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Gradient Ricci Solitons on Multiply Warped Product Manifolds

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Abstract. We consider gradient Ricci solitons on multiply warped product manifolds. We find the necessary and sufficient conditions for multiply warped product manifolds to be gradient Ricci solitons.

1. Introduction

In [7], Ricci solitons are introduced by Hamilton, which are both a natural generalization of Einstein manifolds and a special solution of the Ricci flow. Let (M, g) be a Riemannian manifold endowed with a Riemannian metric g. If there exists a vector field $X \in \chi(M)$ and a constant λ such that the Ricci tensor satisfies the following equation

$$Ric + \frac{1}{2}\boldsymbol{L}_X g = \lambda g, \tag{1}$$

where $\boldsymbol{L}_X$ is the Lie derivative along to X, then (M, g) is called a *Ricci soliton* (see also [8]). The Ricci soliton is said to be shrinking, steady or expanding if $\lambda > 0$, $\lambda = 0$ or $\lambda < 0$, respectively. If $X = grad\psi$ for some function ψ on M, then M is called a *gradient Ricci soliton*. Thus, the equation (1) turns into the following equation

$$Ric + Hess\psi = \lambda g, \tag{2}$$

where $Hess\psi$ denotes the Hessian of ψ and ψ is a potential function [7].

In [10], Petersen and Wylie studied gradient Ricci solitons with maximal symmetry and constructed examples of cohomogeneity one gradient solitons. In [11], the same authors classified 3-dimensional shrinking gradient solitons. In [14], Kim, Lee, Choi and Lee studied gradient Ricci solitons on Riemannian product spaces and warped product spaces. In [15], Shenawy studied conformal and concurrent vector fields of Ricci solitons on warped product manifolds. In [6], Feitosa, Freitas and Gomes obtained a necessary and sufficient condition for constructing a gradient Ricci soliton warped product. In [12], Lee, Kim and Choi studied Ricci solitons and gradient Ricci solitons on the warped product spaces and gradient Yamabe solitons on the Riemannian product spaces. In [13], the same authors studied gradient Ricci solitons on the warped products $M = S^1 \times_f B$ of 1-dimensional circle and Riemannian manifolds and introduced the

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generalized Ricci soliton. In [2], De studied gradient Ricci solitons in para-Sasakian manifolds. In [3], [4], [16] and [19], gradient Ricci solitons in almost contact metric manifolds were studied. Motivated by the above studies, in the present paper, we consider gradient Ricci solitons on multiply warped product manifolds. We find the necessary and sufficient conditions for multiply warped product manifolds to be gradient Ricci solitons.

2. Preliminaries

Let (B, g_B) and (F_i, g_{F_i}) be r and s_i dimensional Riemannian manifolds, respectively, where $i \in \{1, 2, ..., m\}$ and also $M = B \times F_1 \times F_2 \times ... \times F_m$ be an n-dimensional Riemannian manifold, where $n = r + \sum_{i=1}^{m} s_i$. Let $b_i : B \to (0, \infty)$ be smooth functions for $1 \leq i \leq m$. The product manifold $M = B \times_{b_1} F_1 \times_{b_2} F_2 \times ... \times_{b_m} F_m$ endowed with the metric tensor

$$g = \pi^* (g_B) \oplus (b_1 \circ \pi)^2 \sigma_1^* (g_{F_1}) \oplus ... \oplus (b_m \circ \pi)^2 \sigma_m^* (g_{F_m}), \tag{3}$$

where π and σ_i are the natural projection on B and F_i, respectively, is called the *multiply warped product*. The functions $b_i : B \to (0, \infty)$ are called *warping functions* and each manifold (F_i, g_{F_i}) and the manifold (B, g_B) are called *fiber manifolds and* the *base manifold* of the multiply warped product, respectively for $1 \leq i \leq m$ ([5], [17], [18]). We shall denote ∇, ${}^B\nabla$, ${}^{F_i}\nabla$, Ric, ${}^BRic$ and ${}^{F_i}Ric$ the Levi-Civita connections and the Ricci curvatures of the M, B and F_i, respectively.

Now, we give the following lemmas:

Lemma 2.1. *[17] Let $M = B \times_{b_1} F_1 \times_{b_2} F_2 \times ... \times_{b_m} F_m$ be a multiply warped product with metric $g = g_B \oplus b_1^2 g_{F_1} \oplus ... \oplus b_m^2 g_{F_m}$. If $X, Y \in \chi(B)$ and $V \in \chi(F_i)$, $W \in \chi(F_j)$, then*

i) $\nabla_X Y$ is the lift of ${}^B\nabla_X Y$ on B,

ii) $\nabla_X V = \nabla_V X = \frac{X(b_i)}{b_i} V$,

iii) $\nabla_V W = \begin{cases} 0 & \text{if } i \neq j, \\ {}^{F_i}\nabla_V W - \left(\frac{g(V,W)}{b_i}\right) grad_B (b_i) & \text{if } i = j. \end{cases}$

Lemma 2.2. *[17] Let $M = B \times_{b_1} F_1 \times_{b_2} F_2 \times ... \times_{b_m} F_m$ be a multiply warped product with metric $g = g_B \oplus b_1^2 g_{F_1} \oplus ... \oplus b_m^2 g_{F_m}$ and $\varphi : B \to \mathbb{R}$ be a smooth function for any $i \in \{1, 2, ..., m\}$. Then*

i) $grad(\varphi \circ \pi) = grad_B \varphi$,

ii) $\Delta(\varphi \circ \pi) = \Delta_B \varphi + \sum_{i=1}^{m} \frac{s_i}{b_i} g_B (grad_B \varphi, grad_B b_i)$,

where grad and Δ denote the gradient and Laplace-Beltrami operator on M, respectively.

Lemma 2.3. *[17] Let $M = B \times_{b_1} F_1 \times_{b_2} F_2 \times ... \times_{b_m} F_m$ be a multiply warped product with metric $g = g_B \oplus b_1^2 g_{F_1} \oplus ... \oplus b_m^2 g_{F_m}$. If $X, Y \in \chi(B)$ and $V \in \chi(F_i)$, $W \in \chi(F_j)$, then*

i) $Ric(X, Y) = {}^BRic(X, Y) - \sum_{i=1}^{m} \frac{s_i}{b_i} H_B^{b_i}(X, Y)$,

ii) $Ric(X, V) = 0$,

iii) $Ric(V, W) = 0$ if $i \neq j$,

iv)

$$Ric(V, W) = {}^{F_i} Ric(V, W)$$
$$- \left[\frac{\Delta_B b_i}{b_i} + (s_i - 1) \frac{\left\| grad_B b_i \right\|^2}{b_i^2} + \sum_{i=1,\, k \neq i}^{m} s_k \frac{g_B (grad_B b_i, grad_B b_k)}{b_i b_k} \right] g(V, W),$$

where $H_B^{b_i}$ denotes the Hessian operator on B.

Let $\left(M^k, g, grad\psi\right)$ be a gradient Ricci soliton. Using the equation (2), it is easy to see that

$$scal + \Delta\psi = k\lambda, \tag{4}$$

where *scal* denotes scalar curvature of M. Furthermore, by [9], it is well known that

$$2\lambda\psi - \left\|grad\psi\right\|^2 + \Delta\psi = c, \tag{5}$$

for some constant c.

3. The Gradient Ricci Soliton on a Multiply Warped Product

Let ψ be a potential function of a gradient Ricci soliton multiply warped product M as the lifting of a smooth function on B. Assume that $\widetilde{\varphi} = \varphi \circ \pi$ is the lift of a smooth function φ on B. Using the same method in [6], we obtain $\psi = \widetilde{\varphi}$.

Now, we give the following proposition:

Proposition 3.1. *Let $M = B \times_{b_1} F_1 \times_{b_2} F_2 \times ... \times_{b_m} F_m$ be a multiply warped product and φ a smooth function on B. If $(M, g, grad\widetilde{\varphi}, \lambda)$ is a gradient Ricci soliton, then*

$$2\lambda\varphi - \left\|grad_B\varphi\right\|^2 + \Delta_B\varphi + \sum_{i=1}^{m} \frac{s_i}{b_i} grad_B\varphi\,(b_i) = c \tag{6}$$

for some constant c.

Proof. Let $(M, g, grad\widetilde{\varphi}, \lambda)$ be a gradient Ricci soliton. Using the equation (5) and $\psi = \widetilde{\varphi}$, we can write

$$2\lambda\widetilde{\varphi} - \left\|grad\widetilde{\varphi}\right\|^2 + \Delta\widetilde{\varphi} = c \tag{7}$$

for some constant c. From parts i) and ii) of Lemma 2.2, the equation (7) turns into

$$2\lambda\varphi - \left\|grad_B\varphi\right\|^2 + \Delta_B\varphi + \sum_{i=1}^{m} \tfrac{s_i}{b_i} g_B\left(grad_B\varphi, grad_B b_i\right) = c$$

for some constant c. Thus, we obtain the desired result. □

Taking all $b_i = b$ for any $i \in \{1, 2, ..., m\}$ in Proposition 3.1, we can state the following corollary:

Corollary 3.2. *Let $M = B \times_b F_1 \times_b F_2 \times ... \times_b F_m$ be a multiply warped product and φ a smooth function on B. If $(M, g, grad\widetilde{\varphi}, \lambda)$ is a gradient Ricci soliton, then*

$$2\lambda\varphi - \left\|grad_B\varphi\right\|^2 + \Delta_B\varphi + \left(\sum_{i=1}^{m} s_i\right) \frac{grad_B\varphi\,(b)}{b} = c \tag{8}$$

for some constant c.

Proposition 3.3. *Let $M = B \times_{b_1} F_1 \times_{b_2} F_2 \times ... \times_{b_m} F_m$ be a multiply warped product and φ a smooth function on B. If $(M, g, grad\widetilde{\varphi}, \lambda)$ is a gradient Ricci soliton with $s_i > 1$, then*

$${}^{B}Ric + H_B^{\varphi} = \lambda g_B + \sum_{i=1}^{m} \frac{s_i}{b_i} H_B^{b_i}(X, Y) \tag{9}$$

and ${}^{F_i}Ric = \mu \sum_{i=1}^{m} b_i^2 g_{F_i}$ with μ satisfying

$$\mu = \lambda - \frac{grad_B\varphi\,(b_i)}{b_i} + \frac{\Delta_B b_i}{b_i} + (s_i - 1)\frac{\left\|grad_B b_i\right\|^2}{b_i^2} + \sum_{i=1,\, k\neq i}^{m} s_k \frac{grad_B b_i\,(b_k)}{b_i b_k}. \tag{10}$$

Proof. Assume that $(M, g, grad\widetilde{\varphi}, \lambda)$ is a gradient Ricci soliton. Using part i) of Lemma 2.3 and the equation (2), we find

$$\lambda g(X,Y) - Hess\widetilde{\varphi}(X,Y) = {}^{B}Ric(X,Y) - \sum_{i=1}^{m} \frac{s_i}{b_i} H_B^{b_i}(X,Y).$$

For all $X, Y \in \chi(B)$, it is known that $Hess\widetilde{\varphi}(X,Y) = H_B^{\varphi}(X,Y)$, then we can write

$$\lambda g_B(X,Y) - H_B^{\varphi}(X,Y) = {}^{B}Ric(X,Y) - \sum_{i=1}^{m} \frac{s_i}{b_i} H_B^{b_i}(X,Y).$$

This proves the first assertion of the proposition. In a similar way, using part iv) of Lemma 2.3 and the equation (2), we have

$$\lambda g(V,W) - Hess\widetilde{\varphi}(V,W) = {}^{F_i}Ric(V,W) - \left[\frac{\Delta_B b_i}{b_i} + (s_i - 1)\frac{\left\|grad_B b_i\right\|^2}{b_i^2}\right.$$

$$\left. + \sum_{i=1,\, k\neq i}^{m} s_k \frac{g_B(grad_B b_i, grad_B b_k)}{b_i b_k}\right] g(V,W).$$

For all $V, W \in \chi(F_i)$, $1 \leq i \leq m$, we obtain

$$\lambda \sum_{i=1}^{m} b_i^2 g_{F_i}(V,W) - Hess\widetilde{\varphi}(V,W) = {}^{F_i}Ric(V,W)$$

$$- \left[\frac{\Delta_B b_i}{b_i} + (s_i - 1)\frac{\left\|grad_B b_i\right\|^2}{b_i^2} + \sum_{i=1,\, k\neq i}^{m} s_k \frac{g_B(grad_B b_i, grad_B b_k)}{b_i b_k}\right] \sum_{i=1}^{m} b_i^2 g_{F_i}(V,W). \tag{11}$$

Since $grad\widetilde{\varphi} \in \chi(B)$, using part ii) of Lemma 2.1, we find

$$Hess\widetilde{\varphi}(V,W) = g(\nabla_V grad\widetilde{\varphi}, W) = g\left(\frac{grad\widetilde{\varphi}(b_i)}{b_i} V, W\right)$$

$$= \frac{grad_B\varphi(b_i)}{b_i} \sum_{i=1}^{m} b_i^2 g_{F_i}(V,W). \tag{12}$$

Finally, substituting equation (12) into equation (11), we obtain

$$\left[\lambda - \frac{grad_B\varphi(b_i)}{b_i}\right] \sum_{i=1}^{m} b_i^2 g_{F_i}(V,W) = {}^{F_i}Ric(V,W)$$

$$- \left[\frac{\Delta_B b_i}{b_i} + (s_i - 1)\frac{\left\|grad_B b_i\right\|^2}{b_i^2} + \sum_{i=1,\, k\neq i}^{m} s_k \frac{g_B(grad_B b_i, grad_B b_k)}{b_i b_k}\right] \sum_{i=1}^{m} b_i^2 g_{F_i}(V,W).$$

This completes the proof. □

Taking all $b_i = b$ for any $i \in \{1, 2, ..., m\}$ in Proposition 3.3, we can state the following corollary:

Corollary 3.4. *Let $M = B \times_b F_1 \times_b F_2 \times ... \times_b F_m$ be a multiply warped product and φ a smooth function on B. If $(M, g, grad\widetilde{\varphi}, \lambda)$ is a gradient Ricci soliton with $\sum_{i=1}^{m} s_i > 1$, then*

$$^{B}Ric + H_B^{\varphi} = \lambda g_B + \left(\sum_{i=1}^{m} s_i\right) \frac{H_B^b}{b} \tag{13}$$

and $^{F_i}Ric = \mu \sum\limits_{i=1}^{m} g_{F_i}$ *such that*

$$\mu = \lambda b^2 - b \quad grad_B\varphi(b) \quad + b\Delta_B b + \left(\left(\sum_{i=1}^{m} s_i\right) - 1\right) \quad \left\|grad_B b\right\|^2 \quad . \tag{14}$$

Proposition 3.5. *Let (B^r, g_B) be a Riemannian manifold with two smooth functions $b > 0$ and φ satisfying*

$$Ric + Hess\varphi = \lambda g_B + \left(\sum_{i=1}^{m} s_i\right) \frac{Hessb}{b} \tag{15}$$

and

$$2\lambda\varphi - \quad \left\|grad_B\varphi\right\|^2 \quad + \Delta_B\varphi + \left(\sum_{i=1}^{m} s_i\right) \frac{grad_B\varphi(b)}{b} = c \tag{16}$$

for some constant $c, \lambda, s_i \in \mathbb{R}$ with $s_i \neq 0$. Then

$$\lambda b^2 - b \quad grad_B\varphi(b) \quad + b\Delta_B b + \left(\left(\sum_{i=1}^{m} s_i\right) - 1\right) \quad \left\|grad_B b\right\|^2 \quad = \mu, \tag{17}$$

for a constant $\mu \in \mathbb{R}$.

Proof. Let (B^r, g_B) be a Riemannian manifold. Assume that b and φ satisfy the equations (15) and (16). Using the equation (15), we have

$$scal + \Delta_B\varphi = \lambda r + \left(\sum_{i=1}^{m} s_i\right) \frac{\Delta_B b}{b},$$

where *scal* is the scalar curvature of B. Hence, we find

$$d(scal) = -\frac{\left(\sum\limits_{i=1}^{m} s_i\right)}{b^2}(\Delta_B b)\, db + \frac{\left(\sum\limits_{i=1}^{m} s_i\right)}{b} d(\Delta_B b) - d(\Delta_B\varphi). \tag{18}$$

On the other hand, using the equation (15), we calculate

$$divRic = \frac{\left(\sum\limits_{i=1}^{m} s_i\right)}{b} Ric\left(\quad grad_B b \quad ,.\right) + \frac{\left(\sum\limits_{i=1}^{m} s_i\right)}{b} d(\Delta_B b) - \frac{\left(\sum\limits_{i=1}^{m} s_i\right)}{2b^2} d\left(\quad \left\|grad_B b\right\|^2 \quad \right)$$
$$- \quad Ric(grad_B\varphi, .) \quad - d(\Delta_B\varphi). \tag{19}$$

By the use of the equation (15), we obtain

$$Ric(grad_B b, .) \quad = \lambda db + \frac{\left(\sum\limits_{i=1}^{m} s_i\right)}{2b} d\left(\quad \left\|grad_B b\right\|^2 \quad \right) - Hess\varphi\left(\quad grad_B b \quad ,.\right) \tag{20}$$

and

$$Ric\left(grad_B\varphi,.\right) = \lambda d\varphi + \frac{\left(\sum_{i=1}^{m} s_i\right)}{b} Hessb\left(grad_B\varphi\ ,.\right) - \frac{1}{2}\ d\left(\left\|grad_B\varphi\right\|^2\right). \tag{21}$$

Using the equations (20) and (21) into the equation (19), we get

$$divRic = \frac{\left(\sum_{i=1}^{m} s_i\right)}{b}\lambda db + \frac{\left(\sum_{i=1}^{m} s_i\right)\left[\left(\sum_{i=1}^{m} s_i\right)-1\right]}{2b^2}\ d\left(\left\|grad_B b\right\|^2\right) + \frac{\left(\sum_{i=1}^{m} s_i\right)}{b} d\left(\Delta_B b\right)$$

$$-\lambda d\varphi + \frac{1}{2}\ d\left(\left\|grad_B\varphi\right\|^2\right) - d\left(\Delta_B\varphi\right) - \frac{\left(\sum_{i=1}^{m} s_i\right)}{b}\ d\left(grad_B\varphi\left(b\right)\right). \tag{22}$$

By the contracted second Bianchi identity, the equations (18) and (22) give us

$$d\left(b\Delta_B b + \lambda b^2 + \left[\left(\sum_{i=1}^{m} s_i\right) - 1\right]\ \left\|grad_B b\right\|^2\right)$$

$$-\frac{b^2}{\left(\sum_{i=1}^{m} s_i\right)} d\left(\Delta_B\varphi + 2\lambda\varphi - \left\|grad_B\varphi\right\|^2\right) - 2bd\left(grad_B\varphi\left(b\right)\right) = 0. \tag{23}$$

If we take the derivative of the equation (16), then we have

$$-\frac{b^2}{\left(\sum_{i=1}^{m} s_i\right)} d\left(\Delta_B\varphi + 2\lambda\varphi - \left\|grad_B\varphi\right\|^2\right) - bd\left(grad_B\varphi\left(b\right)\right)$$

$$= -\ grad_B\varphi\left(b\right)\ db. \tag{24}$$

Finally, substituting the equation (24) into (23), we obtain

$$d\left(\lambda b^2 + b\Delta_B b + \left(\left(\sum_{i=1}^{m} s_i\right) - 1\right)\ \left\|grad_B b\right\|^2\ - b\ grad_B\varphi\left(b\right)\right) = 0.$$

Thus, we can write

$$\lambda b^2 + b\Delta_B b + \left(\left(\sum_{i=1}^{m} s_i\right) - 1\right)\ \left\|grad_B b\right\|^2\ - b\ grad_B\varphi\left(b\right) = \mu$$

for a constant $\mu \in \mathbb{R}$. This completes the proof. □

Using the above propositions, we can state the following theorems:

Theorem 3.6. *Let $(M = B \times_b F_1 \times_b F_2 \times ... \times_b F_m, g, grad\widetilde{\varphi}, \lambda)$ be a gradient Ricci soliton on a multiply warped product, where $\widetilde{\varphi} = \varphi \circ \pi$. If $(M, g, grad\widetilde{\varphi}, \lambda)$ is expanding $(\lambda < 0)$ or steady $(\lambda = 0)$ with $(\sum_{i=1}^{m} s_i) > 1$ and that its warping function b reaches both maximum and minimum, then $(M, g, grad\widetilde{\varphi}, \lambda)$ must be a Riemannian product.*

Proof. *Let $M = B \times_b F_1 \times_b F_2 \times ... \times_b F_m$ be a multiply warped product. Assume that b reaches both maximum and minimum. By Corollary 3.4, we have ${}^{F_i}Ric = \mu \sum_{i=1}^{m} g_{F_i}$ where*

$$\mu = \lambda b^2 - b \ \ grad_B\varphi(b) \ \ + b\Delta_B b + \left(\sum_{i=1}^{m} s_i - 1\right) \ \left\|grad_B b\right\|^2 \ . \tag{25}$$

Then, by Proposition 3.5, it is known that μ is a constant. Let $p, q \in B^r$ be the points, where b attains its maximum and minimum in B^r. Hence, we can write

$$grad_B b(p) = 0 = grad_B b(q) \quad and\ \Delta_B b(p) \leq 0 \leq \Delta_B b(q). \tag{26}$$

Finally, using the same method of Theorem 1 in [6], we find that b is a constant. This completes the proof. □

Theorem 3.7. *Let $(M = B \times_b F_1 \times_b F_2 \times ... \times_b F_m, g, grad\widetilde{\varphi}, \lambda)$ be a gradient Ricci soliton on a multiply warped product, where $\widetilde{\varphi} = \varphi \circ \pi$. If $(M, g, grad\widetilde{\varphi}, \lambda)$ is a shrinking $(\lambda > 0)$ with compact base and $(\sum_{i=1}^{m} s_i) > 1-$dimensional compact fibers, then $(M, g, grad\widetilde{\varphi}, \lambda)$ must be a compact manifold.*

Proof. Let $(M = B \times_b F_1 \times_b F_2 \times ... \times_b F_m, g, grad\widetilde{\varphi}, \lambda)$ be a gradient Ricci soliton on a multiply warped product. From Theorem 3.6, we have ${}^{F_i}Ric = \mu \sum_{i=1}^{m} g_{F_i}$, where μ is a constant. Using the same method of Theorem 2 in [6], we find

$$\mu vol_\varphi(B^r) = \lambda \int_{B^r} b^2 e^{-\varphi} dB + \left(\left(\sum_{i=1}^{m} s_i\right) - 2\right) \int_{B^r} \ \left\|grad_B b\right\|^2 \ e^{-\varphi} dB.$$

Since $\lambda > 0$ and $\sum_{i=1}^{m} s_i > 1$, we have $\mu > 0$. By the use of the Bonnet-Myers Theorem, F_i are compact for $1 \leq i \leq m$. Thus, $B \times_b F_1 \times_b F_2 \times ... \times_b F_m$ is a compact manifold. This completes the proof. □

Theorem 3.8. *Let (B^r, g_B) be a complete Riemannian manifold with two smooth functions $b > 0$ and φ satisfying the equations (15) and (16). Assume that $\left(F_i^{s_i}, g_{F_i}\right)$ are complete Riemannian manifolds for $1 \leq i \leq m$ with Ricci tensor ${}^{F_i}Ric = \mu \sum_{i=1}^{m} g_{F_i}$ such that μ satisfies the equation (14) and $(\sum_{i=1}^{m} s_i) > 1$. Then $(M = B \times_b F_1 \times_b F_2 \times ... \times_b F_m, g, grad\widetilde{\varphi}, \lambda)$ is a gradient Ricci soliton, where $\widetilde{\varphi} = \varphi \circ \pi$.*

Proof. Assume that (B^r, g_B) is a complete Riemannian manifold with two smooth functions $b > 0$ and φ satisfying the equations (15) and (16). Let $\left(F_i^{s_i}, g_{F_i}\right)$ be complete Riemannian manifolds for $1 \leq i \leq m$ whose Ricci tensor ${}^{F_i}Ric = \mu \sum_{i=1}^{m} g_{F_i}$ with μ satisfying the equation (14) and $\sum_{i=1}^{m} s_i > 1$. Firstly, we can consider the multiply warped product $(B \times_b F_1 \times_b F_2 \times ... \times_b F_m, g)$ with $g = \pi^*(g_B) \oplus (b \circ \pi)^2 \sigma_1^*(g_{F_1}) \oplus ... \oplus (b \circ \pi)^2 \sigma_m^*(g_{F_m})$. By the use of the part i) of Lemma 2.3 and the equation (15), the equation (2) is satisfied for all $X, Y \in \chi(B)$. Then, for $X \in \chi(B)$ and $V \in \chi(F_i)$, using $grad\widetilde{\varphi} \in \chi(B)$, we have

$$Hess\widetilde{\varphi}(X, V) = g(\nabla_X grad\widetilde{\varphi}, V) = 0. \tag{27}$$

In view of the part ii) of Lemma 2.3 and the equation (27), the equation (2) is satisfied. Moreover, for $V \in \chi(F_i)$ and $W \in \chi\left(F_j\right)$ with $i \neq j$, we have

$$Hess\widetilde{\varphi}(V, W) = g(\nabla_V grad\widetilde{\varphi}, W) = 0. \tag{28}$$

Thus, by the use of the part iii) of Lemma 2.3 and the equation (28), the equation (2) is satisfied. Finally, for $V, W \in \chi(F_i)$ and also using the equations (14), ${}^{F_i}Ric = \mu \sum_{i=1}^{m} g_{F_i}$ and the part iv) of Lemma 2.3, we get

$$Ric(V, W) = \ \mu \sum_{i=1}^{m} g_{F_i}(V, W) - \left[b\Delta_B b + (s_i - 1) \ \ \left\|grad_B b\right\|^2\right.$$

$$+\sum_{i=1,\,k\neq i}^{m} s_k \left\|grad_B b\right\|^2 \Bigg] \sum_{i=1}^{m} g_{F_i}(V,W)$$

$$= \left[\mu - b\Delta_B b - (s_i - 1)\left\|grad_B b\right\|^2 - \sum_{i=1,\,k\neq i}^{m} s_k \left\|grad_B b\right\|^2\right] \sum_{i=1}^{m} g_{F_i}(V,W)$$

$$\left[\mu - b\Delta_B b - \left(\left(\sum_{i=1}^{m} s_i\right) - 1\right)\left\|grad_B b\right\|^2\right] \sum_{i=1}^{m} g_{F_i}(V,W)$$

$$= \left(\lambda - \frac{grad_B\varphi(b)}{b}\right) g(V,W). \tag{29}$$

From the equation (12), we have

$$Hess\widetilde{\varphi} = \frac{grad_B\varphi(b)}{b} g(V,W). \tag{30}$$

Substituting the equation (30) into (29), we obtain the equation (2). This proves the theorem. □

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