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Graph Decompositions without Isolates

NATHAN LINIAL*

University of California, Los Angeles, California 90024

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A. Frank (Problem session of the Fifth British Combinatorial Conference, Aberdeen, Scotland, 1975) conjectured that if $G=(V,E)$ is a connected graph with all valencies $\geqslant k$ and $a_1,\ldots,a_k \geqslant 2$ are integers with $\sum a_i = |V|$, then V may be decomposed into subsets $A_1,\ldots,A_k$ so that $|A_i| = a_i$ and the subgraph spanned by A_i in G has no isolated vertices ($i=1,\ldots,k$). The case $k=2$ is proved in Maurer (*J. Combin. Theory Ser. B* **27** (1979), 294–319) along with some extensions. The conjecture for $k=3$ and a result stronger than Maurer's extension for $k=2$ are proved. A related characterization of a k-connected graph is also included in the paper, and a proof of the conjecture for the case $a_1 = a_2 = \cdots = a_{k-1} = 2$.

INTRODUCTION

Graph theoretic terminology is standard; see [1, 2] for definitions. A graph $G=(V,E)$ has order $v=|V|$. For $A, B \subseteq V$ we let

$$E(A,B) = \{[x,y] \in E \mid x \in A, y \in B\},$$
$$E(A) = E(A,A),$$
$$e(A,B) = |E(A,B)|, \qquad e(A) = |E(A)|.$$

Also $\langle A\rangle_G = \langle A\rangle$ is the subgraph of G spanned by A. We let $\delta(G)$ be the smallest valence of vertices in G.

In [3] A. Frank made the following:

Conjecture. *Let $G=(V,E)$ be a connected graph, with $\delta(G)\geqslant k$. Let $a_1,\ldots,a_k \geqslant 2$ be integers with $\sum_1^k a_i = v = |V|$. Then V may be decomposed into $A_1,\ldots,A_k$ so that $|A_i| = a_i$ and $\langle A_i\rangle$ has no isolated vertices $(i=1,\ldots,k)$.*

A graph for which the conclusion of the conjecture holds is said to be k-*decomposable*, so the conjecture says that a connected graph with $\delta \geqslant k$ is k-decomposable.

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This problem for $k=2$ was discussed by Maurer [6] who also considers the computational complexity of finding these, and related, decompositions of graphs. Maurer proved Frank's conjecture for $k=2$ and proved some extensions of this case. Among others he proves

THEOREM M [6]. *Let $G=(V,E)$ be a connected graph, $\delta(G)\geqslant 2$. Let $a_1, a_2 \geqslant 2$ be integers with $a_1+a_2=v=|V|$. Then V may be decomposed into A_1, A_2 so that $|A_i|=a_i$ $(i=1,2)$ one of the $\langle A_i\rangle$ is connected and the other one has no isolated vertices.*

The results of this paper are: a proof of the conjecture for $k=3$ (Theorem 2), a theorem which contains Theorem M, a related characterization of k-connected graphs (Theorem 3), and a proof of the conjecture for $a_1=a_2=\cdots=a_{k-1}=2$.

Our first theorem contains Theorem M. To state it we define a *friendship graph* $F_n=(V_n,E_n)$, where $V_n=\{p\}\cup\{x_i \mid n\geqslant i\geqslant 1\}\cup\{y_i \mid n\geqslant i\geqslant 1\}$, $E_n=\{[x_i,y_i] \mid n\geqslant i\geqslant 1\}\cup\{[p,x_i] \mid n\geqslant i\geqslant 1\}\cup\{[p,y_i] \mid n\geqslant i\geqslant 1\}$. In other words $F_n=K_1+nK_2$. Now we state

THEOREM 1. *Let $G=(V,E)$ be a connected graph, $\delta(G)\geqslant 2$, and let $a_1, a_2\geqslant 2$ be integers such that $|V|=v\geqslant a_1+a_2$. Then unless G is a friendship graph and both a_1, a_2 are odd, there exist $A_1, A_2\subseteq V$ so that $A_1\cap A_2=\phi$, $|A_i|=a_i$ $(i=1,2)$, one of $\langle A_i\rangle$ is connected and the other one has no isolated vertices.*

Proof. Note first that if G is a friendship graph and a_1, a_2 are odd then at least one of the $\langle A_i\rangle$ must have an isolated vertex. We need

LEMMA 1. *Let $G=(V,E)$ be a connected graph, $\delta(G)\geqslant 2$, which is not a friendship graph. Then there are two adjacent vertices $x,y\in V$ so that $\delta(G_{xy})\geqslant 2$, where G_{xy} is the graph obtained from G by contracting x, y to a single vertex.*

Proof. Let us consider the vertices of degree $\geqslant 3$. If there are none G must be a circuit and the lemma holds unless $G=K_3$ which is the friendship graph F_1. If there is just one vertex of degree $\geqslant 3$, G is a collection of circuits having exactly one vertex in common. For such graphs the conclusion of the lemma holds except if all circuits are triangles and the graph is a friendship graph.

If there are two adjacent vertices p, q with $d(p)$, $d(q)\geqslant 3$, then either $\delta(G_{pq})\geqslant 2$ and we let $x=p$, $y=q$, or $\delta(G_{pq})=1$. In the latter case there must be a vertex w with $\Gamma(w)=\{p,q\}$. Let $x=p$, $y=w$ to achieve $\delta(G_{xy})\geqslant 2$.

In the remaining case we can assume that there exist vertices p, q with $d(p)$, $d(q)\geqslant 3$ and such vertices must be nonadjacent. Now let $p=x_0, x_1, \ldots,$

$x_n = q$ be a shortest path between them. We claim that for $x = p$, $y = x_1$ we have $\delta(G_{xy}) \geqslant 2$. Otherwise p and x_1 must have a common neighbour, but since $d(x_1) = 2$, $\Gamma(x_1) = \{p, x_2\}$ and if $p, x_2 \in E$, there is a shorter path from p to q.

We go back to prove the theorem by induction on $c = v - (a_1 + a_2)$. The case $c = 0$ is Theorem M above. So assume $c \geqslant 1$ and G is not a friendship graph. Find x, y which satisfy the lemma and consider $G' = G_{xy}$. If G' is a friendship graph then it is easy to check that the theorem holds. If G' is not a friendship graph we may apply induction:

Let p be the vertex in G' which represents $\{x, y\}$. By the induction hypothesis we may find disjoint subsets A'_1, A'_2 of $V \backslash \{x, y\} \cup \{p\}$ so that $|A'_i| = a_i$ $(i = 1, 2)$, one of $\langle A'_i \rangle_{G'}$ is connected and the other one has no isolated vertices. If $p \notin A'_1 \cup A'_2$, let $A_i = A'_i$ $(i = 1, 2)$ and this satisfies the theorem.

Assume, then, that $p \in A'_1$ and let $A''_1 = A'_1 \backslash \{p\} \cup \{x, y\}$, $A_2 = A'_2$ be sets of vertices in G. They fail to satisfy the theorem only in that $|A''_1| = a_1 + 1$. Since p belongs to a component of $\langle A'_1 \rangle_G$ of order $\geqslant 2$, x, y belong to a component of $\langle A''_1 \rangle_G$ of order $\geqslant 3$. Omitting a non-cut vertex of this component A_1 is obtained and the theorem follows. ∎

Let us state and prove now *the main result*. We prove the conjecture for $k = 3$.

THEOREM 2. *Let $G = (V, E)$ be a connected graph with all valencies $\geqslant 3$. Let a_1, a_2, $a_3 \geqslant 2$ be integers with $a_1 + a_2 + a_3 = v = |V|$. Then there is a decomposition A_1, A_2, A_3 of V so that $|A_i| = a_i$ and $\langle A_i \rangle$ has no isolated vertices $(i = 1, 2, 3)$.*

First we need two technical lemmas:

LEMMA M. *Let G be a graph with $\delta(G) \geqslant 2$. If all components of G are of order $\geqslant 5$ then G is 2-decomposable.*

Proof. Contained in [6, Theorem 4.21].

LEMMA 2. *Assume $\delta(G) \geqslant 3$ implies 3-decomposability for connected graphs G of order $\leqslant v$. Then it implies 3-decomposability also for graphs of order $\leqslant v$ having all components of order $\geqslant 6$.*

Proof. By induction on the order of the graph. Let $c_1 \geqslant \cdots \geqslant c_k \geqslant 6$ be the orders of the components of G and let $a_1 \geqslant a_2 \geqslant a_3 \geqslant 2$ satisfy $a_1 + a_2 + a_3 = v$. If $c_k \leqslant a_1 - 2$ we may continue by induction so assume $c_k \geqslant a_1 - 1$ which readily implies $k \leqslant 3$, and since $k > 1$ we have to check only $k = 2, 3$.

Let $k = 3$ first. Of course $c_3 \leqslant a_1$ but in case of equality we may proceed by induction. So we may assume $c_3 = a_1 - 1$. Similarly $c_2 \geqslant a_2 - 1$ and $c_2 \neq a_2$ imply $c_2 = a_2 - 1$ ($c_2 \geqslant a_2 + 1$ is impossible since $a_1 + a_2 + a_3 = c_1 + c_2 + c_3$, $a_1 \geqslant a_2 \geqslant a_3$, $c_1 \geqslant c_2 \geqslant c_3$). Using the same arguments we remain with two cases:

c_1	c_2	c_3	a_1	a_2	a_3
$a+3$	a	a	$a+1$	$a+1$	$a+1$
a	a	a	$a+1$	$a+1$	$a-2$

with $a \geqslant 6$. Each of these can be handled easily and the details are omitted.

For $k = 2$ we find integers q_1, q_2, q_3 with $a_i - 2 \geqslant q_i \geqslant 2$, or $q_i = a_i$ ($i = 1, 2, 3$) and $\sum q_i = c_1$. Then we decompose c_1 with parameters q_1, q_2, q_3 and c_2 with $a_1 - q_1$, $a_2 - q_2$, $a_3 - q_3$. This yields a solution unless $c_1 = 7$, $a_2 = a_3 = 3$, which can be easily handled. ∎

Proof of Theorem 2. First we show that G may be decomposed into nontrivial stars. Namely, we want to find a set of vertices $R = \{r_1, \ldots, r_m\}$ and nonempty subsets $L_1, \ldots, L_m$ of V with $\Gamma(r_i) \supseteq L_i$ $(1 \leqslant i \leqslant m)$ so that R, $L_1, \ldots, L_m$ is a decomposition of V.

Let $R, L_1, \ldots, L_m$ satisfy the above conditions except that $R \cup (\bigcup_1^m L_i) \neq V$ and let $|R \cup (\bigcup_1^m L_i)|$ be largest possible. Since G is connected there is an $x \in V \backslash (R \cup (\bigcup_1^m L_i))$ with a neighbour in $R \cup (\bigcup_1^m L_i)$. By maximality this neighbour cannot be in R. If $[x, y] \in E$, $y \in L_i$ and $|L_i| \geqslant 2$ we let $L_i' = L_i \backslash y$, $r_{m+1} = y$, $L_{m+1} = \{x\}$ contradicting the maximality. If $L_i = \{y\}$, let $r_i' = y$, $L_i' = \{r_i, x\}$ again contradicting maximality.

Define $S_i = \{r_i\} \cup L_i$, $s_i = |S_i|$ $(m \geqslant i \geqslant 1)$ and assume $s_1 \geqslant \cdots \geqslant s_m$. Among all decompositions into starts $(R, L_1, \ldots, L_m)$ choose one with largest m and among those, choose one with $(s_1, \ldots, s_m)$ lexicographically minimal. These assumptions imply

$$\text{if} \quad s_i \geqslant 4, \qquad \text{then} \quad e(L_i) = 0; \tag{1}$$

$$\text{if} \quad s_i + s_j \geqslant 6,\ i \neq j, \qquad \text{then} \quad e(L_i, L_j) = 0. \tag{2}$$

Besides, if $e(L_i, r_j) \neq 0$, then $s_j \geqslant s_i - 1$. We say that S_j *can be reached* from S_i if there is a sequence $i = i_0, \ldots, i_t = j$ $(t \geqslant 0)$ without repetitions such that $e(L_{i_\nu}, r_{i_{\nu+1}}) \neq 0$ $(\nu = 0, \ldots, t-1)$. The last observation extends to

$$\text{if} \quad S_j \text{ can be reached from } S_i, \qquad \text{then} \quad s_j \geqslant s_i - 1.$$

In the following section we assume $s_1 \geqslant 4$. Consider now all stars $S_1, \ldots, S_p$ with $s_i = s_1$ $(p \geqslant i \geqslant 1)$, and let $P = \{m \geqslant i \geqslant 1 \mid S_i$ can be reached from one of $S_1, \ldots, S_p\}$. Let $H = \langle \bigcup_{i \in P} S_i \rangle$. We claim that H is 3-decomposable. Referring to Lemma 2 we note that all components of H have order $\geqslant 6$. Also all valencies in H are $\geqslant 3$, this can fail for a vertex $x \in L_i$ $(i \in P)$ only if x

has a neighbour in $\bigcup_{i \notin P} L_i$ which by (2) is possible only if $s_1 = 4$, $s_i = 3$, and $[x, y] \in E$, $y \in L_j$, $s_j = 2$. But then we can replace a 4, 3, 2 subsequence of $s_1, \ldots, s_m$ by the lexicographically smaller 3, 3, 3. For r_j, $j \in P$, the condition $d_H(r_j) \geqslant 3$ can fail only if $s_j = 3$, but since $s_1 \geqslant 4$, the edge by which S_j was reached from a larger star ensures that indeed $d_H(r_j) \geqslant 3$.

We want to reduce the proof to the case where $P = \{1, \ldots, m\}$. If $P \subsetneqq \{1, \ldots, m\}$ let $t = \max\{s_j \mid j \notin P\}$. We already know that $|P| \geqslant 3$, $s_i \geqslant s_1 - 1$ $(i \in P)$ and so $\sum_{i \in P} s_i \geqslant 3s_1 - 2$.

If $t \leqslant a_1 - 2$ we can replace a_1, a_2, a_3, the parameters for decomposing, by $a_1 - t$, a_2, a_3, and move to the next largest S_j $(j \notin P)$. If this process can be carried out until all stars not in P are used we finally have to 3-decompose H with parameters a_1', a_2', $a_3' \geqslant 2$, which can be done by induction on v. So consider the first case where it fails. Assume, then, $t \geqslant a_1 - 1$ and use $a_1 \geqslant a_2 \geqslant a_3$, $s_1 \geqslant t + 1$, $\sum_{i \in P} s_i \geqslant 3s_1 - 2$ to write

$$
\begin{aligned}
2s_1 + t + 1 \geqslant 3t + 3 \geqslant 3a_1 &\geqslant a_1 + a_2 + a_3 = v \\
&\geqslant t + \sum_{i \in P} s_i \geqslant t + 3s_1 - 2
\end{aligned}
$$

which implies $3 \geqslant s_1$, a contradiction.

This allows us to assume from now on that $s_m \geqslant s_1 - 1$. Moreover we may assume that $\bigcup_1^m L_i$ is an independent set of vertices, if $s_1 \geqslant 4$. If $s_1 \geqslant 5$ this follows immediately from (1), (2). If $s_1 = 4$, (2) reduces the discussion to a case where some $s_i = 3$, $L_i = \{x, y\}$ and $[x, y] \in E$. But since S_i can be reached from a star on 4 vertices we may transfer vertices and transform S_i to a star on 4 vertices which violates (1).

Besides, we are allowed to assume that $e(r_j, \bigcup_1^m L_i) \geqslant 3$ $(m \geqslant j \geqslant 1)$. Again if $s_1 \geqslant 5$ this is clear and if $s_1 = 4$ and $s_j = 3$, r_j has a neighbour in $\bigcup_{i \neq j} L_i$, since S_j can be reached from other stars.

We claim that we may assume $e(R) = 0$. Otherwise start deleting edges from $E(R)$. On deleting such an edge all valencies in G remain $\geqslant$3 but it may possibly disconnect. So assume that one of these edges is a bridge. By Lemma 2 we may assume that at least one of the components of the graph resulting when this edge is deleted is of order $\leqslant$5. This leads to a short list of possible cases

Sizes of Components	$a_1\, a_2\, a_3$
4, 4	3, 3, 2
5, 4	3, 3, 3
5, 5	4, 4, 2
7, 4	5, 3, 3
7, 5	4, 4, 4

Each one of these may be handled separately. So we may assume

$$E = E\left(R, \bigcup_{1}^{m} L_i\right), \qquad d(x) = 3 \quad \left(x \in \bigcup L_i\right). \tag{3}$$

Consider now the graphs $G_i = G \backslash (\{r_i\} \cup \Gamma(r_i))$ $(m \geqslant i \geqslant 1)$. We want to show that each of them contains a vertex of valence $\leqslant 2$. If $\delta(G_i) \geqslant 3$ for some i, we claim that all components of G_i have order $\geqslant 6$. This follows easily, since G is bipartite and has all valencies $\geqslant 3$. This means that Lemma 2 will be applicable. Let $q = d(r_i) + 1$, if $q \leqslant a_1 - 2$, decompose G_i with parameters $a_1 - q$, a_2, a_3. If $q \geqslant a_3$, consider G_i' which is a graph obtained by adding to G_i $q - a_3$ of the vertices in $\Gamma(r_i)$. $\delta(G_i') \geqslant 2$ and by Lemma M may be decomposed with parameters a_1, a_2.

Assume, then, that the remaining possibility holds, where $a_1 = a_2 = a_3 = q + 1$. Let r_j have neighbours in $\Gamma(r_i)$ and let $A_1 = \{r_i, r_j\} \cup \Gamma(r_i)$. In $G \backslash A_1$ all components are of order $\geqslant 5$ and by Lemma M it can be decomposed with parameters a_2, a_3.

We may put the conclusion of the above paragraph in the form

$$\forall m \geqslant i \geqslant 1, \qquad \exists 1 \leqslant j \neq i \leqslant m \ni |\Gamma(r_j) \backslash \Gamma(r_i)| \leqslant 2, \tag{4}$$

in which case we say that r_i *hits* r_j. In what follows Γ_i stands for $\Gamma(r_i)$. We want to show that there are 4 distinct indices $m \geqslant i_1, i_2, j_1, j_2 \geqslant 1$ so that r_{i_1} hits r_{j_1}, r_{i_2} hits r_{j_2}. By (4) this is not the case only if there is a $m \geqslant t \geqslant 1$ so that all r_i $(m \geqslant i \neq t \geqslant 1)$ hit r_t and only r_t. Let r_t hit r_s. So

$$1 \leqslant |\Gamma_s \backslash \Gamma_t| \leqslant 2, \qquad |\Gamma_t \backslash \Gamma_s| \leqslant 2.$$

Let r_i have a neighbour in $\Gamma_s \backslash \Gamma_t$. Since r_i hits r_t but does not hit r_s it follows that $|\Gamma_t \backslash \Gamma_s| = 2$ and r_i is a neighbour of both vertices in $\Gamma_t \backslash \Gamma_s$. It also follows that $|\Gamma_s \backslash \Gamma_t| = 2$ and r_i is a neighbour of exactly one vertex in $\Gamma_s \backslash \Gamma_t$. But since the vertices in $(\Gamma_s \backslash \Gamma_t) \cup (\Gamma_t \backslash \Gamma_s)$ all have valence 3 (by (3)) this is impossible.

So we have 4 distinct indices $1 \leqslant \alpha, \beta, \gamma, \delta \leqslant m$ so that

$$|\Gamma_\alpha \backslash \Gamma_\beta| \leqslant 2, \qquad |\Gamma_\gamma - \Gamma_\delta| \leqslant 2. \tag{5}$$

Now represent the a_i's as

$$a_i = f_i s + g_i(s-1) + h_i \qquad (i = 1, 2, 3),$$

where $\sum f_i = f - 2$, f being the number of s_i's which are $=s$, and $\sum g_i = g = m - f =$ number of s_i's which are $=s-1$, $\sum h_i = 2s$, $h_i \geqslant 0$. (If $f = 1$, change the roles of s and $s-1$.)

We assign now stars to classes as dictated by these parameters, namely, f_1 s-stars to A_1, etc. Let us say that $h_1, h_2 \leqslant s/2$. Assign S_β to A_1 and S_δ to A_2, only S_α, S_γ are unassigned yet. Now by (5) we transfer h_1 vertices of L_α to A_1 and h_2 vertices of L_γ to A_2. The rests of S_α, S_γ are assigned to A_3 to complete the decomposition.

If $h_1, h_2 \geqslant s/2$, assign S_β, S_δ to A_3 and transfer $s-h_1$, $s-h_2$ vertices from S_α, S_γ respectively to A_3. The remains of S_α, S_γ are assigned to A_1, A_2, respectively.

The only case which needs settling yet is the one where $s_1 \leqslant 3$. Let us say that we have α s_i's equal 2 and β of them equal 3. It is easy to check that if $\alpha \geqslant 4$ and $\beta \geqslant 2$ then a decomposition exists regardless of the values of a_1, a_2, a_3.

So we may assume $\alpha \leqslant 3$ or $\beta \leqslant 2$. The cases are

$$
\begin{array}{ll}
\alpha = 0 & (a_1, a_2, a_3) \equiv (1,1,1) \text{ or } (0,1,2) \text{ or } (2,2,2) \bmod 3, \\
\alpha = 1 & (a_1, a_2, a_3) \equiv (0,1,1) \text{ or } (1,2,2) \bmod 3, \\
\alpha = 2 & (a_1, a_2, a_3) \equiv (2,1,1) \bmod 3, \\
\alpha = 3 & (a_1, a_2, a_3) \equiv (1,1,1) \bmod 3, \\
\beta = 0 & (a_1, a_2, a_3) \equiv (0,1,1) \bmod 2, \\
\beta = 1 & (a_1, a_2, a_3) \equiv (1,1,1) \bmod 2,
\end{array}
$$

of any of their permutations.

If $\beta \leqslant 1$ we can find three 2-stars which can be transformed into two 3-stars, taking care of $\beta \leqslant 1$. If $\alpha \leqslant 3$ we can find two neighbouring 3-stars and transform them into a 4-star and a 2-star, or else transform four 3-stars into two 4-stars and two 2-stars. It is a routine check to validate that the decomposition is achieved in any of these cases. ∎

Together with the conjecture discussed in the present paper Frank made in [3] another conjecture, later proved by Lovász [5] and Györi [4]:

THEOREM LG. *A graph $G=(V,E)$ of order $\geqslant k+1$ is k-connected iff for any k integers $a_1, \ldots, a_k \geqslant 1$ and any k distinct vertices $x_1, \ldots, x_k \in V$, it is possible to decompose V into $A_1, \ldots, A_k$ so that $|A_i| = a_i$, $x_i \in A_i$, $\langle A_i \rangle$ is connected $(i=1, \ldots, k)$.*

This brings to mind the idea that one should try to prove a stronger conjecture than the one discussed in the present paper in which not only $a_1, \ldots, a_k$ are specified but also some vertices $x_1, \ldots, x_k$ in a manner similar to Theorem LG. However, even for the case $a_1 = \cdots = a_{k-1} = 2$ the specification $x_1, \ldots, x_k$ already implies k-connectivity as Theorem 3 shows. The harder part of the theorem is contained in Theorem LG but it seems worth mentioning as it supplies an independent characterization of k-connectivity.

THEOREM 3. *A graph $G=(V,E)$ of order $\geqslant 2k-1$ is k-connected iff for every set $\{x_1,\ldots,x_k\}\subseteq V$, there is a matching of $x_1,\ldots,x_{k-1}$ within $G\backslash\{x_k\}$ so that the vertices which are not in the matching span a connected subgraph of G.*

Proof. The crucial step in the proof is an application of alternating paths, a method which is fundamental in matching theory. See Berge [1, Chap. 8] for several examples of this method.

We assume G to be k-connected, and start by showing that it is possible to match $T=\{x_1,\ldots,x_{k-1}\}$ within $G\backslash x_k$. To show this we employ Hall's theorem [1, p. 134]. Let $X=\{x_1,\ldots,x_k\}$, and for $S\subset T$ let $N(S)$ be the set of those vertices in $V\backslash x_k$ which have a neighbour in S. If T cannot be matched within $G\backslash x_k$, then, by Hall's theorem, $|N(S)|<|S|$ for some $S\subseteq T$. But then the set $W=N(S)\cup(X\backslash S)$ separates S from the rest of the vertices in V. Note that the sets S and W do not exhaust all of V, because together they contain at most $2k-2$ vertices whereas $|V|\geqslant 2k-1$. Therefore W disconnects G, but this is impossible, since

$$|W|=|N(S)|+|X|-|S|<|X|=k.$$

Among all sets Y that can be matched with T in $G\backslash x_k$ we choose one for which the component of x_k in $G\backslash(T\cup Y)$ contains as many vertices as possible. Assume $Y=\{y_1,\ldots,y_{k-1}\}$ and $[x_i,y_i]\in E$ for $i=1,\ldots,k-1$. If $G\backslash(T\cup Y)$ is connected, then the proof is finished, so we assume that it is disconnected.

Let $C_1,\ldots,C_r$ be the components of $G\backslash(T\cup Y)$ and let A_i be the vertex set of C_i. We assume that $r\geqslant 2$, $x_k\in A_1$, and that $|A_1|$ is as large as possible. First we note that $E(A_1,Y)\neq\varnothing$, since otherwise T separates A_1 from Y and therefore from $Y\cup A_2\cup\cdots\cup A_r$, although $|T|=k-1$. Let $Y_1\neq\varnothing$ be the set of those vertices in Y which have a neighbour in A_1. Define

$$\begin{aligned}S=\{x\in T\mid &\text{There is a sequence } x_{\alpha_1},\ldots,x_{\alpha_l}=x \text{ of}\\ &\text{distinct vertices in } T\ (l\geqslant 1) \text{ so that } y_{\alpha_1}\in Y_1 \text{ and}\\ &[x_{\alpha_i},y_{\alpha_{i+1}}]\in E \text{ for } i=1,\ldots,l-1\}.\end{aligned}\qquad(6)$$

We show that

$$E(S,A_i)=\varnothing \qquad \text{for} \quad i=2,\ldots,r. \qquad (7)$$

Suppose on the contrary that for some $x\in S$ there is a $y\in\bigcup_{i=2}^{r}A_i$ such that $[x,y]\in E$. Let $x_{\alpha_1},\ldots,x_{\alpha_l}=x$ be a sequence as in the definition (6). We define $Y' = Y \backslash y_{\alpha_1} \cup y$, and show that T can be matched with Y' in $G \backslash x_k$. For $i \notin \{\alpha_1, \ldots, \alpha_l\}$ we leave x_i and y_i matched. For $j = 1, \ldots, l-1$ we match x_{α_j} with y_{α_j+1}. Note that $[x_{\alpha_j}, y_{\alpha_j+1}] \in E$ by definition (6); $x = x_{\alpha_l}$ is matched with y. However, the component of $G \backslash (T \cup Y')$ which contains x_k includes $A_1 \cup y_{\alpha_1}$, and therefore contains more vertices than the component C_1 of $G \backslash (T \cup Y)$. This contradicts the maximality of $|A_1|$ and proves (7). Denote $S' = \{y_i \mid x_i \in S\}$. We show that $(T \backslash S) \cup S'$ separates $S \cup A_1$ from $(Y \backslash S') \cup \bigcup_{i=2}^{r} A_i$. Since $|(T \backslash S) \cup S'| = k-1$ this is a contradiction which proves the "only if" part of the theorem. Consider A_1 first: evidently, $E(A_1, A_i) = \varnothing$ for $i = 2, \ldots, r$. Also $E(A_1, Y \backslash S') = \varnothing$, since $Y_1 \subseteq S'$. As for S, we have (7). By definition of S we have $E(S, Y) = E(S, S')$ and this part of Theorem 3 is proven.

The "if" part of the theorem is proved as follows: Suppose $S \subseteq V$ is such that $|S| = k-1$ and $G \backslash S$ is disconnected. Let $A_1, \ldots, A_r$ $(r \geqslant 2)$ be the vertex sets of the components of $G \backslash S$. Suppose first that $|A_i| \leqslant k-1$ for some i, and let U be a subset of S having $k - 1 - |A_i|$ vertices. Define $x_1, \ldots, x_{k-1}$ to be the vertices in $A_i \cup U$. Also, let x_k be a vertex in $S \backslash U$. No vertex outside $A_i \cup S$ is adjacent to a vertex of A_i. Therefore, at most $|S \backslash (U \cup x_k)| = |A_i| - 1$ vertices in $V \backslash (A_i \cup U \cup x_k)$ may have a neighbour in A_i. Thus it is impossible to match $A_i \cup U = \{x_1, \ldots, x_{k-1}\}$ within $G \backslash x_k$.

We may assume, then, that $|A_i| \geqslant k$ for $1 \leqslant i \leqslant r$. Now let $x_1, \ldots, x_{k-1}$ be the vertices of S, and x_k a vertex not in S. From the assumption that every component of $G \backslash S$ has $\geqslant k$ vertices it follows that for every matching of S in G (if any), the remaining vertices span a disconnected subgraph of G, a contradiction. ∎

Let us show now that the conjecture holds for the case $a_1 = \cdots = a_{k-1} = 2$. This case is of course a problem on the existence of matchings as was also noted by Frank and Maurer.

THEOREM 4. *Let G be a connected graph of order $\geqslant 2k$ with $\delta(G) \geqslant k$. Then there is a matching $[x_i, y_i]$ $(i = 1, \ldots, k-1\})$ so that the graph $G \backslash (\{x_i \mid i = 1, \ldots, -1\} \cup \{y_i \mid i = 1, \ldots, k-1\})$ has no isolated vertices.*

Proof. That G contains a $(k-1)$ matching is known (see, e.g., [2, Theorem 2.4.2]). Consider a matching $[x_i, y_i]$ $(i = 1, \ldots, k-1)$ for which $G \backslash (\{x_i \mid i = 1, \ldots, k-1\} \cup \{y \mid i = 1, \ldots, k-1\}$ has as few isolated vertices as possible. Let p be an isolated vertex in this subgraph. Identify $V \backslash (\{p\} \cup \{x_i \mid i = 1, \ldots, k-1\} \cup \{y_i \mid i = 1, \ldots, k-1\})$ to a single vertex q. Let H be the resulting graph with vertex set $\{p, q\} \cup \{x_i \mid i = 1, \ldots, k-1\} \cup \{y_i \mid i = 1, \ldots, k-1\}$, and $E = E(H)$. If we can find a perfect matching in H we can translate this back into a $(k-1)$ matching in G with fewer isolated vertices among the vertices which are not in the matching.

We prove that H has a perfect matching by contradiction. If $[p, x_i] \in E$, $[q, y_i] \in E$ for some $k-1 \geqslant i \geqslant 1$, then a perfect matching is obtained by matching $[p, x_i]$, $[q, y_i]$, and $[x_j, y_j]$, $k-1 \geqslant j \neq i \geqslant 1$. Notice that x_i, y_i play exactly the same roles so whenever an assumption on x_i, y_i can be made without loss of generality we will make it with no further comment. We want to show that for $k-1 \geqslant i \geqslant 1$ either $[q, x_i]$, $[q, y_i] \in E$ or $[q, x_i]$, $[q, y_i] \notin E$. Assume that $[q, x_i] \notin E$, $[q, y_i] \in E$. By a previous remark we may assume $[p, x_i] \notin E$. Now $d_H(p) \geqslant k$ and $[q, x_i] \notin E$ implies that $d_H(x_i) \geqslant k$. Therefore both p and x_i have at least $k-1$ neighbours among the $2k-4$ vertices in $\bigcup(\{x_j, y_j\} \mid k-1 \geqslant j \neq i \geqslant 1)$. This implies that for some $j \neq i$, $[x_i, x_j]$, $[p, y_j] \in E$. But now we have the perfect matching $[p, y_j]$, $[x_i, x_j]$, $[q, y_i]$, and $[x_t, y_t]$ $(k-1 \geqslant t \neq i, j \geqslant 1)$, a contradiction.

It follows that there exists a subset $I \subseteq \{1,..., k-1\}$ so that for $i \notin I$, $[q, x_i]$, $[q, y_i] \notin E$ and for $i \in I$, $[q, x_i]$, $[q, y_i] \in E$. By what was said before, $i \in I$ implies $[p, x_i]$, $[p, y_i] \notin E$. Since G is connected there have to be $s \in I$, $t \notin I$ so that $[y_s, y_t] \in E$. We repeat a previous argument to conclude that there is an index $j \neq t$ so that $\{x_t, p\}$ can be matched with $\{x_j, y_j\}$. Now j cannot belong to I and in particular $j \neq s$. Let us say that $[x_t, x_j] \in E$, $[p, y_j] \in E$. Match these pairs and also $[y_s, y_t]$, $[q, x_s]$, and $[x_r, y_r]$ $(k-1 \geqslant r \neq s, t, j \geqslant 1)$ for a perfect matching. ∎

Note added in proof. Theorem 3 was independently proved by E. Győri (*Combinatorica* **1** (1981), 263–273).

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