Graph Filter Banks with M-Channels, Maximal Decimation, and Perfect Reconstruction

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ICASSP - 2016

Caltech

Outline

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What is a Graph Signal ?

$$\boldsymbol{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_i \\ \vdots \\ x_N \end{bmatrix} \in \mathcal{C}^N$$

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$\boldsymbol{V}$ = graph Fourier Basis, $\boldsymbol{V}^{-1}$ = graph Fourier Transform

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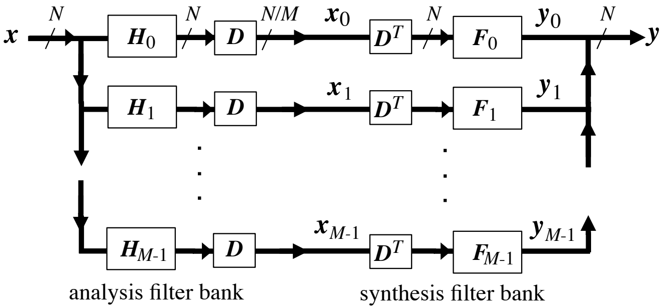
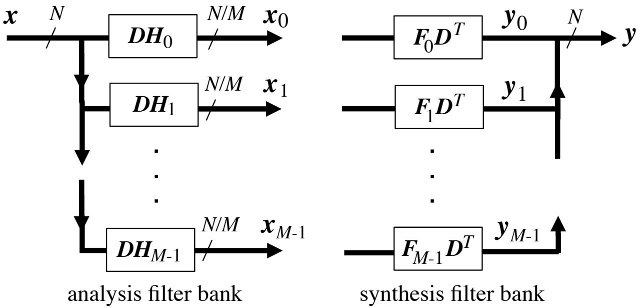
Maximally Decimated M-Channel Filter Banks



M-Channel Filter Banks (Brute-Force)



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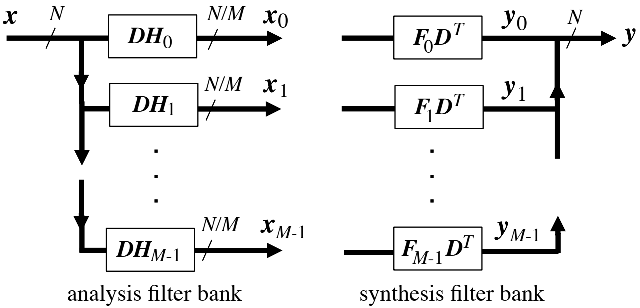

$$\boldsymbol{H}_{anal} = \begin{bmatrix} \boldsymbol{D}\boldsymbol{H}_0 \\ \vdots \\ \boldsymbol{D}\boldsymbol{H}_{M\text{-}1} \end{bmatrix}, \quad \boldsymbol{F}_{syn} = \begin{bmatrix} \boldsymbol{F}_0\boldsymbol{D}^T & \cdots & \boldsymbol{F}_{M\text{-}1}\boldsymbol{D}^T \end{bmatrix}$$

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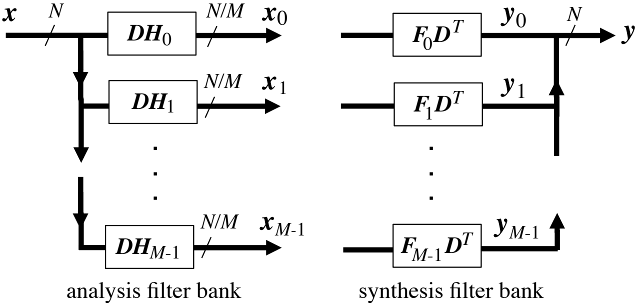

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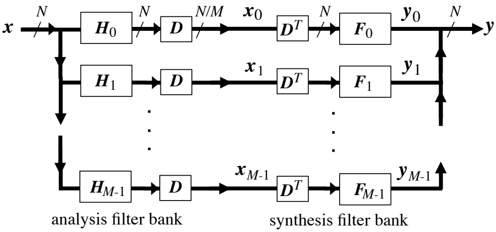
Problems with Brute-Force



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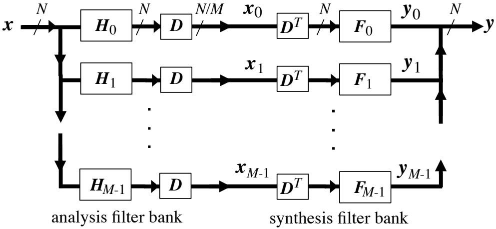

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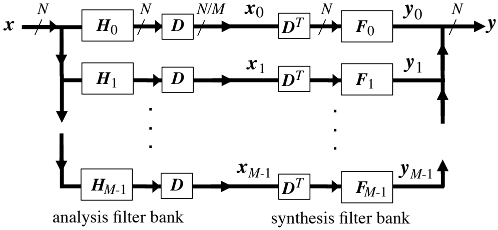

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Filter Design

$\Updownarrow$

Matrix Inversion Problem

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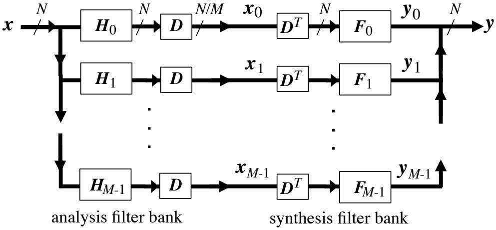

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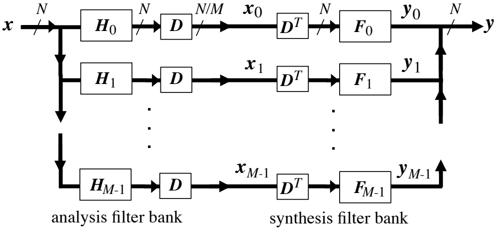

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Computational Complexity = $O(N^2)$

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$$\boldsymbol{H}_k(\boldsymbol{A}) = h_k(0)\,\boldsymbol{A}^0 + h_k(1)\,\boldsymbol{A}^1 + h_k(2)\,\boldsymbol{A}^2 + \ldots + h_k(L)\,\boldsymbol{A}^L$$

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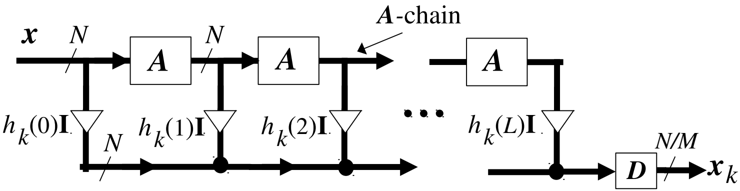

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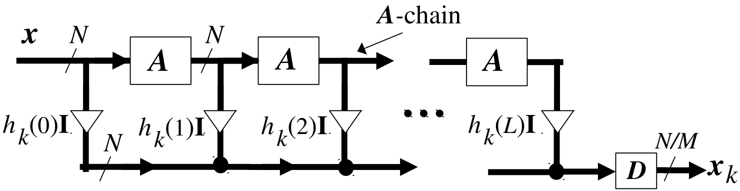

Cost: $LN^2 + LN$

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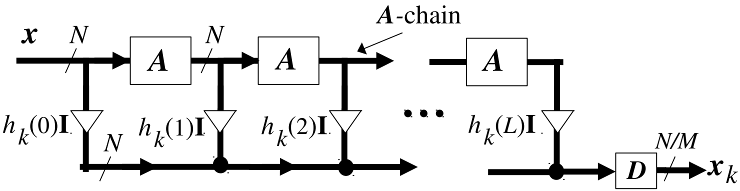

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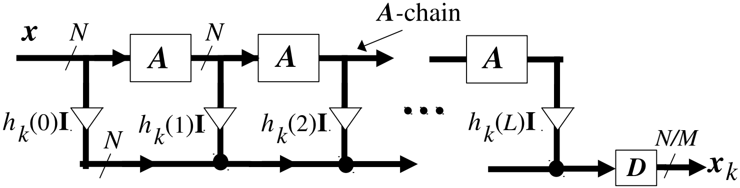

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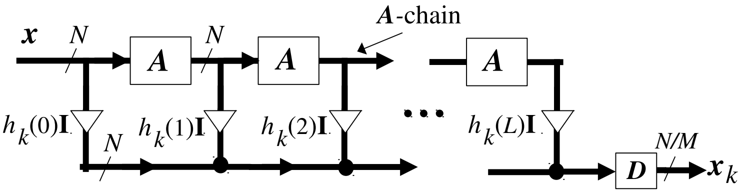

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$L \ll N$

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Keep the first N/M samples

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Definition (Canonical Decimator)

$$\boldsymbol{D} = \begin{bmatrix} \boldsymbol{I}_{N/M} & \mathbf{0}_{N/M} & \cdots & \mathbf{0}_{N/M} \end{bmatrix} \in \mathcal{C}^{(N/M)\times N},$$

which retains the first N/M samples of the given graph signal.

M-fold Expansion

$$\boldsymbol{U} : C^{N/M} \to C^{N}$$

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$$\boldsymbol{U} : \mathcal{C}^{N/M} \rightarrow \mathcal{C}^{N}$$

Given the decimator $\boldsymbol{D} = \left[\begin{array}{cccc} \boldsymbol{I}_{N/M} & \boldsymbol{0}_{N/M} & \cdots & \boldsymbol{0}_{N/M} \end{array} \right]$

$$\boxed{\boldsymbol{U} = \boldsymbol{D}^{T} = \left[\begin{array}{c} \boldsymbol{I}_{N/M} \\ \boldsymbol{0}_{N/M} \\ \vdots \\ \boldsymbol{0}_{N/M} \end{array} \right] \in \mathcal{C}^{N \times (N/M)}}$$

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M-block cyclic graphs

[a] O. Teke and P. P. Vaidyanathan. "Extending Classical Multirate Signal Processing Theory to Graphs". *IEEE Trans. Signal Process. (Submitted).*

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M-block cyclic graphs

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$$\boldsymbol{A}_j \in \mathcal{M}^{N/M}$$

(Under suitable permutation)

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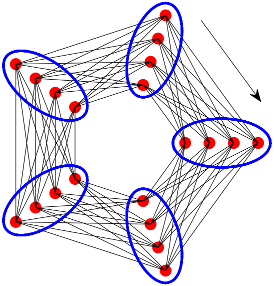

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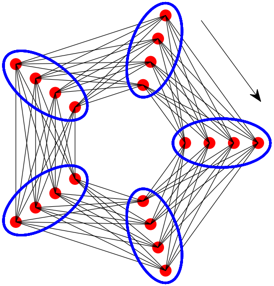

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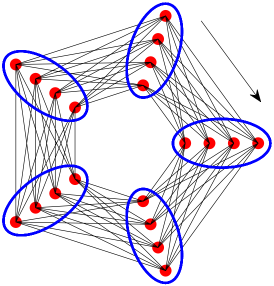

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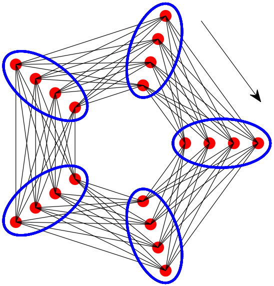

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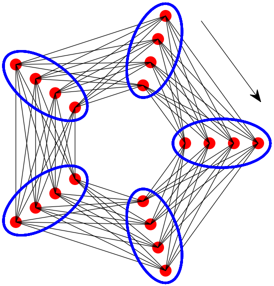

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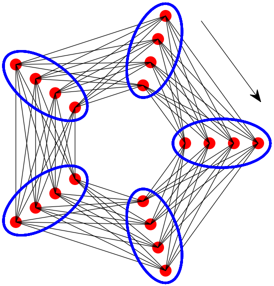

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5. Unique eigenvalue & eigenvector structure [a] [b]

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[b] D. S. Watkins. "Product eigenvalue problems" *SIAM Review*, 2005

Spectrum Folding - Aliasing

Let $\boldsymbol{x}$ be a graph signal

$$\boldsymbol{y} = \boldsymbol{D}^T \, \boldsymbol{D} \, \boldsymbol{x}$$

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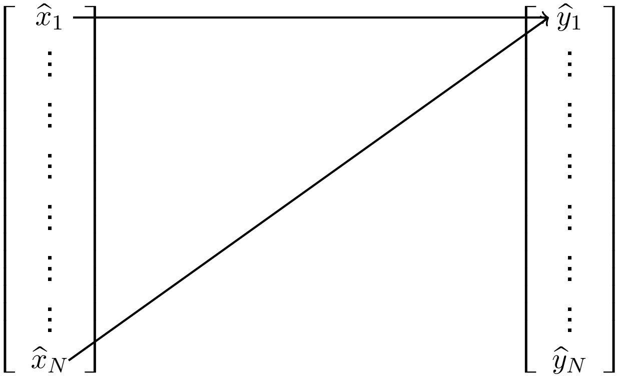
What is the relation between $\widehat{\boldsymbol{x}}$ and $\widehat{\boldsymbol{y}}$?

$$\widehat{\boldsymbol{y}} = \boldsymbol{V}^{\text{-}1}\boldsymbol{D}^{T}\boldsymbol{D}\boldsymbol{V}\ \widehat{\boldsymbol{x}}$$

$$\begin{bmatrix} \widehat{x}_1 \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ \widehat{x}_N \end{bmatrix} \qquad \begin{bmatrix} \widehat{y}_1 \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ \widehat{y}_N \end{bmatrix}$$

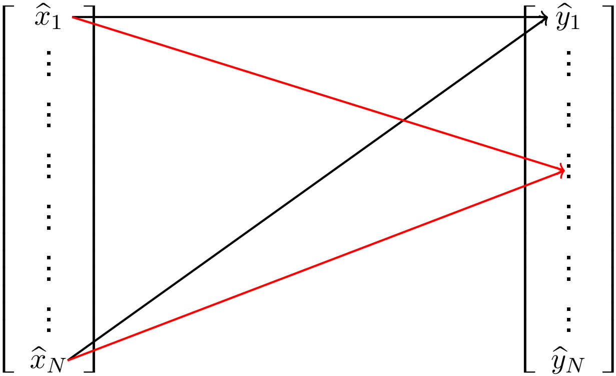
What is the relation between $\widehat{\boldsymbol{x}}$ and $\widehat{\boldsymbol{y}}$?

$$\widehat{\boldsymbol{y}} = \boldsymbol{V}^{-1}\boldsymbol{D}^{T}\boldsymbol{D}\boldsymbol{V}\;\widehat{\boldsymbol{x}}$$



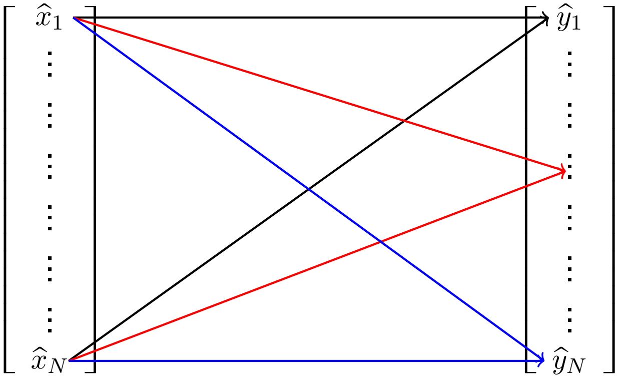
What is the relation between $\widehat{\boldsymbol{x}}$ and $\widehat{\boldsymbol{y}}$?

$$\widehat{\boldsymbol{y}} = \boldsymbol{V}^{-1}\boldsymbol{D}^T\boldsymbol{D}\boldsymbol{V}\ \widehat{\boldsymbol{x}}$$



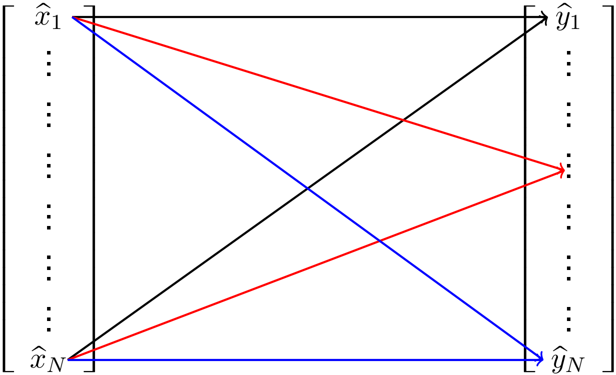
What is the relation between $\widehat{\boldsymbol{x}}$ and $\widehat{\boldsymbol{y}}$?

$$\widehat{\boldsymbol{y}} = \boldsymbol{V}^{-1}\boldsymbol{D}^{T}\boldsymbol{D}\boldsymbol{V}\ \widehat{\boldsymbol{x}}$$



What is the relation between $\widehat{\boldsymbol{x}}$ and $\widehat{\boldsymbol{y}}$?

$$\widehat{\boldsymbol{y}} = \boldsymbol{V}^{-1}\boldsymbol{D}^{T}\boldsymbol{D}\boldsymbol{V}\ \widehat{\boldsymbol{x}}$$



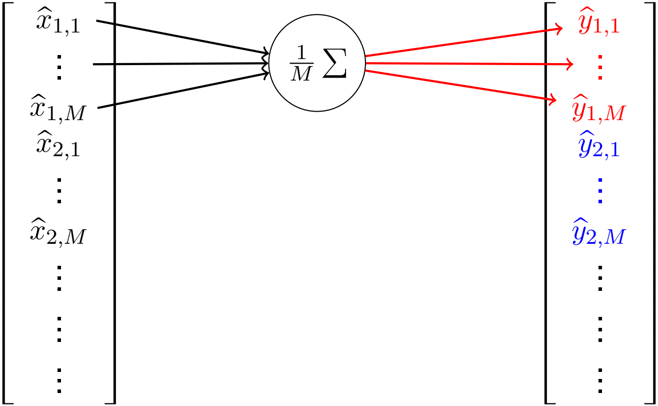
No "simple" relation in general!

Spectrum Folding on M-Block Cyclic Graphs

$$\begin{bmatrix} \widehat{x}_{1,1} \\ \vdots \\ \widehat{x}_{1,M} \\ \widehat{x}_{2,1} \\ \vdots \\ \widehat{x}_{2,M} \\ \vdots \\ \vdots \\ \vdots \end{bmatrix} \qquad \begin{bmatrix} \color{red}{\widehat{y}_{1,1}} \\ \color{red}{\vdots} \\ \color{red}{\widehat{y}_{1,M}} \\ \color{blue}{\widehat{y}_{2,1}} \\ \color{blue}{\vdots} \\ \color{blue}{\widehat{y}_{2,M}} \\ \vdots \\ \vdots \\ \vdots \end{bmatrix}$$

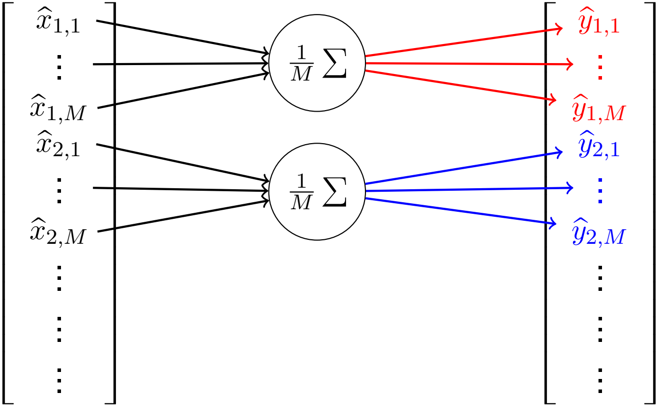
4

Spectrum Folding on M-Block Cyclic Graphs



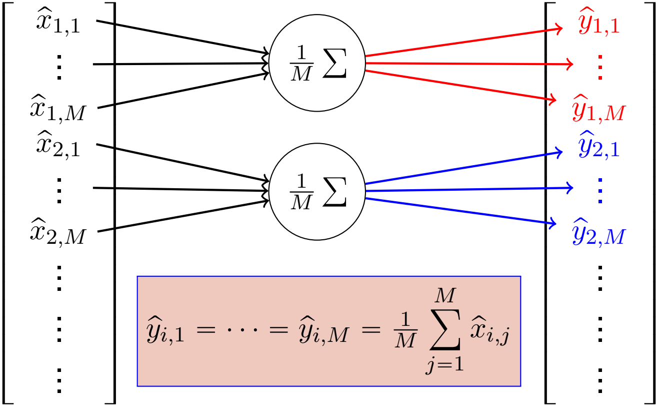
4

Spectrum Folding on M-Block Cyclic Graphs



4

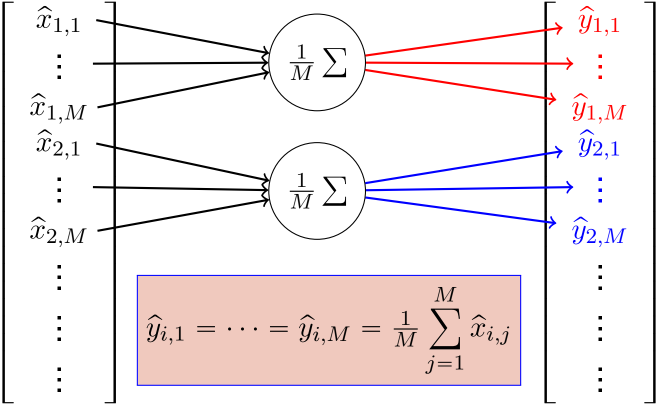
Spectrum Folding on M-Block Cyclic Graphs



$$\widehat{y}_{i,1} = \cdots = \widehat{y}_{i,M} = \frac{1}{M}\sum_{j=1}^{M} \widehat{x}_{i,j}$$

4

Spectrum Folding on M-Block Cyclic Graphs



$$\widehat{y}_{i,1} = \cdots = \widehat{y}_{i,M} = \frac{1}{M}\sum_{j=1}^{M} \widehat{x}_{i,j}$$

$M = 2 \Leftrightarrow$ Bi-partite [4]

[4] S.K. Narang and A. Ortega. "Perfect Reconstruction Two-Channel Wavelet Filter Banks for Graph Structured Data". *IEEE Trans. Signal Process.* 60.6 (2012), pp. 2786–2799.

Band-Limited Signals & Decimation on M-Block Cyclic

Let $\boldsymbol{x}$ be a graph signal

$$\widehat{\boldsymbol{x}} = \begin{bmatrix} \widehat{x}_{1,1} \\ \vdots \\ \widehat{x}_{1,M} \\ \widehat{x}_{2,1} \\ \vdots \\ \widehat{x}_{2,M} \\ \widehat{x}_{3,1} \\ \vdots \\ \vdots \\ \vdots \end{bmatrix}$$

Band-Limited Signals & Decimation on M-Block Cyclic

Let $\boldsymbol{x}$ be a graph signal

$$\widehat{\boldsymbol{x}} = \begin{bmatrix} \widehat{x}_{1,1} \\ \vdots \\ \widehat{x}_{1,M} \\ \widehat{x}_{2,1} \\ \vdots \\ \widehat{x}_{2,M} \\ \widehat{x}_{3,1} \\ \vdots \\ \vdots \\ \vdots \end{bmatrix} \xrightarrow[\text{limited}]{\substack{k^{th} \\ \text{band}}} \begin{bmatrix} \widehat{x}_{1,1} \\ 0 \\ 0 \\ \widehat{x}_{2,1} \\ 0 \\ 0 \\ \widehat{x}_{3,1} \\ \vdots \\ \vdots \\ \vdots \end{bmatrix}$$

Band-Limited Signals & Decimation on M-Block Cyclic

Let $\boldsymbol{x}$ be a graph signal

$$\widehat{\boldsymbol{x}} = \begin{bmatrix} \color{red}{\widehat{x}_{1,1}} \\ \vdots \\ \widehat{x}_{1,M} \\ \color{blue}{\widehat{x}_{2,1}} \\ \vdots \\ \widehat{x}_{2,M} \\ \color{magenta}{\widehat{x}_{3,1}} \\ \vdots \\ \vdots \\ \vdots \end{bmatrix} \xrightarrow[]{\substack{k^{th} \\ band \\ limited}} \begin{bmatrix} \color{red}{\widehat{x}_{1,1}} \\ 0 \\ 0 \\ \color{blue}{\widehat{x}_{2,1}} \\ 0 \\ 0 \\ \color{magenta}{\widehat{x}_{3,1}} \\ \vdots \\ \vdots \\ \vdots \end{bmatrix} \xrightarrow{\boldsymbol{D}^T\boldsymbol{D}}$$

Band-Limited Signals & Decimation on M-Block Cyclic

Let $\boldsymbol{x}$ be a graph signal

$$\widehat{\boldsymbol{x}} = \begin{bmatrix} \widehat{x}_{1,1} \\ \vdots \\ \widehat{x}_{1,M} \\ \widehat{x}_{2,1} \\ \vdots \\ \widehat{x}_{2,M} \\ \widehat{x}_{3,1} \\ \vdots \\ \vdots \\ \vdots \end{bmatrix} \xrightarrow[\text{}]{k^{th}\ band\ limited} \begin{bmatrix} \widehat{x}_{1,1} \\ 0 \\ 0 \\ \widehat{x}_{2,1} \\ 0 \\ 0 \\ \widehat{x}_{3,1} \\ \vdots \\ \vdots \\ \vdots \end{bmatrix} \xrightarrow{\boldsymbol{D}^T\boldsymbol{D}} \widehat{\boldsymbol{y}} = \begin{bmatrix} \widehat{y}_{1,1} \\ \vdots \\ \widehat{y}_{1,M} \\ \widehat{y}_{2,1} \\ \vdots \\ \widehat{y}_{2,M} \\ \widehat{y}_{3,1} \\ \vdots \\ \vdots \\ \vdots \end{bmatrix}$$

Band-Limited Signals & Decimation on M-Block Cyclic

Let $\boldsymbol{x}$ be a graph signal

$$\frac{1}{M}\sum$$

$$\widehat{\boldsymbol{x}} = \begin{bmatrix} \widehat{x}_{1,1} \\ \vdots \\ \widehat{x}_{1,M} \\ \widehat{x}_{2,1} \\ \vdots \\ \widehat{x}_{2,M} \\ \widehat{x}_{3,1} \\ \vdots \\ \vdots \\ \vdots \end{bmatrix} \xrightarrow{\substack{k^{th} \\ band \\ limited}} \begin{bmatrix} \widehat{x}_{1,1} \\ 0 \\ 0 \\ \widehat{x}_{2,1} \\ 0 \\ 0 \\ \widehat{x}_{3,1} \\ \vdots \\ \vdots \\ \vdots \end{bmatrix} \xrightarrow{\boldsymbol{D}^T\boldsymbol{D}} \widehat{\boldsymbol{y}} = \begin{bmatrix} \widehat{y}_{1,1} \\ \vdots \\ \widehat{y}_{1,M} \\ \widehat{y}_{2,1} \\ \vdots \\ \widehat{y}_{2,M} \\ \widehat{y}_{3,1} \\ \vdots \\ \vdots \\ \vdots \end{bmatrix} = \frac{1}{M} \begin{bmatrix} \widehat{x}_{1,1} \\ \vdots \\ \widehat{x}_{1,1} \\ \widehat{x}_{2,1} \\ \vdots \\ \widehat{x}_{2,1} \\ \widehat{x}_{3,1} \\ \vdots \\ \vdots \\ \vdots \end{bmatrix}$$

Interpolation Filter on M-Block Cyclic Graphs

$$\widehat{\boldsymbol{y}} = \frac{1}{M}\begin{bmatrix} \widehat{x}_{1,1} \\ \vdots \\ \widehat{x}_{1,1} \\ \widehat{x}_{2,1} \\ \vdots \\ \widehat{x}_{2,1} \\ \widehat{x}_{3,1} \\ \vdots \\ \vdots \\ \vdots \end{bmatrix}$$

Interpolation Filter on M-Block Cyclic Graphs

$$\widehat{\boldsymbol{y}} = \frac{1}{M}\begin{bmatrix} \widehat{x}_{1,1} \\ \vdots \\ \widehat{x}_{1,1} \\ \widehat{x}_{2,1} \\ \vdots \\ \widehat{x}_{2,1} \\ \widehat{x}_{3,1} \\ \vdots \\ \vdots \\ \vdots \end{bmatrix} \xrightarrow[(interpolation)]{\boldsymbol{z}=\boldsymbol{F}\boldsymbol{y}}$$

Interpolation Filter on M-Block Cyclic Graphs

$$\widehat{\boldsymbol{y}} = \frac{1}{M}\begin{bmatrix} \widehat{x}_{1,1} \\ \vdots \\ \widehat{x}_{1,1} \\ \widehat{x}_{2,1} \\ \vdots \\ \widehat{x}_{2,1} \\ \widehat{x}_{3,1} \\ \vdots \\ \vdots \\ \vdots \end{bmatrix} \xrightarrow[(interpolation)]{\boldsymbol{z}=\boldsymbol{F}\boldsymbol{y}} \widehat{\boldsymbol{z}} = \begin{bmatrix} \widehat{x}_{1,1} \\ 0 \\ 0 \\ \widehat{x}_{2,1} \\ 0 \\ 0 \\ \widehat{x}_{3,1} \\ 0 \\ 0 \\ \vdots \\ \vdots \end{bmatrix} = \widehat{\boldsymbol{x}}$$

Interpolation Filter on M-Block Cyclic Graphs

$$\widehat{\boldsymbol{y}} = \frac{1}{M}\begin{bmatrix} \widehat{x}_{1,1} \\ \vdots \\ \widehat{x}_{1,1} \\ \widehat{x}_{2,1} \\ \vdots \\ \widehat{x}_{2,1} \\ \widehat{x}_{3,1} \\ \vdots \\ \vdots \\ \vdots \end{bmatrix} \xrightarrow[(interpolation)]{\boldsymbol{z}=\boldsymbol{F}\boldsymbol{y}} \widehat{\boldsymbol{z}} = \begin{bmatrix} \widehat{x}_{1,1} \\ 0 \\ 0 \\ \widehat{x}_{2,1} \\ 0 \\ 0 \\ \widehat{x}_{3,1} \\ 0 \\ 0 \\ \vdots \\ \vdots \end{bmatrix} = \widehat{\boldsymbol{x}}$$

Not true for an arbitrary $\boldsymbol{x}$!

For k^{th}-band-limited signals, $\boldsymbol{x}$ can be recovered from $\boldsymbol{D}\boldsymbol{x}$

M-Channel Filter-Banks on M-Block Cyclic Graphs

$$\widehat{\boldsymbol{x}} = \begin{bmatrix} \widehat{x}_{1,1} \\ \vdots \\ \widehat{x}_{1,M} \\ \widehat{x}_{2,1} \\ \vdots \\ \widehat{x}_{2,M} \\ \widehat{x}_{3,1} \\ \vdots \\ \vdots \\ \vdots \end{bmatrix}$$

M-Channel Filter-Banks on M-Block Cyclic Graphs

$$\widehat{\boldsymbol{x}} = \begin{bmatrix} \widehat{x}_{1,1} \\ \vdots \\ \widehat{x}_{1,M} \\ \widehat{x}_{2,1} \\ \vdots \\ \widehat{x}_{2,M} \\ \widehat{x}_{3,1} \\ \vdots \\ \vdots \\ \vdots \end{bmatrix} = \underbrace{\begin{bmatrix} \widehat{x}_{1,1} \\ 0 \\ \vdots \\ 0 \\ \widehat{x}_{2,1} \\ 0 \\ \vdots \\ 0 \\ \widehat{x}_{3,1} \\ 0 \\ \vdots \\ 0 \\ \vdots \end{bmatrix}}_{\widehat{\boldsymbol{x}}_1 = 1^{st}} + \underbrace{\begin{bmatrix} 0 \\ \widehat{x}_{1,2} \\ \vdots \\ 0 \\ 0 \\ \widehat{x}_{2,2} \\ \vdots \\ 0 \\ 0 \\ \widehat{x}_{3,2} \\ \vdots \\ 0 \\ \vdots \end{bmatrix}}_{\widehat{\boldsymbol{x}}_2 = 2^{nd}} + \cdots + \underbrace{\begin{bmatrix} 0 \\ \vdots \\ 0 \\ \widehat{x}_{1,M} \\ 0 \\ \vdots \\ 0 \\ \widehat{x}_{2,M} \\ 0 \\ \vdots \\ 0 \\ \widehat{x}_{3,M} \\ \vdots \end{bmatrix}}_{\widehat{\boldsymbol{x}}_M = M^{th}}$$

M-Channel Filter-Banks (k^{th}-Channel)

$$\widehat{\boldsymbol{x}} = \begin{bmatrix} \widehat{x}_{1,1} \\ \widehat{x}_{1,2} \\ \vdots \\ \widehat{x}_{1,M} \\ \widehat{x}_{2,1} \\ \widehat{x}_{2,2} \\ \vdots \\ \widehat{x}_{2,M} \\ \widehat{x}_{3,1} \\ \widehat{x}_{3,2} \\ \vdots \\ \widehat{x}_{3,M} \\ \vdots \end{bmatrix}$$

M-Channel Filter-Banks (k^{th}-Channel)

$$\widehat{\boldsymbol{x}} = \begin{bmatrix} \widehat{x}_{1,1} \\ \widehat{x}_{1,2} \\ \vdots \\ \widehat{x}_{1,M} \\ \widehat{x}_{2,1} \\ \widehat{x}_{2,2} \\ \vdots \\ \widehat{x}_{2,M} \\ \widehat{x}_{3,1} \\ \widehat{x}_{3,2} \\ \vdots \\ \widehat{x}_{3,M} \\ \vdots \end{bmatrix} \xrightarrow[\widehat{\boldsymbol{x}}_k =]{\boldsymbol{H}_k} \begin{bmatrix} 0 \\ {\color{red}\widehat{x}_{1,k}} \\ \vdots \\ 0 \\ 0 \\ {\color{blue}\widehat{x}_{2,k}} \\ \vdots \\ 0 \\ 0 \\ {\color{magenta}\widehat{x}_{3,k}} \\ \vdots \\ 0 \\ \vdots \end{bmatrix}$$

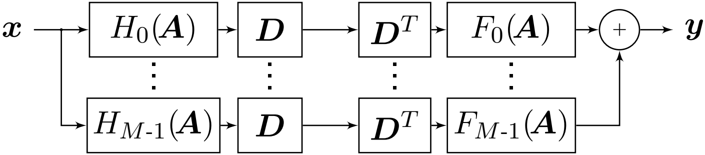
M-Channel Filter-Banks (k^{th}-Channel)

$$\widehat{\boldsymbol{x}} = \begin{bmatrix} \widehat{x}_{1,1} \\ \widehat{x}_{1,2} \\ \vdots \\ \widehat{x}_{1,M} \\ \widehat{x}_{2,1} \\ \widehat{x}_{2,2} \\ \vdots \\ \widehat{x}_{2,M} \\ \widehat{x}_{3,1} \\ \widehat{x}_{3,2} \\ \vdots \\ \widehat{x}_{3,M} \\ \vdots \end{bmatrix} \xrightarrow[\widehat{\boldsymbol{x}}_k =]{\boldsymbol{H}_k} \begin{bmatrix} 0 \\ \widehat{x}_{1,k} \\ \vdots \\ 0 \\ 0 \\ \widehat{x}_{2,k} \\ \vdots \\ 0 \\ 0 \\ \widehat{x}_{3,k} \\ \vdots \\ 0 \\ \vdots \end{bmatrix} \xrightarrow[\widehat{\boldsymbol{y}}_k =]{\boldsymbol{D}^T \boldsymbol{D}} \frac{1}{M} \begin{bmatrix} \widehat{x}_{1,k} \\ \widehat{x}_{1,k} \\ \vdots \\ \widehat{x}_{1,k} \\ \widehat{x}_{2,k} \\ \widehat{x}_{2,k} \\ \vdots \\ \widehat{x}_{2,k} \\ \widehat{x}_{3,k} \\ \widehat{x}_{3,k} \\ \vdots \\ \widehat{x}_{3,k} \\ \vdots \end{bmatrix}$$

M-Channel Filter-Banks (k^{th}-Channel)

$$\hat{\boldsymbol{x}} = \begin{bmatrix} \hat{x}_{1,1} \\ \hat{x}_{1,2} \\ \vdots \\ \hat{x}_{1,M} \\ \hat{x}_{2,1} \\ \hat{x}_{2,2} \\ \vdots \\ \hat{x}_{2,M} \\ \hat{x}_{3,1} \\ \hat{x}_{3,2} \\ \vdots \\ \hat{x}_{3,M} \\ \vdots \end{bmatrix} \xrightarrow[\hat{\boldsymbol{x}}_k =]{\boldsymbol{H}_k} \begin{bmatrix} 0 \\ \hat{x}_{1,k} \\ \vdots \\ 0 \\ 0 \\ \hat{x}_{2,k} \\ \vdots \\ 0 \\ 0 \\ \hat{x}_{3,k} \\ \vdots \\ 0 \\ \vdots \end{bmatrix} \xrightarrow[\hat{\boldsymbol{y}}_k =]{\boldsymbol{D}^T\boldsymbol{D}} \frac{1}{M} \begin{bmatrix} \hat{x}_{1,k} \\ \hat{x}_{1,k} \\ \vdots \\ \hat{x}_{1,k} \\ \hat{x}_{2,k} \\ \hat{x}_{2,k} \\ \vdots \\ \hat{x}_{2,k} \\ \hat{x}_{3,k} \\ \hat{x}_{3,k} \\ \vdots \\ \hat{x}_{3,k} \\ \vdots \end{bmatrix} \xrightarrow[\hat{\boldsymbol{z}}_k =]{\boldsymbol{F}_k} \begin{bmatrix} 0 \\ \hat{x}_{1,k} \\ \vdots \\ 0 \\ 0 \\ \hat{x}_{2,k} \\ \vdots \\ 0 \\ 0 \\ \hat{x}_{3,k} \\ \vdots \\ 0 \\ \vdots \end{bmatrix}$$

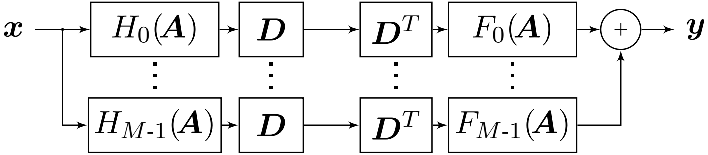
M-Channel FB on M-Block Cyclic Graphs (Details)



5

6

M-Channel FB on M-Block Cyclic Graphs (Details)

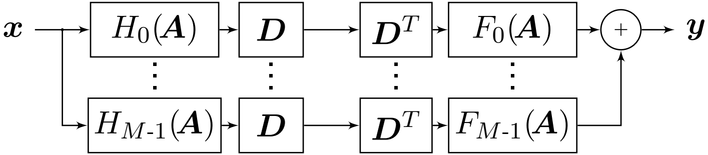


$$\boldsymbol{F}_k = M \ \boldsymbol{H}_k$$

5

6

M-Channel FB on M-Block Cyclic Graphs (Details)



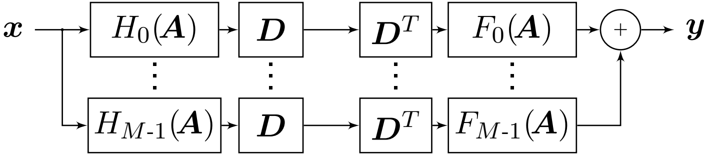
$$\boldsymbol{F}_k = M\ \boldsymbol{H}_k$$

$$\boldsymbol{H}_{k\text{-}1} = \boldsymbol{V}\left(\boldsymbol{I} \otimes \boldsymbol{e}_k \boldsymbol{e}_k{}^T\right)\boldsymbol{V}^{\text{-}1}$$

5

6

M-Channel FB on M-Block Cyclic Graphs (Details)



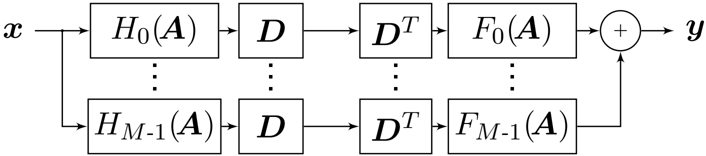
$$\boldsymbol{F}_k = M\ \boldsymbol{H}_k$$

$$\boldsymbol{H}_{k\text{-}1} = \boldsymbol{V}\left(\boldsymbol{I}\otimes \boldsymbol{e}_k{\boldsymbol{e}_k}^T\right)\boldsymbol{V}^{\text{-}1} \qquad H_k(e^{j\omega}) = \begin{cases} 1, & \frac{2\pi k}{M} \leqslant \omega \leqslant \frac{2\pi(k+1)}{M}, \\ 0, & \text{otherwise}, \end{cases}$$

5

6

M-Channel FB on M-Block Cyclic Graphs (Details)



$$\boldsymbol{F}_k = M\ \boldsymbol{H}_k$$

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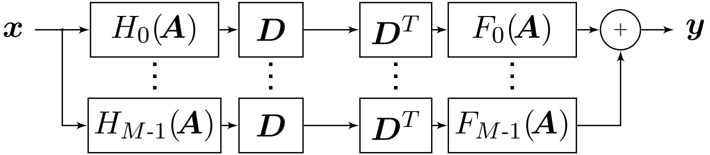
$$\boldsymbol{H}_k \overset{?}{=} H_k(\boldsymbol{A}) = h_k(0)\,\boldsymbol{A}^0 + h_k(1)\,\boldsymbol{A}^1 + \ldots + h_k(L)\,\boldsymbol{A}^L$$

5

6

M-Channel FB on M-Block Cyclic Graphs (Details)



$$\boldsymbol{F}_k = M\ \boldsymbol{H}_k$$

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$$\boldsymbol{H}_k \stackrel{?}{=} H_k(\boldsymbol{A}) = h_k(0)\ \boldsymbol{A}^0 + h_k(1)\ \boldsymbol{A}^1 + \ldots + h_k(L)\ \boldsymbol{A}^L$$

$\boldsymbol{A}$ needs to have distinct eigenvalues.[5] [6]

[5] O. Teke and P. P. Vaidyanathan. "Extending Classical Multirate Signal Processing Theory to Graphs – Part I: Fundamentals". *IEEE Trans. Signal Process. (Submitted).*

[6] A. Sandryhaila and J. M. F. Moura. "Discrete Signal Processing on Graphs". *IEEE Trans. Signal Process.* 61.7 (2013)

Is M-Block Cyclic Property Necessary?

$$M\text{-Block Cyclic }^{7} \rightleftarrows \begin{cases} \text{Eigenvector Property :} & \boldsymbol{v}_{i,j+k} = \boldsymbol{\Omega}^{k}\ \boldsymbol{v}_{i,j} \\ \text{Eigenvalue Property :} & \lambda_{i,j+k} = w^{k}\ \lambda_{i,j} \end{cases}$$

[7] O. Teke and P. P. Vaidyanathan. "Extending Classical Multirate Signal Processing Theory to Graphs – Part I: Fundamentals". *IEEE Trans. Signal Process. (Submitted).*

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For any polynomial $H(\boldsymbol{A})$

$H(\boldsymbol{Q}\boldsymbol{A}\boldsymbol{Q}^{-1}) = \boldsymbol{Q}\, H(\boldsymbol{A})\, \boldsymbol{Q}^{-1}$ for any invertible $\boldsymbol{Q}$

[7] O. Teke and P. P. Vaidyanathan. "Extending Classical Multirate Signal Processing Theory to Graphs – Part I: Fundamentals". *IEEE Trans. Signal Process. (Submitted).*

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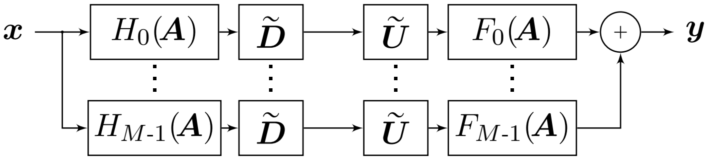

[7] O. Teke and P. P. Vaidyanathan. "Extending Classical Multirate Signal Processing Theory to Graphs – Part I: Fundamentals". *IEEE Trans. Signal Process. (Submitted).*

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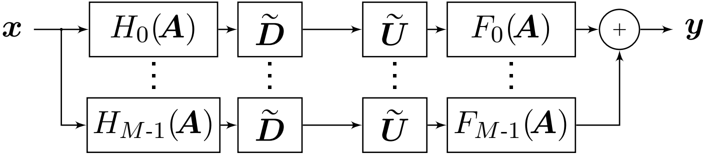
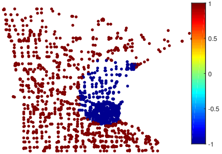


$\widetilde{\boldsymbol{D}}$ and $\widetilde{\boldsymbol{U}}$ have higher complexity, but no restrictive assumptions on $\boldsymbol{A}$.

[7] O. Teke and P. P. Vaidyanathan. "Extending Classical Multirate Signal Processing Theory to Graphs – Part I: Fundamentals". *IEEE Trans. Signal Process. (Submitted).*

Outline

Examples

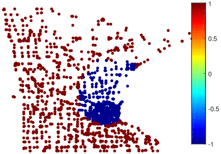


(a) Signal[8] [9]

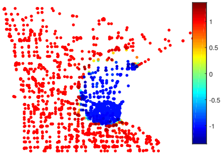
[8] S. Narang and A. Ortega. (2013) Graph bior wavelet toolbox. [Online]. http://biron.usc.edu/wiki/index.php/Graph_Filterbanks

[9] D. K. Hammond, P. Vandergheynst, and R. Gribonval. The spectral graph wavelets toolbox. [Online]. http://wiki.epfl.ch/sgwt

Examples



(i) Signal[8] [9]

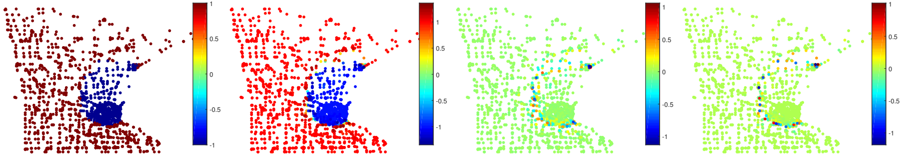


(j) Output of Channel-1

[8] S. Narang and A. Ortega. (2013) Graph bior wavelet toolbox. [Online]. http://biron.usc.edu/wiki/index.php/Graph_Filterbanks

[9] D. K. Hammond, P. Vandergheynst, and R. Gribonval. The spectral graph wavelets toolbox. [Online]. http://wiki.epfl.ch/sgwt

Examples

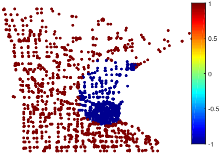


(q) Signal[8] [9] (r) Output of Channel-1 (s) Output of Channel-2 (t) Output of Channel-3

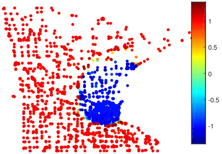
[8] S. Narang and A. Ortega. (2013) Graph bior wavelet toolbox. [Online]. http://biron.usc.edu/wiki/index.php/Graph_Filterbanks

[9] D. K. Hammond, P. Vandergheynst, and R. Gribonval. The spectral graph wavelets toolbox. [Online]. http://wiki.epfl.ch/sgwt

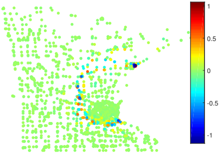
Examples



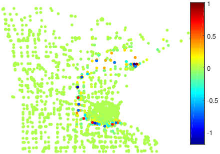
(y) Signal[8] [9]



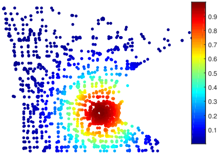
(z) Output of Channel-1



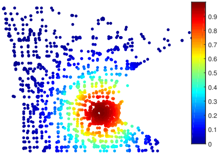
() Output of Channel-2



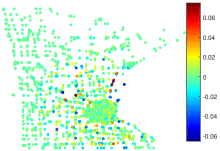
() Output of Channel-3



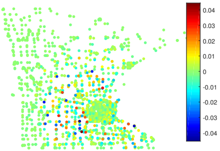
() Signal



() Output of Channel-1



() Output of Channel-2



() Output of Channel-3

[8] S. Narang and A. Ortega. (2013) Graph bior wavelet toolbox. [Online]. http://biron.usc.edu/wiki/index.php/Graph_Filterbanks

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Conclusions

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- Decimator
- M-Block Cyclic Graph
  - Unique Eigenvalue-Eigenvector Structure
- Spectrum Folding
  - Decimation-then-Expansion
  - Bandlimited Signals
  - Interpolation
- M-Channel Filter Banks
- Further Directions
  - How does this compare with alternative ways?
  - From ideal to non-ideal?

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Any questions?

Please email me: oteke@caltech.edu