8-2018

Gradient Ricci Solitons with a Conformal Vector Field

Ramesh Sharma
University of New Haven, rsharma@newhaven.edu

Publisher Citation
Sharma, R. (2018). Gradient Ricci solitons with a conformal vector field. Journal of Geometry, 109(2), 33.

Comments
This is a post-peer-review, pre-copyedit version of an article published in *Journal of Geometry*. The final authenticated version is available online at: http://dx.doi.org/10.1007/s00022-018-0439-x.
Mathematics Subject Classification 53C25 53C44

GRADIENT RICCI SOLITONS WITH A CONFORMAL VECTOR FIELD

Ramesh Sharma

Abstract

We show that a connected gradient Ricci soliton (M, g, f, λ) with constant scalar curvature and admitting a non-homothetic conformal vector field V leaving the potential vector field invariant, is Einstein and the potential function f is constant. For locally conformally flat case and non-homothetic V we show without constant scalar curvature assumption, that f is constant and g has constant curvature.

2010 *Mathematics Subject Classification*: 53C25,53C44

Keywords and phrases: Gradient Ricci soliton, constant scalar curvature, conformal vector field.

1 Introduction

Let M denote a smooth n-dimensional manifold, g a Riemannian metric and X a smooth vector field on M, and λ a real constant. Then the system (M, g, X, λ) is said to define a Ricci soliton if

$$L_X g + 2\,\mathrm{Ric} = 2\lambda g \tag{1}$$

where L denotes the Lie-derivative operator and *Ric* the Ricci tensor of g. Thus a Ricci soliton is a generalization of an Einstein metric for which X is Killing. The Ricci soliton is said to be shrinking, steady, and expanding according as λ is positive, zero, and negative respectively. If the vector field X is the gradient of a smooth function f, i.e. $X = \nabla f$, then (M, g, f, λ) is called a gradient Ricci soliton, in which case the equation (1) becomes

$$\mathrm{Hess}\, f + \mathrm{Ric} = \lambda g \tag{2}$$

where Hess denotes the Hessian operator with respect to g. An important result of Perelman [9] says that a compact Ricci soliton is gradient. The gradient Ricci soliton is said to be trivial when f is constant and g is Einstein.

For a general Ricci soliton vector field X, we have the following formula (Chow et al [1]):

$$L_X S = 2|\operatorname{Ric}|^2 + \Delta S - 2\lambda S \tag{3}$$

for the scalar curvature S, where $\Delta = \operatorname{Tr}.(\text{Hess})$ denotes the Laplacian operator of g.

In [3], Fernández-López and García-Río showed that conformally flat gradient Ricci solitons are locally isometric to a warped product of an interval and a real space form. This result was generalized to include the Lorentzian case by Brozos-Vázquez, García-Río and Gavino-Fernández in [2]. We also note that a Riemannian n-manifold admitting a maximal $\frac{(n+1)(n+2)}{2}$-parameter group of conformal transformations is conformally flat. Therefore it is interesting to examine the effect of the existence of a 1-parameter group of conformal transformations generated by a conformal vector field V on a gradient Ricci soliton. Motivated by this problem, we prove

Theorem 1 *If (M, g, f, λ) is a connected gradient Ricci soliton with constant scalar curvature and admits a non-homothetic conformal vector field V leaving the potential vector field ∇f invariant, then g is Einstein and the potential function f is constant.*

Remark 1. Theorem 1 was motivated by a similar result of Jauregui and Wylie [5]: "A gradient Ricci soliton admitting a non-homothetic conformal vector field V that preserves the gradient 1-form df (i.e. $\nabla_V f$ is constant) is Einstein and f is constant". We note that the hypothesis "$\nabla_V f$ is constant" in the result of Jauregui and Wylie, does not imply the hypothesis "V leaves the potential vector field ∇f invariant)" of Theorem 1. For f constant, g is Einstein (scalar curvature is obviously constant) for which Yano and Nagano [12] proved: "A complete Einstein manifold admitting a complete non-homothetic conformal vector field is isometric to a round sphere." However, if only M is complete and V not necessarily complete, then by a result of Kanai [6] (stated also in Kühnel and Rademacher [7]), M is isometric to one of the following spaces: S^n, E^n, H^n, the warped product $R \times_{\exp} M_*$ where (M_*, g_*) is complete and Ricci flat, or the warped product $R \times_{\cosh} M_*$ where (M_*, g_*) is complete and Einstein with $S_* = -1$.

Remark 2. Constant scalar curvature gradient Ricci Solitons were studied by Petersen and Wylie [10] who showed that a shrinking (respectively, expanding) gradient Ricci soliton with constant scalar curvature S satisfies $0 \leq S \leq n\lambda$ (respectively, $n\lambda \leq S \leq 0$). Also, g is flat if $S = 0$ and Einstein when $S = n\lambda$. Fernández-López and García-Río [4] showed that, if an n-dimensional complete gradient Ricci soliton has constant scalar curvature S then $S \in \{0, \lambda, \ldots (n-1)\lambda, n\lambda\}$. Thus the problem of classifying gradient Ricci solitons with constant scalar curvature is, in general, open.

For the case when V is homothetic, we prove

Proposition 1 *If (M, g, f, λ) is a gradient Ricci soliton with a homothetic vector field V leaving the potential vector field ∇f invariant, then either (i) it is a Gaussian soliton, or (ii) V is Killing. In case (ii), either the soliton is steady or V preserves f.*

A conformal vector field V on a Riemannian manifold (M, g) is defined by

$$L_V g = 2\sigma g \tag{4}$$

where σ is a smooth function on M. V is homothetic when σ is constant, and is Killing when $\sigma = 0$. Denoting the Riemannian connection as well as the gradient operator of g by ∇ we have the following formula:

$$(L_V \nabla)(Y, Z) = (Y\sigma)Z + (Z\sigma)Y - g(Y, Z)\nabla\sigma \tag{5}$$

where Y, Z denote arbitrary smooth vector fields on M. We will follow this notation in the next section.

2 Proofs Of Theorem 1 and Proposition 1

Proof Of Theorem 1. A straightforward computation using the definition (2) provides

$$R(Y, Z)\nabla f + (\nabla_Y Q)Z - (\nabla_Z Q)Y = 0 \tag{6}$$

where R denotes the curvature tensor and Q the Ricci tensor of type (1,1) such that $\mathrm{Ric}(Y, Z) = g(QY, Z)$. Let (e_i) $(i = 1, \ldots, n)$ be a local orthonormal frame on (M, g). Substituting e_i for Y in (6), taking inner product with e_i, summing over i, and using the twice contracted second Bianchi identity: $\mathrm{div}(Q) = \frac{1}{2}dS$ yields the known formula

$$Q(\nabla f) = \frac{1}{2}\nabla S \tag{7}$$

Next, differentiating $|\nabla f|^2$ along an arbitrary vector field, and using equations (2) and (7) gives the known formula

$$|\nabla f|^2 + S - 2\lambda f = c \tag{8}$$

where c is a real constant. As S is constant by hypothesis, equation (7) reduces to

$$Q(\nabla f) = 0. \tag{9}$$

At this point, Lie differentiating the relation: $df = g(\nabla f, .)$ along the conformal vector field V, noting that Lie derivative commutes with exterior derivative d, and using the hypothesis $L_V \nabla f = 0$, we find $d(L_V f) = 2\sigma df$. Applying d on it and using the Poincar'e lemma: $d^2 = 0$ we obtain

$$(d\sigma) \wedge (df) = 0. \tag{10}$$

Let us now express equation (2) in the form

$$\nabla_Y \nabla f + QY = \lambda Y$$

Taking its Lie derivative along V, using the commutation formula (see [11])

$$L_V \nabla_Y Z - \nabla_Y L_V Z - \nabla_{[V,Y]} Z = (L_V \nabla)(Y, Z)$$

with the choice $Z = \nabla f$, along with the hypothesis $L_V \nabla f = 0$ and equations (2) and (5) yields

$$(L_V Q)Y = -g(\nabla f, \nabla \sigma)Y. \tag{11}$$

Now we substitute e_i for Y in (11), take inner product with e_i, sum over i, and use the constant scalar curvature hypothesis in order to obtain

$$g(\nabla f, \nabla \sigma) = 0 \tag{12}$$

The equations (10) and (12) show that

$$(d\sigma \wedge df)(\nabla \sigma, \nabla f) = |\nabla \sigma|^2 |\nabla f|^2 = 0$$

i.e.

$$|\nabla \sigma||\nabla f| = 0. \tag{13}$$

As σ is not constant on M, $\nabla \sigma \neq 0$ on an open subset $\mathcal{U}$ of M. So, from (13), $\nabla f = 0$ on $\mathcal{U}$. Now the g-trace of (2) is $\Delta f + S = n\lambda$ on M. Since

$\Delta f = 0$ on $\mathcal{U}$, we have $S = n\lambda$ on $\mathcal{U}$. By hypothesis, S is constant on M and M is connected, and therefore $S = n\lambda$ on M. Using equation (3) with $X = \nabla f$ gives $|\operatorname{Ric}|^2 = \lambda S$. Hence the identity: $|\operatorname{Ric} - \frac{S}{n} g|^2 = |\operatorname{Ric}|^2 - \frac{S^2}{n}$ provides $\operatorname{Ric} = \lambda g$, i.e. g is Einstein. Thus equation (2) reduces to $\nabla\nabla f = 0$, which implies that $|\nabla f|$ is constant. As $\nabla f = 0$ on $\mathcal{U}$ and M is connected, we conclude that $\nabla f = 0$ on M, and so f is constant on M, completing the proof.

Proof Of Proposition 1. Here we have equation (4) with constant σ. Writing equation (2) as

$$L_{\nabla f} g + 2 \operatorname{Ric} = 2\lambda g,$$

Lie-differentiating it along V and noting that a homothetic vector field preserves the Ricci tensor we get

$$L_V L_{\nabla f} g = 4\lambda\sigma g$$

Using the identity $L_Y L_Z - L_Z L_Y = L_{[Y,Z]}$ and hypothesis $[V, \nabla f] = 0$ in the above equation we find

$$\sigma(L_{\nabla f} g - 2\lambda g) = 0$$

Hence, either (i) $L_{\nabla f} g - 2\lambda g = 0$, or (ii) $\sigma = 0$. Equation in (i) is basically $\nabla\nabla f = \lambda g$, and by a result (Theorem 2, IB) of Okumura [8]), implies that g is flat and hence is a Gaussian soliton. In case (ii), V is Killing and hence $L_V S = 0$. Also, Lie-differentiating (8) along V and noting that $L_V \nabla f = 0$ and $L_V g = 0$ imply $L_V |\nabla f|^2 = 0$ we find that either $\lambda = 0$ or V preserves f. This completes the proof.

3 Conformally Flat Case

Finally, taking into account the result of [3] for a locally conformally flat gradient Ricci soliton as stated in Section 1, we examine this case with the hypothesis $L_V \nabla f = 0$ of Theorem 1, and without constant scalar curvature assumption and prove

Proposition 2 *If (M, g, f, λ) is a locally conformally flat gradient Ricci soliton and admits a non-homothetic conformal vector field V leaving with the potential vector field ∇f invariant, then f is constant and (M, g) has constant curvature.*

Proof. If f is constant, then we are done. So, $\nabla f \neq 0$ on a neighborhood of some point in M. By a result of [3] we know that (M, g) is locally the warped product of an interval I and an $(n-1)$ dimensional manifold N of constant curvature c with metric $g = dt^2 + \psi^2(t)\gamma$, where t is the coordinate on I and ψ is the warping function. Also, f is a function of t. The gradient Ricci soliton equation (2) yields (as mentioned in [2])

$$\ddot{f} = \lambda + (n-1)\frac{\ddot{\psi}}{\psi} \tag{14}$$

$$\psi\dot{\psi}\dot{f} = \lambda\psi^2 - (n-2)c + \psi\ddot{\psi} + (n-2)(\dot{\psi})^2 \tag{15}$$

where an over-dot denotes partial differentiation with respect to t. Let us decompose the conformal vector field V on M as $V = \alpha\partial_t + U^k\partial_k$ where α and U^k depend on t as well as the coordinates x^i on N. The components of conformal Killing equation (4) provide

$$\dot{\alpha} = \sigma \tag{16}$$

$$\partial_i\alpha = -(\partial_t U^k)g_{ik} \tag{17}$$

$$L_U g_{ij} = 2(\sigma - \alpha\frac{\dot{\psi}}{\psi})g_{ij} \tag{18}$$

where $U = U^k\partial_k$. The hypothesis: $L_V \nabla f = [V, \nabla f] = 0$ shows

$$\dot{f}\dot{\alpha} = \alpha\ddot{f} \tag{19}$$

$$\partial_t U^k = 0.$$

Hence $U^k = U^k(x^i)$ and equation (17) implies $\alpha = \alpha(t)$. Equation (19) integrates to $\alpha = \dot{f}$ (up to a constant multiple which can be taken 1). Consequently, (16) assumes the form

$$\sigma = \ddot{f} \tag{20}$$

Equation (18) shows that U is homothetic on (N, γ), i.e. $L_U\gamma = 2k\gamma$ where k is constant such that

$$\ddot{f} - \dot{f}\frac{\dot{\psi}}{\psi} = k. \tag{21}$$

Since (N,γ) has constant curvature c, ${}^{\gamma}\,\mathrm{Ric} = c(n-2)\gamma$. Lie-differentiating it along U provides $ck = 0$. This gives rise to two cases (i) $c = 0$, (ii) $k = 0$. For case (i) equations (14), (15) and (21) give us

$$\frac{\ddot{\psi}}{\psi} - \frac{(\dot{\psi})^2}{\psi^2} = \frac{k}{n-2} \tag{22}$$

which integrates to $\frac{\dot{\psi}}{\psi} = \frac{k}{n-2}t + a$ and further to $\psi = e^{\frac{k}{n-2}t^2+at+b}$ where a, b are arbitrary constants. Using (22) in (15) and differentiating with respect to t we get

$$\ddot{f} = \frac{k}{n-2}[n-1-(\lambda+\frac{k}{n-2})(\frac{k}{n-2}t+a)^{-2}] \tag{23}$$

Comparing it with (14) we get the polynomial equation

$$(n-1)(Kt+a)^4 + \lambda(Kt+a)^2 + K(\lambda+K) = 0.$$

where $K = \frac{k}{n-2}$. The above equation implies that $k = 0$. Hence (23) reduces to $\ddot{f} = 0$, and from (16) we get $\sigma = 0$ contradicting the non-homotheticity of V. Now we examine the case (ii) $k = 0$ for which (21) integrates to $\dot{f} = \psi$. Using this in (14) we have

$$\ddot{\psi} = \frac{\psi}{n-1}(\dot{\psi}-\lambda) \tag{24}$$

Combining this with (15) provides $\psi^2\dot{\psi} = \lambda\psi^2 + (n-1)(\dot{\psi})^2 - (n-1)c$. Differentiating it with respect to t and using (24) gives $\psi^2(\dot{\psi}-\lambda) = 0$. But $\psi \neq 0$ for any t (as g is positive-definite), and so $\dot{\psi} = \lambda$. As already found, $\dot{f} = \psi$. Thus $\ddot{f} = \lambda$ and so from (20) we conclude that $\sigma = \lambda$ contradicting the non-homotheticity of V. This completes the proof.

4 Concluding Remark

The assumption $[V, \nabla f] = 0$ in Theorem 1 and Proposition 2 is needed in the proofs, and is trivially satisfied for constant f in which case g is Einstein.

5 Acknowledgment

The author is thankful to the referee for a valuable question.

References

[1] B. Chow, S.-C. Chu, D. Glickenstein, C. Guenther, J. Isenberg, T. Ivey, D. Knopf, P. Lu, F. Luo and L. Ni, *The Ricci flow: Techniques and Applications, Part I. Geometric Aspects*, Mathematical Surveys and Monographs, **135**, American Mathematical Society, Providence, 2007.

[2] M. Brozos-Vázquez, E. García-Río and S. Gavino-Fernández, *Locally conformally flat Lorentzian gradient Ricci solitons*, J. Geom. Anal. 23 (2013), 1196–1212.

[3] M. Fernández-López and E. García-Río, *A note on locally conformally flat gradient Ricci solitons*, Geom. Dedicata **168** (2014), 1–7.

[4] M. Fernández-López and E. García-Río, *On gradient Ricci solitons with constant scalar curvature*, Proc. Amer. Math. Soc. **144** (2016), 369–378.

[5] J.L. Jauregui and William Wylie, *Conformal diffeomorphisms of gradient Ricci solitons and generalized quasi-Einstein manifolds*, J. Geom. Anal. **25** (2015), 668–708.

[6] M. Kanai, *On a differential equation characterizing a Riemannian structure of a manifold*, Tokyo J. Math. 6 (1983), 143–151.

[7] W. Kühnel and H.-B. Rademacher, *Conformal vector fields on pseudo-Riemannian spaces*, Diff. Geom. Appln. 7 (1997), 237–250.

[8] M. Okumura, *Complete Riemannian manifolds and some vector fields*, Trans. Amer. Math. Soc. **117**(1965), 251–275.

[9] G. Perelman, *The entropy formula for the Ricci flow and its geometric applications*, Preprint, http:arXiv.org abs math.DG/02111159.

[10] P. Petersen and W. Wylie, *On the classification of gradient Ricci solitons*, *Geom. Topol.* **14** (2010), 2277–2300.

[11] K. Yano, *Integral formulas in Riemannian geometry*, Marcel Dekker, New York, 1970.

[12] K. Yano and T. Nagano, *Einstein spaces admitting a one-parameter group of conformal transformations*, Ann. Math. (2) **69** (1959), 451–461.

University Of New Haven, West Haven, CT 06516, USA

E-mail address: rsharma@newhaven.edu