

Graph comparison and its application in network synchronization*

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Abstract—Using the tool of graph comparison from spectral graph theory, we propose new methodologies to guarantee complete synchronization in complex networks. The main idea is to utilize flexibly topological features of a given network so that the eigenvalues of the Laplacian matrix associated with the network can be estimated. The proposed methodologies enable the construction of different coupling-strength combinations in response to different knowledge about sub-networks. The obtained bounds of the network graphs' eigenvalues can be further used to study the robust synchronization problem in face of link failures in networks. Examples are utilized to demonstrate how to apply the methodologies to networks.

I. INTRODUCTION

In the past few decades, synchronization phenomena in various complex networks have attracted much attention [11], [18], [17], [10]. A general approach, called the master stability function method, has been developed in [11] to study the local synchronization problem for linearly coupled chaotic systems. A systematic framework for the study of synchronization of nonlinear dynamical systems with diffusive couplings has been developed in [17]. One key assumption made in quite a number of global synchronization results is that in the network, the synchronized behavior of any two systems is always possible to take place provided that the coupling between the two is sufficiently strong [17]. For networks whose links between nodes may fail to function during the evolution of the networks, researchers have also looked into the robust synchronization problem, for which some robustness metrics have been defined that are functions of all the eigenvalues of the Laplacian matrices of the networks under study [20], [1].

Recently a new general method to study the global synchronization has been proposed in [2], which uses extensively the topological information of the graph that describes the couplings between the systems in a network. A key step is to construct a bound on the total length of all the paths passing through a chosen edge in the graph. This bound can then be exploited to allocate coupling strengths to all the edges in order to achieve global synchronization in the network. As their conditions for network synchronization have been obtained mainly through graph theoretic deduction, we pursue in this paper further along the same line by using recently

reported tools in graph theory, especially the idea of graph comparison from spectral graph theory [14].

In [5], [6], [7], it has been shown that graph embedding is useful in bounding the second smallest eigenvalues of Laplacian matrices. To be more specific, the bounds are obtained by embedding complete graphs into the graph under study. Similar ideas for graph comparison have been reported in [12], [14], where the comparison of combinatorial features can be carried out between two arbitrary graphs with the same vertex set for the purpose of bounding any eigenvalues of Laplacian matrices of the graphs. In this paper, we follow the approach delineated in [12], [14] to study conditions based on graphs for the synchronization in complex networks. By doing so, we prove that the synchronization condition given in [2] for allocating coupling strengths can be explained by comparing the network graph with the complete graph with the same number of vertices. We then look further into the whole spectrums of all the eigenvalues of the networks' Laplacian matrices and thus try to gain new insight into the robust synchronization problem. So the main contribution of the paper lies in a set of new methodologies using graph comparison to study different synchronization problems in complex networks.

The rest of the paper is organized as follows. In Section II, we review some relevant results in the literature on synchronization. Such results are interpreted using concepts from spectral graph theory in Section III. We then in Section IV look into the robust synchronization problem of networks.

II. REVIEW OF A SUFFICIENT CONDITION FOR COMPLETE SYNCHRONIZATION

We consider a network of $n > 1$ coupled identical oscillators whose dynamics are described by

$$\dot{x}_i = f(x_i) + \sum_{j=1}^n \varepsilon_{ij} P x_j, \quad i = 1, \dots, n, \quad (1)$$

where $x_i \in \mathbb{R}^d$ is the state of the i th oscillator, $f(\cdot) : \mathbb{R}^d \rightarrow \mathbb{R}^d$ denotes the identical self-dynamics of each oscillator and is a C^1 function, $\varepsilon_{ij} \geq 0$ describes the strength of the coupling from oscillator j to i , and the diagonal $(0, 1)$ -matrix $P \in \mathbb{R}^{d \times d}$ determines through which components of the states that the oscillators are coupled together. The couplings between the oscillators can be conveniently described by a graph \mathbb{G} with n vertices in which there is an edge from vertex j to i if $\varepsilon_{ij} > 0$. We assume in this paper that the couplings between oscillators are symmetric, namely $\varepsilon_{ij} = \varepsilon_{ji}$ for all $1 \leq i, j \leq n$. We use $\mathcal{E}(\mathbb{G})$ to denote the set of all the edges of \mathbb{G} and assume that there are altogether m edges.

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System (1) has been used widely to study under what conditions the coupled oscillators can achieve asymptotically global and complete synchronization, where for any initial condition, $\|x_i(t) - x_j(t)\| \rightarrow 0$ as $t \rightarrow \infty$ for all i, j [17]. And the synchronization manifold is defined as the linear subspace $\{x : x_i = x_j, \forall i, j\}$. Now we review some useful such conditions. Towards this end, we make one standard technical assumption about system (1).

Assumption 1: For any $x_i \neq x_j$ and sufficiently large positive constant a , it holds that

$$(x_j - x_i)^T \left[\int_0^1 Df(\beta x_j + (1 - \beta)x_i) d\beta - aP \right] (x_j - x_i) < 0,$$

where Df denotes the $d \times d$ Jacobian matrix of f .

Assumption 1 implies that any two coupled oscillators are always able to synchronize when their coupling is sufficiently strong. Here the constant a is determined by both the function f and the projection matrix P .

With this assumption at hand, we can then present some sufficient conditions for synchronization using features of graph \mathbb{G} . Consider a set of paths $\mathcal{P} = \{\mathcal{P}_{ij} | i, j = 1, \dots, n, i \neq j\}$, one for each pair of distinct vertices i and j . We denote the length of the path \mathcal{P}_{ij} by $|\mathcal{P}_{ij}|$, which is the number of edges in \mathcal{P}_{ij} . One of the main results in [2] is as follows.

Theorem 1: [2] Under Assumption 1, the synchronization manifold of system (1) is globally asymptotically stable if

$$\sum_{(i,j) \in \mathcal{E}(\mathbb{G})} \varepsilon_{ij} (x_{i_k} - x_{j_k})^2 > \frac{a}{n} \sum_{i=1}^{n-1} \sum_{j>i}^n (x_{i_k} - x_{j_k})^2$$

for $1 \leq k \leq d$. Here x_{i_k} is the k th component of x_i .

If we label the edges of \mathbb{G} by $1, \dots, m$, then the lower bounds on the coupling strengths of all the edges can be constructed to guarantee that the inequalities in Theorem 1 hold.

Theorem 2: [2] Under the assumptions in Theorem 1, the synchronization manifold of system (1) is globally asymptotically stable if

$$\varepsilon_k > \frac{b_k}{n} a, \quad \text{for } k = 1, \dots, m,$$

where $b_k = \sum_{j>i; k \in \mathcal{P}_{ij}} |\mathcal{P}_{ij}|$ is the sum of the lengths of all those paths \mathcal{P}_{ij} in \mathcal{P} that contain the edge k .

Theorem 2 presents a synchronization condition for allocating coupling strengths that can be checked using combinatorial features of \mathbb{G} , which is appealing as this greatly simplifies computations. This motivates us to further study this condition by introducing more advanced tools in spectral graph theory.

III. GRAPHICAL SYNCHRONIZATION CRITERIA AND GRAPH COMPARISON

In this section, we first explore several graphical synchronization criteria using tools in spectral graph theory, and then rephrase the results reviewed in the previous section. Towards this end, we introduce some notations and discuss

some algebraic properties of graphs. For a symmetric matrix A , by $A \succ 0$ we mean that A is positive definite. And we say $A \succ B$ if $A - B \succ 0$. Similarly, we say $A \succeq B$ if $A - B$ is positive semi-definite. We extend this notation for graphs as follows.

Definition 1: For two undirected graphs \mathbb{G} and \mathbb{H} with the same vertex set $\mathcal{V} = \{1, \dots, n\}$, we say

$$\mathbb{G} \succeq \mathbb{H}$$

if their Laplacian matrices [4] satisfy $L_{\mathbb{G}} \succeq L_{\mathbb{H}}$.

For a graph \mathbb{G} with vertex set \mathcal{V} , we use λ_k , $1 \leq k \leq n$, to denote the k th smallest eigenvalue of $L_{\mathbb{G}}$. For graphs \mathbb{G} and \mathbb{H} with the same vertex set, we consider some multiple $c_{\mathbb{G}}$ of graph \mathbb{G} . Using Courant-Fischer Theorem [14], one can easily prove the following result.

Lemma 1: If \mathbb{G} and \mathbb{H} are the graphs with the same vertex set \mathcal{V} satisfying $c_{\mathbb{G}} \succeq \mathbb{H}$, then $c\lambda_k(\mathbb{G}) \geq \lambda_k(\mathbb{H})$ for all $1 \leq k \leq n$.

In the following, we give a general synchronization criterion under Assumption 1.

Theorem 3: Under Assumption 1, the synchronization manifold of system (1) is globally asymptotically stable if there exists a connected undirected graph \mathbb{G}_0 with the same vertex set of \mathbb{G} such that

$$L_{\mathbb{G}_0} L_{\mathbb{G}} \succ a L_{\mathbb{G}_0}. \quad (2)$$

Proof: Define $x \triangleq [x_1^T, \dots, x_n^T]^T \in \mathbb{R}^{nd}$ and $F(x) \triangleq [f^T(x_1), \dots, f^T(x_n)]^T$. Then system (1) can be written in the compact form

$$\dot{x} = F(x) - (L_{\mathbb{G}} \otimes P)x, \quad (3)$$

where \otimes denotes the Kronecker product [3]. We first prove that it always holds that $x^T (L_{\mathbb{G}_0} \otimes I_d) F(x) \leq ax^T (L_{\mathbb{G}_0} \otimes P)x$.

Let the matrix $B_0 \in \mathbb{R}^{n \times m_0}$ denote the incidence matrix of graph \mathbb{G}_0 , and it is well known that $B_0 B_0^T = L_{\mathbb{G}_0}$ [4]. We label the m_0 edges of graph \mathbb{G}_0 by $e_i = (h_i, t_i)$, $i = 1, \dots, m_0$, where according to the edge orientations defined by B_0 , we call h_i, t_i the head and the tail of e_i respectively. Let \bar{x} denote the vector of pair-wise differences of the states of neighboring oscillators, namely $\bar{x} \triangleq [\bar{x}_{e_1}^T, \dots, \bar{x}_{e_{m_0}}^T]^T$, where $\bar{x}_{e_i} = x_{h_i} - x_{t_i}$ for $i = 1, \dots, m_0$. Then it is easy to check that $\bar{x} = (B_0^T \otimes I_d)x$. From the inequality (13) in the Appendix, we have

$$\begin{aligned} & x^T (L_{\mathbb{G}_0} \otimes I_d) F(x) \\ &= \left((B_0^T \otimes I_d)x \right)^T \left((B_0^T \otimes I_d)F(x) \right) \\ &\leq \left((B_0^T \otimes I_n)x \right)^T (I_m \otimes aP) \left((B_0^T \otimes I_d)x \right) \\ &= ax^T (B_0 B_0^T \otimes P)x \\ &= ax^T (L_{\mathbb{G}_0} \otimes P)x. \end{aligned} \quad (4)$$

We now construct the candidate Lyapunov function for system (3)

$$V = \frac{1}{2} x^T (L_{\mathbb{G}_0} \otimes I_d)x,$$

Then its derivative along the trajectory of system (3) is

$$\begin{aligned}
\frac{dV}{dt} &= x^T (L_{\mathbb{G}_0} \otimes I_d) \left(F(x) - (L_{\mathbb{G}} \otimes P)x \right) \\
&= x^T (L_{\mathbb{G}_0} \otimes I_d) F(x) - x^T (L_{\mathbb{G}_0} L_{\mathbb{G}} \otimes P)x \\
&\leq ax^T (L_{\mathbb{G}_0} \otimes P)x - x^T (L_{\mathbb{G}_0} L_{\mathbb{G}} \otimes P)x \\
&= x^T \left((aL_{\mathbb{G}_0} - L_{\mathbb{G}_0} L_{\mathbb{G}}) \otimes P \right) x.
\end{aligned} \tag{5}$$

Since $L_{\mathbb{G}_0} L_{\mathbb{G}} \succ aL_{\mathbb{G}_0}$, one has $\frac{dV}{dt} \leq 0$. Since $\frac{dV}{dt} = 0$ if and only if $x_i = x_j$ for all $i, j = \{1, \dots, n\}$, we know that the solutions of system (1) asymptotically converge to the synchronization manifold $\{x : x_i = x_j, \forall i, j\}$. \square

Theorem 3 gives a synchronization condition based on graph comparison. One natural idea is then to compare the system graph \mathbb{G} with the complete graph [4]. Let \mathbb{K}_n denote the unweighted, undirected complete graph with n vertices. If we take the graph \mathbb{G}_0 to be \mathbb{K}_n , then one has that the synchronization manifold of system (1) is globally asymptotically stable if $L_{\mathbb{K}_n} L_{\mathbb{G}} \succ aL_{\mathbb{K}_n}$. Note that $L_{\mathbb{K}_n} = nI_n - J$ where J is the n -by- n all-one matrix. So we know $L_{\mathbb{K}_n} L_{\mathbb{G}} = (nI_n - J)L_{\mathbb{G}} = nL_{\mathbb{G}} \succ aL_{\mathbb{K}_n}$. Thus, the synchronization manifold of system (1) is globally asymptotically stable if $L_{\mathbb{G}} \succ \frac{a}{n} L_{\mathbb{K}_n}$. We summarize the analysis in the following theorem.

Theorem 4: Under Assumption 1, the synchronization manifold of system (1) is globally asymptotically stable if

$$\mathbb{G} \succ \frac{a}{n} \mathbb{K}_n. \tag{6}$$

In fact, Theorem 4 and Theorem 1 are one and the same. On one hand, from $\sum_{(i,j) \in \mathcal{E}(\mathbb{G})} \varepsilon_{ij} (x_i - x_j)^2 > \frac{a}{n} \sum_{i=1}^{n-1} \sum_{j>i} (x_i - x_j)^2$, we have $x^T L_{\mathbb{G}} x > \frac{a}{n} x^T L_{\mathbb{K}_n} x$ for all $x = (x_1, x_2, \dots, x_n)^T \in \mathbb{R}^n$. Then it must be true that $x^T (L_{\mathbb{G}} - \frac{a}{n} L_{\mathbb{K}_n}) x > 0$ for all $x \in \mathbb{R}^n$. It follows that $\mathbb{G} \succ \frac{a}{n} \mathbb{K}_n$, namely Theorem 1 implies Theorem 4. On the other, one can easily check that the above argument also holds in the reverse direction.

The implication of Theorem 4 is profound. For any coupled oscillators whose couplings are described by a weighted undirected graph \mathbb{G} , one can always examine whether $\mathbb{G} \succ \frac{a}{n} \mathbb{K}_n$ holds by comparing \mathbb{G} to the complete graph with identical edge weight $\frac{a}{n}$. Now we show that the inequality in Theorem 4 can be stated differently.

Theorem 5: It holds that

$$\mathbb{G} \succ \frac{a}{n} \mathbb{K}_n \Leftrightarrow \lambda_2(\mathbb{G}) > a.$$

Proof: “ \Rightarrow ”: From $\mathbb{G} \succ \frac{a}{n} \mathbb{K}_n$ and Lemma 1, we know $\lambda_2(\mathbb{G}) > \frac{a}{n} \lambda_2(\mathbb{K}_n)$. Since $\lambda_2(\mathbb{K}_n) = n$, it then must be true that $\lambda_2(\mathbb{G}) > a$.

“ \Leftarrow ”: Since the all-one vector $\mathbf{1} = [1, \dots, 1]^T$ is in the kernel of $L_{\mathbb{G}}$ and $L_{\mathbb{K}_n}$. To prove $\mathbb{G} \succ \frac{a}{n} \mathbb{K}_n$, it is equivalent to prove that $x^T (L_{\mathbb{G}} - \frac{a}{n} L_{\mathbb{K}_n}) x > 0$ for any $x \in \mathbb{R}^n$ which is not in the kernel of $L_{\mathbb{G}}$ and $L_{\mathbb{K}_n}$. Furthermore, one can easily see that it suffices to prove that $x^T (L_{\mathbb{G}} - \frac{a}{n} L_{\mathbb{K}_n}) x > 0$ for all the vector $x \in \mathbb{R}^n$ orthogonal to $\mathbf{1}$.

For any vector x orthogonal to $\mathbf{1}$, from Courant-Fischer

theorem [14], one has

$$\lambda_2(\mathbb{G}) \leq \min_{x \perp \mathbf{1}} \frac{x^T \mathbb{G} x}{x^T x}.$$

Thus one has

$$\lambda_2(\mathbb{G}) x^T x \leq x^T L_{\mathbb{G}} x, \quad \forall x \perp \mathbf{1}.$$

Since $\lambda_2(\mathbb{G}) > a$, we know $x^T L_{\mathbb{G}} x > ax^T x$ for all $x \perp \mathbf{1}$. Using the fact that $L_{\mathbb{K}_n} = nI_n - J$, we have

$$\begin{aligned}
x^T (nL_{\mathbb{G}} - aL_{\mathbb{K}_n}) x &= x^T (nL_{\mathbb{G}} - a(nI_n - J)) x \\
&= nx^T L_{\mathbb{G}} x - nax^T x + ax^T Jx \\
&\geq n(x^T L_{\mathbb{G}} x - ax^T x) > 0,
\end{aligned}$$

which implies that $n\mathbb{G} - a\mathbb{K}_n \succ 0$ for all $x \perp \mathbf{1}$, namely $\mathbb{G} \succ \frac{a}{n} \mathbb{K}_n$. \square

Remark 1: In Theorem 3 in [16], a lower bound for $\lambda_2(\mathbb{G}(t))$ has been given to guarantee the synchronization of coupled dynamical oscillators under certain assumptions. In Theorem 5, we have shown the equivalence between graph comparisons and bounding from below the second smallest eigenvalues of the Laplacian matrices of graphs.

Remark 2: The constant a is determined by the oscillators' self-dynamics and the projection matrix P . Hence, if the oscillators' self-dynamics and the projection matrix P are given beforehand, the synchronizability [16], [17] of a coupled dynamical network can be measured by $\lambda_2(\mathbb{G})$.

To apply more tools from spectral graph theory, we need to introduce another equivalent definition of Laplacian matrices of graphs. Following [12], we define the elementary Laplacian $L_{(u,v)}$ to be the Laplacian of the graph with the vertex set \mathcal{V} and only one edge between vertices u and v . Then for an arbitrary graph \mathbb{G} with the edge set \mathcal{E} , its Laplacian matrix can be defined to be

$$L(\mathbb{G}) \triangleq \sum_{(u,v) \in \mathcal{E}} L_{(u,v)}.$$

Two inequalities have been proved in [12] and we list them below.

Lemma 2: [12] Let $c_1, \dots, c_{n-1} > 0$. Then for graph \mathbb{G} with vertex set \mathcal{E} , one has

$$c \left(\sum_{i=1}^{n-1} c_i L_{(i,i+1)} \right) \succeq L_{(1,n)},$$

where $c = \sum_{i=1}^{n-1} \frac{1}{c_i}$.

If we take $c_1 = c_2 = \dots = c_{n-1} = 1$, then Lemma 2 becomes the following result.

Lemma 3: [12] For graph \mathbb{G} with vertex set \mathcal{E} , it holds that

$$(n-1) \left(\sum_{i=1}^{n-1} L_{(i,i+1)} \right) \succeq L_{(1,n)}.$$

Theorem 4 can be further used to give graphical conditions for the synchronization of system (1). Theorem 2 can be rephrased using very intuitive graphical ideas. By doing so,

one can easily derive the result of Theorem 2. *A different proof of Theorem 2:* It holds that

$$\begin{aligned}
\frac{a}{n} L_{\mathbb{K}_n} &= \frac{a}{n} \sum_{i=1}^{n-1} \sum_{j>i} L_{(i,j)} \\
&\succeq \frac{a}{n} \sum_{i=1}^{n-1} \sum_{j>i} \left(|\mathcal{P}_{ij}| \sum_{\substack{k \in \mathcal{P}_{ij} \\ k \in \mathcal{E}(\mathbb{G})}} L_k \right) \\
&\quad \text{(from Lemma 3)} \\
&= \frac{a}{n} \sum_{k=1}^m \left(\sum_{\substack{i=1, j>i \\ k \in \mathcal{P}_{ij}}}^{n-1} |\mathcal{P}_{ij}| \right) L_k \\
&= \frac{a}{n} \sum_{k=1}^m b_k L_k \\
&\prec \sum_{k=1}^m \varepsilon_k L_k = L_{\mathbb{G}},
\end{aligned}$$

where $b_k = \sum_{\substack{i=1, j>i \\ k \in \mathcal{P}_{ij}}}^{n-1} |\mathcal{P}_{ij}|$ has been defined in Theorem 2. And the last inequality holds trivially when $\varepsilon_k > \frac{a}{n} b_k$ for each edge k . Therefore, we arrive at the conclusion in Theorem 2. \square

Remark 3: The relations between different synchronization criteria in this section can be summarized as follows:

- a) Theorem 1 \Leftrightarrow Theorem 4 \Leftrightarrow Theorem 5;
- b) Theorem 2 \Rightarrow Theorem 4.

Now we use a simple example to explain how to use graph comparison to allocate coupling strengths in a connected dynamical network to guarantee its synchronization.

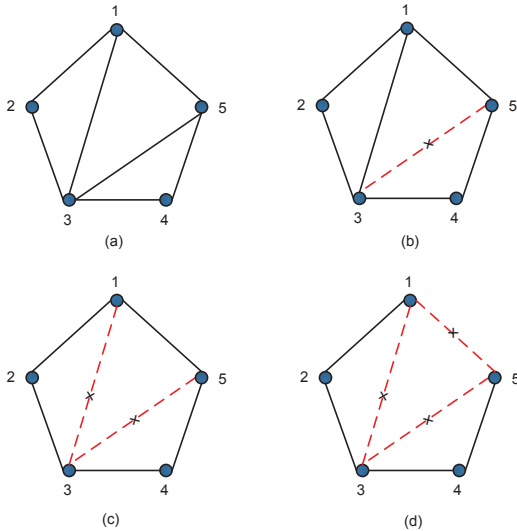


Fig. 1. Network topologies with or without link failures.

We take the graph $\mathbb{G}_5^{(a)}$ shown in Fig. 1(a) as an example. Now we compare the graph $\mathbb{G}_5^{(a)}$ with the complete graph \mathbb{K}_5 . We use edge (1,2) in $\mathbb{G}_5^{(a)}$ to represent the path connecting vertices v_1, v_2 , edge (1,3) for vertices v_1, v_3 ,

edge (1,5) for vertices v_1, v_5 , edge (2,3) for vertices v_2, v_3 , edge (3,4) for vertices v_3, v_4 , edge (3,5) for vertices v_3, v_5 , edge (4,5) for vertices v_4, v_5 , path (1,3,4) for vertices v_1, v_4 , path (2,3,4) for vertices v_2, v_4 , and path (2,3,5) for vertices v_2, v_5 . Then we have

$$\begin{aligned}
L_{\mathbb{K}_5} &= L_{(1,2)} + L_{(1,3)} + L_{(1,4)} + L_{(1,5)} \\
&\quad + L_{(2,3)} + L_{(2,4)} + L_{(2,5)} + L_{(3,4)} + L_{(3,5)} + L_{(4,5)} \\
&\preceq L_{(1,2)} + L_{(1,3)} + [2L_{(1,3)} + 2L_{(3,4)}] + L_{(1,5)} \\
&\quad + L_{(2,3)} + [2L_{(2,3)} + 2L_{(3,4)}] + [2L_{(2,3)} + 2L_{(3,5)}] \\
&\quad + L_{(3,4)} + L_{(3,5)} + L_{(4,5)} \\
&= L_{(1,2)} + 5L_{(2,3)} + 5L_{(3,4)} + L_{(4,5)} + L_{(1,5)} \\
&\quad + 3L_{(1,3)} + 3L_{(3,5)}
\end{aligned}$$

where the inequality sign holds because of Lemma 3. Thus we have $b_{(1,2)} = 1$, $b_{(2,3)} = 5$, $b_{(3,4)} = 5$, $b_{(4,5)} = 1$, $b_{(1,5)} = 1$, $b_{(1,3)} = 3$, $b_{(3,5)} = 3$. Let $\varepsilon_{(i,j)}^{(a)}$ be the coupling strength of edge (i, j) in graph $\mathbb{G}_5^{(a)}$. From Theorem 2, one of the possible sets of bounds in graph $\mathbb{G}_5^{(a)}$ is as follows: $\varepsilon_{(1,2)}^{(a)} = \varepsilon_{(4,5)}^{(a)} = \varepsilon_{(1,5)}^{(a)} \geq \frac{a}{5} = 0.2a$, $\varepsilon_{(1,3)}^{(a)} = \varepsilon_{(3,5)}^{(a)} \geq \frac{3a}{5} = 0.6a$, $\varepsilon_{(2,3)}^{(a)} = \varepsilon_{(3,4)}^{(a)} \geq \frac{5a}{5} = a$.

Remark 4: We have chosen a set of paths in the above computation, and one can in fact arbitrarily choose paths in graph \mathbb{G} connecting distinct pairs of vertices. However, in this example we have chosen the shortest paths between vertices for simplicity. Optimal or suboptimal path selection strategies are of great interest in our future work.

IV. ROBUST SYNCHRONIZATION AGAINST LINK FAILURES

Along the line of graph comparison, we further explore its applications in adjusting adaptively coupling strengths in subnetworks in order to make the overall network robust against link failures in synchronization processes. Robust synchronization against network attacks has been extensively studied in the literature [15], [9], [8]. However, most of the existing work requires the knowledge about the whole network topologies or vertex degree distributions. Few result identifies distributed ways using only local information about subnetworks to synchronize the overall networks under link failures. Thus, it is meaningful to explore whether there is an easy way to keep the network synchronizable by locally adjusting the coupling strengths between oscillators. We show in the sequel that graph comparison is especially useful to find a solution to this challenge problem.

Consider the network shown in Fig. 1(b). Suppose that edge (3,5) is broken and is denoted by a dotted line in Fig. 1(b). We choose the path (3,4,5) for vertices v_3, v_5 . From Lemma 3, one has

$$2L_{(3,4)} + 2L_{(4,5)} \succeq L_{(3,5)}. \quad (7)$$

Thus, in this case we only need to increase the coupling strength assigned to the edges (3,4) and (4,5) by the amount that is equal to or greater than $2\varepsilon_{(3,5)}^{(a)}$. It follows that

$$\varepsilon_{(3,4)}^{(b)} \geq \varepsilon_{(3,4)}^{(a)} + 2\varepsilon_{(3,5)}^{(a)}, \quad \varepsilon_{(4,5)}^{(b)} \geq \varepsilon_{(4,5)}^{(a)} + 2\varepsilon_{(3,5)}^{(a)},$$

and the other coupling strengths $\varepsilon_{(1,2)}^{(b)}, \varepsilon_{(2,3)}^{(b)}, \varepsilon_{(1,5)}^{(b)}, \varepsilon_{(1,3)}^{(b)}$ are just kept the same as those in network $\mathbb{G}_5^{(a)}$. This local adjustment strategy guarantees the following relationship for the two weighted graphs

$$\mathbb{G}_5^{(b)} \succeq \mathbb{G}_5^{(a)}. \quad (8)$$

Using the same procedure, one can locally adjust the coupling strengths in networks $\mathbb{G}_5^{(c)}$ and $\mathbb{G}_5^{(d)}$ to guarantee synchronizability against link failures. From network $\mathbb{G}_5^{(a)}$ to network $\mathbb{G}_5^{(d)}$, the coupling strength adaptation can be done either edge by edge or by directly comparing $\mathbb{G}_5^{(d)}$ with $\mathbb{G}_5^{(a)}$.

Remark 5: The main idea of this coupling strength adaptation is to find another path at a local level in the network to replace the edge corresponding to the failed link. And Lemma 3 is used to determine the changing coupling strengths for the edges on the chosen path. There may be multiple choices for the path with the same two ending vertices of the failed link. However we prefer to choose the shortest path(s) in calculations. This is because a shorter path usually corresponds to smaller changes in the coupling strengths for the edges in the path.

Recently, in [20], [1] robustness metrics in terms of network topologies have been discussed for the synchronization of networked systems. Let $L_{\mathbb{G}}$ be the Laplacian matrix of an undirected connected graph with eigenvalues $0 = \lambda_1 < \lambda_2 \leq \dots \leq \lambda_n$. Then the robustness of the associated networks for synchronization is evaluated by

$$H = \left(\frac{K_f}{2N} \right)^{\frac{1}{2}}, \quad (9)$$

in [20] using the Kirchhoff index [19]

$$K_f = N \sum_{j=2}^n \frac{1}{\lambda_j}. \quad (10)$$

It has been shown that networks with better synchronizability have larger values of λ_2 , while networks with better synchronization robustness have smaller values of K_f or H . However, the structural robustness discussed in [1] deals with the effect of changes in network topologies due to link failures, while functional robustness discussed in [20] addresses how well a system behaves in the presence of noise. In fact, both of the two aspects are inter-related since the relation in (9) holds. These measurements on robustness of networks against link failures and noise all remind us to pay more attention to the eigenvalue spectra of a Laplacian matrices, not just the second-smallest eigenvalue λ_2 .

From Lemma 1 and (8), one has that the eigenvalues of the two weighted graphs satisfy

$$\lambda_k(\mathbb{G}_5^{(b)}) \geq \lambda_k(\mathbb{G}_5^{(a)}),$$

for $k = 1, \dots, 5$. Thus, under our proposed adaptation strategies for coupling-strength allocation, a network performs better synchronizability and better robustness for synchronization, compared with the original network.

Finally, we discuss more general link changes in a network. The results just discussed only consider link failures.

In fact, one can consider the scenarios of a combination of rewiring, deleting or adding links that may take place at the same time in the underlying network. The following lemma shows that graph comparison can be carried out between any two different graphs.

Lemma 4: [13] Let \mathbb{G} and H be undirected connected graphs. Let σ_1 be the least number such that

$$\mathbb{G} \preceq \sigma_1 \mathbb{H}, \quad (11)$$

and let σ_2 be the greatest number such that

$$\sigma_2 \mathbb{H} \preceq \mathbb{G}. \quad (12)$$

Then the condition number $k_f(\mathbb{G}, \mathbb{H}) = \sigma_1/\sigma_2$. Furthermore, if $c_2 \mathbb{H} \preceq \mathbb{G} \preceq c_1 \mathbb{H}$, then $k_f(\mathbb{G}, \mathbb{H}) \leq c_1/c_2$.

This lemma implies that graph comparison can be carried out between arbitrary two different graphs having the same vertex set. Thus coupling strength adjustment still works when a network suffers from a combination of different topological changes. We will look at this problem more carefully in our future work.

V. CONCLUSIONS

In this paper we have presented new ways to allocate coupling strengths using spectral graph theory in order to achieve synchronization in complex networks. The main idea is to bound the second-smallest eigenvalues of Laplacian matrices associated with the given networks by comparing the corresponding network graphs to complete or other graphs with the same vertex sets. The obtained results can simplify the computation and be applied to growing networks. We have also looked at the robustness issues in network synchronization by carrying out graph comparison for bounding the all the eigenvalues of the Laplacian matrices of graphs under comparison.

We are interested in looking into applying the proposed methodologies to networks with directed topologies. The main challenge is then how to deal with the fact that the Laplacian matrices associated with directed graphs are not guaranteed to be positive semi-definite anymore. We are also interested in using the constructed synchronization criteria to develop optimal or sub-optimal solutions to add or delete edges in a network to achieve better synchronizability.

APPENDIX

In this appendix, we show that Assumption 1 implies

$$(x_j - x_i)^T [(f(x_j) - f(x_i)) - aP(x_j - x_i)] < 0, \quad (13)$$

for any $x_i \neq x_j$.

Note that

$$\begin{aligned} f(x_j) - f(x_i) &= \int_0^1 \frac{d}{d\beta} f(\beta x_j + (1 - \beta)x_i) d\beta \\ &= \left[\int_0^1 Df(\beta x_j + (1 - \beta)x_i) d\beta \right] (x_j - x_i), \end{aligned}$$

where Df is the $n \times n$ Jacobian matrix of f . This equation is given in [2]. Then we have

$$(x_j - x_i)^T \left[\int_0^1 Df(\beta x_j + (1 - \beta)x_i) d\beta - aP \right] (x_j - x_i) \\ = (x_j - x_i)^T [(f(x_j) - f(x_i)) - aP(x_j - x_i)].$$

Therefore, Assumption 1 implies the inequality (13).

The inequality (13) guarantees that the individual system $\dot{x}_i = f(x_i)$ can be globally stabilized by the linear state feedback $-aPx_i$. Equivalently, one can check the unique asymptotic behavior of system $\dot{x}_i = f(x_i) - aPx_i$ from the inequality (13). To do this, by using $\frac{1}{2}(x_j - x_i)^T(x_j - x_i)$ as the quadratic Lyapunov function, one can show that for any x_i, x_j with different initial conditions satisfying $\dot{y} = f(y) - aPy$, the two system trajectories coincide asymptotically.

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