

# Granular solids, liquids, and gases

Heinrich M. Jaeger and Sidney R. Nagel

James Franck Institute, The University of Chicago, 5640 S. Ellis Avenue, Chicago, Illinois 60637

Robert P. Behringer

Department of Physics, Duke University, Durham, North Carolina 27706

Granular materials are ubiquitous in the world around us. They have properties that are different from those commonly associated with either solids, liquids, or gases. In this review the authors select some of the special properties of granular materials and describe recent research developments.

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“Who could ever calculate the path of a molecule? How do we know that the creations of worlds are not determined by falling grains of sand?” (Victor Hugo, *Les Misérables*).

## I. INTRODUCTION

Victor Hugo suggested the possibility that patterns created by the movement of grains of sand are in no small part responsible for the shape and feel of the natural world in which we live. No one can seriously doubt that granular materials, of which sand is but one example, are ubiquitous in our daily lives. They play an important role in many of our industries, such as mining, agriculture, and construction. They clearly are also important for geological processes where landslides, erosion, and, on a related but much larger scale, plate tectonics determine much of the morphology of Earth. Practically everything that we eat started out in a granular form, and all the clutter on our desks is often so close to the angle of repose that a chance perturbation will create an avalanche onto the floor. Moreover, Hugo hinted at the extreme sensitivity of the macroscopic world to the precise motion or packing of the individual grains. We may nevertheless think that he has overstepped the bounds of common sense when he related the creation of *worlds* to the movement of simple grains of sand. By the end of this article, we hope to have shown such an enormous richness and complexity to granular motion that Hugo’s metaphor might no longer appear farfetched and could have a literal meaning: what happens to a pile of sand on a table top is relevant to processes taking place on an astrophysical scale.

Granular materials are simple: they are large conglomerations of discrete macroscopic particles. If they are noncohesive, then the forces between them are only

repulsive so that the shape of the material is determined by external boundaries and gravity. If the grains are dry, any interstitial fluid, such as air, can often be neglected in determining many, but not all, of the flow and static properties of the system. Yet despite this seeming simplicity, a granular material behaves differently from any of the other familiar forms of matter—solids, liquids, or gases—and should therefore be considered an additional state of matter in its own right.

In this article, we shall examine in turn the unusual behavior that granular material displays when it is considered to be a solid, liquid, or gas. For example, a sand pile at rest with a slope lower than the angle of repose, as in Fig. 1(a), behaves like a solid: the material remains at rest even though gravitational forces create macroscopic stresses on its surface. If the pile is tilted several degrees above the angle of repose, grains start to flow, as seen in Fig. 1(b). However, this flow is clearly not that of an ordinary fluid because it only exists in a boundary layer at the pile’s surface with no movement in the bulk at all. (Slurries, where grains are mixed with a liquid, have a phenomenology equally complex as the dry powders we shall describe in this article.)

There are two particularly important aspects that contribute to the unique properties of granular materials: ordinary temperature plays no role, and the interactions between grains are dissipative because of static friction and the inelasticity of collisions. We might at first be tempted to view any granular flow as that of a dense gas since gases, too, consist of discrete particles with negligible cohesive forces between them. In contrast to ordinary gases, however, the energy scale  $k_B T$  is insignificant here. The relevant energy scale is the potential energy  $mgd$  of a grain of mass  $m$  raised by its own diameter  $d$  in the Earth’s gravity  $g$ . For typical sand, this energy is at least  $10^{12}$  times  $k_B T$  at room temperature. Because  $k_B T$  is irrelevant, ordinary thermodynamic arguments become useless. For example, many studies have shown (Williams, 1976; Rosato *et al.*, 1987; Fan *et al.*, 1990; Jullien *et al.*, 1992; Duran *et al.*, 1993; Knight *et al.*, 1993; Savage, 1993; Zik *et al.*, 1994; Hill and Kakalios, 1994; Metcalfe *et al.*, 1995) that vibrations or rotations of a granular material will induce particles of different sizes to separate into different regions of the container. Since there are no attractive forces between

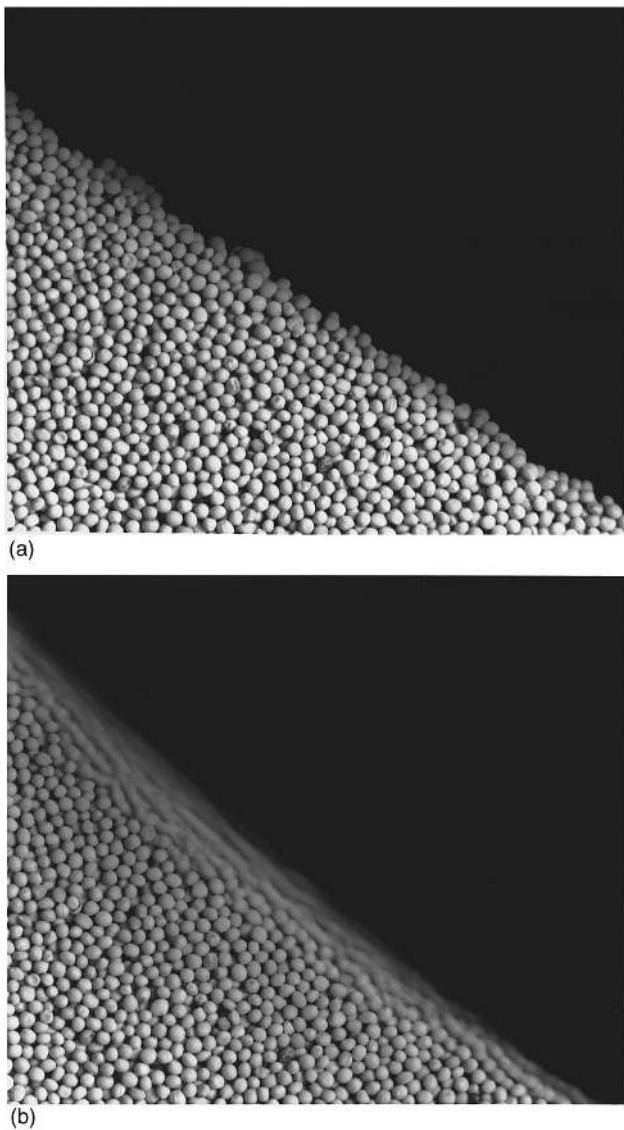


FIG. 1. (a) A pile of mustard seeds that is tilted to an angle lower than the angle of repose. (b) The same pile after the slope has been increased slightly to create an avalanche. Note that the flow occurs only along the surface and that the seeds deeper within the pile do not participate in the motion.

the particles, this separation would at first appear to violate the increase of entropy principle, which normally favors mixing (Zik *et al.*, 1994). In a granular material, on the other hand,  $k_B T \sim 0$  implies that entropy considerations can easily be outweighed by dynamical effects that now become of paramount importance.

An important role of temperature is that it allows a system to explore phase space. In a granular material,  $k_B T \sim 0$  precludes such exploration. Unless perturbed by external disturbances, each metastable configuration of the material will last indefinitely, and no thermal averaging over nearby configurations will take place. Because each configuration has its unique properties, reproducibility of granular behavior, even on large scales and certainly near the static limit where friction is important, is difficult to achieve. Another role of tempera-

ture in ordinary gases or liquids is to provide a microscopic velocity scale. Again, in granular materials this role is completely suppressed, the only velocity scale is imposed by any macroscopic flow itself. It is possible to formulate an effective “granular temperature” in terms of velocity fluctuations around the mean flow velocity (Ogawa, 1978; Savage, 1984; Walton and Braun, 1986; Haff, 1986; Campbell, 1990; Ippolito *et al.*, 1995; Warr and Huntley, 1995; Warr, Huntley, and Jacques, 1995). Yet, as we shall see, such approaches do not always recover thermodynamics or hydrodynamics because of the inelastic nature of each granular collision.

The science of granular media has a long history with much engineering literature devoted to understanding how to deal with these materials. There are many notable names such as Coulomb (1773), who proposed the ideas of static friction, Faraday (1831), who discovered the convective instability in a vibrated powder, and Reynolds (1885), who introduced the notion of dilatancy, which implies that a compacted granular material must expand in order for it to undergo any shear. Over the last decade there has been a resurgence of interest in this field within physics (for overviews see Jaeger and Nagel, 1992; Behringer, 1993; Bideau and Dodds, 1991; Bideau and Hansen, 1993; Jaeger *et al.*, 1994; Mehta, 1994; Mehta and Barker, 1994; Behringer, 1995; Hayakawa, Nishimori, Sasa, and Taguchi, 1995). Sand piles have become a fruitful metaphor for describing many other, and often more microscopic, dissipative dynamical systems. De Gennes (1966), for example, used sand pile avalanches as a macroscopic picture for the motion of flux lines in type-II superconductors. A particularly powerful use of sand as a metaphor lay in the idea of self-organized criticality (Bak *et al.*, 1988), originally described in terms of the avalanches in a sand pile close to its angle of repose. The self-organization paradigm was postulated to have a wide realm of applicability to a variety of natural phenomena. In a similar vein, the physics that has been uncovered in granular materials has clear relevance to what is being done in other areas of condensed matter physics. Slow relaxations are found in vibrated sand piles that bear close similarity to the slow relaxation found in glasses, spin glasses and flux lattices (Jaeger *et al.*, 1989; Duke *et al.*, 1990; Boguslavskii and Drabkin, 1995; Knight *et al.*, 1995). Fluid-like behavior can be induced in these materials, which very much resembles similar phenomena exhibited by conventional liquids (Douady *et al.*, 1989; Fauve *et al.*, 1989; Zik and Stavans, 1991; Melo *et al.*, 1993; Pak and Behringer, 1993; Jaeger *et al.*, 1994; Pak and Behringer, 1994; Melo *et al.*, 1995; Pak *et al.*, 1995). Nonlinear dynamical phenomena are observed that are relevant to breakdown phenomena in semiconductors (Clauss *et al.*, 1990), stick-slip friction on a microscopic scale (Reiter *et al.*, 1994; Radjai *et al.*, 1995), and earthquake dynamics on a macroscopic scale (Carlson *et al.*, 1994).

Despite this interest in granular science, the technology for handling and controlling granular materials is

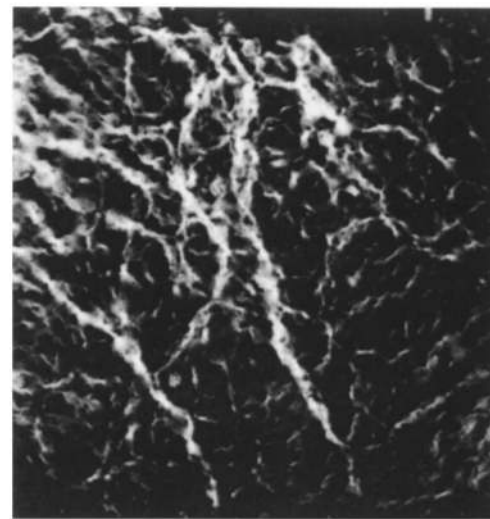
poorly developed. As mentioned above, many of our industries rely on transporting and storing granular materials. These include the pharmaceutical industry that relies on the processing of powders and pills, agriculture and the food processing industry where seeds, grains, and foodstuffs are transported and manipulated, as well as all construction-based industries. Estimates are that we waste 40% (Ennis *et al.*, 1994; Knowlton *et al.*, 1994) of the capacity of many of our industrial plants because of problems related to the transport of these materials. Additional manufacturing processes, e.g., in the automotive industry, rely on casting large metal parts in carefully packed beds of sand. Even a small improvement in our understanding of granular media behavior could have a profound impact on industry.

We turn now to examining some of the particular properties of granular materials that appear under different conditions. The following three sections will explore their unique behavior, contrasting it to that of ordinary solids, liquids, and gases, respectively.

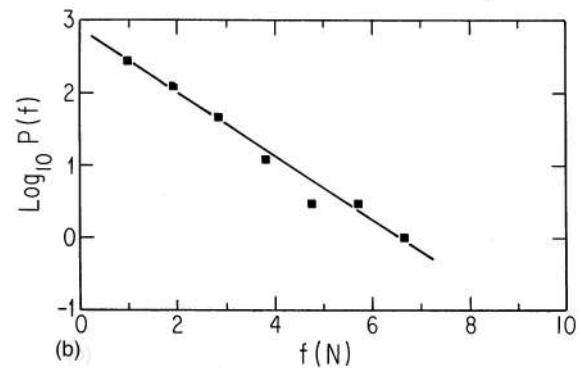
## II. AN UNUSUAL SOLID: SAND AT REST

Already in the resting state, granular materials exhibit a host of unusual behaviors. For example, when the granular material is held in a tall cylindrical container, such as a grain elevator or silo, no height-dependent pressure head occurs as it does with a normal fluid: the pressure at the base of the container does not increase indefinitely as the height of the material inside it is increased. Instead, for a sufficiently tall column, the pressure reaches a maximum value independent of the height. Owing to contact forces between grains and static friction with the sides of the container, the container walls support the extra weight (Janssen, 1895). It is this feature that allows the sand in an hour glass to flow through the orifice at a nearly constant rate; a nearly linear change in filling height over time makes this a useful instrument to measure elapsed time. Underlying this simple, time-averaged flow is, however, a complicated dynamical behavior; see Gallas *et al.* (1993); Wu *et al.* (1993); Pöschel (1994); Peng and Herrmann (1995); Horikawa *et al.* (1995); and Sec. III.

We can investigate the network of forces within the pile in greater detail. One example is shown in Fig. 2(a) for a three-dimensional arrangement of particles. The forces within the pile appear to be very heterogeneous, forming chains along which the stresses are particularly intense (Dantu, 1957; Wakabayashi, 1959; Drescher and de Josselin de Jong, 1972; Ammi *et al.*, 1987; Travers *et al.*, 1987). Shear experiments also attest to the extreme heterogeneity of granular media. For example, when a layer of grains is sheared continually in a narrow annulus, the normal stress measured at one of the boundaries shows large-scale fluctuations with the rms fluctuations comparable to the mean applied stress (Miller *et al.*, 1996). It is not clear from Fig. 2(a) alone what is the distribution of forces within the pile. This can be found by simply placing a piece of carbon paper on the bottom of the container and measuring the areas of



(a)



(b)

FIG. 2. (a) A visualization of the forces in a granular medium as viewed between two crossed-circular polarizers. Three mm Pyrex spheres are surrounded by a mixture of water and glycerol that matches the index of refraction of the Pyrex. A force is exerted on a piston that covers the top surface of the container. The stress-induced birefringence makes those beads that are under stress visible as the bright regions. (b) The distribution of forces  $P$  versus force  $f$  measured at the bottom of a cylindrical container filled with spheres. The bottom of the container was lined with carbon paper and a force was applied to the top surface. The line is an exponential fit to the data:  $P(f) = ce^{-f/f_0}$ . After Liu *et al.* (1995).

the marks left by the forces  $f$  exerted by individual beads. The distribution of forces  $P(f)$  is

$$P(f) = c \exp(-f/f_0), \quad (1)$$

where  $c$  and  $f_0$  are constants (Liu *et al.*, 1995). The fluctuations in  $f$  are large and scale with depth in the same way as the mean force, rather than as its square root as one might have initially expected. Such behavior has been explained (Liu *et al.*, 1995; Coppersmith *et al.*, 1996) in terms of a simple model in which masses placed on a lattice distribute their weights unevenly and randomly to the particles on the layer below them. This model can be solved exactly in a number of different cases, yielding, in agreement with the experiment and simulation, an exponential distribution of large forces

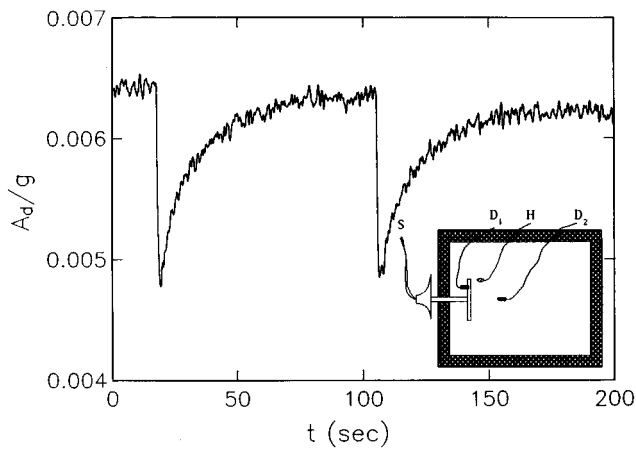


FIG. 3. The transmitted rms magnitude of the acceleration (in units of g) versus time showing the effect of a temperature pulse ( $\Delta T=1$  K) on the sound propagation in a granular medium. The inset shows a schematic view of the apparatus. The source  $S$  was run at 4 kHz. The heater  $H$  was 1 cm from the driving plate. The data show the response of the detector  $D_2$  to two consecutive current pulses separated by 85 seconds. After each pulse, which heats only one bead in the container, the transmission drops by roughly 25%. After Liu and Nagel, (1994).

(see also Radjai *et al.*, 1996). Interestingly, the experiments of Miller *et al.* (1996) also show nearly exponential distributions.

The force chains appearing so clearly in Fig. 2(a) are also important for many of the properties of the granular material such as the transmission of sound (Liu and Nagel, 1992, 1993; Leibig, 1994; Liu, 1994; Melin, 1994; Sinkovits and Sen, 1995). If one initiates a sound wave at one point in the material, the transmitted signal at a second position is sensitive to the exact arrangement of all particles in the container. An example of this extraordinary sensitivity is shown in Fig. 3. By replacing a single bead in the pile with a small carbon resistor, one can measure the effect of a small thermal expansion of a single bead on the transmission of sound (Liu and Nagel, 1994). A one-degree change in the resistor produced by a short current pulse causes a 100 nm thermal expansion, which is small compared to the size of the particles, 5 mm, or the wavelength of the sound, 1 cm. Nevertheless, this perturbation of only a single particle out of the entire bead pack by an amount that is one part in  $10^5$  of all the other obvious length scales in the system can create a reproducible 25% change in the transmission of sound! This dependence on the microscopic arrangement of the particles is again reminiscent of the sentiment expressed by Hugo in the quotation at the beginning of this article: macroscopic phenomena can be affected by the placement and motion of even a single sand grain. If the heater is placed in another position, it will sometimes give an equivalent increase in the signal and sometimes produce no discernible change at all (Liu, 1994). This sensitivity is far greater than what is found in conventional interference effects, where the

perturbation must be comparable to the wavelength. This sensitivity may again be explained by the extreme heterogeneity of the material: if the heater lies on or near a force chain, it can have a much more dramatic effect on the local transmission of the sound wave than if it lies in a region away from any of the chains.

A fundamental issue concerns the packing of granular materials. The random packing of spherical objects was first studied by Stephen Hales, minister of Teddington, who used the dimple patterns on peas that had been expanded in a closed, water-filled vessel to ascertain their geometrical arrangements (Hales, 1727). Depending on the procedure for filling the container, a random assembly of spherical balls can be packed anywhere from a volume fraction of  $\eta=0.55$  to  $\eta=0.64$  (Onoda and Liniger, 1990; Bideau and Dodds, 1991). Through static friction, force chains can hold the sand pile in a metastable configuration between these limits and keep it from collapsing. How does the system pass between these states? Since the energy  $k_B T$  is negligible, the density can only change from disturbances of the container by an external source, for instance, by vibrations. For this situation, Mehta and Edwards (1989) have proposed a new formalism that replaces conventional thermodynamics. They neglect energy (since the particles are assumed to have no interactions aside from a hard sphere repulsion) and replace the Hamiltonian with a volume functional. The entropy is still the logarithm of the number of states at a given volume (Monasson and Pouliquen, 1996), and the other thermodynamic quantities are defined in analogy with ordinary thermodynamics. But now, instead of  $k_B T$ , a new effective temperature emerges that is given by the compactivity of the material. External vibrations unlock the packing, thereby allowing the system to travel slowly through phase space (Barker and Mehta, 1993; Mehta, 1994).

Studies (Knight *et al.*, 1995) of granular material settling under vibrations indicate that the relaxation in these systems is, in fact, logarithmically slow. Even after 100 000 vibration cycles, depending on the vibration intensity, a tube filled with granular material might still undergo significant compaction before reaching a steady state. A variety of models have been proposed to account for this extremely slow settling (Barker and Mehta, 1993; Hong *et al.*, 1994). At present, perhaps the most plausible explanation rests on the idea that the rate of increase in the granular volume fraction is exponentially reduced by excluded volume (Ben-Naim *et al.*, 1996). A simple corresponding picture is that of a parking lot without assigned slots and with a high density of equal-sized, parked cars (*viz.* particles). For the person wishing to park an extra vehicle (or insert an extra particle into the bead pack), the all-too-familiar situation is that there exist large, but not quite large enough, voids between the objects already in place. The question is how many other cars (or particles) have to be moved just a bit for the additional one to fit in? If all densification occurs by random “parking” and “unparking” events, it takes the cooperative motion of many objects (exponential in the density) to open up new slots. As a

result, the approach to the steady-state density is logarithmic in time. Experiments on granular materials as well as simulations of the parking problem indicate that once the steady state is reached there are large density fluctuations (Ben-Naim *et al.*, 1996).

### III. AN UNUSUAL LIQUID: GRANULAR HYDRODYNAMICS

Granular materials can flow like liquids, and there are a variety of theoretical models used to describe such flows. We refer to these models as granular hydrodynamics (even though there is nothing wet here), in the sense that they are continuum theories consisting of partial differential equations, analogous to the Navier-Stokes equations for Newtonian fluids. However, models for granular flow do not have the stature of the Navier-Stokes equations. Those equations arise out of an averaging process over length and time scales that are much larger than typical microscopic scales and much smaller than macroscopic ones. This separation of scales may not occur in granular flows. Indeed, the issue of which are the relevant time and space scales is one of the most important questions to resolve. Slow granular flows of densely packed materials are certainly not ergodic. More rapid flows are complicated by the phenomena of clustering or clumping, which can occur when the coefficient of restitution for particle collisions is less than unity. Even in commercial settings, such as in the flow of coal in a silo, the largest system size may be only a few thousand grain diameters. Since stress chains can easily span 100 grain lengths, there is no compelling reason to believe that the system is homogeneous, and therefore could be characterized by a continuum model. In an ordinary fluid, an observation of the pressure is typically carried out over a time spanning an enormous number of collisions so that the individual impacts of single molecules are averaged out. In many granular flows, the amount of temporal averaging is much weaker.

Dense slow flows and rapid gaslike flows are useful idealizations for the development of models. Owing to rapid energy dissipation, real systems often exhibit both flow types simultaneously in different spatial domains (Drake, 1990); one of the open questions is how to model the transition between the two flows accurately. Kinetic-theory models (Ogawa, 1978; Jenkins and Savage, 1983; Haff, 1983 and 1986; Jenkins, 1987; Lun and Savage, 1987; Savage, 1989; Savage and Jeffrey, 1981; Campbell, 1990) serve to describe granular materials when the density is low. To obtain this state, energy must be continuously supplied, for instance, by shaking. Any dynamical quantity, such as the kinetic energy, is assumed to consist of a slowly varying hydrodynamic part and a fluctuating part.

The other extreme is treated by models for quasistatic plastic deformation (Jackson, 1983), based on Reynolds' principle of dilatancy (Reynolds, 1885; Bagnold, 1966) and on the idea that deformations in compacted materials are typically irreversible. Dilatancy occurs because the grains interlock under applied normal stress, and the

material will only begin to deform for shear stresses above some yield point set by the applied normal stresses. At the yield point, grains begin to slide past each other. For a given density, the locus of points at which this occurs in a space of shear and normal stresses defines the yield surface. Specific models consist of conservation laws in the form of partial differential equations augmented by constitutive models. Thus there is the standard continuity equation for mass conservation, an energy equation, and a momentum equation. The last of these is perhaps the most revealing. It relates the stress tensor  $T_{i,j}$  and the strain-rate tensor (defined here with a minus sign, since granular materials disintegrate under tension)

$$V_{i,j} = -(\partial v_i / \partial x_j + \partial v_j / \partial x_i), \quad (2)$$

where  $v_i$  is the  $i$ th component of the velocity field. In one of the simplest versions of these models (Jackson, 1983; Schaeffer, 1987),

$$T_{i,j} = \sigma(\delta_{i,j} + kV_{i,j}/|V|). \quad (3)$$

Here  $|V|^2 \equiv \sum V_{i,j}^2$  and  $k$  is a constant characteristic for each material. More specifically,  $k/\sqrt{2} = \sin \delta$ , where  $\delta$  is the angle of internal friction. A comparison of this equation with the Navier-Stokes equations reveals that the ordinary viscous terms, proportional to the viscosity and the velocity gradients, have been replaced by shear-rate independent terms. This rate-independent feature is quite remarkable, since it implies that an overall increase in the velocity leaves the stress unchanged. This feature also means that these equations are more complex mathematically than the Navier-Stokes equations and apply only when the material is deforming. Models like that specified by Eq. (3) are used in soil mechanics and in the design of materials-handling devices such as hoppers. However, visualization experiments of flow in thin hoppers (Baxter *et al.*, 1989) using continuous x-ray imaging have revealed a dynamic behavior that is not captured by the standard plasticity models. These experiments show density waves for rough materials but not in smooth, nearly spherical ones. Figure 4 contrasts x-ray images of flow out of a quasi-two-dimensional ( $\sim 1$ -cm-thick) hopper for rough- and smooth-grained materials. In the case shown here for rough grains, the waves propagate upwards (against gravity), but the propagation direction changes sign if the hopper angle is made sufficiently steep. These experiments appear to be inconsistent with theoretical predictions (Jenike, 1961, 1964) and indicate that the grain shape plays a crucial role that requires better understanding. Recent experiments and computer simulations of flow in chute and hopper geometries, where the flow is quite different from what is found in simple fluids, attest to the role of granularity (Lee, 1994b; Pouliquen and Gutfraind, 1996; Pouliquen and Savage, 1996; Zheng and Hill, 1996).

One of the exciting aspects about the present state of the physics of granular media is the vehement debate that still exists about the causes for some of the most prominent behaviors that these materials exhibit when vibrated. We shall briefly discuss two such debates: (i)

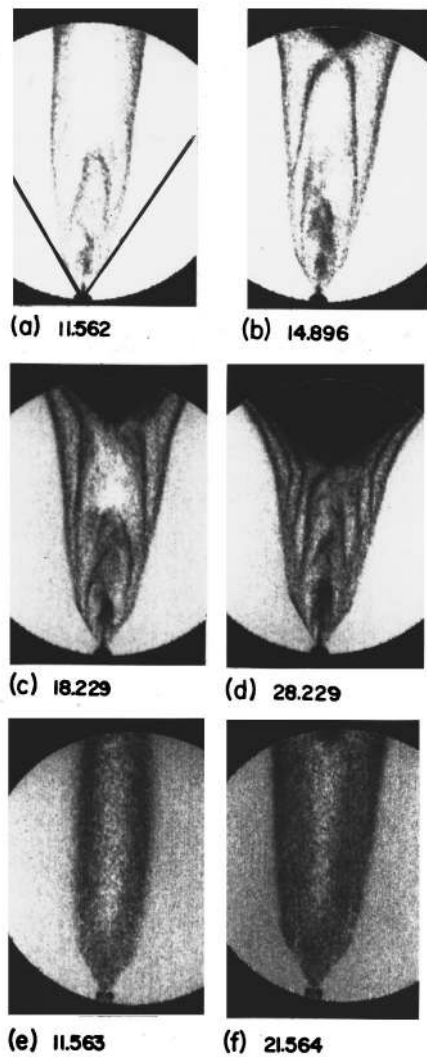


FIG. 4. X-ray images of waves in a hopper of flowing sand, after Baxter *et al.*, (1989). Images (a)–(d) show the waves that form for rough sand (in this case, sieved construction sand); the darker regions correspond to lower density; the numbers under each image indicate the elapsed time in seconds. Images (e)–(f) show two frames from a run using smooth (Ottawa) sand. The waves are absent if the material is smooth, i.e., the particle shape plays a key role. The typical grain sizes of the two materials were identical: 0.6 mm to 0.7 mm.

the cause for vibration-induced convection and heaping, and the role played by interstitial gas; and (ii) the cause of vibration-induced size separation.

Convective flow in vibrated granular material was first observed by Faraday 160 years ago (Faraday, 1831), yet its underlying mechanisms are only partially understood. Both segregation and convection occur when the material is shaken in the vertical direction, typically as

$$z = A \cos(\omega t). \quad (4)$$

When

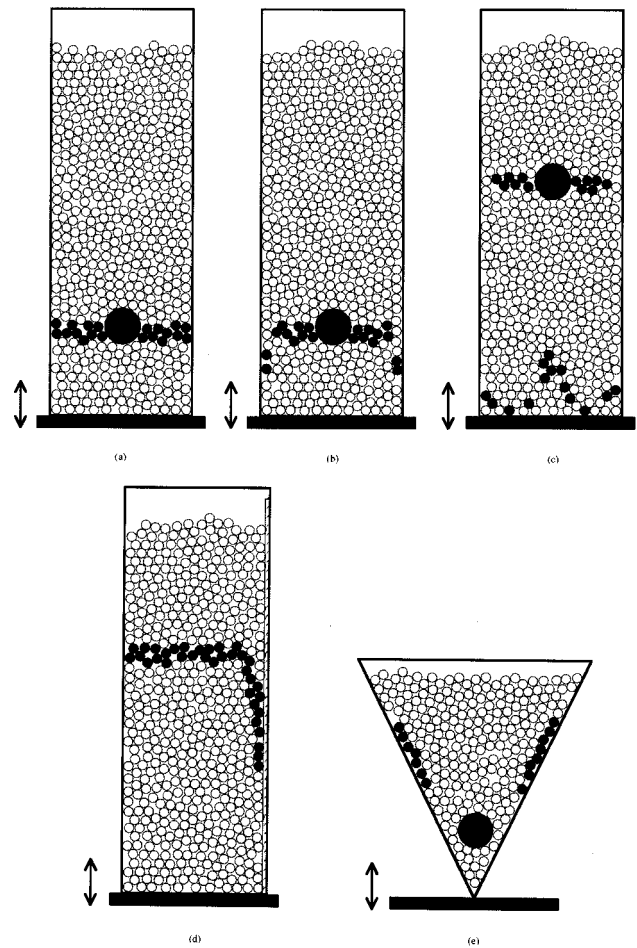


FIG. 5. Schematic drawing of convection rolls in a cylindrical vessel geometry (cross section). (a) Initial configuration. (b) After just a few taps. (c) After several taps and after the beads near the cylinder wall have reached the bottom of the convection roll. (d) Schematic drawing of the situation in a cylindrical cell with smooth frictionless walls except along a vertical stripe on the right-hand side that was coated with rough sand. (e) Cross section of a conical cell geometry. Here the rotation sense of the convection rolls is reversed, the flow occurring upward along the walls. After Knight *et al.*, (1993).

$$\Gamma \equiv A \omega^2 / g \quad (5)$$

is a bit larger than unity, the material rises above the floor of the container for some part of each cycle, dilating in the process, so that a macroscopic flow of grains can occur. This flow takes the form of convection rolls that continuously transport grains, as sketched in Fig. 5. In a typical experiment using cylindrical or rectangular vessels the flow is upwards in the center and downwards in a thin stream along the side walls, leading to the formation of a central heap with a steady avalanche of grains downward (Evesque and Rajchenbach, 1989; Fauve *et al.*, 1989; Laroche *et al.*, 1989; Knight *et al.*, 1993; Pak and Behringer, 1993; Lee, 1994; Pak and Behringer, 1994; Ehrichs *et al.*, 1995; Knight *et al.*, 1995;

Pak *et al.*, 1995). With different boundary conditions, such as side walls that are slanted outward, it is possible to reverse the sense of the convection roll, thus inducing downward flow in the center (Takahashi *et al.*, 1968; Knight *et al.*, 1993; Jaeger *et al.*, 1994; Bourzutschky and Miller, 1995). More generally, container shape, wall and interparticle friction, and internal phase boundaries can combine to reverse the direction of the convective flow (Aoki *et al.*, 1996; Knight, 1996; Van Doorn and Behringer, 1996).

At least three mechanisms have been proposed to explain these states. Savage (1988) considered lateral inhomogeneities in the shaking and found that inelastic collisions of particles lead to upwardly directed pressure gradients that are strongest at the upflow. This mechanism may not be relevant to experiments in which the entire layer is shaken uniformly. A second mechanism involves friction with the walls of the container. Several experiments and numerical simulations have shown a kind of ratchet effect, which produces a thin, rapidly moving boundary layer near the walls and leads to circulating flow (Gallas *et al.*, 1992a,1992b; Herrmann, 1992; Taguchi, 1992a,1992b; Knight *et al.*, 1993; Thompson, 1993; Luding *et al.*, 1994a,1994b; Bourzutschky and Miller, 1995; Ehrichs *et al.*, 1995; Hayakawa, Nishimori, Sasa, and Taguchi, 1995; Hayakawa, Yue, and Hong, 1995; Pöschel *et al.*, 1995; Taguchi and Takayasu, 1995). Recent experiments using magnetic resonance imaging have been able to probe granular motion noninvasively everywhere inside the container (Nakagawa *et al.*, 1993; Ehrichs *et al.*, 1995; Kuperman *et al.*, 1995; Knight *et al.*, 1996) [for direct visualization in quasi-two-dimensional containers see, e.g., Ratkai (1976); Tüzün and Nedderman (1982); Duran *et al.* (1994); Cooke *et al.* (1996)]. From such measurements both the depth dependence of the convection velocity and the detailed shape of the velocity profiles have been obtained (Ehrichs *et al.*, 1995; Knight *et al.*, 1996) (Fig. 6). The experiments show that the fastest flow occurs in the thin boundary layer near the walls. This is very different from what might occur for a conventional fluid, for which the no-slip condition applies, and raises a number of issues about the correct boundary conditions for granular convection and other flows.

A third mechanism for convection and heaping occurs in the presence of interstitial gas. This effect dominates when friction with the container walls is eliminated or reduced (e.g., by choosing periodic boundary conditions that can be realized at least partially in experiments in annular containers and/or by choosing relatively small grains). Faraday (1831) was the first to attribute granular convection to the trapping of gas, and analysis of gas trapping effects has been made by Gutman (1976). More recently, experimenters have tried to clarify the role of gas in granular convection with conflicting results (Evesque and Rajchenbach, 1989; Fauve *et al.*, 1989; Laroche *et al.*, 1989; Pak *et al.* 1995). One set of experiments indicated that the flow stopped when the surrounding pressure was reduced, while another indicated that convection was virtually unchanged for pressures as

low as  $P \approx 4$  Torr. Pak *et al.* (1995) have shed light on this conflict through experiments where the pressure was held fixed at values between atmospheric pressure and vacuum. The convective heap persisted for  $P$  down to 10 Torr. As  $P$  was decreased further, the height  $L$  of the heap steadily diminished. These results apply for grains of diameter up to about 1 mm, and the effect is more pronounced for large oscillation amplitude  $A$ . A theoretical challenge remains to develop a theory that incorporates both the friction and gas effects.

Another key feature of vibrated or flowing granular material is its unique mixing and size-separation (“unmixing”) behavior (for overviews, particularly also of the associated industrial processes, see Williams, 1976 and Fan *et al.*, 1990). When granular materials are shaken, particles of different sizes tend to separate, with the largest particles moving to the top independently of their density (Harwood, 1977; Rosato *et al.*, 1987; Jullien *et al.*, 1992; Duran *et al.*, 1993,1994; Knight *et al.*, 1993; Cooke *et al.*, 1996). Separation phenomena also occur in very long, slowly rotating cylinders with the cylinder axis horizontal (Savage, 1993; Hill and Kakalios, 1994; Zik *et al.*, 1994). Here, particles with different dynamical angles of repose aggregate into sharply delineated regions along the axis. In rotating cylinders or drums with a horizontal axis of rotation, particles flow down the free surface in a succession of avalanches (Jaeger *et al.*, 1989; Rajchenbach, 1990; Bretz *et al.*, 1992; Benza *et al.*, 1993; Evesque, 1993; Morales-Gamboa *et al.*, 1993; Nakagawa *et al.*, 1993; Sen *et al.*, 1994; Baumann *et al.*, 1994; Bouchaud *et al.*, 1994; Clement *et al.*, 1996; Linz and Hänggi, 1995; Frette *et al.*, 1996). Particle motion for more complicated types of agitation, such as horizontal swirling, have also been studied (Scherer *et al.*, 1996). An important question, particularly for industry, is how mixing occurs as a function of the filling fraction of the drum (Hogg *et al.*, 1974). This question was recently addressed both theoretically and experimentally by Metcalfe *et al.* (1995). These authors found that simple geometrical arguments were sufficient to predict the mixing rate and efficiency. Both mixing and unmixing bear directly on such technically important processes as the separation of “fines” (which may or may not be desirable) or the mixing of powdered drugs with a binder, where a well-controlled and homogeneous mixture is highly desirable.

Several mechanisms have been associated with mixing and size separation, including sifting (where small particles fall through the gaps between large particles if the gaps are large enough) and local rearrangements [where large particles will be wedged upwards as smaller grains avalanche into voids beneath them in the dilation phase during each shake (Williams, 1976; Rosato *et al.*, 1987; Fan *et al.*, 1990; Jullien and Meakin, 1992; Duran *et al.*, 1993)]. In vertically shaken systems, however, experiments (Knight *et al.*, 1993) show a direct link between convection and size separation: large particles become entrained in the upward convective flow but, once on the top surface, remain stranded because they cannot follow the smaller grains in the thin layer of downward

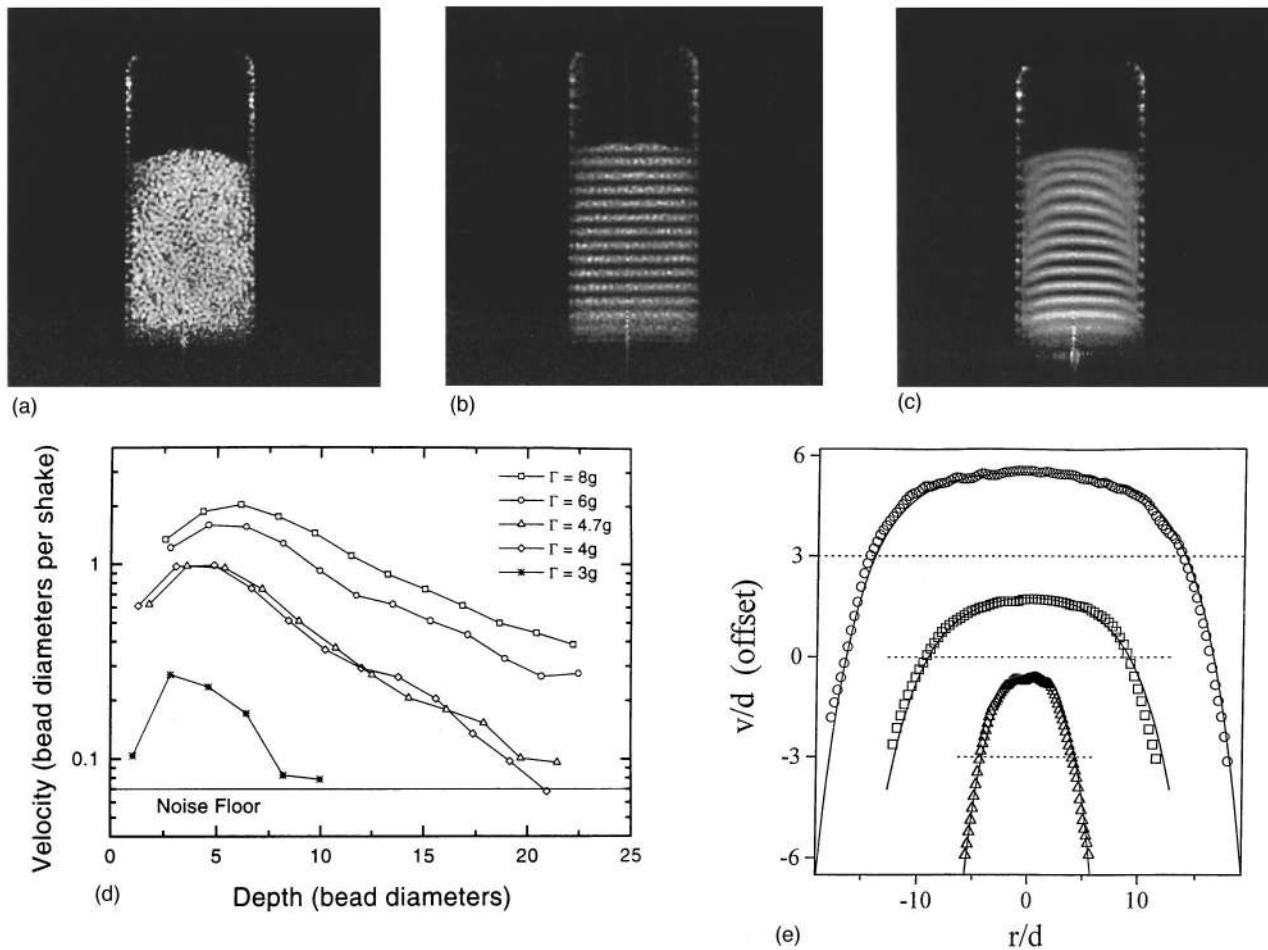


FIG. 6. Magnetic resonance (MR) images of a 1 mm slice through the center of a glass cylinder filled with white poppy seeds (the color table chosen for each of the images is arbitrary). In (a) each grain visible in this image of the stationary cylinder corresponds to one individual seed (roughly 1 mm across). The coating of seeds glued to the inner surface of the container walls is visible above the filling level. In (b) the same slice through the center of the container is imaged, except that here a spatial modulation of the spin polarization in the vertical  $z$  direction is applied. The peaks of this modulation appear as bright bands in the image, serving to label narrow regions in the granular material. In (c), the MR image of the system is prepared in the same way as in (b) but after a single shake of peak acceleration  $\Gamma = 8$  g. The curvature of the horizontal stripes directly gives the flow profile of the seeds (displacement per shake). The layer of seeds glued to the wall provides a marker for the position of the bands prior to the shake. (d) The vertical velocity  $v(z)$  of the central region of each band is plotted as a function of depth  $z$  below the top surface. A straight line on this plot indicates an exponential depth dependence of the velocity. Very close to  $z=0$  the vertical velocity component decreases as particles no longer move straight up but turn sideways towards the container walls. Data for several accelerations,  $\Gamma$ , are shown. After Ehrichs *et al.* (1995). (e) A plot of the velocity  $v(r)$  at a given depth is plotted as a function of radial distance from the central axis of the container. The solid line through the points is a fit of the form  $v(r) = v_c + B[1 - I_0(r/r_0)]$ , where  $I_0$  is the modified Bessel function of order zero and  $v_c$ ,  $B$ , and  $r_c$  are constants. The data shown are normalized by the seed diameter  $d$ . The three curves pertain to three containers of different diameters, which are indicated by the horizontal dotted lines. After Knight *et al.* (1996).

convective flow along the container walls. Recent work indicates that this mechanism drives size separation also in two-dimensional systems (Duran *et al.*, 1994; Cooke *et al.*, 1996). Thus in granular materials, shaking does not induce mixing. In contrast to ordinary liquids, where entropy favors a homogeneously mixed state, dynamics is dominant, and it leads to size separation. Similar behavior has recently been observed in Couette-like shear experiments (Khosropour *et al.*, 1996).

In addition to the convection patterns that exist in the bulk of a vibrated granular material, its free surface can

exhibit several different wave phenomena (Fauve *et al.*, 1989; Melo *et al.*, 1993; Pak *et al.*, 1993; Melo *et al.*, 1995; Brennen *et al.*, 1996; Clement *et al.*, 1996; Metcalf *et al.*, 1996; van Doorn and Behringer, 1996; Wassgren *et al.*, 1996), as well as more complex, and possibly chaotic, states (Dinkelacker *et al.*, 1987; Douady *et al.*, 1989; Pak and Behringer, 1994). The different waves can be either traveling (for material with a steeply sloping heap) or standing (when heaping is weak or nonexistent). Experiments on the former kind of waves show that not only is  $\Gamma$  a relevant parameter but so is the ratio of energies



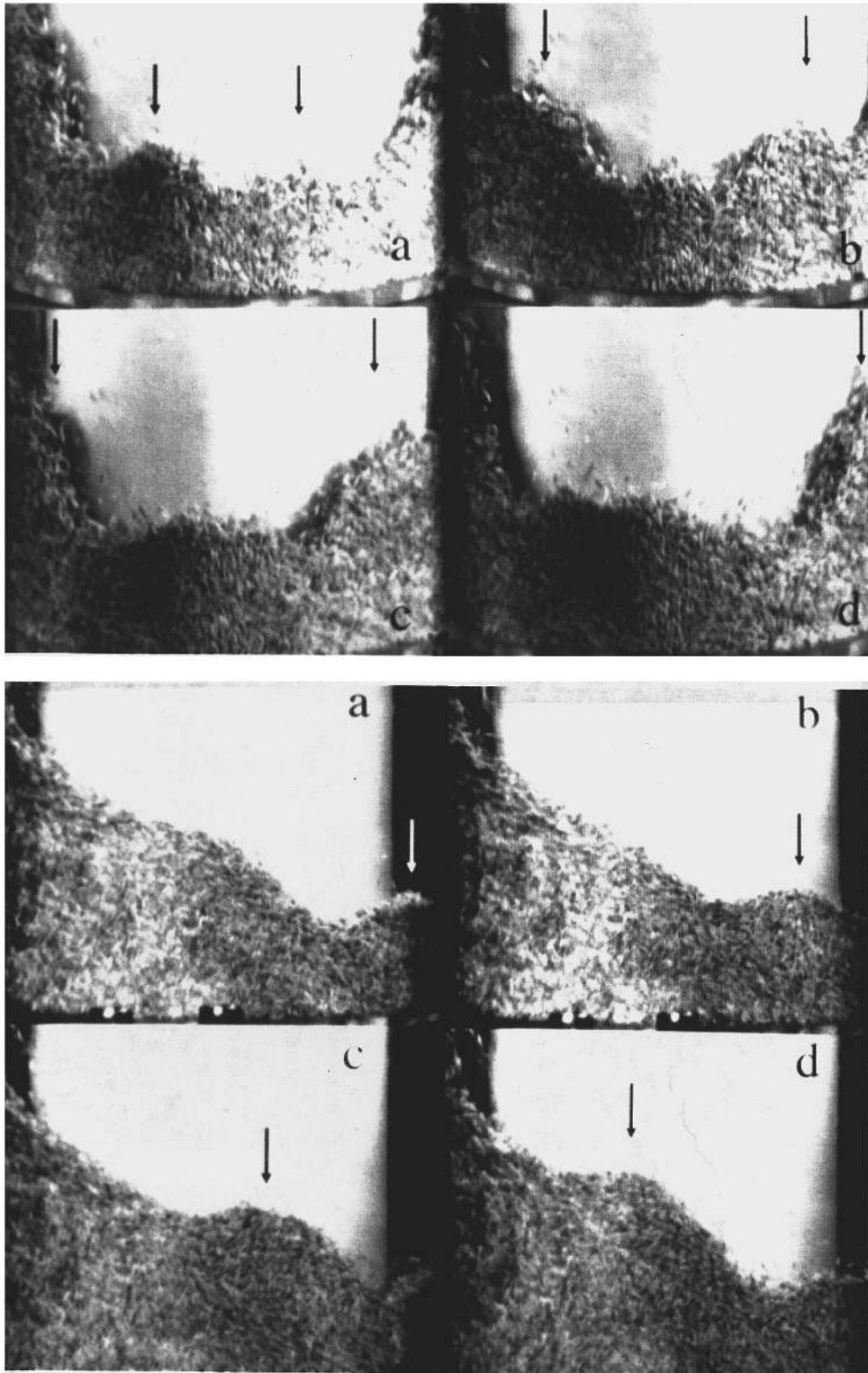


FIG. 7. Side views of traveling waves for shaken sand in an annular container, after Pak and Behringer (1993). The first sequence of images shows a wave propagating up the heap and corresponds to times of 0.0 s, 0.4 s, 0.8 s, and 1.26 s for (a)–(d), respectively. The second sequence is similar but shows the wave pulses forming near the base of the heap, and corresponds to times of 0.0 s, 0.4 s, 0.8 s, and 1.13 s for (a)–(d), respectively.

$mv^2/mgd$ . We show examples of the traveling waves in Fig. 7 for an annular container and of the subharmonic standing waves in Fig. 8. In this last figure, well-defined wave patterns and their superpositions occur that are strikingly familiar from Faraday instabilities in ordinary liquids. In the first two parts of this figure, the waves are confined to a narrow rectangular container. Part (c)

shows the striking patterns that evolve when the container is a large open cylinder. If fine granular materials are shaken with large  $\Gamma$ , bubbling can ensue (Pak and Behringer, 1994), as in Fig. 8(d), resembling fluidized beds. At very high  $\Gamma$ , the resulting state may be considered a kind of granular turbulence (Taguchi, 1995; Taguchi and Takayasu, 1995).

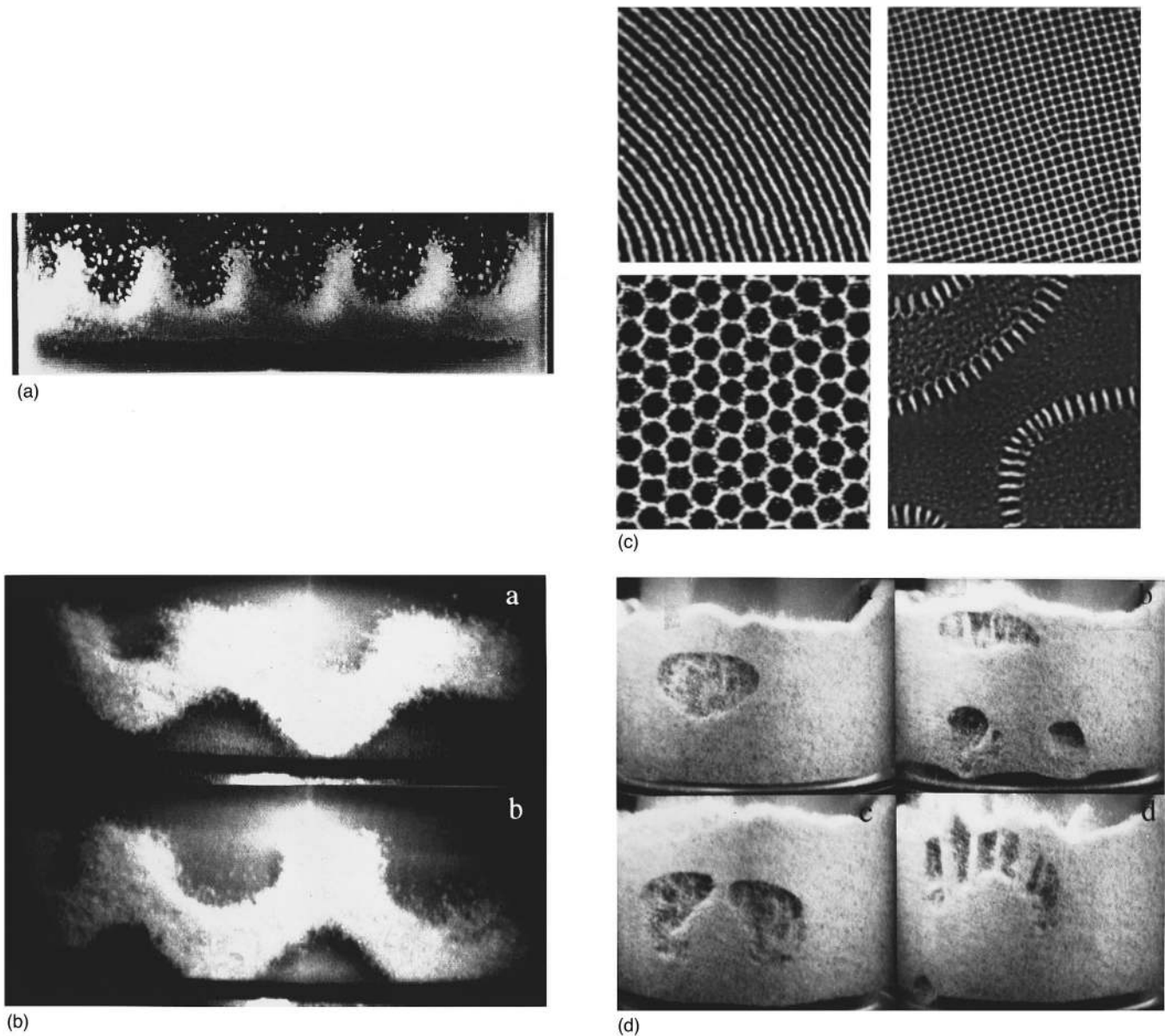


FIG. 8. Complex wave behavior observed in thin layers of granular material shaken vertically. (a) Sideview of a subharmonic standing wave in a quasi one-dimensional rectangular container filled with glass beads at  $f \sim 20$  Hz,  $\Gamma \sim 3.5$ , after Douady *et al.*, (1989). (b) A different standing wave state, in an annular container, after Pak and Behringer (1993). The two snapshots are taken at successive oscillation periods of the container and demonstrate the subharmonic nature of the response, which repeats after two drive cycles rather than one. (c) Thin layers organize into a variety of stationary subharmonic two-dimensional patterns depending on  $f$  and  $\Gamma$ . The pictures show top views of a large cylindrical container filled with brass spheres (165 micron diameter) to a nominal depth of 8 spheres (photograph courtesy of P. Umbanhowar; see Melo *et al.*, 1993 and 1995). Stripe (upper left) and square patterns (upper right) correspond to  $f=40$  Hz,  $\Gamma=3$ . Highly curved interfaces (lower right) between essentially flat featureless plateaus coexist 180 degrees out of phase in different parts of the cell ( $f=40$  Hz,  $\Gamma=5$ ). Hexagonal patterns (lower left) in this picture are produced by driving the system at two frequencies (16 and 18 Hz) for  $\Gamma=2.8$ . (d) In material consisting of small diameter grains that are shaken at large amplitudes, bubbling of air pockets can occur, after Pak and Behringer (1994).

#### IV. AN UNUSUAL GAS: INELASTICITY, CLUSTERING, COLLAPSE

One crucial difference between ordinary gases or liquids and granular media deserves particular attention: interactions between grains are inherently inelastic so that in each collision some energy is lost. As a result, all approaches based on purely elastic interactions or en-

ergy conservation, such as the theory for ideal gases, cannot carry over, with novel features arising for the statistical mechanics of these systems. It is important to remember that any seemingly fluidlike behavior of a granular material is a purely dynamic phenomenon. For example, the surface waves do not arise as a linear response to external energy input but are the consequence of a highly nonlinear hysteretic transition out of the sol-

idlike state. Fluid behavior only sets in above a threshold excitation level, with inelastic collisions bringing the granular medium to rest almost instantly after the energy input is stopped. Drop an individual grain, such as a single marble, onto a glass plate, and it will bounce for quite a while, whereas a loosely filled sack of the very same marbles will stop dead on the plate. This strikingly different collective behavior arises from the exceedingly large number of rapid inelastic collisions by neighboring grains. In fact, we often use this curious behavior for energy absorption in applications including packaging fillers, recoilless hammers, or the common toy, Hacky Sacks. At sufficiently high excitation frequencies, practically all applied energy is dissipated into heat. For example, sound propagation in granular materials decays exponentially for frequencies above 1–2 kHz (Liu, 1994).

The inherent inelasticity of granular collisions leads to complications if we try to apply Newton's Laws to individual grain-grain interactions because we lack a coherent picture of the dissipative forces involved. Questions about the correct velocity dependence of friction forces and about the importance of the impact duration or the inclusion of rotational degrees of freedom have been debated for many years (Bagnold, 1954; Maw *et al.*, 1981; Campbell, 1990; Jaeger *et al.*, 1990; Herrmann, 1992; Walton, 1992; Foerster *et al.*, 1994; Brilliantov *et al.*, 1996; Hertzsch *et al.*, 1995; Luck and Mehta, 1993; Radjai and Roux, 1995); yet these issues still appear very much unresolved. The treatment of shearing collisions that occur off-center and at small relative velocities is particularly uncertain since it involves a crossover from what we usually call "static" to "dynamic" friction. The existence of such a crossover also clearly demonstrates that models based on a single fixed parameter like the coefficient of restitution oversimplify any real collision process.

Since real granular materials are inelastic, energy input from the boundaries, as in an ordinary heat bath, may not be sufficient to thermalize the system. If clustering begins to occur inside the system, this effect may indicate a breakdown of Newtonian hydrodynamics, since such aggregates will not be able to melt away. This clustering in the regime of finite inelasticity has recently become the focus of much interest (Walton, 1992; Goldhirsch and Zanetti, 1993). The conditions under which clustering should appear have been estimated (Goldhirsch and Zanetti, 1993; Esipov and Pöschel, 1995): if a system of linear extent  $L$  is started in a uniform state with grains occupying a volume fraction  $\eta$ , the solutions provided by Newtonian hydrodynamics will become linearly unstable to cluster formation once the product  $\eta L$  exceeds some constant that depends upon the degree of inelasticity. For large enough  $L$  this result implies that the system always becomes unstable towards cluster formation, no matter how small the inelastic contribution to each collision. In Fig. 9, a snapshot taken from a two-dimensional simulation by Goldhirsch and Zanetti (1993), we clearly see the tendency of inelastic collisions to produce particle clustering. Here a

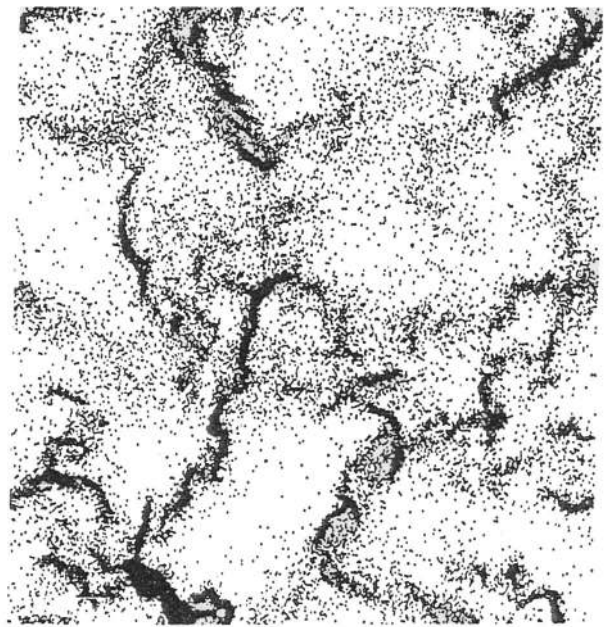


FIG. 9. A typical configuration of 40 000 inelastically colliding particles, exhibiting clustering in two dimensions. Here the coefficient of restitution is 0.6, the time corresponds to 500 collisions per particle, and the average area fraction occupied by particles is 0.05. The system is started with a spatially uniform distribution of initial velocities and allowed to evolve without further energy input, after Goldhirsch and Zanetti (1993).

system of hard discs was started with random initial velocities, in the absence of gravity, and without external forcing.

A particularly exciting development has been the recent recognition that a special type of clustering may occur, called "inelastic collapse." McNamara and Young (1994) showed that inelasticity can lead to an infinite number of collisions in a finite time. (Such a situation of an infinite number of collisions in a finite time occurs whenever a ball bouncing on the ground comes to rest. However, in this case there is an attractive force of gravity pulling the ball back to Earth. In the case of inelastic collapse, there is no attractive force between particles to cause particle collisions. In this case, it is the many-particle dynamics that induces the infinite number of collisions.) In one dimension, such a collision sequence leaves the particles "stuck" together in close contact with no relative motion (McNamara and Young, 1994; Constantin *et al.*, 1995; Du *et al.*, 1995; Grossman and Mungan, 1996). Remarkably, "inelastic collapse" also persists in higher dimensions, where it produces dense chainlike clusters, as shown in Fig. 10. Apparently, even in dimensions higher than one, there is a finite region of phase space where such inelastic collapse can occur (McNamara and Young, 1996; Schörghofer and Zhou, 1996; Zhou and Kadanoff, 1996). The precise relationship between "inelastic collapse" and the phenomenon of clustering, which is the initial signature for a breakdown of ordinary hydrodynamics, needs clarification. One plausible scenario is that once the system forms clusters, the occurrence of inelastic collapse in addition requires that

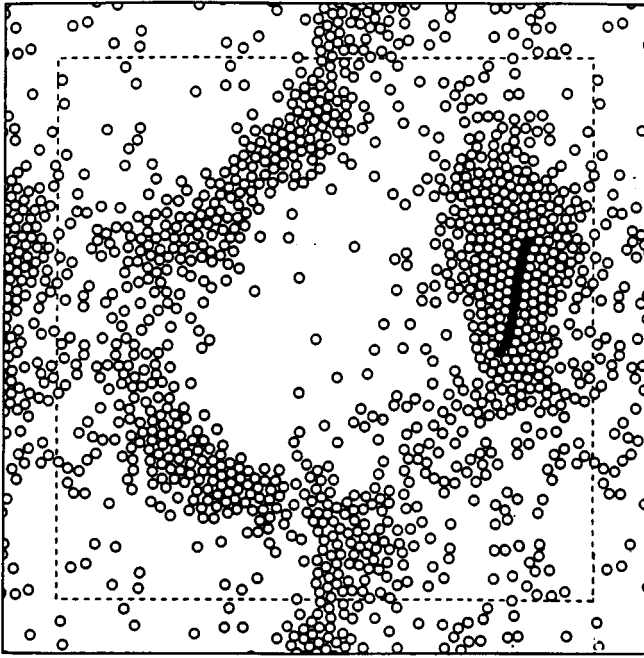


FIG. 10. A two-dimensional simulation of hard disks colliding inelastically in a container with periodic boundary conditions. The line of particles that are solid circles consists of those that have undergone “inelastic collapse.” These particles have, within the capacity of the computer simulation, undergone an infinite number of collisions, after McNamara and Young (1994).

the energy loss per collision must exceed a critical value (Esipov and Pöschel, 1995). Yet a different scenario is also conceivable in which all clusters are transients, eventually terminating in either inelastic collapse or in the formation of shear bands, i.e., regions of locally high shear separating essentially static material (McNamara and Young, 1994 and 1996).

Perhaps the most remarkable aspect of clustering, which is also true for the regime of inelastic collapse, is that it leads to long stringlike grain configurations rather than to a shapeless blob of particles. As noted by Goldhirsch and Zanetti (1993), Fig. 9 in this sense resembles qualitative density maps of the visible universe. We speculate here that the attractive gravitational potential plays the role of a confining container, keeping the density high enough for clusters to form. As in simulations of liquids, the most important part of the interparticle interaction is the strong repulsive one, and the small attractive part can often be discarded if a confining container serves to set the density. Hence on very large scales, the structures created by repeated inelastic collisions may also be responsible for the coagulation observed in “gases” made up of planets. Thus we get back to the quotation from Victor Hugo at the beginning of the article: The motion of grains of sand may indeed be relevant to the creation not just of worlds but of galaxies and the structure and formation of our astronomical landscape.

## V. CONCLUSIONS AND OUTLOOK

We have only been able to touch on some of the distinctive properties of granular materials. In the preceding sections we have shown that these materials act as highly unusual solids, liquids, and gases, depending on how we prepare and excite them. What should be clear is that the physics of granular materials spans a wide variety of phenomena with many possible applications, ranging from the mundane to the celestial. The experimental techniques used to study these systems likewise span a wide range of approaches and sophistication—from the examination of spots left by carbon paper to high speed videography, magnetic resonance imaging, and x-ray tomography. Despite their apparent simplicity these materials display an intriguing range of nonlinear complex behavior, whose unraveling more often than not appears to challenge existing physics wisdom. Many of the new ideas developed within the specific context of granular materials are applicable to a wider range of metastable systems whose thermal energy  $kT$  is irrelevant. Such systems include foams or the tunneling regime for superconducting vortex arrangements. This emerging new field of research within physics raises many pressing, and often controversial, issues that need to be addressed.

From our personal perspective, we clearly see a challenge for new insights from physics to make a strong, technologically relevant impact, not achieved thus far, in spite of the tremendous opportunities and a recognized deficiency in our understanding of real granular materials. In addition, we see a multitude of scientific challenges. For example, for the case of packing, we do not know to what extent the packing history is relevant and, if relevant, how to include it in theories of compaction or stress patterns within the medium. Likewise, when attempting a hydrodynamic approach to granular flow, we are still at a loss as to how to treat the boundaries correctly, while it is obvious that the ordinary hydrodynamic nonslip boundary assumptions are invalid. Of course, it remains to be seen to what extent Newtonian hydrodynamics needs (or can) be modified to describe granular media correctly. Certainly amongst the authors of the present article there is debate over this issue. The debate extends to a related question, namely, whether the idea of inelastic collapse is more than a beautiful theoretical concept or whether it has real experimental ramifications, and whether the difference between inelastic collapse and the more general case of inelastic clustering is experimentally observable.

Our hope is that the recent surge of interest in the basic physics of granular media will produce advances that can then not only lead to improved applications for technological processes but also deepen our understanding of the many related aspects of microscopic and macroscopic physics for which sand has been used as a metaphor.

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