TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 349, Number 10, October 1997, Pages 4021–4051
S 0002-9947(97)02005-9

GRADED LIE ALGEBRAS OF MAXIMAL CLASS

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Abstract. We study graded Lie algebras of maximal class over a field $\mathbf{F}$ of positive characteristic p. A. Shalev has constructed infinitely many pairwise non-isomorphic insoluble algebras of this kind, thus showing that these algebras are more complicated than might be suggested by considering only associated Lie algebras of p-groups of maximal class. Here we construct $|\mathbf{F}|^{\aleph_0}$ pairwise non-isomorphic such algebras, and $\max\{|\mathbf{F}|, \aleph_0\}$ soluble ones. Both numbers are shown to be best possible. We also exhibit classes of examples with a non-periodic structure. As in the case of groups, two-step centralizers play an important role.

1. Introduction

In a recent paper Shalev [Sh2] has shown that graded Lie algebras of maximal class are more complicated than might be suggested by considering only associated Lie algebras of p-groups with maximal class. A nilpotent graded Lie algebra L has maximal class if it has class c and dimension $c + 1$. Thus,

$$L = L_1 \oplus L_2 \oplus \cdots \oplus L_c$$

with $\dim L_1 = 2$ and, for $2 \le i \le c$, $\dim L_i = 1$ and $[L_{i-1}L_1] = L_i$. Shalev shows that over fields with positive characteristic there are such algebras with arbitrarily large soluble length. In this paper we exhibit many more examples, and prove some structure theorems which show that for every field $\mathbf{F}$ with positive characteristic there are $\max(|\mathbf{F}|, \aleph_0)$ isomorphism types of nilpotent graded Lie algebras of maximal class over $\mathbf{F}$.

As usual we include under the heading 'maximal class' algebras L graded by the positive integers; that is,

$$L = L_1 \oplus L_2 \oplus \cdots \oplus L_i \oplus \dots$$

with $\dim L_1 = 2$ and, for $2 \le i$, $\dim L_i = 1$ and $[L_{i-1}L_1] = L_i$. These algebras can be viewed as (projective) limits of nilpotent graded Lie algebras of maximal class. They are just infinite-dimensional in the sense that, correctly interpreted, all their proper quotients are finite-dimensional. Shalev ([Sh2], Theorem 1) showed that there are insoluble algebras of this kind. We show that over every field $\mathbf{F}$ with

Received by the editors March 1, 1996.
1991 *Mathematics Subject Classification.* Primary 17B70, 17B65, 17B05, 17B30, 17B40, 20D15, 20F40.
Key words and phrases. Graded Lie algebras of maximal class, p-groups of maximal class, pro-p-groups of finite coclass, nilpotent Lie algebras.
The first two authors are members of CNR–GNSAGA, Italy, and acknowledge support of MURST, Italy. The third author acknowledges support from CNR-GNSAGA, Italy, and the University of Trento, Italy.

positive characteristic there are $|\mathbf{F}|^{\aleph_0}$ isomorphism types of graded Lie algebras of maximal class (Corollary 9.4).

As with p-groups with maximal class (and indeed finite coclass), the *two-step* centralizers $C(L^i/L^{i+2})$, where L^j is a Lie power, play an important role [Bla]—in fact more so because, in a sense described in Section 3, graded Lie algebras of maximal class are determined by their two-step centralizers (Theorem 3.2). We associate with every graded Lie algebra of maximal class L over a field $\mathbf{F}$ a (coefficient) sequence of elements of the projective line $\mathbf{F}_\infty$ over $\mathbf{F}$. The coefficient sequence of the graded Lie algebra $\mathcal{M}$, of maximal class, which comes from pro-p-groups with maximal class is the constant sequence $(0, \dots)$. All terms of a coefficient sequence except 0 are isolated (Lemma 3.3). There are further strong restrictions on the coefficient sequences (Section 5).

The algebras constructed by Shalev are (just) infinite-dimensional algebras based on some finite-dimensional algebras of Albert and Frank [AF]. Their coefficient sequences are periodic with period length $p^n - 1$, where p is the characteristic. We call a graded Lie algebra of maximal class whose coefficient sequence is ultimately periodic *centralizer periodic*.

The basic construction, which we call *inflation*, is described in Section 6 (we are indebted to Aner Shalev and Georgia Benkart for drawing our attention to the work of Block [Blo] in which this construction is also used – and hence, to its use in the fundamental paper of Zassenhaus [Z]). The inflations of the algebra $\mathcal{M}$ are soluble and have period length a power of the characteristic. Inflation can be reversed by a process we call *deflation* (Section 7). Every just infinite soluble graded Lie algebra of maximal class deflates to $\mathcal{M}$ (Theorem 8.3). On the other hand, deflation is not necessarily reversed by inflation. Still we get that a graded Lie algebra of maximal class is soluble if and only if it is centralizer periodic with period length a power of the characteristic (Theorem 8.2). It follows that there are $\max(|\mathbf{F}|, \aleph_0)$ soluble graded Lie algebras of maximal class over $\mathbf{F}$. And there are the same number of insoluble centralizer periodic ones (Proposition 6.3).

In the work on groups of finite coclass there is the notion of a covered group, one in which all maximal subgroups are two-step centralizers (see [M2] and [Ma]). Some of our results are similar to some in [M2]. There are no groups of maximal class which are covered; in fact they have at most 2 distinct two-step centralizers. There are covered graded Lie algebras of maximal class for every finite (Proposition 6.2) and every countable field with positive characteristic (Theorem 9.2).

These we obtain by repeating the inflation process countably many times, via an inverse limit (Section 9). The same procedure allows us to construct graded Lie algebra of maximal class that are not centralizer periodic. This answers a question of Shalev [Sh3, Problem 28], and is further evidence that graded Lie algebras of finite coclass in positive characteristic are distinctly more complicated than the analogous groups. In characteristic zero, the situation appears to be much better [SZ].

Our results are inspired by, though independent of, computations (Section 10).

2. Preliminaries

An *algebra of maximal class* is a graded Lie algebra

$$L = \bigoplus_{i=1}^{\infty} L_i$$

over a field $\mathbf{F}$, where $\dim(L_1) = 2$, $\dim(L_i) \leq 1$ for $i \geq 2$, and $[L_i L_1] = L_{i+1}$ for all $i \geq 1$, or equivalently, L_1 generates L as an algebra. All subalgebras and ideals will be taken to be graded.

It is sometimes useful to consider the Cartesian product

$$L = \prod_{i=1}^{\infty} L_i$$

instead of the direct product (or sum). When this is done, the algebra acquires a topology by taking the Lie powers

$$L^i = \prod_{j \geq i} L_j,$$

that is, the terms of the lower central series, for $i \geq 2$, as a fundamental system of neighbourhoods of 0. With respect to this topology, L is complete. All subalgebras and ideals will be taken to be graded and closed in this case. We will use the topological setting only in Section 9, where it allows us to consider inverse limits. On the other hand, we use the discrete setting in Section 4, for uniformity with the paper of Shalev [Sh2] we are building upon there.

Algebras of maximal class are either finite dimensional, or just-(infinite dimensional), that is, all their proper factors are finite-dimensional. In fact, if $M \neq 0$ is an ideal of L, take $0 \neq m \in M \cap L_i$, for some i. Then $M \geq \bigoplus_{j \geq i} L_j$, so that M has codimension at most i.

We denote by $\mathbf{F}_q$ the finite field with q elements, for q a prime-power. When $\mathbf{F}$ is a field, we write $\mathbf{F}_\infty = \mathbf{F} \cup \{\infty\}$.

A notation like

$$a_1, a_2^m, (a_3, a_4^n)^\infty,$$

where a_i are arbitrary elements, and m, n are non-negative integers, denotes the sequence

$$a_1, \underbrace{a_2, \dots, a_2}_{m}, a_3, \underbrace{a_4, \dots, a_4}_{n}, a_3, \underbrace{a_4, \dots, a_4}_{n}, a_3, \underbrace{a_4, \dots, a_4}_{n}, \dots .$$

We say such a sequence is *periodic,* a *period* being in this case $a_3, \underbrace{a_4, \dots, a_4}_{n}$, and a *pre-period* being $a_1, \underbrace{a_2, \dots, a_2}_{m}$.

The Jacobi identity in Lie algebras

$$[u[yx]] = [uyx] - [uxy],$$

and its consequence

$$[u[y\underbrace{x \cdots x}_{n}]] = \sum_{i=0}^{n} (-1)^i \binom{n}{i} [u\underbrace{x \cdots x}_{i}\, y\, \underbrace{x \cdots x}_{n-i}],$$

will be used without specific mention.

We will also use several times Lucas' Theorem [L]. Let a and b be two non-negative integers, and p a prime. Write a and b in p-adic form,

$$a = a_n p^n + a_{n-1} p^{n-1} + \cdots + a_1 p + a_0,$$
$$b = b_n p^n + b_{n-1} p^{n-1} + \cdots + b_1 p + b_0,$$

that is, with $0 \leq a_i, b_i < p$, for all i. Then we can compute the binomial coefficient modulo p as

$$\binom{a}{b} \equiv \binom{a_n}{b_n}\binom{a_{n-1}}{b_{n-1}} \dots \binom{a_1}{b_1}\binom{a_0}{b_0} \pmod{p}.$$

This is easily proved by comparing coefficients in the polynomial equality over $\mathbf{F}_p$:

$$(1+t)^a = (1+t^{p^n})^{a_n} \dots (1+t^p)^{a_1}(1+t)^{a_0}.$$

As a simple example, if q is a power of p we have

$$\binom{2q-1}{q} \equiv \binom{q}{q}\binom{q-1}{0} \equiv 1 \pmod{p}.$$

Our standard references for groups and Lie algebras are [Hu] and [Jac].

3. Two-step centralizers

Let $L = \bigoplus_{i \geq 1} L_i$ be an algebra of maximal class over the field $\mathbf{F}$ of positive characteristic p. In this context, the two-step centralizers

$$C_L\left(L^i \Big/ L^{i+2}\right)$$

are equivalent to the centralizers in L_1 of the homogeneous components L_i for $i \geq 2$, as

$$C_L\left(L^i \Big/ L^{i+2}\right) = C_{L_1}(L_i) + L^2.$$

Therefore we define the *two-step centralizers* to be the subspaces of L_1

$$C_i = C_{L_1}(L_i),$$

for $i \geq 2$. Now C_i is a subspace of L_1 of dimension 1, except when L is finite dimensional, of class at most i, so that $C_i = L_1$ in this case. The *sequence of two-step centralizers* is the sequence $(C_i)_{i \geq 2}$. If one of the terms of this sequence equals L_1, then so do all the following ones. In this case we may omit all the terms of the sequence following the first L_1; if we do this, the sequence is infinite if L is infinite dimensional, and has length $c - 1$ if L is finite dimensional of class c. The *abelian algebra of maximal class* $L = L_1$, of dimension 2, has an empty sequence of two-step centralizers.

Now consider the *set* $\mathcal{C} = \{\, C_i : i \geq 2 \,\} \setminus \{L_1\}$ in an algebra L. If $\mathcal{C}$ has n elements, we say that L *has n (distinct) two-step centralizers*. We also number the distinct two-step centralizers in order of occurrence, so C_2 is the first two-step centralizer; if $C_2 = C_3 = \dots = C_{k-1} \neq C_k$, we say that C_k is the second (distinct) two-step centralizer, and so on. It is to be noted that we will apply ordinal numbers to two-step centralizer in this sense only.

Let $M = \mathbf{F}[t]$ be the (abelian) Lie algebra derived from the ring of polynomials. Consider the derivation D of M given by multiplication by t, that is, $fD = tf$ for $f \in M$. Let $\mathcal{M}$ be the Lie algebra $\mathbf{F}x + M$, where x induces the derivation D on M. Write $y = 1 \in M$. Let $\mathcal{M}_1 = \mathbb{F}x + \mathbb{F}y$. It is easily seen that $\mathcal{M}_i = \mathbf{F}t^{i-1}$, for $i > 1$, and that $C_{\mathcal{M}_1}(\mathcal{M}_i) = \mathbf{F}y$ for $i \geq 2$. Note that $\mathcal{M}$ has a presentation with generators x and y, and relators $[y\underbrace{x \cdots x}_{i}y]$, for $i \geq 1$. We recall that a Lie algebra L is said to be metabelian when $[LL]$ is abelian. When $\mathbf{F}$ is a prime field $\mathbf{F}_p$, this is the graded Lie algebra associated to the unique infinite pro-p-group of maximal class. We first have

3.1 Lemma. *If all the two-step centralizers distinct from L_1 coincide in the algebra L of maximal class, then L is a factor of $\mathcal{M}$.*

Proof. Let $C_2 = \mathbf{F}y$ and let $x \in L_1 \setminus \mathbf{F}y$. Then L is generated by $\{x, y\}$ and satisfies the defining relations of $\mathcal{M}$, and is thus a quotient of it. □

It follows that every metabelian algebra is a quotient of $\mathcal{M}$, and $\mathcal{M}$ is the unique infinite-dimensional metabelian algebra of maximal class. In fact, let L be a metabelian algebra of maximal class, and $\mathbf{F}y = C_2$. Choose $x \in L_1 \setminus \mathbf{F}y$, so that $L_2 = \mathbf{F} \cdot [yx]$, and $[yxy] = 0$. We have, by induction on i,

$$0 = [[y \underbrace{x \cdots x}_{i}][yx]] = [y \underbrace{x \cdots x}_{i} yx] - [y \underbrace{x \cdots x}_{i+1} y] = -[y \underbrace{x \cdots x}_{i+1} y],$$

for all $i \geq 1$, so that $\mathbf{F}y$ is the unique two-step centralizer. We call $\mathcal{M}$ *the infinite-dimensional metabelian algebra of maximal class*.

It follows immediately from the standard theory of p-groups of maximal class ([Hu, III.14]) that the family of algebras of maximal class associated with the pro-p-groups of maximal class consists of the infinite-dimensional metabelian algebra of maximal class over $\mathbf{F}_p$, of its quotients, and, for $p > 3$, of one algebra for each even dimension $6, 8, \ldots, p+1$. The latter algebras are associated with the exceptional p-groups of maximal class, those that have two distinct two-step centralizers.

One can actually show that the sequence of two-step centralizers of an algebra L of maximal class determines the isomorphism type of L in all cases. To formulate this precisely, we have to be able to compare the two-step centralizer sequences of two algebras, and say when they are the same. We do this by attaching a canonical form to the sequence of two-step centralizers.

Choose a non-zero element y in the first two-step centralizer of a non-abelian algebra L of maximal class. If there are no other distinct two-step centralizers, we have already seen that L is determined. Suppose then that L has at least two distinct two-step centralizers, and choose a non-zero element x in the second two-step centralizer. If there is a third distinct two-step centralizer, then one can assume, possibly replacing x, say, with a scalar multiple of it, that the third distinct two-step centralizer contains the element $y - x$. Now all two-step centralizers contain a uniquely determined element of the form $y - \alpha x$, for some $\alpha \in \mathbf{F}_\infty$, where we understand $y - \infty x = x$. This procedure amounts to introducing an affine coordinate on a projective line.

Define the *coefficient sequence* of L to be the (normalised) sequence $(\alpha_i)_{i \geq 2}$ in $\mathbf{F}_\infty$, where $C_{L_1}(L_i) = \mathbf{F} \cdot (y - \alpha_i x)$. A simple calculation shows that the coefficient sequence depends only on L. We say that two algebras of maximal class have the same sequences of two-step centralizers when their coefficient sequences, determined as above, are the same. We have

3.2 Theorem. *An algebra of maximal class is determined up to isomorphism by its sequence of two-step centralizers.*

Proof. We show that the coefficient sequence of the algebra L uniquely determines a presentation of L, or equivalently its adjoint representation on L^2. If all two-step centralizers of L coincide, we have just seen that L is a factor algebra of the unique infinite-dimensional metabelian one. So assume L has at least two distinct two-step centralizers, and write the two-step centralizers in the canonical form described above.

We first have that L is generated by x and y. We have $L_2 = \mathbf{F}u_2 \neq 0$, where $u_2 = [yx]$. We define recursively, if $L_i \neq 0$,

$$u_i = \begin{cases} [u_{i-1}x] & \text{if } [u_{i-1}x] \neq 0, \\ [u_{i-1}y] & \text{if } [u_{i-1}x] = 0, \end{cases}$$

so that $L_i = \mathbf{F}u_i$ for all $i \geq 2$. This completely determines a presentation of L in terms of x, y and the u_i. In fact, if $\mathbf{F}x \neq C_{i-1} = \mathbf{F} \cdot (y - \alpha x)$, for some $\alpha \in \mathbf{F}$, we have

$$[u_{i-1}y] = [u_{i-1}, \alpha x + y - \alpha x] = \alpha[u_{i-1}x] = \alpha u_i,$$

so that the action of x and y on the u_i is completely determined in any case. □

We will say that an algebra of maximal class is *periodic* if its sequence of two-step centralizers is (ultimately) periodic. For instance, the infinite-dimensional metabelian algebra of maximal class is periodic. And a finite-dimensional algebra L of maximal class is also periodic, as the elements of its sequence of two-step centralizers ultimately equal L_1.

It follows from the above theorem that over a field $\mathbf{F}$ there are at most $|\mathbf{F}|^{\aleph_0}$ algebras of maximal class, and $\max\{|\mathbf{F}|, \aleph_0\}$ periodic ones. We will see that these bounds are actually attained over every field $\mathbf{F}$ of positive characteristic (6.2 and 9.4(1)).

We now need the following simple lemma.

3.3 Lemma. *If $C_2 = C_{i-1} \neq C_i$, and the class of L is at least $i+2$, then $C_{i+1} = C_2$.*

Proof. Let $C_2 = \mathbf{F}y$, and $C_i = \mathbf{F}z$, for some $y, z \in L_1$. We have $L_2 = \mathbf{F} \cdot [zy]$, so that $[zyy] = 0$. If $0 \neq u \in L_{i-1}$, we have from our assumptions on the two-step centralizers

$$0 = [u[zyy]] = [uzyy] - 2[uyzy] + [uyyz] = [uzyy].$$

Now $[uzy] \neq 0$, so that $L_{i+1} = \mathbf{F} \cdot [uzy]$, and our claim follows. □

There is another method which is very useful for describing algebras of maximal class, especially those with two distinct two-step centralizers. In the sequence of two-step centralizers of an algebra of maximal class, the first two-step centralizer occurs in contiguous subsequences, separated by isolated occurrences of a different two-step centralizer. We call each of these subsequences a *constituent* of the algebra. More precisely, a constituent is a subsequence

$$(C_i, C_{i+1}, \ldots, C_j)$$

of the sequence of two-step centralizers such that $C_2 = C_i = C_{i+1} = \cdots = C_j$, but $C_{j+1} \notin \{ C_2, L_1 \}$ and either $i = 2$, or $C_{i-1} \neq C_2$. We make more precise statements on the pattern of constituents in Section 5. Note that if L has exactly two distinct two-step centralizers, the sequence of its constituent lengths determines the sequence of two-step centralizers, and thus the isomorphism type of L. In the case of two distinct two-step centralizers, the constituent sequence turns out indeed to be the most practical way of describing the algebra.

We mention at this point that it will sometimes be convenient to add formally a first element C_1 to the sequence of two-step centralizers of an algebra of maximal class. This will be a one-dimensional subspace of L_1, which is not necessarily a two-step centralizer. If $C_1 = C_2$, the length of the first constituent thus has to be increased by one.

4. The algebras of Albert-Frank-Shalev

Albert and Frank [AF] have constructed certain simple algebras over a prime field $\mathbf{F}_p$. They depend on four parameters a, b, n and p, where $0 \leq a < b < n$, so that $n \geq 2$. The algebra of Albert and Frank $S = AF(a, b, n, p)$ corresponding to these parameters has dimension $p^n - 1$, and is described below. Shalev [Sh2] shows that the algebras S can be given a cyclic grading

$$S = \bigoplus_{i=1}^{p^n-1} S_i,$$

where $\dim(S_i) = 1$ for all i, and that there is a (non-singular) derivation D of S such that $S_i D = S_{i+1}$ cyclically. Let $S_i = \mathbf{F}_p u_i$. Consider the algebra obtained by extending S by D, and tensoring with the polynomial ring $\mathbf{F}_p[t]$,

$$(S + \mathbf{F}_p D) \otimes \mathbf{F}_p[t],$$

and let $L = AFS(a, b, n, p)$ be the subalgebra spanned by $u_1 \otimes t, D \otimes t$. It is easy to see that L is an algebra of maximal class, with

$$L_1 = \mathbf{F}_p \cdot (u_1 \otimes t) + \mathbf{F}_p \cdot (D \otimes t), \qquad L_i = \mathbf{F}_p \cdot (u_i \otimes t^i) \quad \text{for } i > 1,$$

where subscripts are taken cyclically.

In this section we shall show that the algebras of Albert-Frank-Shalev have two distinct two-step centralizers, and we will describe their constituents, as defined in the previous section. We claim that the algebra $AFS(a, b, n, p)$ has a periodic sequence of constituent lengths

$$\begin{cases} \left(2p^b - 2, (p^b - 1)^{p^{n-b}-2}\right)^{\infty} & \text{if } a = 0, \\ \left(2p^a - 1, (p^a - 1)^{p^{b-a}-2}, 2p^a - 2, \right. & \\ \qquad \left. \left((p^a - 1)^{p^{b-a}-2}, 2p^a - 1\right)^{p^{n-b}-2}, (p^a - 1)^{p^{b-a}-2}\right)^{\infty} & \text{if } a > 0. \end{cases}$$

Here, as explained in the preliminaries, an exponent s denotes repetition s times of the basic pattern. If $s = 0$, the pattern is omitted altogether. Note that we make use of the possibility of regarding the first constituent in two different ways.

For instance the algebras $AFS(0, 1, n, 3)$ have constituent length sequence

$$\begin{array}{ll} (4, 2)^{\infty} & \text{for } n = 2, \\ (4, 2^7)^{\infty} & \text{for } n = 3, \\ (4, 2^{25})^{\infty} & \text{for } n = 4. \end{array}$$

The algebras $AFS(1, 2, n, 3)$ have constituent length sequence

$$\begin{array}{ll} (5, 2, 4, 2, 5, 2)^{\infty} & \text{for } n = 3, \\ (5, 2, 4, (2, 5)^7, 2)^{\infty} & \text{for } n = 4, \\ (5, 2, 4, (2, 5)^{25}, 2)^{\infty} & \text{for } n = 5. \end{array}$$

The simple Lie algebras $AF(a, b, n, p)$ of Albert and Frank [AF] are defined as follows. Let a, b, n, p be as above. Let $\overline{S}$ be a vector space of dimension $p^n - 1$ over $\mathbf{F}_{p^n}$, having a basis $\{e_x \mid x \in \mathbf{F}_{p^n}^*\}$ indexed by the non-zero elements of $\mathbf{F}_{p^n}$ (we also put $e_0 = 0$). We make $\overline{S}$ into a Lie algebra over $\mathbf{F}_{p^n}$ by defining a Lie product on the basis elements and extending linearly, via the formula

$$[e_x, e_y] = f(x, y)\, e_{x+y},$$

where $f : \mathbf{F}_{p^n} \times \mathbf{F}_{p^n} \to \mathbf{F}_{p^n}$ is the bilinear form defined by

$$f(x, y) = x^{p^a} y^{p^b} - x^{p^b} y^{p^a}.$$

It is clear from the definition that $AF(a, b, n, p)$ depends on n and p, and on the cosets $a + n\mathbf{Z}$, $b + n\mathbf{Z}$. Therefore it makes sense to define $AF(a, b, n, p)$ for all unordered pairs of integers $\{a, b\}$ with $a \not\equiv b \pmod{n}$. So for instance we can regard the algebra $AFS(0, b, n, p)$ as $AFS(b, n, n, p)$.

We will make a sharper statement later.

It is known that $\overline{S}$ is a simple Lie algebra over $\mathbf{F}_{p^n}$, but Shalev shows in [Sh2] that $\overline{S}$ is actually defined over the prime field $\mathbf{F}_p$. Indeed, another basis of $\overline{S}$ is given by $\{f_0, \dots, f_{p^n-2}\}$, where

$$f_i = \sum_{x \in \mathbf{F}_{p^n}^*} x^i e_x,$$

for $0 \le i < p^n - 1$, and, according to Proposition 2.4 of [Sh2] (but see also below), the multiplication constants of $\overline{S}$ with respect to this basis belong to the prime field. Hence

$$S = \sum_{i=0}^{p^n-2} \mathbf{F}_p f_i$$

is an $\mathbf{F}_p$-Lie subalgebra of $\overline{S}$, which we shall denote by $AF(a, b, n, p)$.

As remarked in [Sh2], it is convenient to put

$$u_j = f_{j-p^a-p^b},$$

for all j (viewed modulo $p^n - 1$), because $S_j = \mathbf{F}_p u_j$ form a cyclic grading of S. Also, S has a derivation D, such that $u_j D = u_{j+1}$ for all j.

In terms of the u_j, Proposition 2.4 of [Sh2] asserts that

$$\text{(Rel)} \qquad \begin{cases} [u_{p^a+p^b}, u_{p^a}] = u_{2p^a+p^b}, \\ [u_{p^a+p^b}, u_{p^b}] = -u_{p^a+2p^b}, \\ [u_{p^a+p^b}, u_j] = 0 & \text{otherwise.} \end{cases}$$

Note that these relations, which describe $\operatorname{ad}(u_{p^a+p^b})$ on S, together with the relations $u_j D = u_{j+1}$ for all j, form a finite presentation (in terms of the generators $u_0, \dots, u_{p^n-2}$, and D) for the split extension of S by D, as a Lie algebra over $\mathbf{F}_p$ (because L is generated as a Lie algebra by $u_{p^a+p^b}$ and D). This yields a proof of the fact, mentioned above, that $\overline{S}$ is defined over the prime field.

A remarkable fact is that, once n and p are fixed, the isomorphism type of $AF(a, b, n, p)$ depends only on the distance between the two subsets $a + n\mathbf{Z}$ and $b + n\mathbf{Z}$ of $\mathbf{Z}$, that is to say, on

$$\min\{b - a, n + a - b\}.$$

More precisely, for any a, b we shall set up an isomorphism

$$\delta : AF(a, b, n, p) \to AF(a - 1, b - 1, n, p).$$

In fact, let $\overline{S} = AF(a, b, n, p)$ be as above, and let $\overline{S}' = AF(a - 1, b - 1, n, p)$ have a basis $\{e'_x \mid x \in \mathbf{F}_{p^n}^*\}$, with Lie multiplication given by

$$[e'_x, e'_y] = f'(x, y)\, e'_{x+y},$$

where

$$f'(x,y) = x^{p^{a-1}} y^{p^{b-1}} - x^{p^{b-1}} y^{p^{a-1}}.$$

Thus $f(x,y) = f'(x^p, y^p)$, and one easily checks that the map

$$\overline{\delta} : e_x \mapsto e'_{x^p}$$

defines by linear extension a Lie algebra isomorphism of $\overline{S}$ onto $\overline{S}'$. Furthermore, if f'_i and u'_i denote the elements of $\overline{S}'$ defined in the same way as f_i and u_i do for $\overline{S}$, then $\overline{\delta}(f_{ip}) = f'_i$, and $\overline{\delta}(u_{ip}) = u'_i$. Hence $\overline{\delta}$ restricts to the desired isomorphism δ, which sends u_{ip} to u'_i. Furthermore, if D' denotes the derivation of $S' = AF(a-1, b-1, n, p)$ such that $u'_j D' = u'_{j+1}$ for all j, then the derivation of S which corresponds to D' under the isomorphism δ is D^p. This will be of relevance in Section 7, when we shall describe the process of deflation for algebras of maximal class. It is perhaps worth mentioning, however, that our formulas for the constituent lengths sequence will show that the algebras of maximal class $AFS(a,b,n,p)$ and $AFS(a-1, b-1, n, p)$ are not isomorphic, unless $a = 0$, $b = 1$ and $n = 2$.

Now we proceed to the determination of the two-step centralizers, and hence the constituent lengths, of $L = AFS(a,b,n,p)$. By the very definition of L, this reduces to finding the action of $\mathrm{ad}(u_1)$ on the basis $\{u_j\}$ of the simple Lie algebra $S = AF(a,b,n,p)$. In fact, if we define the coefficients $\alpha_j \in \mathbf{F}_p$ via

$$[u_j, u_1] = \alpha_j u_{j+1}$$

(where subscripts are taken cyclically modulo $p^n - 1$), then

$$C_{S_1 + \mathbf{F}_p D}(S_i) = \mathbf{F}_p \cdot (u_1 - \alpha_i D),$$

and, in turn, for $i \geq 2$,

$$C_{L_1}(L_i) = \mathbf{F}_p \cdot (u_1 \otimes t - \alpha_i D \otimes t).$$

We also put

$$[u_j, u_{p^a}] = \beta_j u_{j+p^a},$$

and, recalling that in characteristic p the p-th power of a derivation is itself a derivation, we find that

$$[u_{p^a+p^b}, u_j] = [u_{p^a}, u_j] D^{p^b} - [u_{p^a}, u_{j+p^b}].$$

From formulas (Rel) we deduce that

$$\beta_{j+p^b} - \beta_j = \begin{cases} 1 & \text{if } j = p^a, \\ -1 & \text{if } j = p^b, \\ 0 & \text{otherwise.} \end{cases}$$

Starting from the known value $\beta_{p^a} = 0$, we may find all coefficients β_j recursively, for example by the following argument. If the coefficients β_j are arranged cyclically, with β_j immediately followed by β_{j+p^b}, then this cyclic sequence will split into a sequence of 0's, starting with β_{2p^b} and ending with β_{p^a}, followed by a sequence of 1's. We shall compute the value of $\beta_{p^b+p^a+j}$ for $0 \leq j < p^n - 1$ (note that we know $\beta_{p^b+p^a} = 1$). We shall write

$$j = ip^b + r,$$

with $0 \leq i < p^{n-b}$ and $0 \leq r < p^b$ (note that the pair of values $i = p^{n-b} - 1$, $r = p^b - 1$ is never attained, since $j < p^n - 1$). We observe that the smallest non-negative multiple of p^b which is congruent to j modulo $p^n - 1$ is

$$ip^b + r + r(p^n - 1) = p^b(i + rp^{n-b}).$$

In particular, for $j = 2p^b - (p^b + p^a) = p^b - p^a$, such a multiple of p^b will be $p^b(p^n - p^{n-b+a})$, and for $j = p^a - (p^b + p^a) + (p^n - 1)$, it will be $p^b(p^n - 2)$. It follows from our previous discussion that $\beta_{p^b+p^a+j} = 0$ if and only if

$$p^n - p^{n-b+a} \leq i + rp^{n-b} \leq p^n - 2,$$

which is equivalent to

$$p^b - p^a \leq r \leq p^b - \frac{i+2}{p^{n-b}}.$$

Now the inequality on the right is actually superfluous, as $j < p^n - 1$. We conclude that

$$\text{(B)} \qquad \beta_{p^b+p^a+j} = \begin{cases} 0 & \text{if } r \geq p^b - p^a, \\ 1 & \text{otherwise.} \end{cases}$$

Now the case $a = 0$ is easily settled by noting that $\alpha_j = \beta_j$ for all j then. Explicitly, we have $\alpha_1 = 0$, and $\alpha_{kp^b} = 0$ for $2 \leq k < p^{n-b}$, while the remaining α_j are 1. Hence $AFS(0, b, n, p)$ has only two distinct two-step centralizers, namely $C_2 = \mathbf{F}_p \cdot (u_1 - D)$, and $C_{2p^b} = \mathbf{F}_p u_1$. The constituent lengths can be found by counting the number of 1's between two consecutive 0's, in the (cyclic) sequence (α_j). Thus we see that $AFS(0, b, n, p)$ has constituent lengths sequence

$$(2p^b - 2, (p^b - 1)^{p^{n-b}-2})^\infty.$$

We shall now turn to the case $a > 0$. A relation between the coefficients (α_j) and (β_j) can be found by computing the commutator $[u_{p^a+1}, u_j]$ in two different ways, namely

$$[u_{p^a+1}, u_j] = [u_{j+p^a}, u_1] - [u_j, u_1]D^{p^a},$$

and

$$[u_{p^a+1}, u_j] = [u_{j+1}, u_{p^a}] - [u_j, u_{p^a}]D.$$

This gives us the relation

$$\alpha_{j+p^a} - \alpha_j = \beta_{j+1} - \beta_j,$$

for all j. After a change of indices, and taking into account our previous computation of the coefficients β_j, we obtain

$$\alpha_{p^b+2p^a-1+j} - \alpha_{p^b+p^a-1+j} = \beta_{p^b+p^a+j} - \beta_{p^b+p^a-1+j} = \begin{cases} 1 & \text{if } r = 0, \\ -1 & \text{if } r = p^b - p^a, \\ 0 & \text{otherwise.} \end{cases}$$

It is not difficult to deduce that

$$\alpha_{p^b+2p^a-1+j} = \begin{cases} \alpha_{p^b+2p^a-1} & \text{if } r = 0, p^a, \dots, (p^{b-a} - 2)p^a, \\ \alpha_{p^b+2p^a-1} - 1 & \text{otherwise.} \end{cases}$$

Since we know that

$$0 = \alpha_{p^a+p^b} = \alpha_{p^a+p^b+(p^n-1)} = \alpha_{p^b+2p^a-1+(p^n-p^a)},$$

we conclude that

$$\alpha_{p^b+2p^a-1+j} = \begin{cases} 1 & \text{if } r = 0, p^a, \dots, (p^{b-a}-2)p^a, \\ 0 & \text{otherwise.} \end{cases}$$

This proves that $AFS(a,b,n,p)$ has exactly two distinct two-step centralizers. Note that the above formula specializes to (B) when $a = 0$. However, in contrast to the case $a = 0$, when $a > 0$ we have $\alpha_2 = 0$ (since α_2 can be written as $\alpha_{p^b+2p^a-1+j}$, for some $0 \leq j < p^n - 1$ with $r = p^b - 2p^a + 2$), and thus the first two-step centralizer is $C_2 = \mathbf{F}_p u_1$, and the second distinct two-step centralizer is $C_{2p^b} = \mathbf{F}_p \cdot (u_1 - D)$. By counting the number of 0's between consecutive 1's in the sequence (α_j), one sees that the constituent length sequence for $AFS(a,b,n,p)$ when $a > 0$ is

$$\begin{aligned} &2p^a - 2, (p^a-1)^{p^{b-a}-2}, 2p^a - 2, \\ &(((p^a-1)^{p^{b-a}-2}, 2p^a - 1)^{p^{n-b}-1}, (p^a-1)^{p^{b-a}-2}, 2p^a - 2)^{\infty}. \end{aligned}$$

Here we take advantage of the possibility of taking formally an extra centralizer C_1 to write the constituent length sequence in the more convenient purely periodic form stated at the beginning of this section.

5. Constituents

Let $L = \bigoplus_{i \geq 1} L_i$ be an algebra of maximal class over the field $\mathbf{F}$ of positive characteristic p. In this section we describe the possible lengths of constituents. Let $C_2 = \mathbf{F}y$ be the first two-step centralizer in L. If there are no more two-step centralizers distinct from C_2, we have seen that L is uniquely determined. Thus suppose there are at least two distinct two-step centralizers. Assume that $C_k = \mathbf{F}x$ is the first occurrence of the second distinct two-step centralizer. Therefore

$$\mathbf{F}y = C_2 = C_3 = \cdots = C_{k-1} \neq C_k = \mathbf{F}x,$$

and L has class at least $k+1$. We begin by showing

5.1 Lemma. *k is even. If the class of L is at least $k+2$, then $2p$ divides k.*

The first part of the lemma reflects a well-known property of exceptional groups of maximal class ([Hu], III.14.9 and III.14.12), and relates to [M2, Lemma 3.1].

Proof. We have $L_i = \mathbf{F} \cdot [y \underbrace{x \cdots x}_{i-1}]$ for $i = 2, \dots, k$, and $L_{k+1} = \mathbf{F} \cdot [y \underbrace{x \cdots x}_{k-1} y]$. If $k = 2h+1$ is odd, then

$$0 = [[y \underbrace{x \cdots x}_{h}][y \underbrace{x \cdots x}_{h}]] = (-1)^h [y \underbrace{x \cdots x}_{k-1} y].$$

Therefore k is even. Thus we have

$$\begin{aligned}0 &= [[y \underbrace{x \cdots x}_{k/2}][y \underbrace{x \cdots x}_{k/2}]]\\ &= \pm \binom{k/2}{k/2-1} [y \underbrace{x \cdots x}_{k-1} yx]\\ &= \pm \frac{k}{2} [y \underbrace{x \cdots x}_{k-1} yx].\end{aligned}$$

Since $C_k \neq C_2$, we have by Lemma 3.3 that C_2 centralizes $L_{k+1} = \mathbf{F} \cdot [y \underbrace{x \cdots x}_{k-1} y]$.

Since by assumption L_{k+1} is not central, we have in fact $C_2 = C_{k+1}$, and $[y \underbrace{x \cdots x}_{k-1} yx] \neq 0$. From the above calculations we obtain that p divides $k/2$. □

We note that because the characteristic has to divide k we have

5.2 Corollary. *An algebra of maximal class over a field of characteristic zero is centre-by-metabelian.*

Because of Lemma 5.1, we can write the length of the first constituent in the form $2qn - 2$, where $q = p^h$, with $h > 0$, and p does not divide n.

5.3 Lemma. *If p is odd, L has first constituent of length $2qn-2$, and L has class at least $2qn+q+1$, then the second constituent has length $q-1$.*

Proof. First we have, for $0 \leq j < q-1$,

$$[y \underbrace{x \cdots x}_{2qn-1} y \underbrace{x \cdots x}_{j} y] = 0.$$

To prove this, work by induction on j. The case $j = 0$ is given by Lemma 3.3. For $j > 0$ write $j+1 = \beta p^t$, with $\beta \not\equiv 0 \pmod p$. Note that as $j < q-1$, we have $p^t < q$, so that $j + p^t < 2q - 1$, and $[y \underbrace{x \cdots x}_{j+p^t} y] = 0$. Therefore

$$\begin{aligned}0 &= [[y \underbrace{x \cdots x}_{2qn-p^t-1}][y \underbrace{x \cdots x}_{j+p^t} y]]\\ &= (-1)^{p^t} \binom{j+p^t}{p^t} [y \underbrace{x \cdots x}_{2qn-1} y \underbrace{x \cdots x}_{j} y].\end{aligned}$$

This implies $[y \underbrace{x \cdots x}_{2qn-1} y \underbrace{x \cdots x}_{j} y] = 0$, as

$$\binom{j+p^t}{p^t} = \binom{\beta p^t + p^t - 1}{p^t} \equiv \beta \not\equiv 0 \pmod p,$$

by Lucas' Theorem.

Finally we show that $[y \underbrace{x \cdots x}_{2qn-1} y \underbrace{x \cdots x}_{q}] = 0$. Since the class of L is at least $2qn + q + 1$, we will obtain that

$$\mathbf{F}x = C_{2qn} \neq \mathbf{F}y = C_{2qn+1} = \cdots = C_{2qn+q-1} \neq C_{2qn+q} = \mathbf{F}x,$$

so that the second constituent has length $q-1$ as claimed. Because $qn+(q-1)/2$ is an integer, and

$$\begin{aligned} 0 &= [[y \underbrace{x\cdots x}_{qn+(q-1)/2}][y \underbrace{x\cdots x}_{qn+(q-1)/2}]] \\ &= \pm\binom{qn+(q-1)/2}{q}[y\underbrace{x\cdots x}_{2qn-1}y\underbrace{x\cdots x}_{q}], \end{aligned}$$

Lucas' Theorem gives

$$\binom{qn+(q-1)/2}{q} \equiv n \not\equiv 0 \pmod p,$$

so that our claim is proved. □

5.4 Lemma. *Suppose $n>1$.*

1. *If p is odd, then L has class at most $2qn+q$.*
2. *If $p=2$, then L has class at most $2qn+2q-1$.*

Part (1) of this lemma is related to part (a) of Corollary 4.4 of [M2].

Proof. As $n>1$, we have $[y\underbrace{x\cdots x}_{2q-1}y]=0$, so that

$$\begin{aligned} 0 &= [[y \underbrace{x\cdots x}_{2qn-q-1}][y\underbrace{x\cdots x}_{2q-1}y]] \\ &= (-1)^q\binom{2q-1}{q}[y\underbrace{x\cdots x}_{2qn-1}y\underbrace{x\cdots x}_{q-1}y]. \end{aligned}$$

Since $\binom{2q-1}{q}=\binom{q+q-1}{q}\equiv 1 \pmod p$, we obtain $[y\underbrace{x\cdots x}_{2qn-1}y\underbrace{x\cdots x}_{q-1}y]=0$. This concludes the proof when p is odd, in view of Lemma 5.3.

Suppose now that $p=2$. The only value $j<2q-1$ such that q divides $j+1$ is $j=q-1$. Since we have

$$[y\underbrace{x\cdots x}_{2qn-1}y\underbrace{x\cdots x}_{q-1}y]=0$$

here, the proof of Lemma 5.3 shows that

$$[y\underbrace{x\cdots x}_{2qn-1}y\underbrace{x\cdots x}_{j}y]=0$$

for all $j<2q-1$.

Now

$$\begin{aligned} 0 &= [[y \underbrace{x\cdots x}_{q(n+1)-1}][y \underbrace{x\cdots x}_{q(n+1)-1}]] \\ &= (-1)^{qn-q}\binom{q(n+1)-1}{qn-q}[y\underbrace{x\cdots x}_{2qn-1}y\underbrace{x\cdots x}_{2q-1}], \end{aligned}$$

so that $L_{2qn+2q}=0$, as

$$\binom{q(n+1)-1}{qn-q}=\binom{qn+q-1}{q(n-1)}\equiv\binom{n}{n-1}\equiv n\not\equiv 0 \pmod 2. \quad □$$

We may summarize the above in the following theorem.

5.5 Theorem. *Let L be an algebra of maximal class over a field of characteristic p. If the class of L is large enough, then the first constituent has length of the form $2p^h - 2$, for some $h \geq 1$. If p is odd, then the second constituent has length $p^h - 1$.*

This theorem, and Lemma 5.3 above, are related to Theorem 4.1 of [M2].

Because of this result, we will confine ourselves from now on to algebras of maximal class whose first constituent has length $2q-2$, where $q = p^h$ is a power of the characteristic p. We are therefore covering in particular the infinite-dimensional case. The number $q = p^h$ will play an important role in the description of the structure of L: we call it the *parameter* of the algebra L.

Note that the Albert-Frank-Shalev algebra $AFS(h, h+1, n, 2)$, for $1 \leq h < n-1$, has parameter $q = 2^h$ and second constituent of length $2q-2$, so that the restriction on the length of the second constituent of Theorem 5.5 really does not apply to $p = 2$.

We can now state

5.6 Proposition. *Let L be an algebra of maximal class with parameter $q = p^h$. Then the lengths of the constituents of L can only assume the values*

$$2q-1, \qquad \textit{and} \qquad 2q - p^s - 1, \quad \textit{for} \;\; 0 \leq s \leq h.$$

We will show in the next section that all these numbers actually occur as constituent lengths in some algebra of maximal class.

Proof. Let L be an algebra of maximal class with parameter $q = p^h$. Let the first two-step centralizer be $C_2 = \mathbf{F}y$ and the second one be $C_{2q} = \mathbf{F}x$.

First note that $q-1$ is a lower bound for the length of a constituent. This is basically a calculation we have already employed. In fact, proceed by induction on the occurrence of constituents, the base being provided by the statement about the length of the first constituent. Consider the constituent starting at C_{i+m+1}, where $m \geq q-1$ is the length of the previous constituent $(C_i, C_{i+1}, \dots, C_{i+m-1})$. Let $0 \neq u \in L_{i-1}$, or $u = x$ if $i = 2$. We have thus $L_{i+m} = \mathbf{F} \cdot [uy\underbrace{x \cdots x}_{m}]$. We claim that

$$[uy\underbrace{x \cdots x}_{m}y\underbrace{x \cdots x}_{j}y] = 0 \quad \text{for } 1 \leq j < q-1,$$

so that by the definition of constituents $[uy\underbrace{x \cdots x}_{m}y\underbrace{x \cdots x}_{j+1}] \neq 0$ for $1 \leq j < q$. Proceeding by induction on j, write $j+1 = \beta p^t$, with $\beta \not\equiv 0 \pmod{p}$. Since $j+1 < q$, we have $p^t < q$, so that $j + p^t < 2q-1$. Also, $1 \leq p^t \leq p^{h-1} < q-1 \leq m$. Thus we have $[y\underbrace{x \cdots x}_{j+p^t}y] = 0$, and

$$\begin{aligned} 0 &= [[uy\underbrace{x \cdots x}_{m-p^t}][y\underbrace{x \cdots x}_{j+p^t}y]] \\ &= (-1)^{p^t}\binom{j+p^t}{p^t}[uy\underbrace{x \cdots x}_{m}y\underbrace{x \cdots x}_{j}y], \end{aligned}$$

where the binomial coefficient is congruent to $\beta \not\equiv 0 \pmod{p}$. It may be worth noting that the situation is more delicate here than it was in the similar argument in the proof of Lemma 5.3, as we might have that both $[uy\underbrace{x\cdots x}_{m}y]$ and $[uy\underbrace{x\cdots x}_{m+1}]$ are non-zero here.

Next note that $2q-1$ is an upper bound for the length of a constituent. As above, consider the constituent starting at C_{i+m+1}, where $m \geq q-1$ is the length of the previous constituent $(C_i, C_{i+1}, \dots, C_{i+m-1})$. Let $0 \neq u \in L_{i-1}$, and write $v = [uy\underbrace{x\cdots x}_{m}]$. We have thus $L_{i+m} = \mathbf{F}v$. We want to show that the following constituent has length at most $2q-1$.

We have $[y\underbrace{x\cdots x}_{2q}] = 0$. Thus if $[vx] = 0$, we have

$$0 = [v[y\underbrace{x\cdots x}_{2q}]] = [vy\underbrace{x\cdots x}_{2q}],$$

so that one of the two-step centralizers $C_{i+m+1}, \dots, C_{i+m+2q}$ must be $\mathbf{F}x$.

So suppose $[vz] = 0$ for some $z \in L_1 \setminus \{\mathbf{F}y, \mathbf{F}x\}$. We have $[y\underbrace{z\cdots z}_{2q-1}x] = 0$, so that

$$0 = [v[y\underbrace{z\cdots z}_{2q-1}x]] = [v[y\underbrace{z\cdots z}_{2q-1}]x] - [vx[y\underbrace{z\cdots z}_{2q-1}]].$$

Suppose, by way of contradiction, that $C_{i+m+1} = \cdots = C_{i+m+2q} = \mathbf{F}y$. Thus

$$[vx[y\underbrace{z\cdots z}_{2q-1}]] = \sum_{j=0}^{2q-1} (-1)^j \binom{2q-1}{j} [vx\underbrace{z\cdots z}_{j}y\underbrace{z\cdots z}_{2q-1-j}] = 0.$$

It follows that

$$0 = [v[y\underbrace{z\cdots z}_{2q-1}]x] = [vy\underbrace{z\cdots z}_{2q-1}x],$$

so that either one of the centralizers $C_{i+m+1}, \dots, C_{i+m+2q-1}$ is $\mathbf{F}z$, or $C_{i+m+2q} = \mathbf{F}x$, a contradiction.

Now suppose by induction $(C_i, C_{i+1}, \dots, C_{i+m-1})$ is a constituent of length $m = 2q - p^s - 1$, where we admit formally $s = -\infty$ and $p^{-\infty} = 0$, to take care of the case $m = 2q-1$. We want to show that the length of the next constituent can only assume one of the stated values. Proceeding by way of contradiction, we may thus assume that the next constituent has length n, with $q-1 < n < 2q-2$.

Write $n+1 = \beta p^t$, with $\beta \not\equiv 0 \pmod{p}$, so that we have $p^t < q$. Suppose $n + p^t = (\beta+1)p^t - 1 \geq 2q - 1 = 2p^h - 1$, so that $\beta \geq 2p^{h-t} - 1$. Since $\beta p^t = n + 1 < 2q - 1$, we have also $\beta < 2p^{h-t}$, so that the only possibility is $\beta = 2p^{h-t} - 1$, and $n = 2q - p^t - 1$, one of the allowed values.

Therefore we may assume $n + p^t < 2q - 1$, so that $[y\underbrace{x\cdots x}_{n+p^t}y] = 0$. We have also $2q - p^s - p^t - 1 \geq 0$. Take as above $0 \neq u \in L_{i-1}$. We get

$$\begin{aligned} 0 &= [[uy\underbrace{x\cdots x}_{2q-p^s-p^t-1}][y\underbrace{x\cdots x}_{n+p^t}y]] \\ &= (-1)^{p^t}\binom{n+p^t}{p^t}[uy\underbrace{x\cdots x}_{2q-p^s-1}y\underbrace{x\cdots x}_{n}y], \end{aligned}$$

a contradiction. $\square$

When p is odd, we can actually make a slightly stronger statement, which extends Lemma 5.3.

5.7 Lemma. *If p is odd, and the class of L is large enough, then a constituent of length $m > q-1$ is followed by a constituent of length $q-1$.*

Proof. If $m > q-1$, we have $m \geq 2q - q/p - 1 > q$, as $q > 2$, so that, in the notation already employed,

$$\begin{aligned} 0 &= [[uy\underbrace{x\cdots x}_{m-q}][y\underbrace{x\cdots x}_{2q}]] \\ &= (-1)^q \binom{2q}{q} [uy\underbrace{x\cdots x}_{m}y\underbrace{x\cdots x}_{q}], \end{aligned}$$

where the binomial coefficient is congruent to $2 \not\equiv 0 \pmod{p}$. ☐

6. Inflation

In this section we describe a construction procedure for algebras of maximal class. The arguments we employ here are self-contained, but the construction itself goes back to Zassenhaus [Z] and Block [Blo].

Let $\mathbf{F}$ be a field of positive characteristic p. Let ε be an element with minimal polynomial X^p over $\mathbf{F}$, and consider the extension $\mathbf{F}[\varepsilon]$. Now let M be a graded Lie algebra over $\mathbf{F}$, and consider the tensor product

$$M^{\uparrow} = M \otimes_{\mathbf{F}} \mathbf{F}[\varepsilon],$$

which can be seen as a Lie algebra over $\mathbf{F}$ or $\mathbf{F}[\varepsilon]$. Given a derivation $D \in \mathrm{Der}_{\mathbf{F}}(M)$ of degree 1, this extends naturally to an element of $\mathrm{Der}_{\mathbf{F}[\varepsilon]}(M^{\uparrow})$, which we also call D. We now construct a derivation $E \in \mathrm{Der}_{\mathbf{F}}(M^{\uparrow})$, with the property that $E^p = D$. It is well known that in characteristic p the p-th power of a derivation is again a derivation.

Consider a graded basis $\mathcal{B}$ of M over $\mathbf{F}$. Then the elements

$$b \otimes \frac{\varepsilon^i}{i!}$$

form a basis for $M^{\uparrow}$ over $\mathbf{F}$, as b ranges in $\mathcal{B}$, and $p > i \geq 0$. If w is the original grading on M, we see that

$$w'(b \otimes \frac{\varepsilon^i}{i!}) = pw(b) - i$$

defines a grading of $M^{\uparrow}$.

Let ∂ be the "standard" derivation of $\mathbf{F}[\varepsilon]$ defined by $\varepsilon\partial = 1$, so that $\varepsilon^i\partial = i\varepsilon^{i-1}$ for $i > 0$, and $1\partial = 0$. Consider the element of $\mathrm{Der}_{\mathbf{F}}(M^{\uparrow})$

$$E = D \otimes \frac{\varepsilon^{p-1}}{(p-1)!} + 1 \otimes \partial,$$

in the notation of [Blo, Theorem 7.1], that is, E acts on the basis $b \otimes \varepsilon^i/i!$ by

$$E : \begin{cases} b \otimes \dfrac{\varepsilon^i}{i!} \mapsto b \otimes \dfrac{\varepsilon^{i-1}}{(i-1)!} & \text{if } i > 0, \\ b \otimes 1 \mapsto bD \otimes \dfrac{\varepsilon^{p-1}}{(p-1)!} \ . \end{cases}$$

It is easy to verify that E has degree 1, and that $E^p = D$. We call E the *p-th root of D*, and write $E = D^{1/p}$.

What is important here is that the inflation of an algebra of maximal class is an algebra of maximal class. Let L be an algebra of maximal class over the field $\mathbf{F}$, and let M be any maximal ideal of L. Choose an element $s \in L_1 \setminus M$, and consider the derivation $D = \mathrm{ad}(s) \in \mathrm{Der}_{\mathbf{F}}(M)$. We define the *inflation of L at M* to be the algebra ${}^M L$ which is the extension of $M^{\uparrow}$ by an element s' inducing on $M^{\uparrow}$ the derivation $E = D^{1/p}$. As above, $M^{\uparrow}$ is a graded algebra.

Take an element $0 \neq t \in M \cap L_1$, and define

$$t' = t \otimes \frac{\varepsilon^{p-1}}{(p-1)!} \in M^{\uparrow}.$$

As above, E is a derivation of $M^{\uparrow}$ of degree 1. To show that ${}^M L$ is of maximal class, it suffices thus to show that s' and t' generate ${}^M L$, or equivalently, $[{}^M L_k, {}^M L_1] = {}^M L_{k+1}$ for all $k \geq 1$. In fact we have, for b in a graded basis of M,

$$[b \otimes \frac{\varepsilon^i}{i!}, s'] = b \otimes \frac{\varepsilon^{i-1}}{(i-1)!},$$

and thus $[{}^M L_k, s'] = {}^M L_{k+1}$, if $i > 0$, or equivalently if

$$k = w'(b \otimes \frac{\varepsilon^i}{i!}) = pw(b) - i \not\equiv 0 \pmod{p}.$$

If $i = 0$, we have

$$[b \otimes 1, s'] = [bs] \otimes \frac{\varepsilon^{p-1}}{(p-1)!}.$$

If this is zero, then $[bs] = 0$ in L, so that either $\mathbf{F}b$ is the centre of L, and $\mathbf{F}b \otimes 1$ is the centre of ${}^M L$, or $C_{w(b)} = \mathbf{F}s$. In the latter case $[bt] \neq 0$, so that

$$[b \otimes 1, t'] = [b \otimes 1, t \otimes \frac{\varepsilon^{p-1}}{(p-1)!}] = [bt] \otimes \frac{\varepsilon^{p-1}}{(p-1)!}$$

is non-zero. So in all cases we have $[{}^M L_k, {}^M L_1] = {}^M L_{k+1}$ for all $k \geq 1$.

Observe that if L has class c then ${}^M L$ has class cp.

The above construction may appear to depend on the choice of s, but all choices lead to isomorphic algebras. In fact, suppose without loss of generality we choose $s+t$ instead of s. The underlying ideal being in both cases M, it is enough to show that the elements $s' + t'$ and $(s+t)'$ induce the same derivation on $M^{\uparrow}$. Here the element $(s+t)'$ induces on $M^{\uparrow}$ the derivation $\mathrm{ad}(s+t)^{1/p}$. Now if $i > 0$ one has

$$[b \otimes \frac{\varepsilon^i}{i!}, s' + t'] = b \otimes \frac{\varepsilon^{i-1}}{(i-1)!} + [bt] \otimes \frac{\varepsilon^{i+p-1}}{i!} = b \otimes \frac{\varepsilon^{i-1}}{(i-1)!} = [b \otimes \frac{\varepsilon^i}{i!}, (s+t)'],$$

as $i + p - 1 \geq p$. We also have

$$\begin{aligned}
[b \otimes 1, s' + t'] &= [b \otimes 1, s'] + [b \otimes 1, t'] \\
&= [b, s] \otimes \frac{\varepsilon^{p-1}}{(p-1)!} + [b, t] \otimes \frac{\varepsilon^{p-1}}{(p-1)!} \\
&= [b, s+t] \otimes \frac{\varepsilon^{p-1}}{(p-1)!} \\
&= [b \otimes 1, (s+t)'].
\end{aligned}$$

Next we describe how two-step centralizers behave under inflation. It is handy, when going from an algebra L to its inflation ${}^{M}L$, to consider the linear map $L_1 \to {}^{M}L_1$ defined by $\alpha s + \beta t \mapsto \alpha s' + \beta t'$, for $\alpha, \beta \in \mathbf{F}$, in the above notation. This allows an easy description of ${}^{M}L$ via its sequence of two-step centralizers. To do this, we first take advantage of the possibility of prepending a first element C_1 to the sequence of two-step centralizers of L. We choose here formally $C_1 = C_{L_1}(L_1) = M \cap L_1 = \mathbf{F}t$, in the above notation. It is now easy to see

6.1 Lemma. 1. *If* $i \equiv 0 \pmod{p}$*, then*

$$C_{{}^{M}L_1}({}^{M}L_i) = C_{L_1}(L_{i/p})'.$$

2. *If* $i \not\equiv 0 \pmod{p}$*, and* $i \geq 2$*, then*

$$C_{{}^{M}L_1}({}^{M}L_i) = C_{L_1}(L_2)' = \mathbf{F}t'.$$

Note that the element t determines $M = \mathbf{F}t + \bigoplus_{i \geq 2} L_i$, and conversely M determines $\mathbf{F}t = M \cap L_1$. We may thus speak equivalently of inflating L at M or at $\mathbf{F}t$ or at t. We also write ${}^{M}L = {}^{t}L$.

Also note that if we identify the elements of L_1 with the elements of ${}^{M}L_1 = {}^{t}L_1$ via the prime operator, $\mathbf{F}t$ is the first two-step centralizer in ${}^{M}L$. If $\mathbf{F}t$ was already the first two-step centralizer in L, the number and order of occurrence of distinct two-step centralizers in L and ${}^{M}L$ is the same. If $\mathbf{F}t$ was a centralizer other than the first one, the number of distinct two-step centralizers in L and ${}^{M}L$ is the same, with $\mathbf{F}t$ moving to the first place in the order of occurrence. Finally, if $\mathbf{F}t$ was not a two-step centralizer in L, then ${}^{M}L$ has one two-step centralizer more than L, with the first two-step centralizer in L occurring as the second distinct one in ${}^{M}L$, and so on.

This identification makes it possible to consider inflating for instance the abelian algebra L of maximal class with respect to a finite sequence $t_n, t_{n-1}, \dots, t_1$ of non-zero elements in L_1. In particular, if the t_i are pairwise linearly independent, so that $\mathbf{F}$ has at least $n-1$ elements, one sees with Lemma 6.1 that in the resulting algebra N of maximal class the i-th two-step centralizer is $\mathbf{F}t_i$, and that it occurs first as $C_N(N_{2p^{i-1}})$

Now in analogy with the case of groups (see [Ma]), we say that an algebra L of maximal class is *covered* if the set $\{ C_i : i \geq 2 \}$ consists of all one-dimensional subspaces of L_1. If $\mathbf{F}$ is a finite field with $n-1$ elements, we can construct covered algebras, starting with the abelian algebra L of maximal class, and inflating all n subspaces of dimension 1 of L_1 to two-step centralizers. The covered algebra constructed this way has class p^n. It is however not hard to see that the smallest covered factor of this algebra has class $2p^{n-1}+1$.

We have obtained

6.2 Proposition. *Over a field of positive characteristic p and order n there are covered algebras of maximal class of class* $2p^n + 1$.

We return to this theme in Section 9.

Now consider inflating the abelian algebra of maximal class L with respect to four distinct subspaces of L of dimension 1, for fields with at least 3 elements. We obtain an algebra of class p^4 with four distinct two-step centralizers. We have noted that we may put these four two-step centralizers in canonical form as

$$\mathbf{F}y, \mathbf{F}x, \mathbf{F} \cdot (y - x), \mathbf{F} \cdot (y - \alpha x),$$

where x and y form a basis of $L = L_1$, and the value $\alpha \in \mathbf{F} \setminus \{0, 1\}$ is uniquely determined by the isomorphism type of the algebra. Therefore two algebras of this kind are isomorphic if and only if they have the same value of α. We thus see that if the field $\mathbf{F}$ of characteristic p is infinite, there are infinitely many pairwise non-isomorphic algebras of maximal class of class p^4 which have a factor of class $2p^3 + 1$ with four distinct two-step centralizers.

We also have

6.3 Proposition. *Over a field* $\mathbf{F}$ *of positive characteristic there are* $\max\{|\mathbf{F}|, \aleph_0\}$ *periodic, insoluble algebras of maximal class.*

Proof. The algebras of Albert-Frank-Shalev deal with the case of $\mathbf{F}$ finite. If $\mathbf{F}$ is infinite, it suffices to start with an algebra of Albert-Frank-Shalev, and inflate it twice with respect to suitable elements, to obtain an insoluble algebra with four distinct two-step centralizers. By the above, we get $|\mathbf{F}|$ pairwise non-isomorphic algebras. □

There is another 'projective' description of the transformation of two-step centralizers when going from L to ${}^{M}L$ which is sometimes useful, and deals with the coefficient sequence. Let this sequence be $(\alpha_i)_{i \geq 2}$ for L. We recall that we have set the first two distinct two-step centralizers to be $\mathbf{F}y$ and $\mathbf{F}x$, and the third distinct one, if it exists, to be $\mathbf{F} \cdot (y - x)$. The two-step centralizer C_i can then be written uniquely as $\mathbf{F}(y - \alpha_i \cdot x)$, for $\alpha_i \in \mathbf{F}_\infty$, where we understand $y - \infty \cdot x = x$. Here too we have chosen $C_1 = \mathbf{F}t$ as above. Now if $(\alpha'_i)_{i \geq 2}$ is the coefficient sequence for ${}^{M}L$, we have as above $\alpha'_i = 0$ if $i \not\equiv 0 \pmod{p}$ and $i \geq 2$, whereas

$$\alpha'_{pi} = 1 - \frac{\beta}{\alpha_i},$$

if t is a nonzero multiple of $y - \beta x$, where $\beta \neq 0, \infty$. If $\beta = 0$, then $\alpha'_{pi} = \alpha_i$, whereas for $\beta = \infty$ we have

$$\alpha'_{pi} = \frac{1}{\alpha_i}\,.$$

It is also easy to see what happens to the length of constituents under inflation.

The inflation of an algebra L with parameter q with respect to the first two-step centralizer has parameter pq, and a constituent of length m in L becomes a constituent of length $pm + p - 1 = p(m + 1) - 1$ in the inflation.

In the inflation with respect to an element which is not already a two-step centralizer, the parameter becomes p, and all constituents except the first one have length $p - 1$.

In the inflation with respect to a two-step centralizer which is not the first one, the parameter is also p, and the length of a constituent other than the first one is either $p - 1$ or $2p - 1$.

In the particular case of an algebra L with two distinct two-step centralizers $\mathbf{F}y$, $\mathbf{F}x$, inflation replaces a constituent of length m with a subsequence of constituent lengths $2p - 1, (p - 1)^{m-1}$, and the first constituent by the subsequence $2p - 2$, $(p - 1)^{2q-3}$, where q is the parameter of L.

Hence

6.4 Lemma. *In an inflated algebra over a field of positive characteristic* p*, all constituents except the first one have length* $m \equiv -1 \pmod{p}$*.*

We are now able to prove the following converse of Proposition 5.6.

6.5 Theorem. *Let p be a prime number, $h \geq 1$, and $0 \leq s \leq h$. Then there exists an algebra of maximal class with two distinct two-step centralizers over the field $\mathbf{F}_p$ with parameter p^h, and a constituent other than the first one of length $2p^h - p^s - 1$.*

Proof. Given the ambiguity on the length of the first constituent, we will realize $2q-1$ and $2q-2$ as lengths of constituents which are not the first ones.

In fact, proceed by induction on h. For $h = 1$, the algebra $AFS(1, 2, 3, p)$ has constituents of all possible lengths $p-1, 2p-2, 2p-1$. Assume we have algebras $L^{-\infty}, L^0, \dots, L^h$ with two distinct two-step centralizers such that L^s has parameter p^h, and at least one constituent of length $2p^h - p^s - 1$, where we understand $p^{-\infty} = 0$. By inflating L^s with respect to the first two-step centralizer, we obtain an algebra with parameter p^{h+1}, and a constituent of length

$$p(2p^h - p^s - 1 + 1) - 1 = 2p^{h+1} - p^{s+1} - 1.$$

This leaves out only constituents of length $2p^{h+1} - 2$. But such a constituent appears in the algebra $AFS(h+1, h+2, h+3, p)$. Note that our inflation method can provide algebras with at most three different constituent lengths, as in the Albert-Frank-Shalev algebras we use as starting points. $\square$

7. Deflation

An algebra L of maximal class can be recovered from any of its inflations $N = {}^{M}L$. In fact, choose any element $z \in N_1$ which does not lie in the first two-step centralizer. Let ζ denote an element that induces on $\bigoplus_{i\geq 1} N_{ip}$ the derivation $\mathrm{ad}(z)^p$. Then the algebra

$$N^{\downarrow} = \mathbf{F}\zeta + \bigoplus_{i\geq 1} N_{ip}$$

is clearly isomorphic to L. Actually, the definition of $N^{\downarrow}$ makes sense for every algebra N of maximal class; we call $N^{\downarrow}$ the *deflation* of N.

We claim that deflation does not depend on the choice of the element z. In fact, let $\mathbf{F}y$ be the first two-step centralizer in N. To prove our claim, we will show that

$$\mathrm{ad}(z+y)^p = \mathrm{ad}(z)^p + \mathrm{ad}([y \underbrace{z \cdots z}_{p-1}]),$$

as derivations of $\bigoplus_{i\geq 1} N_{ip}$. In fact, let $0 \neq u \in N_{ip}$. Because the minimum length of a constituent is $p-1$, we have that

$$[u \underbrace{z \cdots z}_{i}] \neq 0 \qquad \text{and} \qquad [u \underbrace{z \cdots z}_{i-1} y] = 0$$

for all $1 \leq i \leq p$, except possibly for exactly one value i_0, for which we have $[u \underbrace{z \cdots z}_{i_0 - 1} y] \neq 0$, whereas we do not make any assumption about $[u \underbrace{z \cdots z}_{i_0}]$. If there is no such i_0, we thus have

$$\begin{aligned} u\,\mathrm{ad}([y\underbrace{z\cdots z}_{p-1}]) &= [u[y\underbrace{z\cdots z}_{p-1}]] \\ &= \sum_{i=0}^{p-1}(-1)^i\binom{p-1}{i}[u\underbrace{z\cdots z}_{i}y\underbrace{z\cdots z}_{p-1-i}] \\ &= 0, \end{aligned}$$

so that

$$\begin{aligned} u\,\mathrm{ad}(z+y)^p &= [u,\underbrace{z+y,\dots,z+y}_{p}] \\ &= [u\underbrace{z\cdots z}_{p}] = u\,\mathrm{ad}(z)^p \\ &= u\big(\mathrm{ad}(z)^p + \mathrm{ad}([y\underbrace{z\cdots z}_{p-1}])\big). \end{aligned}$$

In the other case

$$\begin{aligned} u\,\mathrm{ad}([y\underbrace{z\cdots z}_{p-1}]) &= (-1)^{i_0-1}\binom{p-1}{i_0-1}[u\underbrace{z\cdots z}_{i_0-1}y\underbrace{z\cdots z}_{p-i_0}] \\ &= [u\underbrace{z\cdots z}_{i_0-1}y\underbrace{z\cdots z}_{p-i_0}], \end{aligned}$$

so that

$$\begin{aligned} u\,\mathrm{ad}(z+y)^p &= [u,\underbrace{z+y,\dots,z+y}_{p}] \\ &= [u\underbrace{z\cdots z}_{p}] + [u\underbrace{z\cdots z}_{i_0-1}y\underbrace{z\cdots z}_{p-i_0}] \\ &= u(\mathrm{ad}(z)^p + \mathrm{ad}([y\underbrace{z\cdots z}_{p-1}])). \end{aligned}$$

It also follows that $N^{\downarrow}$ is of maximal class. We have $N_1^{\downarrow} = \mathbf{F}z' + \mathbf{F}y'$, and $N_i^{\downarrow} = N_{ip}$, for $i \geq 2$. Now if all the two-step centralizers

$$C_{ip}, C_{ip+1}, \dots, C_{(i+1)p-1}$$

are different from $\mathbf{F}z$, then for $0 \neq u \in N_{ip}$ we have

$$0 \neq [uz'] = [u\underbrace{z\cdots z}_{p}] \in N_{(i+1)p} = N_{i+1}^{\downarrow}.$$

If $C_{ip+i_0-1} = \mathbf{F}z \neq \mathbf{F}y$, for some $1 \leq i_0 \leq p$, then

$$0 \neq [u\underbrace{z\cdots z}_{i_0-1}y\underbrace{z\cdots z}_{p-i_0}] = [u[y\underbrace{z\cdots z}_{p-1}]] = [uy'] \in N_{(i+1)p} = N_{i+1}^{\downarrow}.$$

Note that in this case exactly one of the above two-step centralizers is distinct from $\mathbf{F}y$, as constituents have length at least $p-1$.

It is not difficult to compare two-step centralizers in L and its deflation $L^{\downarrow}$. As in the case of inflation, define a linear map

$$\begin{aligned} {}' : & L_1 \longrightarrow L_1^{\downarrow}, \\ & \alpha z + \beta y \longmapsto \alpha \zeta + \beta [y \underbrace{z \cdots z}_{p-1}], \end{aligned}$$

for $\alpha, \beta \in \mathbf{F}$. Let $C_i = C_{L_1}(L_i)$ and $C_i' = C_{L_1^{\downarrow}}(L_i^{\downarrow})$.

We have

7.1 Proposition. 1. *If all the two-step centralizers $C_{ip}, C_{ip+1}, \dots, C_{(i+1)p-1}$ equal $C_2 = \mathbf{F}y$, for $i \geq 2$, then*

$$C_{L_1^{\downarrow}}(L_i^{\downarrow}) = \mathbf{F}y'.$$

2. *If there is a $1 \leq i_0 \leq p$ such that $C_{ip+i_0-1} = \mathbf{F} \cdot (z - \alpha y)$, for some α, then*

$$C_{L_1^{\downarrow}}(L_i^{\downarrow}) = \mathbf{F} \cdot (z' - \alpha y').$$

Proof. Note that, as above, exactly one of the above two-step centralizers is thus distinct from $\mathbf{F}y$.

The first possibility is clear. As to the second one, we have, for $0 \neq u \in C_{ip}$,

$$\begin{aligned} [u, z' - \alpha y'] &= [uz'] - \alpha [uy'] \\ &= [u \underbrace{z \cdots z}_{p}] - \alpha [u \underbrace{z \cdots z}_{i_0 - 1} y \underbrace{z \cdots z}_{p - i_0}] \\ &= [u \underbrace{z \cdots z}_{i_0 - 1}, z - \alpha y, \underbrace{z \cdots z}_{p - i_0}] \\ &= 0. \quad \square \end{aligned}$$

Note the following immediate consequences

7.2 Corollary. 1. *In going from L to $L^{\downarrow}$, no centralizer can disappear except possibly the first one.*

2. *Suppose that in going from L to $L^{\downarrow}$ the first centralizer disappears. Then L has parameter p, and all constituents except the first one have length $p - 1$.*
3. *Suppose that in going from L to $L^{\downarrow}$ the first centralizer becomes the i-th one, for some $i > 1$. Then L has parameter p, and all constituents, distinct from the first one, leading to the first occurrences of the second, third, …, i-th centralizer in L have length $p - 1$.*

In general, it is not true that inflation is the inverse of deflation. It can happen, as we now show, that ${}^{t}(L^{\downarrow}) \ncong L$ for all $0 \neq t \in L_1$. In fact, by our observations above, an inflated algebra with parameter q and two distinct two-step centralizers has no constituents, except the first one, of length $2q - 2$.

Now it follows readily from the theory we have developed for the Albert-Frank-Shalev algebras that

7.3 Proposition. *For $0 \leq a < b < n$, and p a prime, we have*

$$AFS(a, b, n, p)^{\downarrow} = AFS(a - 1, b - 1, n, p),$$

where we understand $AFS(0, b, n, p) = AFS(b, n, n, p)$ in the left-hand term.

Therefore all algebras of Albert-Frank-Shalev can be obtained via deflation from another algebra of this kind. However, they have more than one constituent of length $2q-2$, while, as we have shown above, all algebras obtained by inflation have only one constituent of length $2q-2$.

On the other hand, we have

7.4 Proposition. *If an algebra of maximal class L with parameter q has at most one constituent of length $2q-2$, then it can be recovered from its deflation $N = L^{\downarrow}$ with a suitable inflation.*

Proof. In fact, all constituents except the first thus have lengths congruent to $p-1$ modulo p. Hence values i for which $C_i \neq C_2$ are spaced apart by multiples of p. The first such value is $2q$, where q is the parameter of the algebra. Since this is itself a multiple of p, we obtain that $C_i = C_2 = \mathbf{F}y$ in L, for $i \not\equiv 0 \pmod{p}$. It follows that $L \cong {}^{y'}(L^{\downarrow})$, as the two algebras have the same sequence of two-step centralizers. □

We can use deflation to show

7.5 Lemma. *Let L be an infinite-dimensional algebra of maximal class over a field of positive characteristic p. Suppose the i-th distinct two-step centralizer appears for the first time as $C_{L_1}(L_j)$. Then $j \geq 2p^{i-1}$, and the bound is attained.*

We consider infinite-dimensional algebras to avoid the exceptional behaviour of the second distinct two-step centralizer in low-dimensional algebras, as in those coming from the exceptional groups of maximal class ([Hu, III.14.24]).

Proof. We proceed by induction on i. The case $i = 1$ is clear and the case $i = 2$ is settled in Lemma 5.1. Suppose L is an infinite-dimensional algebra of maximal class in which the i-th distinct two-step centralizer T, for some $i > 2$, occurs as early as possible as $T = C_{L_1}(L_k)$, and suppose by way of contradiction that $k < 2p^{i-1}$. In the deflated algebra $L^{\downarrow}$, it is readily seen that the earliest occurrence of T as a two-step centralizer is as $T = C_{L_1^{\downarrow}}(L^{\downarrow}_{\lceil k/p \rceil})$, with $\lceil k/p \rceil \leq 2p^{i-2}$. If T is the i-th distinct two-step centralizer in $L^{\downarrow}$, we get a contradiction, as $\lceil k/p \rceil < k$.

So suppose T is the $(i-1)$-th distinct two-step centralizer in $L^{\downarrow}$. By the description of centralizers in $L^{\downarrow}$, this can only happen if all occurrences of the first centralizer in L prior to T vanish during deflation, and in turn this can only happen if all constituents in L leading to T, except the first one, have length exactly $p-1$, so that L has parameter p. It follows that k is a multiple of p. By induction, we have $\lceil k/p \rceil = 2p^{i-2}$, so that $2p^{i-1} - p < k < 2p^{i-1}$, and k is not a multiple of p, a contradiction. □

8. Soluble algebras

In this section we will be concerned with the soluble algebras of maximal class. A finite-dimensional algebra of maximal class is nilpotent, and thus soluble. *We will therefore take all algebras of maximal class in this section to be infinite-dimensional.*

Let $L = \bigoplus_{i=1}^{\infty} L_i$ be an algebra of maximal class over a field $\mathbf{F}$ of positive characteristic p. Note first that if L is soluble, then it is abelian-by-(finite dimensional), as the last non-trivial term of the derived series will be an abelian ideal of finite codimension, since L is just-(infinite dimensional).

We begin by proving

8.1 Proposition. *Let L be an infinite-dimensional, soluble algebra of maximal class. Let $M = \bigoplus_{i>k} L_i$ be the maximal abelian ideal of L, or $M = C_L\left(\bigoplus_{i\geq 2} L_i\right)$ if L is metabelian. Then M is a free k-dimensional module over a polynomial ring $\mathbf{F}[t]$, and L/M embeds into $\mathrm{End}_{\mathbf{F}[t]}(M)$.*

Proof. We employ an argument modelled on one from [Sh1].

If L is metabelian, then L is the unique infinite-dimensional metabelian algebra of maximal class, as constructed in Section 3. So suppose $M = \bigoplus_{i>k} L_i$, with $k \geq 2$.

Let $0 \neq \tau \in L_k$. Since M is the maximal abelian ideal, τ does not centralize M. Let z be any element of L, and $m \in M$. Since $[\tau z] \in M$, and M is abelian, we have

$$0 = [m[\tau z]] = [m\tau z] - [mz\tau].$$

We claim $[L_{k+1}\tau] \neq 0$. Assume, by way of contradiction, that $[L_{k+1}\tau] = 0$. Let $0 \neq m_0 \in L_{k+1}$, and $0 \neq m \in L_{k+s+1}$, for some s. Then there are $z_1, \ldots, z_s \in L_1$ such that $m = [m_0, z_1, \ldots, z_s]$. Thus

$$\begin{aligned} 0 &= [m_0, \tau, z_1, \ldots, z_s] \\ &= [m_0, z_1, \tau, \ldots, z_s] \\ &= \ldots \\ &= [m_0, z_1, \ldots, z_s, \tau] \\ &= [m\tau], \end{aligned}$$

so that τ centralizes M, a contradiction. It follows that $[L_{k+1}\tau] = L_{2k+1}$, and more generally, by the same argument, that $[L_s\tau] = L_{s+k}$, for all $s > k$. Completing the proof is now a straightforward matter. The Lie algebra morphism

$$\begin{aligned} \varphi : &L \longrightarrow \mathrm{End}_{\mathbf{F}}(M), \\ &z \mapsto (u \mapsto [uz]), \end{aligned}$$

has kernel M. The element $t = \varphi(\tau)$ centralizes $\varphi(L)$ in $\mathrm{End}_{\mathbf{F}}(M)$, so that $\varphi(L) \subseteq \mathrm{End}_{\mathbf{F}[t]}(M)$. Now if f is a polynomial with coefficients in $\mathbf{F}$, it is easy to see that $uf(t) = 0$ for some $0 \neq u \in M$ implies $f = 0$. Therefore $\mathbf{F}[t]$ is a polynomial ring, and M is a finite-dimensional, torsion-free $\mathbf{F}[t]$-module. □

Clearly if $C_s = \mathbf{F}z$, for some $s > k$, then

$$[L_{s+k}z] = [L_s\tau z] = [L_s z\tau] = 0,$$

that is, $C_{s+k} = C_s$. It follows that the sequence of two-step centralizers of L is periodic, with period length k, starting with C_{k+1}. Therefore there are at most $\max\{|\mathbf{F}|, \aleph_0\}$ soluble algebras of maximal class.

We can actually make a stronger statement on the periodicity of soluble algebras of maximal class.

8.2 Theorem. *Let L be a non-metabelian algebra of maximal class over a field $\mathbf{F}$ of positive characteristic p. The following are equivalent:*

1. *L is soluble.*
2. *L is periodic, of period a power of p, namely*

$$C_{L_1}(L_i) = C_{L_1}(L_{i+p^m}),$$

for some m, and for all $i > k$, for a suitable k. Furthermore, if k and m in (2) are chosen to be minimal, then $k = p^m$, and L^{k+1} is the maximal abelian ideal of L.

Proof. Without loss of generality, we may assume L is uncovered; tensoring with a suitable extension field will ensure this. Let $z \in L_1$ be an element that is not in any two-step centralizer, $y \neq 0$ an element in the first centralizer, and define $u_i = [y \underbrace{z \cdots z}_{i-1}]$, for $i \geq 2$, so that $L_i = \mathbf{F} u_i$, for $i \geq 2$. For $i \geq 2$, let $[u_i y] = \alpha_i u_{i+1}$, for some $\alpha_i \in \mathbf{F}$. Clearly L is periodic if and only if the sequence $(\alpha_i)_{i\geq 2}$ is periodic.

Suppose L is soluble, and let L^{k+1} be its maximal abelian ideal. Let $p^m \geq k$. Then for $i > k$ we have

$$\begin{aligned} 0 = [u_i u_{p^m+1}] &= [u_i [y \underbrace{z \cdots z}_{p^m}]] \\ &= [u_i y \underbrace{z \cdots z}_{p^m}] - [u_i \underbrace{z \cdots z}_{p^m} y] \\ &= (\alpha_i - \alpha_{i+p^m})\, u_{i+p^m+1}, \end{aligned}$$

so the sequence $(\alpha_i)_{i\geq 2}$ is periodic, of period a power of p. Note a consequence of Proposition 8.1: if L^{k+1} is the maximal abelian ideal of L, then

$$C_L(L^i) = L^{k+1}, \qquad \text{for all } i > k.$$

Take the smallest m such that there is a k for which

$$\alpha_i = \alpha_{i+p^m} \qquad \text{for all } i > k,$$

and take k to be minimal with this property. Then the above calculation, read backwards, shows that

$$[u_i u_{p^m+1}] = 0 \quad \text{for } i > k, \qquad \text{and} \quad [u_k u_{p^m+1}] \neq 0.$$

By means of

$$[u_i u_{j+1}] = [u_i [u_j z]] = [u_i u_j z] - [u_i z u_j]$$

it is easy to prove by induction on j that

$$[u_i u_j] = 0 \qquad \text{for all } i > k \text{ and } j > p^m,$$

and that

$$[u_k u_j] = -[u_k u_{p^m+1}] \neq 0 \qquad \text{for all } j > p^m.$$

The latter equality yields $k \leq p^m$, and the former shows that L^{p^m+1} is abelian, whence L is soluble.

Furthermore, we deduce that

$$C_L(L^{p^m+1}) = L^{k+1},$$

and hence L^{k+1} is the maximal abelian ideal of L. But then k is a period of L, and thus it is a multiple of the minimum period p^m. As $k \leq p^m$, we obtain $k = p^m$, concluding the proof. ∎

Note that the codimension of the maximal abelian ideal of a soluble algebra decreases under deflation, unless the algebra is metabelian.

In fact, let this codimension be $k+1$ in an algebra L. If L is not metabelian, then $k > 1$. In the deflated algebra $L^{\downarrow}$, the codimension of the maximal abelian ideal is at most
$$k' + 1 = \left\lceil \frac{k+1}{p} \right\rceil.$$
This is less than $k+1$. In fact we have
$$\left\lceil \frac{k+1}{p} \right\rceil \le \frac{k+1}{p} + 1.$$
Now $(k+1)/p + 1 < k+1$ if and only if $(k+1) + p < (k+1)p$, that is, $k(p-1) > 1$, which always holds.

This implies

8.3 Proposition. *Soluble algebras of maximal class reduce to a metabelian one under repeated deflation.*

Conversely, we can construct soluble algebras of maximal class by repeatedly inflating a metabelian one. In fact, one has the easy

8.4 Lemma. *If L is a soluble algebra of maximal class, then so are all of its inflations.*

Proof. This can be seen directly from the definition of inflation, or exploiting the characterization of soluble algebras of maximal class in terms of their periodicity. □

To provide an example, let us construct a set of cardinality $\aleph_0$ of infinite-dimensional soluble algebras with two distinct two-step centralizers, for $p = 3$. Here we make use of the description of inflation we have given in the previous section. We start with the metabelian one. We first inflate it with respect to any element not in the first two-step centralizer, to get an algebra with constituent length sequence $5, 2^{\infty}$. Now we inflate with respect to the first, resp. to the second, two-step centralizer to get algebras with constituent length sequences $16, 8^{\infty}$, resp. $4, 2^3, (5, 2)^{\infty}$. Repeating the argument, we get four algebras with constituent length sequences
$$52, 26^{\infty}, \qquad 4, 2^{15}, (5, 2^7)^{\infty}, \qquad 16, 8^3, (17, 8)^{\infty}, \qquad 4, 2^3, (5, 2)^3, (5, 2^4, 5, 2)^{\infty},$$
and so on. Note that in writing the constituent length sequence this way, although the period is correctly displayed, periodicity may occur earlier than the writing appears to suggest.

In the next section, we will construct some more infinite-dimensional soluble algebras of maximal class with parameter q, and exactly one constituent, distinct from the first one, of length $2q - 2$.

9. Limits

In this section we illustrate a way of constructing further infinite-dimensional algebras of maximal class by using inverse limits.

A first straightforward application of this method allows us to construct soluble algebras of maximal class with two distinct centralizers and parameter $q = p^h$, which have exactly two constituents of length $2q - 2$. By Lemma 6.4 such algebras cannot be obtained by inflation.

Let $N = AFS(h, h+k, k+n, p)$, for $h, k, n \geq 1$. Here we consider h and k fixed, and n variable. Consider the factor algebra $M^{(n)}$ of N with class $p^{h+k+n} + 2p^h$; it has constituent length sequence

$$2p^h - 2, (p^h - 1)^{p^k - 2}, 2p^h - 2, ((p^h - 1)^{p^k - 2}, 2p^h - 1)^{p^n - 1}.$$

Since the constituent length sequence of an algebra of maximal class with two distinct centralizers determines the sequence of two-step centralizers, and thus its isomorphism type, these algebras $M^{(n)}$ form an inverse system. The limit algebra has constituent length sequence

$$2p^h - 2, (p^h - 1)^{p^k - 2}, 2p^h - 2, ((p^h - 1)^{p^k - 2}, 2p^h - 1)^{\infty},$$

and is thus periodic, with period p^{h+k}. By Theorem 8.2, it is soluble.

For instance, for $p = 3$ we get the following algebras with two distinct two-step centralizers, besides those already constructed in the previous section. For $h = 1$ and $k = 1$ we get the algebra with constituent length sequence $4, 2, 4, (2, 5)^{\infty}$, and period 9. This algebra inflates to the two algebras $16, 8, 14, (8, 17)^{\infty}$ and $4, 2^3, 5, 2, 5, 2^3, (5, 2, 5, 2^4)^{\infty}$, both of period 27. For $h = 1$ and $k = 2$, and $h = 2$ and $k = 1$, we get two more algebras of period 27, with constituent length sequences

$$4, 2^7, 4, (2^7, 5)^{\infty} \qquad \text{and} \qquad 16, 8, 16, (8, 17)^{\infty},$$

and so on. In the same way, for all primes p and for each field $\mathbf{F}$ of characteristic p we are able to construct

$$2^{m-1} + \sum_{i=1}^{m-1} 2^{m-i-1} i$$

pairwise non-isomorphic, infinite-dimensional, soluble algebras of maximal class with two distinct two-step centralizers and period of length p^m.

We now describe a slightly more complicated process of taking limits of algebras of maximal class. This involves repeating the inflation process countably many times.

Let L be an algebra of maximal class, and $0 \neq t \in L_1$. Recall that if we identify L_1 and ${}^tL_1$ via the prime operator, we can describe the centralizer sequence of ${}^tL$ in terms of that for L.

This identification allows us, as seen before, to define the inflation of an algebra L with respect to a finite subsequence $(t_1, \dots, t_n)$ of a sequence $\tau = (t_i)_{i \geq 1}$ of non-zero elements of L_1 as follows:

$${}^{(t_1,\dots,t_n)}L = {}^{t_1}\left({}^{(t_2,\dots,t_n)}L\right) = {}^{t_1}\left({}^{t_2} \dots \left({}^{t_n}L\right) \dots\right).$$

It follows from our previous observations that the sequence of two-step centralizers of L up to class p^n is completely determined by the sequence $(t_1, \dots, t_n)$ alone. Therefore the isomorphism type of the class p^n factor ${}^{(t_1,\dots,t_n)}L^{(n)}$ of ${}^{(t_1,\dots,t_n)}L$ is also determined by $(t_1, \dots, t_n)$, and so ${}^{(t_1,\dots,t_n)}L^{(n)}$ is isomorphic to the class p^n factor of the algebra

$${}^{(t_1,\dots,t_{n+1})}L = {}^{t_1}\left({}^{(t_2,\dots,t_{n+1})}L\right).$$

We obtain that ${}^{(t_1,\dots,t_n)}L^{(n)}$ is a homomorphic image of the class p^{n+1} factor algebra ${}^{(t_1,\dots,t_{n+1})}L^{(n+1)}$ of the algebra ${}^{(t_1,\dots,t_{n+1})}L$. It follows that the algebras ${}^{(t_1,\dots,t_n)}L^{(n)}$, with the described homomorphisms, form an inverse system.

Now consider an infinite sequence $\tau = (t_i)_{i\geq 1}$ of non-zero elements of L_1. We define the algebra ${}^{\tau}L$ as the limit of the inverse system ${}^{(t_1,\dots,t_n)}L^{(n)}$.

It is easily seen that distinct sequences may lead to isomorphic algebras. To ensure a one-to-one correspondence between sequences τ and isomorphism types of algebras ${}^{\tau}L$, we consider a set of normalized sequences, by going projective as before. Fix a basis u, v for $L = L_1$, and consider the set $\mathcal{T}$ of sequences $\tau = (t_i)_{i\geq 1}$ with the following properties:

1. For each $i \geq 1$, we have $t_i = u - \alpha_i v$ for some $\alpha_i \in \mathbf{F}_\infty$. Here we understand as always $u - \infty v = v$.
2. $t_1 = u$.
3. If not all t_i equal u, then the second element in order of occurrence in τ is v.
4. If not all t_i equal u or v, then the third element in order of occurrence in τ is $u - v$.

The two-step centralizers in ${}^{\tau}L$ are $\mathbf{F}t_1, \dots, \mathbf{F}t_n, \dots$ in this order of occurrence. $\mathbf{F}t_n$, for $n \geq 1$, occurs for the first time as a two-step centralizer as $C_{{}^{\tau}L_1}({}^{\tau}L_{2p^{n-1}})$. It follows that ${}^{\tau}L$ uniquely determines the sequence $\tau \in \mathcal{T}$. Also, it follows from the descriptions of two-step centralizers in an inflated algebra, and from our normalization of sequences τ, that $\tau \in \mathcal{T}$ uniquely determines the sequence of two-step centralizers, and thus the isomorphism type, of ${}^{\tau}L$. We thus have

9.1 Proposition. *There is a one-to-one correspondence between sequences $\tau \in \mathcal{T}$ and isomorphism types of algebras ${}^{\tau}L$.*

A presentation for an algebra ${}^{\tau}L$, where $\tau = (t_i)_{i\geq 1}$, can be read off, as described in Section 3, from the following description of the two-step centralizers $C_i = C_{{}^{\tau}L_1}({}^{\tau}L_i)$:

$$C_i = \mathbf{F}t_j \quad \text{if} \quad \begin{cases} i \equiv 0 \pmod{p^{j-1}} \quad \text{and} \quad i \neq p^{j-1}, \\ \qquad \text{and} \\ i \not\equiv 0 \pmod{p^j} \qquad \text{or} \quad i = p^j. \end{cases}$$

As a first example, if we take $\tau = (u, u, u, \dots)$, we obtain that ${}^{\tau}L$ is the infinite-dimensional metabelian algebra of maximal class.

More interestingly, with this construction we are able to prove

9.2 Theorem. *There are covered algebras of maximal class over every field of positive characteristic which is at most countable.*

This we obtain simply by taking τ to be a sequence $(u, v, u-v, ...)$ which includes generators for all 1-dimensional subspaces of L_1.

We now see which sequences $\tau \in \mathcal{T}$ correspond to periodic algebras. Note first that if $\tau = (t_1, t_2, \dots) \in \mathcal{T}$, then

$$({}^{\tau}L)^{\downarrow} = {}^{\tau^{\downarrow}}L,$$

where $\tau^{\downarrow} = (t_2, t_3, \dots)$. Here it is not necessarily true that $\tau^{\downarrow} \in \mathcal{T}$. Conversely, we also have

$${}^{t_1}\left({}^{(t_2,t_3,\dots)}L\right) = {}^{(t_1,t_2,\dots)}L.$$

Suppose now ${}^{\tau}L$ is periodic, with (not necessarily minimal) period k. It is easily seen that the deflated algebra $({}^{\tau}L)^{\downarrow}$ has period $k/(k, p)$. After a finite number m of deflation steps, we will obtain an algebra N with period $k' = k/p^m$ coprime to p. By the above remark, the resulting algebra is still of the form $N = {}^{\sigma}L$, for some sequence σ, and thus it is obtained by inflation. But we have seen that in an inflated algebra which is not metabelian, all constituents have lengths congruent to $-1 \pmod p$. Counting also the trailing two-step centralizer of each constituent, we see that the period of an inflated algebra must have length a power of p. This contradicts $(k', p) = 1$, unless $k' = 1$. If $k' = 1$, σ is a constant sequence, and N is the infinite-dimensional metabelian algebra of maximal class. By the above remarks, ${}^{\tau}L$ is obtained from L by a finite number of inflation steps, and it is therefore soluble. Also, τ is an ultimately constant sequence. Clearly there are $\max\{|\mathbf{F}|, \aleph_0\}$ such sequences. We have obtained

9.3 Proposition. *For $\tau \in \mathcal{T}$, the following are equivalent*

1. *${}^{\tau}L$ is periodic.*
2. *${}^{\tau}L$ is soluble.*
3. *τ is ultimately constant.*

9.4 Corollary. *Over every field $\mathbf{F}$ of positive characteristic there are*

1. *$|\mathbf{F}|^{\aleph_0}$ (non-periodic) algebras of maximal class, and*
2. *$\max\{|\mathbf{F}|, \aleph_0\}$ infinite-dimensional, soluble algebras of maximal class.*

As mentioned in the Introduction, the existence of non-periodic algebras of maximal class answers a question of Shalev [Sh3, Problem 28].

A particularly interesting class of examples is provided by the algebras of maximal class, which we are tempted to call *fractal*, that are transformed back to themselves by a finite number of deflation steps. The algebras of Albert-Frank-Shalev are examples of periodic algebras with this property: this follows easily from Proposition 7.3. The algebras we construct here are non-periodic, and have the additional property that they are transformed back to themselves also under a suitable finite sequence of *inflation* steps.

The simplest example of this kind is ${}^{\tau}L$, with $\tau = (u, v, u, v, \dots)$. Here $({}^{\tau}L)^{\downarrow}$ is isomorphic to ${}^{\tau}L$, and ${}^{\tau}L$ inflates to itself with respect to the second distinct two-step centralizer $\mathbf{F}v$. To construct this algebra, one starts by inflating the abelian algebra of maximal class with respect to two independent elements, and then keeps inflating with respect to the *second* distinct two-step centralizer. For the prime $p = 3$, this algebra has constituent length sequence

$$4, 2^3, (5, 2)^3, (5, 2^4, 5, 2)^3, (5, 2^4, (5, 2)^4, 5, 2^4, 5, 2)^3, \dots.$$

In fact, we obtain this sequence by starting with an algebra of class 7 with two distinct two-step centralizers with just one constituent of length 4, and applying repeatedly the transformation, described in the section on inflation, that replaces a constituent of length m with the subsequence $2p - 1, (p - 1)^{m-1}$. This example can be generalized to the algebras ${}^{\tau_n}L$, with

$$\tau_n = (\underbrace{u, \dots, u}_{n}, \underbrace{v, \dots, v}_{n}, \underbrace{u, \dots, u}_{n}, \underbrace{v, \dots, v}_{n}, \dots),$$

for $n \geq 1$. Such an algebra reduces to itself after n deflation steps, and no fewer.

Other examples of algebras that deflate to themselves are obtained by considering sequences like

$$\tau = (u, v, u - v, u, v, u - v, \dots).$$

10. Computations

None of the results of this paper rely on computer calculations. Still, a lot of computational work went into the writing of this paper. We report on it briefly here.

Our first computer calculations made use of a simple-minded GAP program [S+] to calculate the constituent lengths in some of the Albert-Frank-Shalev algebras. This lead to the results in Section 4.

We made use of the ANU p-Quotient Program [HNO] to compute with graded Lie algebras over a prime field, as explained elsewhere [CMNS].

To simplify the exploration of algebras of maximal class with two distinct two-step centralizers, we built a small `C` preprocessor, called `ppi` (for `p`repare `p`Q `i`nput), which produces the input needed by the ANU p-Quotient Program from a description of the sequence of constituent lengths. This allowed us to explore these algebras to class 127 (the limit under the then current version of the program). The theory developed in this paper is able to account for the algebras we observed in this way. We constructed `ppi` so that, in producing the presentation to be used as input file by the ANU p-Quotient Program, it exploits our theory. Therefore the final output acted as a confirmation of the correctness of our results, and actually allowed us to spot a couple of mistakes in earlier versions of this paper.

References

[AF] A. A. Albert and M. S. Frank, *Simple Lie algebras of characteristic p*, Rend. Sem. Mat. Univ. Politec. Torino **14** (1954/5), 117–139. MR **18:**52a

[Bla] N. Blackburn, *On a special class of p-groups*, Acta Math. **100** (1958), 45–92. MR **21:**1349

[Blo] Richard E. Block, *Determination of the differentiably simple rings with a minimal ideal*, Ann. of Math. **90** (1969), 433–459. MR **40:**4319

[CMNS] A. Caranti, S. Mattarei, M. F. Newman and C. M. Scoppola, *Thin groups of prime-power order and thin Lie algebras*, Quart. J. Math. Oxford Ser. (2) **47** (1996), 279–296. CMP 97:02

[CN^3] F. Celler, M. F. Newman, W. Nickel and A. C. Niemeyer, *An algorithm for computing quotients of prime-power order for finitely presented groups and its implementation in GAP*, Research Report 127, Australian National University, Canberra, 1993.

[HNO] George Havas, M.F. Newman and E.A. O'Brien, *ANU p-Quotient Program (Version 1.2), written in* `C`, available from `maths.anu.edu.au` by `anonymous ftp` in the directory `pub/PQ`, as a share library with `GAP 3.4` and as part of `Magma`, 1995.

[Hu] B. Huppert, *Endliche Gruppen I*, Springer, Berlin, 1967. MR **37:**302

[Jac] Nathan Jacobson, *Lie algebras*, Dover, New York, 1979. MR **80k:**17001

[L] Édouard Lucas, *Sur les congruences des nombres eulériens et des coefficients différentiels des fonctions trigonométriques, suivant un module premier*, Bull. Soc. Math. France **6** (1878), 49–54.

[Ma] Avinoam Mann, *Space groups and groups of prime power order. VII. Powerful p-groups and uncovered p-groups*, Bull. London Math. Soc. **24** (1992), 271–276. MR **93c:**20041

[M2] S. McKay, *On the structure of a special class of p-groups II*, Quart. J. Math. Oxford Ser. (2) **41** (1990), 431–448. MR **91k:**20027

[S+] Martin Schönert et al., *GAP – Groups, Algorithms, and Programming*, Release 3.4, Lehrstuhl D für Mathematik, Rheinisch-Westfälische Technische Hochschule, Aachen, Germany, 1995.

[Sh1] Aner Shalev, *The structure of finite p-groups: constructive proof of the coclass conjectures*, Invent. Math. **115** (1994), 315–345. MR **95j:**20022b

[Sh2] Aner Shalev, *Simple Lie algebras and Lie algebras of maximal class*, Arch. Math. (Basel) **63** (1994), 297–301. MR **95j:**17025

[Sh3] Aner Shalev, *Finite p-groups*, Proceedings of the NATO Advanced Study Institute on Finite and Locally Finite Groups, Istanbul, Turkey, 1994, NATO ASI Series, vol. 471, Kluwer Academic Publishers, Dordrecht/Boston/London, 1995, pp. 401–450. MR **96f:**20002

[SZ] Aner Shalev and E. I. Zelmanov, *Narrow Lie algebras I: a coclass theory and a characterization of the Witt algebra*, J. Algebra **189** (1997), 294–330.

[Z] Hans Zassenhaus, *Über Liesche Ringe mit Primzahlcharakteristik*, Abh. Math. Sem. Univ. Hamburg **13** (1939), 1–100.

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