at a term $\mathcal{S}_\infty^{1,n}[F,Y]$ and the fact that ϱ is a projector. The coordinate expression (34) results from a direct computation

$$\begin{aligned}
-d_H\Xi &= -d_H[\theta^A F_A^\lambda + \theta^A_\nu F_A^{\lambda\nu} + \cdots + \theta^A_{\nu_s\cdots\nu_1} F_A^{\lambda\nu_s\cdots\nu_1} \\
&\quad + \theta^A_{\nu_{s+1}\nu_s\cdots\nu_1} \wedge F_A^{\lambda\nu_{s+1}\nu_s\cdots\nu_1} + \cdots] \wedge \omega_\lambda = [\theta^A d_\lambda F_A^\lambda + \theta^A_\nu(F_A^\nu + d_\lambda F_A^{\lambda\nu}) \\
&\quad + \cdots + \theta^A_{\nu_{s+1}\nu_s\cdots\nu_1}(F_A^{\nu_{s+1}\nu_s\cdots\nu_1} + d_\lambda F_A^{\lambda\nu_{s+1}\nu_s\cdots\nu_1}) + \cdots] \wedge \omega \\
&= [\theta^A d_\lambda F_A^\lambda + \theta^A_\nu(\partial_A^\nu\mathcal{L}) + \cdots + \theta^A_{\nu_{s+1}\nu_s\cdots\nu_1}(\partial_A^{\nu_{s+1}\nu_s\cdots\nu_1}\mathcal{L}) + \cdots] \wedge \omega \\
&= \theta^A(d_\lambda F_A^\lambda - \partial_A\mathcal{L}) \wedge \omega + dL = -\delta L + dL.
\end{aligned}$$

□

The form Ξ_L (34) provides a global Lepage equivalent of a graded Lagrangian L.

Given a graded Lagrangian system $(\mathcal{S}_\infty^*[F;Y], L)$, by its infinitesimal transformations are meant contact graded derivations of a real GCR $\mathcal{S}_\infty^0[F;Y]$. They constitute a $\mathcal{S}_\infty^0[F;Y]$-module $\mathfrak{d}\mathcal{S}_\infty^0[F;Y]$ which is a real Lie superalgebra with respect to the Lie superbracket (7). The following holds [17, 19].

Theorem 17. *The derivation module $\mathfrak{d}\mathcal{S}_\infty^0[F;Y]$ is isomorphic to the $\mathcal{S}_\infty^0[F;Y]$-dual $(\mathcal{S}_\infty^1[F;Y])^*$ of a module of graded one-forms $\mathcal{S}_\infty^1[F;Y]$. It follows that a* DBGA *$\mathcal{S}_\infty^*[F;Y]$ is the minimal Chevalley–Eilenberg differential calculus over a real* GCR *$\mathcal{S}_\infty^0[F;Y]$.*

Let $\vartheta\rfloor\phi$, $\vartheta \in \mathfrak{d}\mathcal{S}^0_\infty[F;Y]$, $\phi \in \mathcal{S}^1_\infty[F;Y]$, denote the corresponding interior product. Extended to a DBGA $\mathcal{S}^*_\infty[F;Y]$, it obeys the rule

$$\vartheta\rfloor(\phi\wedge\sigma) = (\vartheta\rfloor\phi)\wedge\sigma + (-1)^{|\phi|+[\phi][\vartheta]}\phi\wedge(\vartheta\rfloor\sigma), \quad \phi,\sigma \in \mathcal{S}^*_\infty[F;Y].$$

Restricted to the coordinate chart (3) of $J^\infty Y$, the algebra $\mathcal{S}^*_\infty[F;Y]$ is a free $\mathcal{S}^0_\infty[F;Y]$-module generated by one-forms dx^λ, θ^A_Λ. Due to the isomorphism stated in Theorem 17, any graded derivation $\vartheta \in \mathfrak{d}\mathcal{S}^0_\infty[F;Y]$ takes the local form

$$\vartheta = \vartheta^\lambda\partial_\lambda + \vartheta^A\partial_A + \sum_{0<|\Lambda|}\vartheta^A_\Lambda\partial^\Lambda_A, \tag{34}$$

where $\partial^\Lambda_A\rfloor dy^B_\Sigma = \delta^B_A\delta^\Lambda_\Sigma$ up to permutations of multi-indices Λ and Σ. Every graded derivation ϑ (34) yields a graded Lie derivative

$$\mathbf{L}_\vartheta\phi = \vartheta\rfloor d\phi + d(\vartheta\rfloor\phi), \quad \mathbf{L}_\vartheta(\phi\wedge\sigma) = \mathbf{L}_\vartheta(\phi)\wedge\sigma + (-1)^{[\vartheta][\phi]}\phi\wedge\mathbf{L}_\vartheta(\sigma),$$

of a DBGA $\mathcal{S}^*_\infty[F;Y]$.

A graded derivation ϑ (34) is called contact if a Lie derivative $\mathbf{L}_\vartheta$ preserves an ideal of contact graded forms of a DBGA $\mathcal{S}^*_\infty[F;Y]$. It takes the form

$$\vartheta = \upsilon_H + \upsilon_V = \upsilon^\lambda d_\lambda + \left[\upsilon^A\partial_A + \sum_{|\Lambda|>0} d_\Lambda(\upsilon^A - s^A_\mu\upsilon^\mu)\partial^\Lambda_A\right], \tag{35}$$

where υ_H and υ_V denotes the horizontal and vertical parts of ϑ [17, 19]. A glance at the expression (35) shows that a contact graded derivation ϑ is an infinite-order jet prolongation of its restriction

$$\upsilon = \upsilon^\lambda\partial_\lambda + \upsilon^A\partial_A \tag{36}$$

to a GCR $S^0[F;Y]$. Since coefficients ϑ^λ and ϑ^i depend on jet coordinates y^i_Λ, $0 < |\Lambda|$, in general, one calls υ (36) a generalized vector field.

Theorem 18. *A corollary of the decomposition* (33) *is that the Lie derivative of a graded Lagrangian along any contact graded derivation* (35) *obeys the first variational formula*

$$\mathbf{L}_\vartheta L = \upsilon_V\rfloor\delta L + d_H(h_0(\vartheta\rfloor\Xi_L)) + d_V(\upsilon_H\rfloor\omega)\mathcal{L}, \tag{37}$$

where Ξ_L is the Lepage equivalent (34) *of L* [6, 17].

A contact graded derivation ϑ (35) is called a variational symmetry of a graded Lagrangian L if the Lie derivative $\mathbf{L}_\vartheta L$ is d_H-exact, i.e. $\mathbf{L}_\vartheta L = d_H\sigma$.

Lemma 19. *A glance at the expression* (37) *shows the following*: (i) *A contact graded derivation ϑ is a variational symmetry only if it is projected onto X.* (ii) *Any projectable contact graded derivation is a variational symmetry of a variationally trivial graded Lagrangian.* (iii) *A contact graded derivation ϑ is a variational symmetry if and only if its vertical part υ_V* (35) *is well.* (iv) *It is a variational symmetry if and only if a graded density $\upsilon_V\rfloor\delta L$ is d_H-exact.*

Note that generalized symmetries of differential equations and Lagrangians of even variables have been intensively studied [3, 26, 30].

Theorem 20. *If a contact graded derivation ϑ (35) is a variational symmetry of a graded Lagrangian L, the first variational formula (37) restricted to* Ker δL *leads to the weak conservation law*

$$0 \approx d_H(h_0(\vartheta\rfloor\Xi_L) - \sigma).$$

For the sake of brevity, the common symbol υ further stands for the generalized graded vector field υ (36), a contact graded derivation ϑ determined by υ, and a Lie derivative $\mathbf{L}_\vartheta$.

A vertical contact graded derivation $\upsilon = \upsilon^A\partial_A$ is said to be nilpotent if $\upsilon(\upsilon\phi) = 0$ for any horizontal graded form $\phi \in S^{0,*}_\infty[F,Y]$. It is nilpotent only if it is odd and if and only if the equality $\upsilon(\upsilon^A) = 0$ holds for all υ^A [17].

Remark 21. For the sake of convenience, right derivations $\overleftarrow{\upsilon} = \overleftarrow{\partial}_A\upsilon^A$ also are considered. They act on graded functions and differential forms ϕ on the right by the rules

$$\overleftarrow{\upsilon}(\phi) = d\phi\lfloor\overleftarrow{\upsilon} + d(\phi\lfloor\overleftarrow{\upsilon}), \quad \overleftarrow{\upsilon}(\phi\wedge\phi') = (-1)^{[\phi']}\overleftarrow{\upsilon}(\phi)\wedge\phi' + \phi\wedge\overleftarrow{\upsilon}(\phi'),$$
$$\theta_{\Lambda A}\lfloor\overleftarrow{\partial}^{\Sigma B} = \delta^A_B\delta^\Sigma_\Lambda.$$

7. Noether Identities

Let $(\mathcal{S}^*_\infty[F;Y], L)$ be a graded Lagrangian system. Describing its Noether identities, we follow the general analysis of Noether identities of differential operators on fiber bundles [33].

Without a loss of generality, let a Lagrangian L be even. Its Euler–Lagrange operator δL (31) takes its values into a graded vector bundle

$$\overline{VF} = V^*F \underset{F}{\otimes} \overset{n}{\bigwedge} T^*X \to F, \tag{38}$$

where V^*F is the vertical cotangent bundle of $F \to X$. It however is not a vector bundle over Y. Therefore, we restrict our consideration to the case of a pull-back composite bundle

$$F = Y \underset{X}{\times} F^1 \to Y \to X, \tag{39}$$

where $F^1 \to X$ is a vector bundle.

Remark 22. Let us introduce the following notation. Given the vertical tangent bundle VE of a fiber bundle $E \to X$, by its density-dual bundle is meant a fiber bundle

$$\overline{VE} = V^*E \underset{E}{\otimes} \overset{n}{\bigwedge} T^*X. \tag{40}$$

If $E \to X$ is a vector bundle, we have

$$\overline{VE} = \overline{E} \underset{X}{\times} E, \quad \overline{E} = E^* \underset{X}{\otimes} \overset{n}{\bigwedge} T^*X,$$

where $\overline{E}$ is called the density-dual of E. Let $E = E^0 \oplus_X E^1$ be a graded vector bundle over X. Its graded density-dual is defined as $\overline{E} = \overline{E}^1 \oplus_X \overline{E}^0$. In these terms, we treat a composite bundle F as a graded vector bundle over Y possessing only an odd part. The density-dual $\overline{VF}$ (40) of the vertical tangent bundle VF of $F \to X$ is $\overline{VF}$ (38). If F is the pull-back bundle (39), then

$$\overline{VF} = \left(\left(\overline{F}^1 \underset{Y}{\oplus} V^*Y \right) \underset{Y}{\otimes} \overset{n}{\bigwedge} T^*X \right) \underset{Y}{\oplus} F^1 \tag{41}$$

is a graded vector bundle over Y. Given a graded vector bundle $E = E^0 \oplus_Y E^1$ over Y, we consider a composite bundle $E \to E^0 \to X$ and a DBGA

$$\mathcal{P}^*_\infty[E; Y] = \mathcal{S}^*_\infty[E; E^0]. \tag{42}$$

Lemma 23. *One can associate to any graded Lagrangian system* $(\mathcal{S}^*_\infty[F;Y], L)$ *the chain complex* (43) *whose one-boundaries vanish on* $\operatorname{Ker}\delta L$.

Proof. Let us consider the density-dual $\overline{VF}$ (41) of the vertical tangent bundle $VF \to F$, and let us enlarge an original algebra $\mathcal{S}^*_\infty[F;Y]$ to the DBGA $\mathcal{P}^*_\infty[\overline{VF};Y]$ (42) with a local generating basis $(s^A, \overline{s}_A)$, $[\overline{s}_A] = ([A]+1) \operatorname{mod} 2$. Following the physical terminology [4, 21], we agree to call its elements $\overline{s}_A$ the antifields of antifield number $\operatorname{Ant}[\overline{s}_A] = 1$. A DBGA $\mathcal{P}^*_\infty[\overline{VF};Y]$ is endowed with a nilpotent right-graded derivation $\overline{\delta} = \overleftarrow{\partial}^A \mathcal{E}_A$, where $\mathcal{E}_A$ are the variational derivatives (31). Then we have a chain complex

$$0 \leftarrow \operatorname{Im}\overline{\delta} \overset{\overline{\delta}}{\longleftarrow} \mathcal{P}^{0,n}_\infty[\overline{VF};Y]_1 \overset{\overline{\delta}}{\longleftarrow} \mathcal{P}^{0,n}_\infty[\overline{VF};Y]_2 \tag{43}$$

of graded densities of antifield number ≤ 2. Its one-boundaries $\overline{\delta}\Phi$, $\Phi \in \mathcal{P}^{0,n}_\infty[\overline{VF};Y]_2$, by very definition, vanish on $\operatorname{Ker}\delta L$. □

Any one-cycle

$$\Phi = \sum_{0\leq|\Lambda|} \Phi^{A,\Lambda} \overline{s}_{\Lambda A} \omega \in \mathcal{P}^{0,n}_\infty[\overline{VF};Y]_1 \tag{44}$$

of the complex (43) is a differential operator on a fiber bundle $\overline{VF}$ such that it is linear on fibers of $\overline{VF} \to F$ and its kernel contains the graded Euler–Lagrange operator δL (31), i.e.

$$\overline{\delta}\Phi = 0, \qquad \sum_{0\leq|\Lambda|} \Phi^{A,\Lambda} d_\Lambda \mathcal{E}_A \omega = 0. \tag{45}$$

These equalities are Noether identities of an Euler–Lagrange operator δL [6, 8, 33].

In particular, one-chain Φ (44) are necessarily Noether identities if they are boundaries. Therefore, these Noether identities are called trivial. Accordingly, non-trivial Noether identities modulo the trivial ones are associated to elements of the first homology $H_1(\overline{\delta})$ of the complex (43). A Lagrangian L is called degenerate if there are non-trivial Noether identities.

Non-trivial Noether identities can obey first-stage Noether identities. In order to describe them, let us assume that the module $H_1(\overline{\delta})$ is finitely generated. Namely, there exists a graded projective $C^\infty(X)$-module $\mathcal{C}_{(0)} \subset H_1(\overline{\delta})$ of finite rank possessing a local basis $\{\Delta_r\omega\}$:

$$\Delta_r\omega = \sum_{0\leq|\Lambda|} \Delta_r^{A,\Lambda}\overline{s}_{\Lambda A}\omega, \quad \Delta_r^{A,\Lambda} \in \mathcal{S}^0_\infty[F;Y], \tag{46}$$

such that any element $\Phi \in H_1(\overline{\delta})$ factorizes as

$$\Phi = \sum_{0\leq|\Xi|} \Phi^{r,\Xi} d_\Xi \Delta_r\omega, \quad \Phi^{r,\Xi} \in \mathcal{S}^0_\infty[F;Y], \tag{47}$$

through elements (46) of $\mathcal{C}_{(0)}$. Thus, all non-trivial Noether identities (45) result from Noether identities

$$\overline{\delta}\Delta_r = \sum_{0\leq|\Lambda|} \Delta_r^{A,\Lambda} d_\Lambda \mathcal{E}_A = 0, \tag{48}$$

called the complete Noether identities.

Lemma 24. *If the homology $H_1(\overline{\delta})$ of the complex* (43) *is finitely generated in the above-mentioned sense, this complex can be extended to the one-exact chain complex* (50) *with a boundary operator whose nilpotency conditions are equivalent to the complete Noether identities* (48).

Proof. By virtue of Serre–Swan Theorem 7, a graded module $\mathcal{C}_{(0)}$ is isomorphic to a module of sections of the density-dual $\overline{E}_0$ of some graded vector bundle $E_0 \to X$. Let us enlarge $\mathcal{P}^*_\infty[\overline{VF};Y]$ to a DBGA

$$\overline{\mathcal{P}}^*_\infty\{0\} = \mathcal{P}^*_\infty\left[\overline{VF} \underset{Y}{\oplus} \overline{E}_0; Y\right], \tag{49}$$

possessing the local generating basis $(s^A, \overline{s}_A, \overline{c}_r)$ where $\overline{c}_r$ are antifields of Grassmann parity $[\overline{c}_r] = ([\Delta_r]+1)\bmod 2$ and antifield number $\mathrm{Ant}[\overline{c}_r] = 2$. The DBGA (49) is provided with an odd right-graded derivation $\delta_0 = \overline{\delta} + \overleftarrow{\partial}^r\Delta_r$ which is nilpotent if and only if the complete Noether identities (48) hold. Then δ_0 is a boundary operator of a chain complex

$$0 \leftarrow \mathrm{Im}\,\overline{\delta} \xleftarrow{\overline{\delta}} \mathcal{P}^{0,n}_\infty[\overline{VF};Y]_1 \xleftarrow{\delta_0} \overline{\mathcal{P}}^{0,n}_\infty\{0\}_2 \xleftarrow{\delta_0} \overline{\mathcal{P}}^{0,n}_\infty\{0\}_3 \tag{50}$$

of graded densities of antifield number ≤ 3. Let $H_*(\delta_0)$ denote its homology. We have $H_0(\delta_0) = H_0(\overline{\delta}) = 0$. Furthermore, any one-cycle Φ up to a boundary takes the form (47) and, therefore, it is a δ_0-boundary

$$\Phi = \sum_{0\leq|\Sigma|} \Phi^{r,\Xi} d_\Xi \Delta_r\omega = \delta_0\left(\sum_{0\leq|\Sigma|} \Phi^{r,\Xi}\overline{c}_{\Xi r}\omega\right).$$

Hence, $H_1(\delta_0) = 0$, i.e. the complex (50) is one-exact. $\square$

Let us consider the second homology $H_2(\delta_0)$ of the complex (50). Its two-chains read

$$\Phi = G + H = \sum_{0\leq|\Lambda|} G^{r,\Lambda}\overline{c}_{\Lambda r}\omega + \sum_{0\leq|\Lambda|,|\Sigma|} H^{(A,\Lambda)(B,\Sigma)}\overline{s}_{\Lambda A}\overline{s}_{\Sigma B}\omega. \tag{51}$$

Its two-cycles define first-stage Noether identities

$$\delta_0\Phi = 0, \qquad \sum_{0\leq|\Lambda|} G^{r,\Lambda} d_\Lambda \Delta_r \omega = -\overline{\delta} H. \tag{52}$$

Conversely, let the equality (52) hold. Then it is a cycle condition of the two-chains (51).

The first-stage Noether identities (52) are trivial either if a two-cycle Φ (51) is a δ_0-boundary or its summand G vanishes on $\operatorname{Ker}\delta L$. Therefore, non-trivial first-stage Noether identities fails to exhaust the second homology $H_2(\delta_0)$ the complex (50) in general.

Lemma 25. *Non-trivial first-stage Noether identities modulo the trivial ones are identified with elements of the homology $H_2(\delta_0)$ if and only if any $\overline{\delta}$-cycle $\phi \in \overline{\mathcal{P}}^{0,n}_\infty\{0\}_2$ is a δ_0-boundary.*

Proof. It suffices to show that, if the summand G of a two-cycle Φ (51) is $\overline{\delta}$-exact, then Φ is a boundary. If $G = \overline{\delta}\Psi$, let us write

$$\Phi = \delta_0\Psi + (\overline{\delta} - \delta_0)\Psi + H. \tag{53}$$

Hence, the cycle condition (52) reads

$$\delta_0\Phi = \overline{\delta}((\overline{\delta} - \delta_0)\Psi + H) = 0.$$

Since any $\overline{\delta}$-cycle $\phi \in \overline{\mathcal{P}}^{0,n}_\infty\{0\}_2$, by assumption, is δ_0-exact, then $(\overline{\delta} - \delta_0)\Psi + H$ is a δ_0-boundary. Consequently, Φ (53) is δ_0-exact. Conversely, let $\Phi \in \overline{\mathcal{P}}^{0,n}_\infty\{0\}_2$ be a $\overline{\delta}$-cycle, i.e.

$$\overline{\delta}\Phi = 2\Phi^{(A,\Lambda)(B,\Sigma)}\overline{s}_{\Lambda A}\overline{\delta}\overline{s}_{\Sigma B}\omega = 2\Phi^{(A,\Lambda)(B,\Sigma)}\overline{s}_{\Lambda A} d_\Sigma \mathcal{E}_B \omega = 0.$$

It follows that $\Phi^{(A,\Lambda)(B,\Sigma)}\overline{\delta}\overline{s}_{\Sigma B} = 0$ for all indices (A,Λ). Omitting a $\overline{\delta}$-boundary term, we obtain

$$\Phi^{(A,\Lambda)(B,\Sigma)}\overline{s}_{\Sigma B} = G^{(A,\Lambda)(r,\Xi)} d_\Xi \Delta_r.$$

Hence, Φ takes the form $\Phi = G'^{(A,\Lambda)(r,\Xi)} d_\Xi \Delta_r \overline{s}_{\Lambda A}\omega$. Then there exists a three-chain $\Psi = G'^{(A,\Lambda)(r,\Xi)}\overline{c}_{\Xi r}\overline{s}_{\Lambda A}\omega$ such that

$$\delta_0\Psi = \Phi + \sigma = \Phi + G''^{(A,\Lambda)(r,\Xi)} d_\Lambda \mathcal{E}_A \overline{c}_{\Xi r}\omega. \tag{54}$$

Owing to the equality $\overline{\delta}\Phi = 0$, we have $\delta_0\sigma = 0$. Thus, σ in the expression (54) is $\overline{\delta}$-exact δ_0-cycle. By assumption, it is δ_0-exact, i.e. $\sigma = \delta_0\psi$. Consequently, a $\overline{\delta}$-cycle Φ is a δ_0-boundary $\Phi = \delta_0\Psi - \delta_0\psi$. □

A degenerate Lagrangian system is called reducible if it admits non-trivial first-stage Noether identities.

If the condition of Lemma 25 is satisfied, let us assume that non-trivial first-stage Noether identities are finitely generated as follows. There exists a graded projective $C^\infty(X)$-module $\mathcal{C}_{(1)} \subset H_2(\delta_0)$ of finite rank possessing a local basis $\{\Delta_{r_1}\omega\}$:

$$\Delta_{r_1}\omega = \sum_{0\leq|\Lambda|} \Delta_{r_1}^{r,\Lambda}\overline{c}_{\Lambda r}\omega + h_{r_1}\omega, \tag{55}$$

such that any element $\Phi \in H_2(\delta_0)$ factorizes as

$$\Phi = \sum_{0\leq|\Xi|} \Phi^{r_1,\Xi} d_\Xi \Delta_{r_1}\omega, \quad \Phi^{r_1,\Xi} \in \mathcal{S}^0_\infty[F;Y], \tag{56}$$

through elements (55) of $\mathcal{C}_{(1)}$. Thus, all non-trivial first-stage Noether identities (52) result from the equalities

$$\sum_{0\leq|\Lambda|} \Delta_{r_1}^{r,\Lambda} d_\Lambda \Delta_r + \overline{\delta} h_{r_1} = 0, \tag{57}$$

called the complete first-stage Noether identities.

Lemma 26. *The one-exact complex* (50) *of a reducible Lagrangian system is extended to the two-exact one* (58) *with a boundary operator whose nilpotency conditions are equivalent to the complete Noether identities* (48) *and the complete first-stage Noether identities* (57).

Proof. By virtue of Serre–Swan Theorem 7, a graded module $\mathcal{C}_{(1)}$ is isomorphic to a module of sections of the density-dual $\overline{E}_1$ of some graded vector bundle $E_1 \to X$. Let us enlarge the DBGA $\overline{\mathcal{P}}^*_\infty\{0\}$ (49) to a DBGA

$$\overline{\mathcal{P}}^*_\infty\{1\} = \mathcal{P}^*_\infty\left[\overline{VF}\underset{Y}{\oplus}\overline{E}_0\underset{Y}{\oplus}\overline{E}_1;Y\right],$$

possessing a local generating basis $\{s^A, \overline{s}_A, \overline{c}_r, \overline{c}_{r_1}\}$ where $\overline{c}_{r_1}$ are first-stage Noether antifields of Grassmann parity $[\overline{c}_{r_1}] = ([\Delta_{r_1}] + 1)\bmod 2$ and antifield number $\mathrm{Ant}[\overline{c}_{r_1}] = 3$. This DBGA is provided with an odd right-graded derivation $\delta_1 = \delta_0 + \overleftarrow{\partial}^{r_1}\Delta_{r_1}$, which is nilpotent if and only if the complete Noether identities (48) and the complete first-stage Noether identities (57) hold. Then δ_1 is a boundary operator of a chain complex

$$0 \leftarrow \mathrm{Im}\,\overline{\delta} \xleftarrow{\overline{\delta}} \mathcal{P}^{0,n}_\infty[\overline{VF};Y]_1 \xleftarrow{\delta_0} \overline{\mathcal{P}}^{0,n}_\infty\{0\}_2 \xleftarrow{\delta_1} \overline{\mathcal{P}}^{0,n}_\infty\{1\}_3 \xleftarrow{\delta_1} \overline{\mathcal{P}}^{0,n}_\infty\{1\}_4 \tag{58}$$

of graded densities of antifield number ≤ 4. Let $H_*(\delta_1)$ denote its homology. It is readily observed that

$$H_0(\delta_1) = H_0(\overline{\delta}), \quad H_1(\delta_1) = H_1(\delta_0) = 0.$$

By virtue of the expression (56), any two-cycle of the complex (58) is a boundary

$$\Phi = \sum_{0\leq|\Xi|} \Phi^{r_1,\Xi} d_\Xi \Delta_{r_1}\omega = \delta_1 \left(\sum_{0\leq|\Xi|} \Phi^{r_1,\Xi} \overline{c}_{\Xi r_1}\omega \right).$$

It follows that $H_2(\delta_1) = 0$, i.e. the complex (58) is two-exact. □

If the third homology $H_3(\delta_1)$ of the complex (58) is not trivial, its elements correspond to second-stage Noether identities which the complete first-stage ones satisfy, and so on. Iterating the arguments, one comes to the following.

A degenerate graded Lagrangian system $(\mathcal{S}^*_\infty[F;Y], L)$ is called N-stage reducible if it admits finitely generated non-trivial N-stage Noether identities, but no non-trivial $(N+1)$-stage ones. It is characterized as follows [7, 8].

- There are graded vector bundles $E_0, \ldots, E_N$ over X, and a DBGA $\mathcal{P}^*_\infty[\overline{VF}; Y]$ is enlarged to a DBGA

$$\overline{\mathcal{P}}^*_\infty\{N\} = \mathcal{P}^*_\infty\left[\overline{VF} \underset{Y}{\oplus} \overline{E}_0 \underset{Y}{\oplus} \cdots \underset{Y}{\oplus} \overline{E}_N; Y\right], \tag{59}$$

  with the local generating basis $(s^A, \overline{s}_A, \overline{c}_r, \overline{c}_{r_1}, \ldots, \overline{c}_{r_N})$ where $\overline{c}_{r_k}$ are Noether k-stage antifields of antifield number $\mathrm{Ant}[\overline{c}_{r_k}] = k+2$.
- The DBGA (59) is provided with the nilpotent right-graded derivation

$$\delta_{\mathrm{KT}} = \delta_N = \overline{\delta} + \sum_{0\leq|\Lambda|} \overleftarrow{\partial}^r \Delta^{A,\Lambda}_r \overline{s}_{\Lambda A} + \sum_{1\leq k\leq N} \overleftarrow{\partial}^{r_k} \Delta_{r_k}, \tag{60}$$

$$\Delta_{r_k}\omega = \sum_{0\leq|\Lambda|} \Delta^{r_{k-1},\Lambda}_{r_k} \overline{c}_{\Lambda r_{k-1}}\omega$$
$$+ \sum_{0\leq|\Sigma|,|\Xi|} \left(h^{(r_{k-2},\Sigma)(A,\Xi)}_{r_k} \overline{c}_{\Sigma r_{k-2}} \overline{s}_{\Xi A} + \cdots \right)\omega \in \overline{\mathcal{P}}^{0,n}_\infty\{k-1\}_{k+1}, \tag{61}$$

  of antifield number -1. The index $k = -1$ here stands for $\overline{s}_A$. The nilpotent derivation δ_{KT} (60) is called the Koszul–Tate operator.
- With this graded derivation, the module $\overline{\mathcal{P}}^{0,n}_\infty\{N\}_{\leq N+3}$ of densities of antifield number $\leq (N+3)$ is decomposed into the exact Koszul–Tate chain complex

$$0 \leftarrow \mathrm{Im}\,\overline{\delta} \overset{\overline{\delta}}{\longleftarrow} \mathcal{P}^{0,n}_\infty[\overline{VF};Y]_1 \overset{\delta_0}{\longleftarrow} \overline{\mathcal{P}}^{0,n}_\infty\{0\}_2 \overset{\delta_1}{\longleftarrow} \overline{\mathcal{P}}^{0,n}_\infty\{1\}_3$$
$$\cdots \overset{\delta_{N-1}}{\longleftarrow} \overline{\mathcal{P}}^{0,n}_\infty\{N-1\}_{N+1} \overset{\delta_{\mathrm{KT}}}{\longleftarrow} \overline{\mathcal{P}}^{0,n}_\infty\{N\}_{N+2} \overset{\delta_{\mathrm{KT}}}{\longleftarrow} \overline{\mathcal{P}}^{0,n}_\infty\{N\}_{N+3} \tag{62}$$

  which satisfies the following homology regularity condition.

Condition 27. Any $\delta_{k<N}$-cycle $\phi \in \overline{\mathcal{P}}^{0,n}_\infty\{k\}_{k+3} \subset \overline{\mathcal{P}}^{0,n}_\infty\{k+1\}_{k+3}$ is a δ_{k+1}-boundary.

Note that the exactness of the complex (62) means that any $\delta_{k<N}$-cycle $\phi \in \mathcal{P}^{0,n}_\infty\{k\}_{k+3}$, is a δ_{k+2}-boundary, but not necessarily a δ_{k+1}-one.

- The nilpotentness $\delta^2_{\mathrm{KT}} = 0$ of the Koszul–Tate operator (60) is equivalent to complete non-trivial Noether identities (48) and complete non-trivial ($k \leq N$)-stage Noether identities

$$\sum_{0\leq|\Lambda|} \Delta^{r_{k-1},\Lambda}_{r_k} d_\Lambda \left(\sum_{0\leq|\Sigma|} \Delta^{r_{k-2},\Sigma}_{r_{k-1}} \overline{c}_{\Sigma r_{k-2}} \right) = -\overline{\delta} \left(\sum_{0\leq|\Sigma|,|\Xi|} h^{(r_{k-2},\Sigma)(A,\Xi)}_{r_k} \overline{c}_{\Sigma r_{k-2}} \overline{s}_{\Xi A} \right). \tag{63}$$

This item means the following.

Proposition 28. *Any δ_k-cocycle $\Phi \in \mathcal{P}^{0,n}_\infty\{k\}_{k+2}$ is a k-stage Noether identity, and vice versa.*

Proof. Any $(k+2)$-chain $\Phi \in \mathcal{P}^{0,n}_\infty\{k\}_{k+2}$ takes the form

$$\Phi = G + H = \sum_{0\leq|\Lambda|} G^{r_k,\Lambda} \overline{c}_{\Lambda r_k} \omega + \sum_{0\leq\Sigma,0\leq\Xi} (H^{(A,\Xi)(r_{k-1},\Sigma)} \overline{s}_{\Xi A} \overline{c}_{\Sigma r_{k-1}} + \cdots)\omega. \tag{64}$$

If it is a δ_k-cycle, then

$$\sum_{0\leq|\Lambda|} G^{r_k,\Lambda} d_\Lambda \left(\sum_{0\leq|\Sigma|} \Delta^{r_{k-1},\Sigma}_{r_k} \overline{c}_{\Sigma r_{k-1}} \right) + \overline{\delta} \left(\sum_{0\leq\Sigma,0\leq\Xi} H^{(A,\Xi)(r_{k-1},\Sigma)} \overline{s}_{\Xi A} \overline{c}_{\Sigma r_{k-1}} \right) = 0 \tag{65}$$

are the k-stage Noether identities. Conversely, let the condition (65) hold. Then it can be extended to a cycle condition as follows. It is brought into the form

$$\delta_k \left(\sum_{0\leq|\Lambda|} G^{r_k,\Lambda} \overline{c}_{\Lambda r_k} + \sum_{0\leq\Sigma,0\leq\Xi} H^{(A,\Xi)(r_{k-1},\Sigma)} \overline{s}_{\Xi A} \overline{c}_{\Sigma r_{k-1}} \right) = -\sum_{0\leq|\Lambda|} G^{r_k,\Lambda} d_\Lambda h_{r_k} + \sum_{0\leq\Sigma,0\leq\Xi} H^{(A,\Xi)(r_{k-1},\Sigma)} \overline{s}_{\Xi A} d_\Sigma \Delta_{r_{k-1}}.$$

A glance at the expression (61) shows that a term in the right-hand side of this equality belongs to $\mathcal{P}^{0,n}_\infty\{k-2\}_{k+1}$. It is a δ_{k-2}-cycle, then a δ_{k-1}-boundary $\delta_{k-1}\Psi$ in accordance with Condition 27. Then the equality (65) is a $\overline{c}_{\Sigma r_{k-1}}$-dependent part of a cycle condition

$$\delta_k \left(\sum_{0\leq|\Lambda|} G^{r_k,\Lambda} \overline{c}_{\Lambda r_k} + \sum_{0\leq\Sigma,0\leq\Xi} H^{(A,\Xi)(r_{k-1},\Sigma)} \overline{s}_{\Xi A} \overline{c}_{\Sigma r_{k-1}} - \Psi \right) = 0,$$

but $\delta_k \Psi$ does not make a contribution to this condition. □

Proposition 29. *Any trivial k-stage Noether identity is a δ_k-boundary* $\Phi \in \mathcal{P}^{0,n}_\infty\{k\}_{k+2}$.

Proof. The k-stage Noether identities (65) are trivial either if a δ_k-cycle Φ (64) is a δ_k-boundary or its summand G vanishes on $\operatorname{Ker}\delta L$. Let us show that, if the summand G of Φ (64) is $\overline{\delta}$-exact, then Φ is a δ_k-boundary. If $G = \overline{\delta}\Psi$, one can write

$$\Phi = \delta_k\Psi + (\overline{\delta} - \delta_k)\Psi + H.$$

Hence, the δ_k-cycle condition reads

$$\delta_k\Phi = \delta_{k-1}((\overline{\delta} - \delta_k)\Psi + H) = 0.$$

By virtue of Condition 27, any δ_{k-1}-cycle $\phi \in \overline{\mathcal{P}}^{0,n}_\infty\{k-1\}_{k+2}$ is δ_k-exact. Then $(\overline{\delta} - \delta_k)\Psi + H$ is a δ_k-boundary. Consequently, Φ (64) is δ_k-exact. □

Note that all non-trivial k-stage Noether identities (65), by assumption, factorize as

$$\Phi = \sum_{0\le|\Xi|} \Phi^{r_k,\Xi} d_\Xi \Delta_{r_k}\omega, \quad \Phi^{r_1,\Xi} \in \mathcal{S}^0_\infty[F;Y],$$

through the complete ones (63).

It may happen that a graded Lagrangian system possesses non-trivial Noether identities of any stage. However, we restrict our consideration to N-reducible Lagrangian systems.

8. Second Noether Theorems

Different variants of the second Noether theorem have been suggested in order to relate reducible Noether identities and gauge symmetries [4, 6, 18]. The inverse second Noether Theorem 32, that we formulate in homology terms, associates to the Koszul–Tate complex (62) of non-trivial Noether identities the cochain sequence (71) with the ascent operator $\mathbf{u}$ (72) whose components are non-trivial gauge and higher-stage gauge symmetries.

Remark 30. Let us use the following notation. Given the DBGA $\overline{\mathcal{P}}^*_\infty\{N\}$ (59), we consider a DBGA

$$\mathcal{P}^*_\infty\{N\} = \mathcal{P}^*_\infty\left[F \underset{Y}{\oplus} E_0 \underset{Y}{\oplus} \cdots \underset{Y}{\oplus} E_N; Y\right], \tag{66}$$

possessing a local generating basis $(s^A, c^r, c^{r_1}, \ldots, c^{r_N})$, $[c^{r_k}] = ([\overline{c}_{r_k}] + 1) \bmod 2$, and a DBGA

$$P^*_\infty\{N\} = \mathcal{P}^*_\infty\left[\overline{VF} \underset{Y}{\oplus} E_0 \oplus \cdots \underset{Y}{\oplus} E_N \underset{Y}{\oplus} \overline{E}_0 \underset{Y}{\oplus} \cdots \underset{Y}{\oplus} \overline{E}_N; Y\right], \tag{67}$$

with a local generating basis $(s^A, \overline{s}_A, c^r, c^{r_1}, \ldots, c^{r_N}, \overline{c}_r, \overline{c}_{r_1}, \ldots, \overline{c}_{r_N})$. Following the physical terminology, we call their elements c^{r_k} the k-stage ghosts of ghost number

$\mathrm{gh}[c^{r_k}] = k+1$ and antifield number $\mathrm{Ant}[c^{r_k}] = -(k+1)$. A $C^\infty(X)$-module $\mathcal{C}^{(k)}$ of k-stage ghosts is the density-dual of a module $\mathcal{C}_{(k)}$ of k-stage antifields. The DBGAs $\overline{\mathcal{P}}^*_\infty\{N\}$ (59) and $\mathcal{P}^*_\infty\{N\}$ (66) are subalgebras of $P^*_\infty\{N\}$ (67). The Koszul–Tate operator δ_{KT} (60) is naturally extended to a graded derivation of a DBGA $P^*_\infty\{N\}$.

Remark 31. Any graded differential form $\phi \in \mathcal{S}^*_\infty[F;Y]$ and any finite tuple (f^Λ), $0 \leq |\Lambda| \leq k$, of local graded functions $f^\Lambda \in \mathcal{S}^0_\infty[F;Y]$ obey the following relations [19]:

$$\sum_{0\leq|\Lambda|\leq k} f^\Lambda d_\Lambda \phi \wedge \omega = \sum_{0\leq|\Lambda|} (-1)^{|\Lambda|} d_\Lambda(f^\Lambda)\phi\wedge\omega + d_H\sigma, \tag{68}$$

$$\sum_{0\leq|\Lambda|\leq k} (-1)^{|\Lambda|} d_\Lambda(f^\Lambda\phi) = \sum_{0\leq|\Lambda|\leq k} \eta(f)^\Lambda d_\Lambda\phi,$$

$$\eta(f)^\Lambda = \sum_{0\leq|\Sigma|\leq k-|\Lambda|} (-1)^{|\Sigma+\Lambda|} \frac{(|\Sigma+\Lambda|)!}{|\Sigma|!|\Lambda|!} d_\Sigma f^{\Sigma+\Lambda}, \tag{69}$$

$$\eta(\eta(f))^\Lambda = f^\Lambda. \tag{70}$$

Theorem 32. *Given a Koszul–Tate complex* (62), *the module of graded densities* $\mathcal{P}^{0,n}_\infty\{N\}$ *is decomposed into a cochain sequence*

$$0 \to \mathcal{S}^{0,n}_\infty[F;Y] \xrightarrow{\mathbf{u}} \mathcal{P}^{0,n}_\infty\{N\}^1 \xrightarrow{\mathbf{u}} \mathcal{P}^{0,n}_\infty\{N\}^2 \xrightarrow{\mathbf{u}} \cdots, \tag{71}$$

$$\mathbf{u} = u + u^{(1)} + \cdots + u^{(N)} = u^A\partial_A + u^r\partial_r + \cdots + u^{r_{N-1}}\partial_{r_{N-1}}, \tag{72}$$

graded in a ghost number. Its ascent operator $\mathbf{u}$ (72) *is an odd-graded derivation of ghost number* 1 *where* u (77) *is a variational symmetry of a graded Lagrangian* L *and the graded derivations* $u_{(k)}$ (80), $k = 1,\ldots,N$, *obey the relations* (79).

Proof. Given the Koszul–Tate operator (60), let us extend an original grade Lagrangian L to a Lagrangian

$$L_e = L + L_1 = L + \sum_{0\leq k\leq N} c^{r_k}\Delta_{r_k}\omega = L + \delta_{\mathrm{KT}}\left(\sum_{0\leq k\leq N} c^{r_k}\overline{c}_{r_k}\omega\right) \tag{73}$$

of zero antifield number. It is readily observed that a Koszul–Tate operator δ_{KT} is an exact symmetry of the extended Lagrangian $L_e \in P^{0,n}_\infty\{N\}$ (73). Since a graded derivation δ_{KT} is vertical, it follows from the first variational formula (37) that

$$\left[\frac{\overleftarrow{\delta}\mathcal{L}_e}{\delta\overline{s}_A}\mathcal{E}_A + \sum_{0\leq k\leq N}\frac{\overleftarrow{\delta}\mathcal{L}_e}{\delta\overline{c}_{r_k}}\Delta_{r_k}\right]\omega = \left[\upsilon^A\mathcal{E}_A + \sum_{0\leq k\leq N}\upsilon^{r_k}\frac{\delta\mathcal{L}_e}{\delta c^{r_k}}\right]\omega = d_H\sigma,$$

$$\upsilon^A = \frac{\overleftarrow{\delta}\mathcal{L}_e}{\delta\overline{s}_A} = u^A + w^A = \sum_{0\leq|\Lambda|} c^r_\Lambda \eta(\Delta^A_r)^\Lambda + \sum_{1\leq i\leq N}\sum_{0\leq|\Lambda|} c^{r_i}_\Lambda \eta(\overleftarrow{\partial}^A(h_{r_i}))^\Lambda,$$

$$\upsilon^{r_k} = \frac{\overleftarrow{\delta}\mathcal{L}_e}{\delta\overline{c}_{r_k}} = u^{r_k} + w^{r_k} = \sum_{0\leq|\Lambda|} c^{r_{k+1}}_\Lambda \eta(\Delta^{r_k}_{r_{k+1}})^\Lambda + \sum_{k+1<i\leq N}\sum_{0\leq|\Lambda|} c^{r_i}_\Lambda \eta(\overleftarrow{\partial}^{r_k}(h_{r_i}))^\Lambda. \tag{74}$$

The equality (74) is split into a set of equalities

$$\frac{\overleftarrow{\delta}(c^r\Delta_r)}{\delta\overline{s}_A}\mathcal{E}_A\omega = u^A\mathcal{E}_A\omega = d_H\sigma_0, \tag{75}$$

$$\left[\frac{\overleftarrow{\delta}(c^{r_k}\Delta_{r_k})}{\delta\overline{s}_A}\mathcal{E}_A + \sum_{0\leq i<k}\frac{\overleftarrow{\delta}(c^{r_k}\Delta_{r_k})}{\delta\overline{c}_{r_i}}\Delta_{r_i}\right]\omega = d_H\sigma_k, \tag{76}$$

where $k = 1,\ldots,N$. A glance at the equality (75) shows that, by virtue of the first variational formula (37), an odd-graded derivation

$$u = u^A\frac{\partial}{\partial s^A}, \quad u^A = \sum_{0\leq|\Lambda|} c^r_\Lambda \eta(\Delta^A_r)^\Lambda, \tag{77}$$

of $\mathcal{P}^0\{0\}$ is a variational symmetry of a graded Lagrangian L. Every equality (76) falls into a set of equalities graded by the polynomial degree in antifields. Let us consider that of them which is linear in antifields $\overline{c}_{r_{k-2}}$. We have

$$\frac{\overleftarrow{\delta}}{\delta\overline{s}_A}\left(c^{r_k}\sum_{0\leq|\Sigma|,|\Xi|} h^{(r_{k-2},\Sigma)(A,\Xi)}_{r_k}\overline{c}_{\Sigma r_{k-2}}\overline{s}_{\Xi A}\right)\mathcal{E}_A\omega$$
$$+\frac{\overleftarrow{\delta}}{\delta\overline{c}_{r_{k-1}}}\left(c^{r_k}\sum_{0\leq|\Sigma|}\Delta^{r'_{k-1},\Sigma}_{r_k}\overline{c}_{\Sigma r'_{k-1}}\right)\sum_{0\leq|\Xi|}\Delta^{r_{k-2},\Xi}_{r_{k-1}}\overline{c}_{\Xi r_{k-2}}\omega = d_H\sigma_k.$$

This equality is brought into the form

$$\sum_{0\leq|\Xi|}(-1)^{|\Xi|}d_\Xi\left(c^{r_k}\sum_{0\leq|\Sigma|} h^{(r_{k-2},\Sigma)(A,\Xi)}_{r_k}\overline{c}_{\Sigma r_{k-2}}\right)\mathcal{E}_A\omega$$
$$+u^{r_{k-1}}\sum_{0\leq|\Xi|}\Delta^{r_{k-2},\Xi}_{r_{k-1}}\overline{c}_{\Xi r_{k-2}}\omega = d_H\sigma_k.$$

Using the relation (68), we obtain an equality

$$\sum_{0\leq|\Xi|} c^{r_k}\sum_{0\leq|\Sigma|} h^{(r_{k-2},\Sigma)(A,\Xi)}_{r_k}\overline{c}_{\Sigma r_{k-2}}d_\Xi\mathcal{E}_A\omega$$
$$+u^{r_{k-1}}\sum_{0\leq|\Xi|}\Delta^{r_{k-2},\Xi}_{r_{k-1}}\overline{c}_{\Xi r_{k-2}}\omega = d_H\sigma'_k. \tag{78}$$

A variational derivative of both its sides with respect to $\overline{c}_{r_{k-2}}$ leads to a relation

$$\sum_{0\leq|\Sigma|} d_\Sigma u^{r_{k-1}}\partial^\Sigma_{r_{k-1}} u^{r_{k-2}} = \overline{\delta}(\alpha^{r_{k-2}}),$$
$$\alpha^{r_{k-2}} = -\sum_{0\leq|\Sigma|}\eta(h^{(r_{k-2})(A,\Xi)}_{r_k})^\Sigma d_\Sigma(c^{r_k}\overline{s}_{\Xi A}), \tag{79}$$

which an odd-graded derivation

$$u^{(k)} = u^{r_{k-1}}\partial_{r_{k-1}} = \sum_{0\leq|\Lambda|} c^{r_k}_\Lambda \eta(\Delta^{r_{k-1}}_{r_k})^\Lambda \partial_{r_{k-1}}, \quad k=1,\ldots,N, \tag{80}$$

satisfies. Graded derivations u (77) and $u^{(k)}$ (80) are assembled into the ascent operator $\mathbf{u}$ (72) of the cochain sequence (71). □

A glance at the expression (77) shows that a variational symmetry u is a linear differential operator on a $C^\infty(X)$-module $\mathcal{C}^{(0)}$ of ghosts. Therefore, it is a gauge symmetry of a graded Lagrangian L which is associated to the complete Noether identities (48) [18, 19]. This association is unique due to the following direct second Noether theorem.

Theorem 33. *A variational derivative of the equality* (75) *with respect to ghosts* c^r *leads to the equality*

$$\delta_r(u^A\mathcal{E}_A\omega) = \sum_{0\leq|\Lambda|}(-1)^{|\Lambda|}d_\Lambda(\eta(\Delta^A_r)^\Lambda\mathcal{E}_A) = \sum_{0\leq|\Lambda|}(-1)^{|\Lambda|}\eta(\eta(\Delta^A_r))^\Lambda d_\Lambda\mathcal{E}_A = 0,$$

which reproduces the complete Noether identities (48) *by means of the relation* (70).

Moreover, the gauge symmetry u (77) is complete in the following sense. Let

$$\sum_{0\leq|\Xi|} C^R G^{r,\Xi}_R d_\Xi \Delta_r\omega$$

be some projective $C^\infty(X)$-module of finite rank of non-trivial Noether identities (47) parametrized by the corresponding ghosts C^R. We have the equalities

$$\begin{aligned}
0 &= \sum_{0\leq|\Xi|} C^R G^{r,\Xi}_R d_\Xi\left(\sum_{0\leq|\Lambda|}\Delta^{A,\Lambda}_r d_\Lambda\mathcal{E}_A\right)\omega \\
&= \sum_{0\leq|\Lambda|}\left(\sum_{0\leq|\Xi|}\eta(G^r_R)^\Xi C^R_\Xi\right)\Delta^{A,\Lambda}_r d_\Lambda\mathcal{E}_A\omega + d_H(\sigma) \\
&= \sum_{0\leq|\Lambda|}(-1)^{|\Lambda|}d_\Lambda\left(\Delta^{A,\Lambda}_r\sum_{0\leq|\Xi|}\eta(G^r_R)^\Xi C^R_\Xi\right)\mathcal{E}_A\omega + d_H\sigma
\end{aligned}$$

$$= \sum_{0\le|\Lambda|} \eta(\Delta_r^A)^\Lambda d_\Lambda \left(\sum_{0\le|\Xi|} \eta(G_R^r)^\Xi C_\Xi^R\right) \mathcal{E}_A\omega + d_H\sigma$$

$$= \sum_{0\le|\Lambda|} u_r^{A,\Lambda} d_\Lambda \left(\sum_{0\le|\Xi|} \eta(G_R^r)^\Xi C_\Xi^R\right) \mathcal{E}_A\omega + d_H\sigma.$$

It follows that a graded derivation

$$d_\Lambda \left(\sum_{0\le|\Xi|} \eta(G_R^r)^\Xi C_\Xi^R\right) u_r^{A,\Lambda}\partial_A$$

is a variational symmetry of a graded Lagrangian L and, consequently, its gauge symmetry parametrized by ghosts C^R. It factorizes through the gauge symmetry (77) by putting ghosts

$$c^r = \sum_{0\le|\Xi|} \eta(G_R^r)^\Xi C_\Xi^R.$$

Turn now to the relation (79). For $k = 1$, it takes the form

$$\sum_{0\le|\Sigma|} d_\Sigma u^r \partial_r^\Sigma u^A = \overline{\delta}(\alpha^A)$$

of a first-stage gauge symmetry condition on $\operatorname{Ker}\delta L$ which the non-trivial gauge symmetry u (77) satisfies. Therefore, one can treat an odd-graded derivation

$$u^{(1)} = u^r \frac{\partial}{\partial c^r}, \quad u^r = \sum_{0\le|\Lambda|} c_\Lambda^{r_1} \eta(\Delta_{r_1}^r)^\Lambda,$$

as a first-stage gauge symmetry associated to the complete first-stage Noether identities

$$\sum_{0\le|\Lambda|} \Delta_{r_1}^{r,\Lambda} d_\Lambda \left(\sum_{0\le|\Sigma|} \Delta_r^{A,\Sigma}\overline{s}_{\Sigma A}\right) = -\overline{\delta}\left(\sum_{0\le|\Sigma|,|\Xi|} h_{r_1}^{(B,\Sigma)(A,\Xi)}\overline{s}_{\Sigma B}\overline{s}_{\Xi A}\right).$$

Iterating the arguments, one comes to the relation (79) which provides a k-stage gauge symmetry condition, associated to the complete k-stage Noether identities (63).

Theorem 34. *Conversely, given the k-stage gauge symmetry condition* (79), *a variational derivative of the equality* (78) *with respect to ghosts c^{r_k} leads to an equality, reproducing the k-stage Noether identities* (63) *by means of the relations* (69) *and* (70).

This is a higher-stage extension of the direct second Noether theorem to reducible gauge symmetries. The odd-graded derivation $u_{(k)}$ (80) is called the

k-stage gauge symmetry. It is complete as follows. Let

$$\sum_{0\leq|\Xi|} C^{R_k} G^{r_k,\Xi}_{R_k} d_\Xi \Delta_{r_k}\omega$$

be a projective $C^\infty(X)$-module of finite rank of non-trivial k-stage Noether identities (47) factorizing through the complete ones (63) and parametrized by the corresponding ghosts C^{R_k}. One can show that it defines a k-stage gauge symmetry factorizing through $u^{(k)}$ (80) by putting k-stage ghosts

$$c^{r_k} = \sum_{0\leq|\Xi|} \eta(G^{r_k}_{R_k})^\Xi C^{R_k}_\Xi.$$

The odd-graded derivation $u_{(k)}$ (80) is said to be the complete non-trivial k-stage gauge symmetry of a Lagrangian L. Thus, components of the ascent operator $\mathbf{u}$ (72) are complete non-trivial gauge and higher-stage gauge symmetries.

Appendix A

We quote the following generalization of the abstract de Rham theorem [24]. Let

$$0 \to S \xrightarrow{h} S_0 \xrightarrow{h^0} S_1 \xrightarrow{h^1} \cdots \xrightarrow{h^{p-1}} S_p \xrightarrow{h^p} S_{p+1}, \quad p>1,$$

be an exact sequence of sheaves of Abelian groups over a paracompact topological space Z, where the sheaves S_q, $0\leq q<p$, are acyclic, and let

$$0 \to \Gamma(Z,S) \xrightarrow{h_*} \Gamma(Z,S_0) \xrightarrow{h^0_*} \Gamma(Z,S_1) \xrightarrow{h^1_*} \cdots \xrightarrow{h^{p-1}_*} \Gamma(Z,S_p) \xrightarrow{h^p_*} \Gamma(Z,S_{p+1}) \qquad \text{(A.1)}$$

be the corresponding cochain complex of sections of these sheaves.

Theorem A.1. *The q-cohomology groups of the cochain complex* (A.1) *for* $0\leq q\leq p$ *are isomorphic to the cohomology groups* $H^q(Z,S)$ *of* Z *with coefficients in the sheaf* S [16, 37].

Appendix B

The proof of Theorems 13 and 14 falls into the following three steps [17, 19, 34].

(I) We start with showing that the complexes (29) and (30) are locally exact.

Lemma B.1. *If* $Y=\mathbb{R}^{n+k}\to\mathbb{R}^n$, *the complex* (29) *is acyclic.*

Proof. Referring to [4] for the proof, we summarize a few formulas. Any horizontal graded form $\phi\in\mathcal{S}^{0,*}_\infty[F;Y]$ admits a decomposition

$$\phi=\phi_0+\widetilde\phi, \quad \widetilde\phi=\int_0^1 \frac{d\lambda}{\lambda}\sum_{0\leq|\Lambda|} s^A_\Lambda \partial^\Lambda_A\phi, \qquad \text{(B.1)}$$

where ϕ_0 is an exterior form on $\mathbb{R}^{n+k}$. Let $\phi\in\mathcal{S}^{0,m<n}_\infty[F;Y]$ be d_H-closed. Then its component ϕ_0 (B.1) is an exact exterior form on $\mathbb{R}^{n+k}$ and $\widetilde\phi=d_H\xi$, where ξ is

given by the following expressions. Let us introduce an operator

$$D^{+\nu}\widetilde{\phi} = \int_0^1 \frac{d\lambda}{\lambda} \sum_{0\leq k} k\delta^\nu_{(\mu_1}\delta^{\alpha_1}_{\mu_2}\cdots\delta^{\alpha_{k-1}}_{\mu_k)}\lambda s^A_{(\alpha_1\cdots\alpha_{k-1})}\partial_A^{\mu_1\cdots\mu_k}\widetilde{\phi}(x^\mu,\lambda s^A_\Lambda, dx^\mu).$$

The relation $[D^{+\nu}, d_\mu]\widetilde{\phi} = \delta^\nu_\mu\widetilde{\phi}$ holds, and it leads to a desired expression

$$\xi = \sum_{k=0} \frac{(n-m-1)!}{(n-m+k)!} D^{+\nu} P_k \partial_\nu \rfloor \widetilde{\phi},$$
$$P_0 = 1, \quad P_k = d_{\nu_1}\cdots d_{\nu_k} D^{+\nu_1}\cdots D^{+\nu_k}. \tag{B.2}$$

Now let $\phi \in \mathcal{S}^{0,n}_\infty[F;Y]$ be a graded density such that $\delta\phi = 0$. Then its component ϕ_0 (B.1) is an exact n-form on $\mathbb{R}^{n+k}$ and $\widetilde{\phi} = d_H\xi$, where ξ is given by the expression

$$\xi = \sum_{|\Lambda|\geq 0}\sum_{\Sigma+\Xi=\Lambda} (-1)^{|\Sigma|} s^A_\Xi d_\Sigma \partial_A^{\mu+\Lambda}\widetilde{\phi}\omega_\mu. \tag{B.3}$$

We also quote the homotopy operator (5.107) in [30] which leads to the expression

$$\xi = \int_0^1 I(\phi)(x^\mu, \lambda s^A_\Lambda, dx^\mu)\frac{d\lambda}{\lambda},$$
$$I(\phi) = \sum_{|\Lambda|,\mu} \frac{\Lambda_\mu + 1}{n-m+|\Lambda|+1} d_\Lambda \left[\sum_{|\Xi|}(-1)^\Xi \frac{(\mu+\Lambda+\Xi)!}{(\mu+\Lambda)!\Xi!} s^A d_\Xi \partial_A^{\mu+\Lambda+\Xi}(\partial_\mu\rfloor\phi)\right], \tag{B.4}$$

where $\Lambda! = \Lambda_{\mu_1}!\cdots\Lambda_{\mu_n}!$, and Λ_μ denotes a number of occurrences of the index μ in Λ [30]. The graded forms (B.3) and (B.4) differ in a d_H-exact graded form. □

Lemma B.2. *If* $Y = \mathbb{R}^{n+k} \to \mathbb{R}^n$, *the complex* (30) *is exact.*

Proof. The fact that a d_H-closed graded form $\phi \in \mathcal{S}^{1,m<n}_\infty[F;Y]$ is d_H-exact is derived from Lemma B.1 as follows. We write

$$\phi = \sum \phi^\Lambda_A \wedge \theta^A_\Lambda, \tag{B.5}$$

where $\phi^\Lambda_A \in \mathcal{S}^{0,m}_\infty[F;Y]$ are horizontal graded m-forms. Let us introduce additional variables $\overline{s}^A_\Lambda$ of the same Grassmann parity as s^A_Λ. Then one can associate to each graded $(1,m)$-form ϕ (B.5) a unique horizontal graded m-form

$$\overline{\phi} = \sum \phi^\Lambda_A \overline{s}^A_\Lambda, \tag{B.6}$$

whose coefficients are linear in variables $\overline{s}^A_\Lambda$, and vice versa. Let us put a modified total differential

$$\bar{d}_H = d_H + dx^\lambda \wedge \sum_{0<|\Lambda|} \overline{s}^A_{\lambda+\Lambda}\overline{\partial}^\Lambda_A,$$

acting on graded forms (B.6), where $\overline{\partial}^\Lambda_A$ is the dual of $d\,\overline{s}^A_\Lambda$. Comparing the equalities

$$\bar{d}_H\overline{s}^A_\Lambda = dx^\lambda s^A_{\lambda+\Lambda}, \quad d_H\theta^A_\lambda = dx^\lambda \wedge \theta^A_{\lambda+\Lambda},$$

one can easily justify that $\overline{d_H\phi} = \bar{d}_H\overline{\phi}$. Let the graded $(1,m)$-form ϕ (B.5) be d_H-closed. Then the associated horizontal graded m-form $\overline{\phi}$ (B.6) is $\bar{d}_H$-closed and, by virtue of Lemma B.1, it is $\bar{d}_H$-exact, i.e. $\overline{\phi} = \bar{d}_H\overline{\xi}$, where $\overline{\xi}$ is a horizontal graded $(m-1)$-form given by the expression (B.2) depending on additional variables $\overline{s}^A_\Lambda$. A glance at this expression shows that, since $\overline{\phi}$ is linear in variables $\overline{s}^A_\Lambda$, so is $\overline{\xi} = \sum \xi^\Lambda_A \overline{s}^A_\Lambda$. It follows that $\phi = d_H\xi$ where $\xi = \sum \xi^\Lambda_A \wedge \theta^A_\Lambda$. It remains to prove the exactness of the complex (30) at the last term $\varrho(\mathcal{S}^{1,n}_\infty[F;Y])$. If

$$\varrho(\sigma) = \sum_{0\leq|\Lambda|} (-1)^{|\Lambda|}\theta^A \wedge [d_\Lambda(\partial^\Lambda_A \rfloor \sigma)] = \sum_{0\leq|\Lambda|} (-1)^{|\Lambda|}\theta^A \wedge [d_\Lambda \sigma^\Lambda_A]\omega = 0, \quad \sigma \in \mathcal{S}^{1,n}_\infty,$$

a direct computation gives

$$\sigma = d_H\xi, \quad \xi = -\sum_{0\leq|\Lambda|}\sum_{\Sigma+\Xi=\Lambda} (-1)^{|\Sigma|}\theta^A_\Xi \wedge d_\Sigma \sigma^{\mu+\Lambda}_A \omega_\mu. \qquad \square$$

(II) Let us now prove Theorems 13 and 14 for a DBGA $\mathcal{Q}^*_\infty[F;Y]$. Similarly to $\mathcal{S}^*_\infty[F;Y]$, the sheaf $\mathfrak{Q}^*_\infty[F;Y]$ and a DBGA $\mathcal{Q}^*_\infty[F;Y]$ are decomposed into Grassmann-graded variational bicomplexes. We consider their subcomplexes

$$0 \to \mathbb{R} \to \mathfrak{Q}^0_\infty[F;Y] \xrightarrow{d_H} \mathfrak{Q}^{0,1}_\infty[F;Y] \xrightarrow{d_H} \cdots \xrightarrow{d_H} \mathfrak{Q}^{0,n}_\infty[F;Y] \xrightarrow{\delta} \varrho(\mathfrak{Q}^{1,n}_\infty[F;Y]), \tag{B.7}$$

$$0 \to \mathfrak{Q}^{1,0}_\infty[F;Y] \xrightarrow{d_H} \mathfrak{Q}^{1,1}_\infty[F;Y] \xrightarrow{d_H} \cdots \xrightarrow{d_H} \mathfrak{Q}^{1,n}_\infty[F;Y] \xrightarrow{\varrho} \varrho(\mathfrak{Q}^{1,n}_\infty[F;Y]) \to 0, \tag{B.8}$$

$$0 \to \mathbb{R} \to \mathcal{Q}^0_\infty[F;Y] \xrightarrow{d_H} \mathcal{Q}^{0,1}_\infty[F;Y] \xrightarrow{d_H} \cdots \xrightarrow{d_H} \mathcal{Q}^{0,n}_\infty[F;Y] \xrightarrow{\delta} \Gamma(\varrho(\mathfrak{Q}^{1,n}_\infty[F;Y])), \tag{B.9}$$

$$0 \to \mathcal{Q}^{1,0}_\infty[F;Y] \xrightarrow{d_H} \mathcal{Q}^{1,1}_\infty[F;Y] \xrightarrow{d_H} \cdots \xrightarrow{d_H} \mathcal{Q}^{1,n}_\infty[F;Y] \xrightarrow{\varrho} \Gamma(\varrho(\mathfrak{Q}^{1,n}_\infty[F;Y])) \to 0. \tag{B.10}$$

By virtue of Lemmas B.1 and B.2, the complexes (B.7) and (B.8) are acyclic. The terms $\mathfrak{Q}^{*,*}_\infty[F;Y]$ of the complexes (B.7) and (B.8) are sheaves of $\mathcal{Q}^0_\infty[F;Y]$-modules. Since $J^\infty Y$ admits the partition of unity just by elements of $\mathcal{Q}^0_\infty[F;Y]$, these sheaves are fine and, consequently, acyclic. By virtue of abstract de Rham Theorem A.1, cohomology of the complex (B.9) equals the cohomology of $J^\infty Y$ with coefficients in the constant sheaf $\mathbb{R}$ and, consequently, the de Rham cohomology of Y in accordance with the isomorphisms (4). Similarly, the complex (B.10) is proved to be exact.

Due to monomorphisms $\mathcal{O}^*_\infty \to \mathcal{S}^*_\infty[F;Y] \to \mathcal{Q}^*_\infty[F;Y]$ this proof gives something more.

Theorem B.3. *Every d_H-closed graded form $\phi \in \mathcal{Q}^{0,m<n}_\infty[F;Y]$ falls into the sum*

$$\phi = h_0\sigma + d_H\xi, \quad \xi \in \mathcal{Q}^{0,m-1}_\infty[F;Y], \tag{B.11}$$

where σ is a closed m-form on Y. Any δ-closed $\phi \in \mathcal{Q}^{0,n}_\infty[F;Y]$ is the sum

$$\phi = h_0\sigma + d_H\xi, \quad \xi \in \mathcal{Q}^{0,n-1}_\infty[F;Y], \tag{B.12}$$

where σ is a closed n-form on Y.

(III) It remains to prove that cohomology of the complexes (29) and (30) equals that of the complexes (B.9) and (B.10).

Let the common symbol D stand for d_H and δ. Bearing in mind the decompositions (B.11) and (B.12), it suffices to show that, if an element $\phi \in \mathcal{S}^*_\infty[F;Y]$ is D-exact in an algebra $\mathcal{Q}^*_\infty[F;Y]$, then it is so in an algebra $\mathcal{S}^*_\infty[F;Y]$.

Lemma B.1 states that, if Y is a contractible bundle and a D-exact graded form ϕ on $J^\infty Y$ is of finite jet order $[\phi]$ (i.e. $\phi \in \mathcal{S}^*_\infty[F;Y]$), there exists a graded form $\varphi \in \mathcal{S}^*_\infty[F;Y]$ on $J^\infty Y$ such that $\phi = D\varphi$. Moreover, a glance at the expressions (B.2) and (B.3) shows that a jet order $[\varphi]$ of φ is bounded by an integer $N([\phi])$, depending only on a jet order of ϕ. Let us call this fact the finite exactness of an operator D. Lemma B.1 shows that the finite exactness takes place on $J^\infty Y|_U$ over any domain $U \subset Y$. Let us prove the following.

Lemma B.4. *Given a family $\{U_\alpha\}$ of disjoint open subsets of Y, let us suppose that the finite exactness takes place on $J^\infty Y|_{U_\alpha}$ over every subset U_α from this family. Then, it is true on $J^\infty Y$ over the union $\cup_\alpha U_\alpha$ of these subsets.*

Proof. Let $\phi \in \mathcal{S}^*_\infty[F;Y]$ be a D-exact graded form on $J^\infty Y$. The finite exactness on $(\pi^\infty_0)^{-1}(\cup U_\alpha)$ holds since $\phi = D\varphi_\alpha$ on every $(\pi^\infty_0)^{-1}(U_\alpha)$ and $[\varphi_\alpha] < N([\phi])$. □

Lemma B.5. *Suppose that the finite exactness of an operator D takes place on $J^\infty Y$ over open subsets U, V of Y and their non-empty overlap $U \cap V$. Then, it is also true on $J^\infty Y|_{U\cup V}$.*

Proof. Let $\phi = D\varphi \in \mathcal{S}^*_\infty[F;Y]$ be a D-exact form on $J^\infty Y$. By assumption, it can be brought into the form $D\varphi_U$ on $(\pi^\infty_0)^{-1}(U)$ and $D\varphi_V$ on $(\pi^\infty_0)^{-1}(V)$, where φ_U and φ_V are graded forms of bounded jet order. Let us consider their difference $\varphi_U - \varphi_V$ on $(\pi^\infty_0)^{-1}(U \cap V)$. It is a D-exact graded form of bounded jet order $[\varphi_U - \varphi_V] < N([\phi])$ which, by assumption, can be written as $\varphi_U - \varphi_V = D\sigma$ where σ is also of bounded jet order $[\sigma] < N(N([\phi]))$. Lemma B.6 below shows that $\sigma = \sigma_U + \sigma_V$ where σ_U and σ_V are graded forms of bounded jet order on

$(\pi_0^\infty)^{-1}(U)$ and $(\pi_0^\infty)^{-1}(V)$, respectively. Then, putting

$$\varphi'|_U = \varphi_U - D\sigma_U, \quad \varphi'|_V = \varphi_V + D\sigma_V,$$

we have a graded form ϕ, equal to $D\varphi'_U$ on $(\pi_0^\infty)^{-1}(U)$ and $D\varphi'_V$ on $(\pi_0^\infty)^{-1}(V)$, respectively. Since the difference $\varphi'_U - \varphi'_V$ on $(\pi_0^\infty)^{-1}(U \cap V)$ vanishes, we obtain $\phi = D\varphi'$ on $(\pi_0^\infty)^{-1}(U \cup V)$ where

$$\varphi' = \begin{cases} \varphi'|_U = \varphi'_U, \\ \varphi'|_V = \varphi'_V \end{cases}$$

is of bounded jet order $[\varphi'] < N(N([\phi]))$. □

Lemma B.6. *Let U and V be open subsets of a bundle Y and $\sigma \in \mathfrak{G}^*_\infty$ a graded form of bounded jet order on $(\pi_0^\infty)^{-1}(U \cap V) \subset J^\infty Y$. Then, σ is decomposed into a sum $\sigma_U + \sigma_V$ of graded forms σ_U and σ_V of bounded jet order on $(\pi_0^\infty)^{-1}(U)$ and $(\pi_0^\infty)^{-1}(V)$, respectively.*

Proof. By taking a smooth partition of unity on $U \cup V$ subordinate to a cover $\{U, V\}$ and passing to a function with support in V, one gets a smooth real function f on $U \cup V$ which equals 0 on a neighborhood of $U \backslash V$ and 1 on a neighborhood of $V \backslash U$ in $U \cup V$. Let $(\pi_0^\infty)^* f$ be the pull-back of f onto $(\pi_0^\infty)^{-1}(U \cup V)$. A graded form $((\pi_0^\infty)^* f)\sigma$ equals 0 on a neighborhood of $(\pi_0^\infty)^{-1}(U)$ and, therefore, can be extended by 0 to $(\pi_0^\infty)^{-1}(U)$. Let us denote it as σ_U. Accordingly, a graded form $(1 - (\pi_0^\infty)^* f)\sigma$ has an extension σ_V by 0 to $(\pi_0^\infty)^{-1}(V)$. Then, $\sigma = \sigma_U + \sigma_V$ is a desired decomposition because σ_U and σ_V are of the jet order which does not exceed that of σ. □

To prove the finite exactness of D on $J^\infty Y$, it remains to choose an appropriate cover of Y. A smooth manifold Y admits a countable cover $\{U_\xi\}$ by domains U_ξ, $\xi \in \mathbf{N}$, and its refinement $\{U_{ij}\}$, where $j \in \mathbf{N}$ and i runs through a finite set, such that $U_{ij} \cap U_{ik} = \emptyset$, $j \neq k$ [22]. Then Y has a finite cover $\{U_i = \cup_j U_{ij}\}$. Since the finite exactness of an operator D takes place over any domain U_ξ, it also holds over any member U_{ij} of the refinement $\{U_{ij}\}$ of $\{U_\xi\}$ and, in accordance with Lemma B.4, over any member of a finite cover $\{U_i\}$ of Y. Then by virtue of Lemma B.5, the finite exactness of D takes place on $J^\infty Y$ over Y.

Similarly, one can show that, restricted to $\mathcal{S}^{k,n}_\infty[F;Y]$, the operator ϱ remains exact.

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