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Gradient Yamabe Solitons on Multiply Warped Product Manifolds

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ABSTRACT

We consider gradient Yamabe solitons on multiply warped product manifolds. We find the necessary and sufficient conditions for multiply warped product manifolds to be gradient Yamabe solitons.

Keywords: *Yamabe soliton; gradient Yamabe soliton; multiply warped product.*
AMS Subject Classification (2010): *Primary: 53C25 ; Secondary: 53C21; 58E11.*

1. Introduction

Let (M, g) be a semi-Riemannian manifold. If there exists a smooth vector field $X \in \chi(M)$ and a real number ρ such that the scalar curvature of M satisfies the following equation

$$(scal_g - \rho)\, g = \frac{1}{2} \boldsymbol{L}_X g, \tag{1.1}$$

where $\boldsymbol{L}_X$ is the Lie derivative along to X, then (M, g) is called a *Yamabe soliton*. The Yamabe soliton is said to be shrinking, steady or expanding if $\rho > 0$, $\rho = 0$ or $\rho < 0$, respectively. If $X = grad\varphi$ for some smooth function φ on M, then we say that $(M, g, grad\varphi, \rho)$ is called a *gradient Yamabe soliton* with potential function φ. Thus, the equation (1.1) turns into

$$(scal_g - \rho)\, g = Hess\varphi, \tag{1.2}$$

where $Hess\varphi$ denotes the Hessian of φ. When φ is constant, a gradient Yamabe soliton turns into a trivial Yamabe soliton. If ρ is a differentiable function on M then we obtain an *almost Yamabe soliton*. In particular, for gradient vector field, we obtain an *almost gradient Yamabe soliton* [1].

In [11], Ma and Miquel studied the scalar curvature of Yamabe solitons. In [2], Cao, Sun and Zhang studied every complete nontrivial gradient Yamabe soliton admits a special global warped product structure with a one-dimensional base. In [7], He studied a complete gradient steady Yamabe soliton on warped product. In [12], Neto and Tenenblat studied gradient Yamabe solitons conformal to an n-dimensional pseudo-Euclidean space. In [16], Tokura, Adriano, Pina and Barboza studied gradient Yamabe soliton on warped product manifolds with limited warping function, and for the compact base. For other developments about Ricci solitons and gradient Ricci soliton; [3], [5], [8], [9], [10], [13], [17]. Motivated by the above studies, in the present paper, we consider gradient Yamabe solitons on multiply warped product manifolds. We find the necessary and sufficient conditions for multiply warped product manifolds to be gradient Yamabe solitons.

2. Preliminaries

Let (B, g_B) and (F_i, g_{F_i}) be r and s_i dimensional semi-Riemannian manifolds, respectively, where $i \in \{1, 2, ..., m\}$ and also $M = B \times F_1 \times F_2 \times ... \times F_m$ be an $n = r + t$ dimensional semi-Riemannian manifold, where $t = \sum_{i=1}^{m} s_i$. Let $b_i : B \to (0, \infty)$ be positive smooth functions for $1 \leq i \leq m$. The product manifold $M =$

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$B \times_{b_1} F_1 \times_{b_2} F_2 \times ... \times_{b_m} F_m$ endowed with the metric tensor

$$\overline{g} = \pi^* (g_B) \oplus (b_1 \circ \pi)^2 \sigma_1^* (g_{F_1}) \oplus ... \oplus (b_m \circ \pi)^2 \sigma_m^* (g_{F_m}), \tag{2.1}$$

where π and σ_i are the natural projection on B and F_i, respectively, is called the *multiply warped product*. The functions $b_i : B \to (0, \infty)$ are called *warping functions* and each manifolds (F_i, g_{F_i}) and the manifold (B, g_B) are called *fiber manifolds and* the *base manifold* of the multiply warped product, respectively for $1 \leq i \leq m$ ([4], [14], [15]). We shall denote $scal$, $scal_B$, $scal_{F_i} = \lambda_{F_i}$ the scalar curvatures of the M, B and F_i, respectively.

Now, we give the following lemmas:

Lemma 2.1. *[14] Let $M = B \times_{b_1} F_1 \times_{b_2} F_2 \times ... \times_{b_m} F_m$ be a multiply warped product with metric $\overline{g} = g_B \oplus b_1^2 g_{F_1} \oplus ... \oplus b_m^2 g_{F_m}$. If $X, Y \in \chi(B)$ and $V \in \chi(F_i)$, $W \in \chi(F_j)$, then*
i) $\nabla_X Y$ is the lift of ${}^B\nabla_X Y$ on B,
ii) $\nabla_X V = \nabla_V X = \frac{X(b_i)}{b_i} V$,
iii) $$\nabla_V W = \begin{cases} 0 & \text{if } i \neq j, \\ {}^{F_i}\nabla_V W - \left(\frac{g(V,W)}{b_i}\right) grad_B (b_i) & \text{if } i = j. \end{cases}$$

Lemma 2.2. *[14]Let $M = B \times_{b_1} F_1 \times_{b_2} F_2 \times ... \times_{b_m} F_m$ be a multiply warped product with metric $\overline{g} = g_B \oplus b_1^2 g_{F_1} \oplus ... \oplus b_m^2 g_{F_m}$. Then*

$$scal = scal_B - 2\sum_{i=1}^m s_i \frac{\Delta_B b_i}{b_i} + \sum_{i=1}^m \frac{scal_{F_i}}{b_i^2} - \sum_{i=1}^m s_i (s_i - 1) \frac{\|grad_B b_i\|^2}{b_i^2}$$
$$- \sum_{i=1}^m \sum_{i=1,\, l\neq i}^m s_i s_l \frac{g_B (grad_B b_i, grad_B b_l)}{b_i b_l}.$$

3. The gradient Yamabe soliton on multiply warped product

Let φ be a potential function of a gradient Yamabe soliton on multiply warped product $(M, \overline{g})$.

Proposition 3.1. *Let $M = B \times_{b_1} F_1 \times_{b_2} F_2 \times ... \times_{b_m} F_m$ be a multiply warped product with $\overline{g} = g_B \oplus b_1^2 g_{F_1} \oplus ... \oplus b_m^2 g_{F_m}$. If the multiply warped product $(M, \overline{g})$ is a gradient Yamabe soliton with potential function $\varphi : M \longrightarrow \mathbb{R}$, and there exists a pair of orthogonal vectors $(X_\imath, X_\jmath)$ of the base, such that $Hess_B (\varphi) (X_\imath, X_\jmath) \neq 0$ then the potential function φ depends only on the base.*

Proof. Assume that $M = B \times_{b_1} F_1 \times_{b_2} F_2 \times ... \times_{b_m} F_m$ is a gradient Yamabe soliton with potential function $\varphi : M \longrightarrow \mathbb{R}$. Using the equation (1.2) and Lemma 2.2, we have

$$\left\{ scal_B - 2\sum_{i=1}^m s_i \frac{\Delta_B b_i}{b_i} + \sum_{i=1}^m \frac{scal_{F_i}}{b_i^2} - \sum_{i=1}^m s_i (s_i - 1) \frac{\|grad_B b_i\|^2}{b_i^2} \right.$$
$$\left. - \sum_{i=1}^m \sum_{i=1,\, l\neq i}^m s_i s_l \frac{g_B (grad_B b_i, grad_B b_l)}{b_i b_l} - \rho \right\} g_B (X_\imath, X_\jmath) = Hess\varphi (X_\imath, X_\jmath), \tag{3.1}$$

$$\left\{ scal_B - 2\sum_{i=1}^m s_i \frac{\Delta_B b_i}{b_i} + \sum_{i=1}^m \frac{scal_{F_i}}{b_i^2} - \sum_{i=1}^m s_i (s_i - 1) \frac{\|grad_B b_i\|^2}{b_i^2} \right.$$
$$\left. - \sum_{i=1}^m \sum_{i=1,\, l\neq i}^m s_i s_l \frac{g_B (grad_B b_i, grad_B b_l)}{b_i b_l} - \rho \right\} \overline{g} (X_\imath, Y_\jmath) = Hess\varphi (X_\imath, Y_\jmath), \tag{3.2}$$

$$\left\{ scal_B - 2\sum_{i=1}^m s_i \frac{\Delta_B b_i}{b_i} + \sum_{i=1}^m \frac{scal_{F_i}}{b_i^2} - \sum_{i=1}^m s_i (s_i - 1) \frac{\|grad_B b_i\|^2}{b_i^2} \right.$$
$$\left. - \sum_{i=1}^m \sum_{i=1,\, l\neq i}^m s_i s_l \frac{g_B (grad_B b_i, grad_B b_l)}{b_i b_l} - \rho \right\} \left(\sum_{i=1}^m b_i^2 g_{F_i} (Y_\imath, Y_\jmath) \right)$$

$$= Hess\varphi\left(Y_\imath, Y_\jmath\right), \tag{3.3}$$

where $X_1, X_2, ..., X_r \in \chi(B)$ and $Y_1, Y_2, ..., Y_t \in \chi(F_i)$ for $1 \leq \imath \leq r$ and $1 \leq \jmath \leq t$. Using the equation $\overline{g}\left(X_\imath, Y_\jmath\right) = 0$, we obtain $Hess\varphi\left(X_\imath, Y_\jmath\right) = 0$.

From Lemma 2.1 in [6], we can write

$$\varphi\left(X, Y\right) = z(x) + b_i(x)\nu_i\left(y\right) \tag{3.4}$$

where $z : B \longrightarrow \mathbb{R}$ and $\upsilon_i : F_i \longrightarrow \mathbb{R}$. Then, we have $Hess\varphi\left(X_\imath, X_\jmath\right) = Hess_B\varphi\left(X_\imath, X_\jmath\right)$. Using the last equation in (3.1), we obtain

$$\left\{ scal_B - 2\sum_{i=1}^{m} s_i \frac{\Delta_B b_i}{b_i} + \sum_{i=1}^{m} \frac{scal_{F_i}}{b_i^2} - \sum_{i=1}^{m} s_i\left(s_i - 1\right)\frac{\left\|grad_B b_i\right\|^2}{b_i^2} - \sum_{i=1}^{m}\sum_{i=1,\, l\neq i}^{m} s_i s_l \frac{g_B\left(grad_B b_i, grad_B b_l\right)}{b_i b_l} - \rho \right\} g_B\left(X_\imath, X_\jmath\right) = Hess_B\varphi\left(X_\imath, X_\jmath\right). \tag{3.5}$$

Substituting the equation (3.4) in (3.5), we find

$$\left\{ scal_B - 2\sum_{i=1}^{m} s_i \frac{\Delta_B b_i}{b_i} + \sum_{i=1}^{m} \frac{scal_{F_i}}{b_i^2} - \sum_{i=1}^{m} s_i\left(s_i - 1\right)\frac{\left\|grad_B b_i\right\|^2}{b_i^2} - \sum_{i=1}^{m}\sum_{i=1,\, l\neq i}^{m} s_i s_l \frac{g_B\left(grad_B b_i, grad_B b_l\right)}{b_i b_l} - \rho \right\} g_B\left(X_\imath, X_\jmath\right) = Hess_B z + \nu_i Hess_B b_i.$$

By the use of the equation $Hess_B\left(\varphi\right)\left(X_\imath, X_\jmath\right) = Hess_B b_i\left(X_\imath, X_\jmath\right) \neq 0$, we can write

$$\nu_i = -\frac{Hess_B z\left(X_\imath, X_\jmath\right)}{Hess_B b_i\left(X_\imath, X_\jmath\right)}. \tag{3.6}$$

From the equation (3.6), we obtain that the potential function φ depends only on the base. □

From Proposition 3.1, we consider a multiply warped product gradient Yamabe soliton $\left(M = B \times_{b_1} F_1 \times_{b_2} F_2 \times ... \times_{b_m} F_m, \overline{g}, grad\varphi, \rho\right)$, splitting of the form φ

$$\varphi(X, Y) = \varphi_0(X) + \sum_{k=1}^{m} \varphi_k(Y), \tag{3.7}$$

where $\varphi_0 \in C^\infty\left(B\right)$ and $\varphi_k \in C^\infty\left(F_i\right)$.

Now, we give the following Theorem:

Theorem 3.1. *Let $\left(M = B \times_{b_1} F_1 \times_{b_2} F_2 \times ... \times_{b_m} F_m, \overline{g}, grad\varphi, \rho\right)$ be a gradient Yamabe soliton on multiply warped product given by $\varphi(X, Y) = \varphi_0(X) + \sum_{k=1}^{m} \varphi_k(Y)$, then one of the following cases occurs*

(1) *M is the Riemannian product between a trivial gradient Yamabe soliton and m gradient Yamabe solitons,*

(2) *M is the Riemannian product between $(m + 1)$ gradient Yamabe solitons,*

(3) *M is the multiply warped product between an almost gradient Yamabe soliton and m trivial gradient Yamabe solitons.*

Proof. Assume that $M = B \times_{b_1} F_1 \times_{b_2} F_2 \times ... \times_{b_m} F_m$ is a gradient Yamabe soliton with potential function $\varphi(X, Y) = \varphi_0(X) + \sum_{k=1}^{m} \varphi_k(Y)$. For $X_1, X_2, ..., X_r \in \chi(B)$ and $Y_1, Y_2, ..., Y_t \in \chi(F_i)$ where $1 \leq \imath \leq r$ and $1 \leq \jmath \leq t$, we have $Hess\left(\varphi\right)\left(X_\imath, Y_\jmath\right) = 0$. Using part ii) of Lemma 2.1, we find

$$Hess(\varphi)\left(X_\imath, Y_\jmath\right) = X_\imath\left(Y_\jmath\left(\varphi\right)\right) - \frac{X_\imath\left(b_i\right)}{b_i} Y_\jmath\left(\varphi\right) = 0. \tag{3.8}$$

Using the equation (3.7) in (3.8), we obtain

$$X_\imath\left(Y_\jmath\left(\varphi\right)\right) - \frac{X_\imath\left(b_i\right)}{b_i} Y_\jmath\left(\varphi\right) = 0 - \sum_{k=1}^{m} \frac{X_\imath\left(b_i\right)}{b_i} Y_\jmath(\varphi_k) = 0. \tag{3.9}$$

Then, b_i are constant or $\varphi(X,Y) = \varphi_0(X)+$ constant. We investigate the proof in three cases:

(1) Let b_i be constant or $\varphi(X,Y) = \varphi_0(X)+$ constant. In this case, $M = B \times_{b_1} F_1 \times_{b_2} F_2 \times ... \times_{b_m} F_m$ is a Riemannian product and we have

$$\left(scal_B + \sum_{i=1}^{m} \frac{scal_{F_i}}{b_i^2} - \rho\right) g_B\left(X_\imath, X_\jmath\right) = Hess_B\varphi_0\left(X_\imath, X_\jmath\right), \tag{3.10}$$

$$\left(scal_B + \sum_{i=1}^{m} \frac{scal_{F_i}}{b_i^2} - \rho\right) \overline{g}\left(X_\imath, Y_\jmath\right) = Hess\varphi\left(X_\imath, Y_\jmath\right) = 0, \tag{3.11}$$

$$\left(scal_B + \sum_{i=1}^{m} \frac{scal_{F_i}}{b_i^2} - \rho\right) \left(\sum_{i=1}^{m} b_i^2 g_{F_i}\left(Y_\imath, Y_\jmath\right)\right) = Hess\varphi\left(Y_\imath, Y_\jmath\right). \tag{3.12}$$

From part iii) of Lemma 2.1, we can write

$$\begin{aligned} Hess\varphi\left(Y_\imath, Y_\jmath\right) &= Y_\imath\left(Y_\jmath\left(\varphi\right)\right) - \left({}^{M}\nabla_{Y_\imath} Y_\jmath\right)\varphi \\ &= Y_\imath\left(Y_\jmath\left(\varphi\right)\right) + \frac{g(Y_\imath, Y_\jmath)}{b_i} grad_B\left(b_i\right)\left(\varphi\right) - {}^{F_i}\nabla_{Y_\imath} Y_\jmath\left(\varphi\right) \\ &= Hess_{F_i}\varphi_k\left(Y_\imath, Y_\jmath\right) + \frac{\sum\limits_{i=1}^{m} b_i^2 g_{F_i}\left(Y_\imath, Y_\jmath\right)}{b_i} grad_B\left(b_i\right)\left(\varphi\right). \end{aligned} \tag{3.13}$$

Using the equation (3.13) in (3.12), we obtain

$$\begin{aligned} &\left(scal_B + \sum_{i=1}^{m} \frac{scal_{F_i}}{b_i^2} - \rho\right) \left(\sum_{i=1}^{m} b_i^2 g_{F_i}\left(Y_\imath, Y_\jmath\right)\right) \\ &= Hess_{F_i}\varphi_k\left(Y_\imath, Y_\jmath\right) + \frac{\sum\limits_{i=1}^{m} b_i^2 g_{F_i}\left(Y_\imath, Y_\jmath\right)}{b_i} grad_B\left(b_i\right)\left(\varphi\right). \end{aligned} \tag{3.14}$$

Since $scal_{F_i}$ is a constant on B, we find from (3.10) that B is a gradient Yamabe soliton of the form $\left(B, g_B, grad\varphi_0, -\sum\limits_{i=1}^{m} \frac{scal_{F_i}}{b_i^2} + \rho\right)$. Moreover, since $\varphi(X,Y) = \varphi_0(X)+$ constant, we have from the equation (3.14) that F_i are trivial gradient Yamabe solitons of the form $\left(F_i, g_{F_i}, grad0, -\left(\sum\limits_{i=1}^{m} b_i^2\right) scal_B + \left(\sum\limits_{i=1}^{m} b_i^2\right)\rho\right)$. This proves the first assertion of the theorem.

(2) Let b_i be constant or $\varphi(X,Y) = \varphi_0(X)+ \sum\limits_{k=1}^{m} \varphi_k$, where φ_k is not necessarily constant. In this case, $M = B \times_{b_1} F_1 \times_{b_2} F_2 \times ... \times_{b_m} F_m$ is a Riemannian product and we have

$$\left(scal_B + \sum_{i=1}^{m} \frac{scal_{F_i}}{b_i^2} - \rho\right) g_B\left(X_\imath, X_\jmath\right) = Hess_B\varphi_0\left(X_\imath, X_\jmath\right), \tag{3.15}$$

$$\left(scal_B + \sum_{i=1}^{m} \frac{scal_{F_i}}{b_i^2} - \rho\right) \overline{g}\left(X_\imath, Y_\jmath\right) = Hess\varphi\left(X_\imath, Y_\jmath\right) = 0, \tag{3.16}$$

$$\begin{aligned} &\left(scal_B + \sum_{i=1}^{m} \frac{scal_{F_i}}{b_i^2} - \rho\right) \left(\sum_{i=1}^{m} b_i^2 g_{F_i}\left(Y_\imath, Y_\jmath\right)\right) = Hess_{F_i}\varphi_k\left(Y_\imath, Y_\jmath\right) \\ &+ \frac{\sum\limits_{i=1}^{m} b_i^2 g_{F_i}\left(Y_\imath, Y_\jmath\right)}{b_i} grad_B\left(b_i\right)\left(\varphi\right). \end{aligned} \tag{3.17}$$

Since $scal_{F_i}$ is a constant on B, we find from (3.15) that B is a gradient Yamabe soliton of the form $\left(B, g_B, grad\varphi_0, -\sum_{i=1}^{m}\frac{scal_{F_i}}{b_i^2}+\rho\right)$. From the equation (3.17), we obtain that F_i are gradient Yamabe solitons of the form $\left(F_i, g_{F_i}, grad\varphi_k, -\left(\sum_{i=1}^{m} b_i^2\right)scal_B+\left(\sum_{i=1}^{m} b_i^2\right)\rho\right)$. This proves the second assertion of the theorem.

(3) Let b_i be non constant or $\varphi(X,Y)=\varphi_0(X)+$ constant. In this case, we can write

$$\left\{scal_B-2\sum_{i=1}^{m}s_i\frac{\Delta_B b_i}{b_i}+\sum_{i=1}^{m}\frac{scal_{F_i}}{b_i^2}-\sum_{i=1}^{m}s_i\left(s_i-1\right)\frac{\left\|grad_B b_i\right\|^2}{b_i^2}\right.$$
$$\left.-\sum_{i=1}^{m}\sum_{i=1,\,l\neq i}^{m}s_is_l\frac{g_B\left(grad_B b_i, grad_B b_l\right)}{b_ib_l}-\rho\right\}g_B\left(X_\imath,X_\jmath\right)=Hess_B\varphi_0\left(X_\imath,X_\jmath\right), \tag{3.18}$$
$$\left\{scal_B-2\sum_{i=1}^{m}s_i\frac{\Delta_B b_i}{b_i}+\sum_{i=1}^{m}\frac{scal_{F_i}}{b_i^2}-\sum_{i=1}^{m}s_i\left(s_i-1\right)\frac{\left\|grad_B b_i\right\|^2}{b_i^2}\right.$$
$$\left.-\sum_{i=1}^{m}\sum_{i=1,\,l\neq i}^{m}s_is_l\frac{g_B\left(grad_B b_i, grad_B b_l\right)}{b_ib_l}-\rho\right\}\overline{g}\left(X_\imath,Y_\jmath\right)=Hess\varphi\left(X_\imath,Y_\jmath\right)=0, \tag{3.19}$$
$$\left\{scal_B-2\sum_{i=1}^{m}s_i\frac{\Delta_B b_i}{b_i}+\sum_{i=1}^{m}\frac{scal_{F_i}}{b_i^2}-\sum_{i=1}^{m}s_i\left(s_i-1\right)\frac{\left\|grad_B b_i\right\|^2}{b_i^2}\right.$$
$$\left.-\sum_{i=1}^{m}\sum_{i=1,\,l\neq i}^{m}s_is_l\frac{g_B\left(grad_B b_i, grad_B b_l\right)}{b_ib_l}-\rho\right\}\left(\sum_{i=1}^{m}b_i^2g_{F_i}\left(Y_\imath,Y_\jmath\right)\right)=Hess\varphi\left(Y_\imath,Y_\jmath\right). \tag{3.20}$$

From the equation (3.13) and $\varphi(X,Y)=\varphi_0(X)+$ constant, we find

$$\left\{scal_B-2\sum_{i=1}^{m}s_i\frac{\Delta_B b_i}{b_i}+\sum_{i=1}^{m}\frac{scal_{F_i}}{b_i^2}-\sum_{i=1}^{m}s_i\left(s_i-1\right)\frac{\left\|grad_B b_i\right\|^2}{b_i^2}\right.$$
$$\left.-\sum_{i=1}^{m}\sum_{i=1,\,l\neq i}^{m}s_is_l\frac{g_B\left(grad_B b_i, grad_B b_l\right)}{b_ib_l}-\rho\right\}\left(\sum_{i=1}^{m}b_i^2g_{F_i}\left(Y_\imath,Y_\jmath\right)\right)$$
$$=\frac{\sum_{i=1}^{m}b_i^2g_{F_i}\left(Y_\imath,Y_\jmath\right)}{b_i}grad_B\left(b_i\right)\left(\varphi_0\right). \tag{3.21}$$

Since b_i are positive for $1\leq i\leq m$, by equation (3.21) we obtain

$$\left(\sum_{i=1}^{m}scal_{F_i}-\psi\right)\left(\sum_{i=1}^{m}b_i^2g_{F_i}\left(Y_\imath,Y_\jmath\right)\right)=0,$$

where

$$\psi=\left(\sum_{i=1}^{m}b_i^2\right)\left(scal_B-2\sum_{i=1}^{m}s_i\frac{\Delta_B b_i}{b_i}-\sum_{i=1}^{m}s_i\left(s_i-1\right)\frac{\left\|grad_B b_i\right\|^2}{b_i^2}\right.$$
$$\left.-\sum_{i=1}^{m}\sum_{i=1,\,l\neq i}^{m}s_is_l\frac{g_B\left(grad_B b_i, grad_B b_l\right)}{b_ib_l}-\frac{grad_B\left(b_i\right)\left(\varphi_1\right)}{b_i}-\rho\right).$$

Since ψ depend only on B, we have that ψ is constant on F_i, then by equation (3.21),we have that F_i are a trivial gradient Yamabe solitons. Moreover, by equation (3.18) we have that (B,g_B) is a gradient almost Yamabe soliton of the form

$$\left(B, g_B, grad\varphi_0, -\left[\sum_{i=1}^{m}\frac{scal_{F_i}}{b_i^2}-2\sum_{i=1}^{m}s_i\frac{\Delta_B b_i}{b_i}-\sum_{i=1}^{m}s_i\left(s_i-1\right)\frac{\left\|grad_B b_i\right\|^2}{b_i^2}\right.\right.$$

$$-\sum_{i=1}^{m}\sum_{i=1,\,l\neq i}^{m}s_is_l\frac{g_B\left(grad_Bb_i,grad_Bb_l\right)}{b_ib_l}-\rho\Bigg]\Bigg)\,.$$

This proves the third assertion of the theorem. □

The above theorem denotes us that if we take the potential function depending only on the base then the fibers F_i are constant scalar curvature. Then, we will take a gradient Yamabe soliton with potential function of the form $\varphi(X,Y)=\varphi_0(X)+$ constant on a multiply warped product, the base conformal to an r-dimensional pseudo-Euclidean space, and the fibers chosen to be scalar constant spaces. Let $(\mathbb{R}^r,g_{\mathbb{R}})$ be the pseudo-Euclidean space, $r\geq 3$ with coordinates $X=(X_1,...,X_r)$, $g_{\imath\jmath}=\delta_{\imath\jmath}\epsilon_{\imath\jmath}$ and let $M=(\mathbb{R}^r,\overline{g})\times_{b_i}F_i^{s_i}$ be a multiply warped product where $\widetilde{g}=\frac{1}{\phi^2}g_{\mathbb{R}}$, F_i a semi-Riemannian scalar constant manifolds with curvatures λ_{F_i} , $t=\sum\limits_{i=1}^{m}s_i\geq 1$, $b_i,\phi,\varphi:\mathbb{R}^r\to\mathbb{R}$, smooth functions, and b_i are positive functions. Then, we obtain necessary and sufficient conditions for the multiply warped product to be a gradient Yamabe soliton.

Now, we give the following Theorem:

Theorem 3.2. *Let $(\mathbb{R}^r,g_{\mathbb{R}})$ be a pseudo-Euclidean space, $r\geq 3$ with coordinates, $X=(X_1,...,X_r)$ and $g_{\imath\jmath}=\delta_{\imath\jmath}\epsilon_{\imath\jmath}$, let $M=(\mathbb{R}^r,\widetilde{g})\times_{b_1}F_1\times_{b_2}F_2\times...\times_{b_m}F_m$, be a multiply warped product where $\widetilde{g}=\frac{1}{\phi^2}g_{\mathbb{R}}$ and F_i semi-Riemannian scalar-constant manifolds with curvatures λ_{F_i}, $t=\sum\limits_{i=1}^{m}s_i\geq 1$, $b_i,\phi,\varphi:\mathbb{R}^r\to\mathbb{R}$, smooth functions, and b_i are positive functions. Then, the multiply warped product $(M,\overline{g})$ is a gradient Yamabe soliton with potential function φ if and only if the functions b_i,ϕ,φ satisfy*

$$\varphi_{x_\imath x_\jmath}+\frac{\phi_{x_\imath}}{\phi}\varphi_{x_\jmath}+\frac{\phi_{x_\jmath}}{\phi}\varphi_{x_\imath}=0\qquad \imath\neq\jmath, \tag{3.22}$$

$$\begin{aligned}&\Bigg\{(r-1)\left(2\phi\sum_k\epsilon_k\phi_{x_kx_k}-r\sum_k\epsilon_k\phi_{x_k}^2\right)\\&-2\sum_{i=1}^{m}s_i\frac{\phi^2\sum_k\epsilon_k\left(b_i\right)_{x_kx_k}-(n-2)\,\phi\sum_k\epsilon_k\phi_{x_k}\left(b_i\right)_{x_k}}{b_i}\\&+\sum_{i=1}^{m}\frac{\lambda_{F_i}}{b_i^2}-\sum_{i=1}^{m}s_i\left(s_i-1\right)\frac{\left(\phi^2\sum_k\epsilon_k\left(b_i^2\right)_{x_k}\right)}{b_i^2}\\&-\sum_{i=1}^{m}\sum_{i=1,\,l\neq i}^{m}s_is_l\frac{\left(\phi^2\sum_{k=1}^{r}\epsilon_k\left(b_i\right)_{x_k}\left(b_l\right)_{x_k}\right)}{b_ib_l}-\rho\Bigg\}\frac{\epsilon_i}{\phi^2}\\&=\varphi_{x_\imath x_\imath}+2\frac{\phi_{x_\imath}}{\phi}\varphi_{x_\imath}-\epsilon_i\sum_k\epsilon_k\frac{\phi_{x_k}}{\phi}\varphi_{x_k},\end{aligned} \tag{3.23}$$

$$\begin{aligned}&\Bigg\{(r-1)\left(2\phi\sum_k\epsilon_k\phi_{x_kx_k}-r\sum_k\epsilon_k\phi_{x_k}^2\right)\\&-2\sum_{i=1}^{m}s_i\frac{\phi^2\sum_k\epsilon_k\left(b_i\right)_{x_kx_k}-(n-2)\,\phi\sum_k\epsilon_k\phi_{x_k}\left(b_i\right)_{x_k}}{b_i}\\&+\sum_{i=1}^{m}\frac{\lambda_{F_i}}{b_i^2}-\sum_{i=1}^{m}s_i\left(s_i-1\right)\frac{\left(\phi^2\sum_k\epsilon_k\left(b_i^2\right)_{x_k}\right)}{b_i^2}\\&-\sum_{i=1}^{m}\sum_{i=1,\,l\neq i}^{m}s_is_l\frac{\left(\phi^2\sum_k\epsilon_k\left(b_i\right)_{x_k}\left(b_l\right)_{x_k}\right)}{b_ib_l}-\rho\Bigg\}=\frac{\phi^2}{b_i}\sum_k\epsilon_k\phi_{x_k}\left(b_i\right)_{x_k}.\end{aligned} \tag{3.24}$$

Proof. Assume that $M = (\mathbb{R}^r, \widetilde{g}) \times_{b_1} F_1 \times_{b_2} F_2 \times ... \times_{b_m} F_m$ is multiply warped product with gradient Yamabe soliton structure. So, we have

$$(scal_{\overline{g}} - \rho)\,\overline{g} = Hess_{\overline{g}}\varphi. \tag{3.25}$$

For $X_1, X_2, ..., X_r \in \chi(B)$ and $Y_1, Y_2, ..., Y_t \in \chi(F_i)$, we can write

$$\left\{ scal_{\widetilde{g}} - 2\sum_{i=1}^{m} s_i \frac{\Delta_{\widetilde{g}} b_i}{b_i} + \sum_{i=1}^{m} \frac{scal_{F_i}}{b_i^2} - \sum_{i=1}^{m} s_i (s_i - 1) \frac{\left\|grad_{\widetilde{g}} b_i\right\|^2}{b_i^2} - \sum_{i=1}^{m} \sum_{i=1,\, l\neq i}^{m} s_i s_l \frac{\widetilde{g}\left(grad_{\widetilde{g}} b_i, grad_{\widetilde{g}} b_l\right)}{b_i b_l} - \rho \right\} \widetilde{g}(X_\imath, X_\jmath) = Hess_{\overline{g}}\varphi(X_\imath, X_\jmath), \tag{3.26}$$

$$\left\{ scal_{\widetilde{g}} - 2\sum_{i=1}^{m} s_i \frac{\Delta_{\widetilde{g}} b_i}{b_i} + \sum_{i=1}^{m} \frac{scal_{F_i}}{b_i^2} - \sum_{i=1}^{m} s_i (s_i - 1) \frac{\left\|grad_{\widetilde{g}} b_i\right\|^2}{b_i^2} - \sum_{i=1}^{m} \sum_{i=1,\, l\neq i}^{m} s_i s_l \frac{\widetilde{g}\left(grad_{\widetilde{g}} b_i, grad_{\widetilde{g}} b_l\right)}{b_i b_l} - \rho \right\} \overline{g}(X_\imath, Y_\jmath) = Hess_{\overline{g}}\varphi(X_\imath, Y_\jmath) = 0, \tag{3.27}$$

$$\left\{ scal_{\widetilde{g}} - 2\sum_{i=1}^{m} s_i \frac{\Delta_{\widetilde{g}} b_i}{b_i} + \sum_{i=1}^{m} \frac{scal_{F_i}}{b_i^2} - \sum_{i=1}^{m} s_i (s_i - 1) \frac{\left\|grad_{\widetilde{g}} b_i\right\|^2}{b_i^2} - \sum_{i=1}^{m} \sum_{i=1,\, l\neq i}^{m} s_i s_l \frac{\widetilde{g}\left(grad_{\widetilde{g}} b_i, grad_{\widetilde{g}} b_l\right)}{b_i b_l} - \rho \right\} \left(\sum_{i=1}^{m} b_i^2 g_{F_i}(Y_\imath, Y_\jmath) \right) = Hess_{\overline{g}}\varphi(Y_\imath, Y_\jmath). \tag{3.28}$$

It is well known that for the conformal metric $\widetilde{g} = \frac{1}{\phi^2} g_{\mathbb{R}}$, the Christofel symbol is given by

$$\overline{\Gamma}^k_{\imath\jmath} = 0,\ \overline{\Gamma}^\imath_{\imath\jmath} = -\frac{\phi_{x_\jmath}}{\phi},\ \ \overline{\Gamma}^k_{\imath\imath} = \epsilon_\imath \epsilon_k \frac{\phi_{x_k}}{\phi},\ \overline{\Gamma}^\imath_{\imath\imath} = -\frac{\phi_{x_\imath}}{\phi}.$$

Then we get

$$Hess_{\overline{g}}\varphi_{\imath\jmath} = \varphi_{x_\imath x_\jmath} + \frac{\phi_{x_\imath}}{\phi}\varphi_{x_\jmath} + \frac{\phi_{x_\jmath}}{\phi}\varphi_{x_\imath}, \qquad \imath \neq \jmath. \tag{3.29}$$

$$Hess_{\overline{g}}\varphi_{\imath\imath} = \varphi_{x_\imath x_\imath} + 2\frac{\phi_{x_\imath}}{\phi}\varphi_{x_\imath} - \epsilon_\imath \sum_k \epsilon_k \frac{\phi_{x_k}}{\phi}\varphi_{x_k}, \qquad \imath = \jmath. \tag{3.30}$$

From [16], the Ricci curvature is given by

$$Ric_{\widetilde{g}} = \frac{1}{\phi^2}\left\{ (r-2)\phi Hess_{g_{\mathbb{R}}}\phi + \left[\phi \Delta_{g_{\mathbb{R}}}\phi - (r-1)\left|\nabla_{g_{\mathbb{R}}}\phi\right|^2\right] g_{\mathbb{R}} \right\}.$$

Thus, we have the scalar curvature on conformal metric

$$scal_{\widetilde{g}} = (r-1)\left(2\phi \Delta_{g_{\mathbb{R}}}\phi - r\left|\nabla_{g_{\mathbb{R}}}\phi\right|^2\right) = (r-1)\left(2\phi \sum_k \epsilon_k \phi_{x_k x_k} - r\sum_k \epsilon_k \phi^2_{x_k}\right). \tag{3.31}$$

Since $\varphi : \mathbb{R}^r \to \mathbb{R}$, we obtain

$$Hess_{\overline{g}}\varphi(X_\imath, X_\jmath) = Hess_{\widetilde{g}}\varphi(X_\imath, X_\jmath), \ \ \forall \imath, \jmath \tag{3.32}$$

On the other hand, we have

$$\left\{ \begin{array}{c} scal_{F_i} g_{F_i} = \lambda_{F_i} g_{F_i} \\ \overline{g}(Y_\imath, Y_\jmath) = \sum_{i=1}^{m} b_i^2 g_{F_i}(Y_\imath, Y_\jmath) \\ \Delta_{\widetilde{g}} b_i = \phi^2 \sum_k \epsilon_k (b_i)_{x_k x_k} - (n-2)\,\phi \sum_k \epsilon_k \phi_{x_k} (b_i)_{x_k} \\ \sum_{i=1}^{m} \widetilde{g}\left(grad_{\widetilde{g}} b_i, grad_{\widetilde{g}} b_i\right) = \phi^2 \sum_{i=1}^{m} \sum_k \epsilon_k \left(b_i^2\right)_{x_k} \\ \sum_{i=1}^{m} \sum_{i=1,\, l\neq i}^{m} \widetilde{g}\left(grad_{\widetilde{g}} b_i, grad_{\widetilde{g}} b_l\right) = \phi^2 \sum_{i=1}^{m} \sum_{i=1,\, l\neq i}^{m} \sum_k \epsilon_k (b_i)_{x_k} (b_l)_{x_k}. \end{array} \right. \tag{3.33}$$

Substituting the equations the equations (3.30), (3.31), (3.33) in equation (3.26), we find

$$\Big\{(r-1)\left(2\phi\sum_k \epsilon_k\phi_{x_kx_k} - r\sum_k \epsilon_k\phi_{x_k}^2\right)$$

$$-2\sum_{i=1}^m s_i\frac{\phi^2\sum_k \epsilon_k (b_i)_{x_kx_k} - (n-2)\,\phi\sum_k \epsilon_k\phi_{x_k}(b_i)_{x_k}}{b_i}$$

$$+\sum_{i=1}^m \frac{\lambda_{F_i}}{b_i^2} - \sum_{i=1}^m s_i(s_i-1)\frac{\left(\phi^2\sum_k \epsilon_k (b_i^2)_{x_k}\right)}{b_i^2}$$

$$-\sum_{i=1}^m\sum_{i=1,\,l\neq i}^m s_is_l\frac{\left(\phi^2\sum_k \epsilon_k (b_i)_{x_k}(b_l)_{x_k}\right)}{b_ib_l} - \rho\Big\}\frac{\epsilon_\imath}{\phi^2}$$

$$= \varphi_{x_\imath x_\imath} + 2\frac{\phi_{x_\imath}}{\phi}\varphi_{x_\imath} - \epsilon_\imath\sum_k \epsilon_k\frac{\phi_{x_k}}{\phi}\varphi_{x_k}.$$

The last equation is the equation (3.23). Similary, replacing the equation (3.29) into the equation (3.27), we obtain

$$\varphi_{x_\imath x_\jmath} + \frac{\phi_{x_\imath}}{\phi}\varphi_{x_\jmath} + \frac{\phi_{x_\jmath}}{\phi}\varphi_{x_\imath} = 0,$$

which is the equation (3.22). Similar way of equation (3.13), we get

$$Hess_{\overline{g}}\varphi(Y_\imath, Y_\jmath) = Y_\imath(Y_\jmath(\varphi)) - \left({}^M\nabla_{Y_\imath}Y_\jmath\right)\varphi$$

$$= Hess_{F_i}\varphi_k(Y_\imath, Y_\jmath) + \frac{\sum_{i=1}^m b_i^2 g_{F_i}(Y_\imath, Y_\jmath)}{b_i} grad_{\overline{g}}(b_i)(\varphi)$$

$$= \frac{\sum_{i=1}^m b_i^2 g_{F_i}(Y_\imath, Y_\jmath)}{b_i}\phi^2\sum_k \epsilon_k (b_i)_{x_k}\phi_{x_k} \tag{3.34}$$

Then substituting equations (3.31), (3.33) and (3.34) in equation (3.28), we find

$$\Big\{(r-1)\left(2\phi\sum_k \epsilon_k\phi_{x_kx_k} - r\sum_k \epsilon_k\phi_{x_k}^2\right)$$

$$-2\sum_{i=1}^m s_i\frac{\phi^2\sum_k \epsilon_k (b_i)_{x_kx_k} - (n-2)\,\phi\sum_k \epsilon_k\phi_{x_k}(b_i)_{x_k}}{b_i}$$

$$+\sum_{i=1}^m \frac{\lambda_{F_i}}{b_i^2} - \sum_{i=1}^m s_i(s_i-1)\frac{\left(\phi^2\sum_k \epsilon_k (b_i^2)_{x_k}\right)}{b_i^2}$$

$$-\sum_{i=1}^m\sum_{i=1,\,l\neq i}^m s_is_l\frac{\left(\phi^2\sum_k \epsilon_k (b_i)_{x_k}(b_l)_{x_k}\right)}{b_ib_l} - \rho\Big\} = \frac{\phi^2}{b_i}\sum_k \epsilon_k\phi_{x_k}(b_i)_{x_k}.$$

The converse is a straightforward computation. This completes the proof. □

In order to obtain solutions for equations in Theorem 3.2, we consider b_i, ϕ and φ invariant under the action of an $(r-1)$-dimensional translation group, and $\xi = \sum_{k=1}^{r} \alpha_k x_k,\ \alpha_k \in \mathbb{R}$ be a basic invariant for the $(r-1)$-dimensional translation group, then we get

Theorem 3.3. *Let $(\mathbb{R}^r, g_{\mathbb{R}})$ be a pseudo-Euclidean space, $r \geq 3$ with coordinates, $X = (X_1, ..., X_r)$ and $g_{ij} = \delta_{ij}\epsilon_{ij}$, let $M = (\mathbb{R}^r, \widetilde{g}) \times_{b_1} F_1 \times_{b_2} F_2 \times ... \times_{b_m} F_m$, be a multiply warped product where $\widetilde{g} = \frac{1}{\phi^2} g_{\mathbb{R}}$ and F_i semi-Riemannian scalar-constant manifolds with curvatures λ_{F_i}, $t = \sum_{i=1}^{m} s_i \geq 1$, $b_i, \phi, \varphi : \mathbb{R}^r \to \mathbb{R}$, smooth functions and $b_i > 0$. Consider the function $b_i(\xi), \phi(\xi)$ and $\varphi(\xi)$, where $\xi = \sum_{k=1}^{r} \alpha_k x_k$, $\alpha_k \in \mathbb{R}$ and $\sum_{k=1}^{r} \epsilon_k \alpha_k^2 = \epsilon_{k_0}$ or $\sum_{k=1}^{r} \epsilon_k \alpha_k^2 = 0$. Then, the multiply warped product $(M, \overline{g})$ is a gradient Yamabe soliton with potential function φ if and only if the functions b_i, ϕ, φ satisfy*

$$\varphi'' + 2\frac{\phi'}{\phi}\varphi' = 0, \tag{3.35}$$

$$\begin{aligned} &\epsilon_{k_0}\Bigg\{(r-1)\left(2\phi\phi'' - r(\phi')^2\right) - 2\sum_{i=1}^{m} s_i \frac{\left(\phi^2 (b_i)'' - (n-2)\phi\phi'(b_i)'\right)}{b_i} \\ &- \sum_{i=1}^{m} s_i(s_i-1)\frac{\left(\phi^2 (b_i')^2\right)}{b_i^2} - \sum_{i=1}^{m}\sum_{i=1,\, l\neq i}^{m} s_i s_l \frac{\left(\phi^2 (b_i)'(b_l)'\right)}{b_i b_l} + \phi\phi'\varphi'\Bigg\} \\ &= \rho - \sum_{i=1}^{m} \frac{\lambda_{F_i}}{b_i^2}, \end{aligned} \tag{3.36}$$

$$\begin{aligned} &\epsilon_{k_0}\Bigg\{(r-1)\left(2\phi\phi'' - r(\phi')^2\right) - 2\sum_{i=1}^{m} s_i \frac{\left(\phi^2 (b_i)'' - (n-2)\phi\phi'(b_i)'\right)}{b_i} \\ &- \sum_{i=1}^{m} s_i(s_i-1)\frac{\left(\phi^2 (b_i')^2\right)}{b_i^2} - \sum_{i=1}^{m}\sum_{i=1,\, l\neq i}^{m} s_i s_l \frac{\left(\phi^2 (b_i)'(b_l)'\right)}{b_i b_l} - \frac{\left(\sum_{i=1}^{m} b_i^2\right)}{b_i}\phi^2 (b_i')\varphi'\Bigg\} \\ &= \rho - \sum_{i=1}^{m} \frac{\lambda_{F_i}}{b_i^2}, \end{aligned} \tag{3.37}$$

where $\epsilon_{k_0} = \sum_{k=1}^{r} \epsilon_k \alpha_k^2$. Moreover,

$$\left(\varphi'' + 2\frac{\phi'}{\phi}\varphi'\right) = 0$$

and

$$\rho - \sum_{i=1}^{m} \frac{\lambda_{F_i}}{b_i^2} = 0,$$

where $\sum_{k=1}^{r} \epsilon_k \alpha_k^2 = 0$.

Proof. Assume that $b_i(\xi), \phi(\xi)$ and $\varphi(\xi)$ are function of ξ, where $\xi = \sum_k \alpha_k x_k$, $\alpha_k \in \mathbb{R}$ and $\sum_{k=1}^{r} \epsilon_k \alpha_k^2 = \epsilon_{k_0}$ or $\sum_{k=1}^{r} \epsilon_k \alpha_k^2 = 0$, then we obtain

$$\phi_{x_\imath} = \phi'\alpha_\imath,\ \ \phi_{x_\imath x_\jmath} = \phi''\alpha_\imath\alpha_\jmath,\ \ (b_i)_{x_\imath} = (b_i)'\alpha_\imath, \tag{3.38}$$

$$(b_i)_{x_\imath x_\jmath} = (b_i)''\alpha_\imath\alpha_\jmath,\ \varphi_{x_\imath} = \varphi'\alpha_\imath,\ \varphi_{x_\imath x_\jmath} = \varphi''\alpha_\imath\alpha_\jmath. \tag{3.39}$$

Replacing the equations (3.38) and (3.39) in (3.22), we find

$$\left(\varphi'' + 2\frac{\phi'}{\phi}\varphi'\right)\alpha_\imath\alpha_\jmath = 0, \quad \forall\, \imath \neq \jmath. \tag{3.40}$$

If there exist $\imath, \jmath, \imath \neq \jmath$ such that $\alpha_\imath\alpha_\jmath \neq 0$, then we have from the equation (3.40)

$$\varphi'' + 2\frac{\phi'}{\phi}\varphi' = 0. \tag{3.41}$$

Similarly, using the equations (3.38) and (3.39) in the equations (3.23) and (3.24), we obtain

$$\begin{aligned}
&\Bigg\{(r-1)\left(2\phi\phi''\sum_k \epsilon_k\alpha_k^2 - r(\phi')^2\sum_k \epsilon_k\alpha_k^2\right) \\
&-2\sum_{i=1}^m s_i\frac{\left(\phi^2(b_i)''\sum_k \epsilon_k\alpha_k^2 - (n-2)\phi\phi'(b_i)'\sum_k \epsilon_k\alpha_k^2\right)}{b_i} \\
&+\sum_{i=1}^m \frac{\lambda_{F_i}}{b_i^2} - \sum_{i=1}^m s_i(s_i-1)\frac{\left(\phi^2(b_i')^2\sum_k \epsilon_k\alpha_k^2\right)}{b_i^2} \\
&-\sum_{i=1}^m\sum_{i=1,\, l\neq i}^m s_is_l\frac{\left(\phi^2(b_i)'(b_l)'\sum_k \epsilon_k\alpha_k^2\right)}{b_ib_l} - \rho\Bigg\}\frac{\epsilon_\imath}{\phi^2} \\
&= \varphi''\alpha_\imath^2 + 2\frac{\phi'}{\phi}\varphi'\alpha_\imath^2 - \epsilon_\imath\frac{\phi'}{\phi}\varphi'\sum_k \epsilon_k\alpha_k^2
\end{aligned} \tag{3.42}$$

and

$$\begin{aligned}
&\Bigg\{(r-1)\left(2\phi\phi''\sum_k \epsilon_k\alpha_k^2 - r(\phi')^2\sum_k \epsilon_k\alpha_k^2\right) \\
&-2\sum_{i=1}^m s_i\frac{\left(\phi^2(b_i)''\sum_k \epsilon_k\alpha_k^2 - (n-2)\phi\phi'(b_i)'\sum_k \epsilon_k\alpha_k^2\right)}{b_i} \\
&+\sum_{i=1}^m \frac{\lambda_{F_i}}{b_i^2} - \sum_{i=1}^m s_i(s_i-1)\frac{\left(\phi^2(b_i')^2\sum_k \epsilon_k\alpha_k^2\right)}{b_i^2} \\
&-\sum_{i=1}^m\sum_{i=1,\, l\neq i}^m s_is_l\frac{\left(\phi^2(b_i)'(b_l)'\sum_k \epsilon_k\alpha_k^2\right)}{b_ib_l} - \rho\Bigg\} \\
&= \frac{\left(\sum_{i=1}^m b_i^2\right)}{b_i}\phi^2(b_i')\varphi'\sum_k \epsilon_k\alpha_k^2.
\end{aligned} \tag{3.43}$$

By the use of the equation (3.41) and $\sum_k \epsilon_k\alpha_k^2 = \epsilon_{k_0}$ in (3.42), we find

$$\epsilon_{k_0}\Bigg\{(r-1)\left(2\phi\phi'' - r(\phi')^2\right) - 2\sum_{i=1}^m s_i\frac{\left(\phi^2(b_i)'' - (n-2)\phi\phi'(b_i)'\right)}{b_i}$$

$$-\sum_{i=1}^{m} s_i\left(s_i-1\right)\frac{\left(\phi^2\left(b_i'\right)^2\right)}{b_i^2}-\sum_{i=1}^{m}\sum_{i=1,\,l\neq i}^{m} s_i s_l\frac{\left(\phi^2\left(b_i\right)'\left(b_l\right)'\right)}{b_i b_l}+\phi\phi'\varphi'\Bigg\}=\rho-\sum_{i=1}^{m}\frac{\lambda_{F_i}}{b_i^2}.$$

In the same way, using equation (3.41) and $\sum_k \epsilon_k\alpha_k^2=\epsilon_{k_0}$ in (3.43), we get

$$\epsilon_{k_0}\Bigg\{(r-1)\left(2\phi\phi''-r\left(\phi'\right)^2\right)-2\sum_{i=1}^{m} s_i\frac{\left(\phi^2\left(b_i\right)''-(n-2)\,\phi\phi'\left(b_i\right)'\right)}{b_i}$$
$$-\sum_{i=1}^{m} s_i\left(s_i-1\right)\frac{\left(\phi^2\left(b_i'\right)^2\right)}{b_i^2}-\sum_{i=1}^{m}\sum_{i=1,\,l\neq i}^{m} s_i s_l\frac{\left(\phi^2\left(b_i\right)'\left(b_l\right)'\right)}{b_i b_l}$$
$$-\frac{\left(\sum_{i=1}^{m} b_i^2\right)}{b_i}\phi^2\left(b_i'\right)\varphi'\Bigg\}=\rho-\sum_{i=1}^{m}\frac{\lambda_{F_i}}{b_i^2}.$$

Hence, if we take $\sum_k \epsilon_k\alpha_k^2=\epsilon_{k_0}$, then we obtain the equations (3.35), (3.36) and (3.37).

Now, taking $\sum_k \epsilon_k\alpha_k^2=0$ in equations (3.35), (3.36) and (3.37), we have

$$\left(\varphi''+2\frac{\phi'}{\phi}\varphi'\right)=0 \tag{3.44}$$

and

$$\rho-\sum_{i=1}^{m}\frac{\lambda_{F_i}}{b_i^2}=0. \tag{3.45}$$

Now, we need to consider the case $\alpha_{k_0}=1$ and $\alpha_k=0$ for $k\neq k_0$. In this case, equation (3.40) is trivially satisfied and since equation (3.43) does not depend on the index $\imath$, we have that equation (3.43) is equivalent to equation (3.37). Finally, we need to show the validity of equation (3.35) and (3.36). Taking $\imath=k_0$, that is, $\alpha_{k_0}=1$, in (3.42), we obtain

$$\Bigg\{(r-1)\left(2\phi\phi''\epsilon_{k_0}-r\left(\phi'\right)^2\epsilon_{k_0}\right)-2\sum_{i=1}^{m} s_i\frac{\left(\phi^2\left(b_i\right)''\epsilon_{k_0}-(n-2)\,\phi\phi'\left(b_i\right)'\epsilon_{k_0}\right)}{b_i}$$
$$+\sum_{i=1}^{m}\frac{\lambda_{F_i}}{b_i^2}-\sum_{i=1}^{m} s_i\left(s_i-1\right)\frac{\left(\phi^2\left(b_i'\right)^2\epsilon_{k_0}\right)}{b_i^2}-\sum_{i=1}^{m}\sum_{i=1,\,l\neq i}^{m} s_i s_l\frac{\left(\phi^2\left(b_i\right)'\left(b_l\right)'\epsilon_{k_0}\right)}{b_i b_l}-\rho\Bigg\}\frac{\epsilon_{k_0}}{\phi^2}$$
$$=\varphi''+\frac{\phi'}{\phi}\varphi'$$

and for $\imath=k_0$, that is, $\alpha_\imath=0$, we get

$$\Bigg\{(r-1)\left(2\phi\phi''\epsilon_{k_0}-r\left(\phi'\right)^2\epsilon_{k_0}\right)-2\sum_{i=1}^{m} s_i\frac{\left(\phi^2\left(b_i\right)''\epsilon_{k_0}-(n-2)\,\phi\phi'\left(b_i\right)'\epsilon_{k_0}\right)}{b_i}$$
$$+\sum_{i=1}^{m}\frac{\lambda_{F_i}}{b_i^2}-\sum_{i=1}^{m} s_i\left(s_i-1\right)\frac{\left(\phi^2\left(b_i'\right)^2\epsilon_{k_0}\right)}{b_i^2}-\sum_{i=1}^{m}\sum_{i=1,\,l\neq i}^{m} s_i s_l\frac{\left(\phi^2\left(b_i\right)'\left(b_l\right)'\epsilon_{k_0}\right)}{b_i b_l}-\rho\Bigg\}\frac{\epsilon_\imath}{\phi^2}$$
$$=-\epsilon_\imath\epsilon_{k_0}\frac{\phi'}{\phi}\varphi'.$$

However, the last equation are equivalent to equations (3.35) and (3.36). This completes the proof. □

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