Graph homomorphisms: structure and symmetry

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Abstract

This paper is the first part of an introduction to the subject of graph homomorphism in the mixed form of a course and a survey. We give the basic definitions, examples and uses of graph homomorphisms and mention some results that consider the structure and some parameters of the graphs involved. We discuss vertex transitive graphs and Cayley graphs and their rather fundamental role in some aspects of graph homomorphisms. Graph colourings are then explored as homomorphisms, followed by a discussion of various graph products.

1 Introduction

Homomorphisms provide a way of simplifying the structure of objects one wishes to study while preserving much of it that is of significance. Most mathematicians remember the isomorphism theorems we learn in a first course on group theory, and certainly anyone involved in some way with mathematics or computer science knows about integers modulo some n. It is not surprising that homomorphisms also appeared in graph theory, and that they have proven useful in many areas.

We do not claim to provide the definitive survey but only an introductory course. Of necessity, we had to omit most proofs and, sadly, even many results and some aspects of the subject matter. But we do try to provide an extensive bibliography and the interested reader will be able to find the missing pieces with little trouble. For the rest, we will all have to wait for the book that Hell and Nešetřil are reportedly writing. The course/survey is divided into two parts. The present paper is concerned with structure and symmetry, the sequel [56] deals with computational aspects of graph homomorphisms and surveys the computer science roots of the subject.

Graph homomorphisms in the current sense were first studied by Sabidussi in the late fifties and early sixties, with results published in the paper on *Graph derivatives* [109] and used by him in, among others, [108]. This was followed by much activity of what has become known as the *Prague school* of category theory. There, people around Hedrlín and Pultr pursued homomorphisms of relational systems in general, and graph homomorphisms in particular, in the sixties and the seventies, with many of the results collected in *Combinatorial, Algebraic and Topological Representations of Groups, Semigroups and Categories* by Pultr and Trnková [102]. Some of this work was continued by Hell in his Ph.D. thesis [65] in 1972 with a study of retracts of graphs. In computer science, graph homomorphisms have been studied as *interpretations*, especially in relation to grammars ([93, 92], for example), and as "general colourings". Welzl's homomorphism density theorem 2.33 (Section 2.5) arose in this context. Others took an algorithmic approach such as *How difficult is it to recognize if a graph is homomorphic to a given one?*, usually – but not always – proving the problem *NP*-complete. This is discussed in more detail in [56].

Graph homomorphisms are mostly used as tools, especially in connection with colouring problems. For example, the chromatic difference sequence of a graph studied by Albertson, Berman, Collins, Tardif, Zhou and others ([3, 5, 119, 132, 134]), relies on them. Since odd cycles must map to odd cycles of at most equal length, people have studied homomorphisms into them, for example [5, 25, 49]. But some of the rare structural results stem from considering graphs which have the property that there are homomorphisms from the cartesian product $G\Box G$ into G.

Many of the concepts of graph homomorphisms work equally well for finite and for infinite graphs (clearly infinite graphs homomorphically equivalent to finite ones "behave" essentially like them). But the important idea of a core of a graph does not easily generalize to infinite graphs and its considerations lead to undecidability results (these are discussed [56]). This is the main reason for limiting the present paper to finite graphs.

2 Basics

2.1 Basic definitions

Unless otherwise indicated, graphs in this paper will be simple, loopless and finite. We shall assume the basics of graph theory and, unless otherwise stated, use the notation of [19] for graphs and of [106] for groups. We will denote the vertex set of a graph G by $V(G)$ and its edge set by $E(G)$; the edge between u and v will be denoted by $[u, v]$. The order of a graph G, i.e. the number of vertices, will be denoted by $|G|$. Most of the time we shall confuse an equivalence class of graphs under isomorphisms with a particular representative of it. Thus, the complete graph K_n will usually be on the vertex set $\{0, 1, \ldots, n-1\} = [n]$. The complement of a graph G will be denoted by $\overline{G}$. We will denote by P_n and by C_n the path and the cycle on n vertices. Functions will be composed from right to left, $g \circ h(x) = gh(x) = g(h(x))$, and we define $\phi(S) = \{\phi(s) : \ s \in S\}$.

Definition 2.1 Let G and H be graphs. A function $\phi : V(G) \longrightarrow V(H)$ is a *homomorphism* from G to H if it preserves edges, that is, if for any edge $[u, v]$ of G, $[\phi(u), \phi(v)]$ is an edge of H. We write simply $\phi : G \longrightarrow H$.

It is usual to think of graphs as sets of vertices with a binary relation. This is consistent with writing $\phi : G \longrightarrow H$ for a homomorphism which maps vertices to vertices. When

$\phi : G \longrightarrow H$ is a homomorphism, it induces a mapping

$$\phi_E : E(G) \longrightarrow E(H),\ \phi_E([u,v]) = [\phi(u), \phi(v)],$$

and we can define the graph $\phi(G) = (\phi(V(G)), \phi_E(E(G)))$, called the *homomorphic image* of G in H.

In group theory, there are trivial homomorphisms from one object to another. Any group can be mapped by a homomorphism into any other by simply sending all its elements to the identity of the target group. In fact, the study of kernels is important in algebra. In the context of graphs without loops, the notion of a homomorphism is far more restrictive. Indeed, there need not be a homomorphism between two graphs, and these cases are as much a part of the theory as those where homomorphisms do exist (there are other categories where homomorphisms do not always exist between two objects, e.g. the category of bounded lattices or that of semigroups, and the interested reader will want to consult [102]).

A surjective homomorphism is often called an *epimorphism*, an injective one a *monomorphism* and a bijective homomorphism is sometimes called a *bimorphism*. Note that unlike in group theory, the inverse of a bijective homomorphism need not be a homomorphism. For example, any bijection from $\overline{K}_n$ to K_n is a bimorphism. A homomorphism from a graph G to itself is called an *endomorphism*. The identity endomorphism on a graph G will be denoted by id_G. The set of endomorphisms of a graph G is a semigroup under composition.

Definition 2.2 A homomorphism $\phi : G \longrightarrow H$ is called *faithful* if $\phi(G)$ is an induced subgraph of H. It will be called *full* if $[u,v] \in E(G)$ if and only if $[\phi(u), \phi(v)] \in E(H)$, that is, when $\phi^{-1}(x) \cup \phi^{-1}(y)$ induces a complete bipartite graph whenever $[x,y] \in E(H)$.

In other words, a homomorphism $\phi : G \longrightarrow H$ is faithful when there is an edge between any two pre-images $\phi^{-1}(u)$ and $\phi^{-1}(v)$ such that $[u,v]$ is an edge of H. When a faithful homomorphism ϕ is bijective, it is full since each $\phi^{-1}(u)$ is a singleton, and we have that $[\phi^{-1}(u), \phi^{-1}(v)]$ is an edge in G if and only if $[u,v]$ is an edge in H. Thus a faithful bijective homomorphism is an *isomorphism* and in this case we write $G \cong H$. An isomorphism from G to G is an *automorphism* of G; the set of automorphisms of a graph forms a group under composition. We shall denote the group of automorphisms of G by $\mathrm{Aut}(G)$.

A surjective faithful homomorphisms ϕ is sometimes called *complete*. When $\phi : G \longrightarrow H$ is a complete homomorphism, H is a homomorphic image of G. Similarly, G is isomorphic to its image under ϕ when ϕ is injective and faithful.

Observation 2.3 *Suppose that $\phi : G \longrightarrow H$ is a graph homomorphism. Then ϕ is an isomorphism if and only if ϕ is bijective and if ϕ^{-1} is also a homomorphism. In particular, if $G = H$ then ϕ is an automorphism if and only if it is bijective.*

Example 2.4 Let P_n be the path $u_0 \dots u_{n-1}$ and K_2 the complete graph with vertices $0, 1$. It is easy to see that the mappings described below are homomorphisms, and that ϕ_2 is onto and ϕ_3 is one-to-one.

- $\phi_1(u_i) = u_i$ when $i = 0, \dots, n-2$, $\phi_1(u_{n-1}) = u_{n-3}$ (from P_n to P_n).
- $\phi_2(u_{2k}) = 0,\ \phi_2(u_{2k+1}) = 1$ for $k = 0, \dots, \lfloor \frac{n}{2} \rfloor$ (from P_n to K_2).

- $\phi_3(0) = u_i,\ \phi_3(1) = u_{i+1}$, for any $0 \leq i < n-1$ (from K_2 to P_n).

In fact, K_2 is the homomorphic image of any bipartite graph. In particular, there always is a homomorphism from the cycle C_{2k} onto K_2 for any k.

Example 2.5 Let $C_m = u_0 \dots u_{m-1}$ and $C_n = v_0 \dots v_{n-1}$ be odd cycles, with $m > n$. There is a homomorphism from C_m to C_n, for example, $\phi : C_m \longrightarrow C_n$ given by $\phi(u_i) = v_i$ for $0 \leq i \leq n-1$, and $u_{n+j} = v_{j(\bmod 2)}$. On the other hand, there is no homomorphism from C_n into C_m.

Observation 2.6 *Let G and H be non-bipartite graphs and $\phi : G \longrightarrow H$ a homomorphism. Then*

(1) *the length of a shortest odd cycle in G (the* odd girth *of G) is at least the odd girth of H;*

(2) *the size $\omega(G)$ of a largest clique in G (the* clique number *of G) is at least $\omega(H)$.*

Example 2.7 A *proper k-colouring* of a graph G is an assignment of colours to the vertices of G in such a way that adjacent vertices get different colours. It is useful to think of a proper k-colouring as a function $c : V \longrightarrow [k]$, with $[k] = \{0, 1, \dots, k-1\}$. Indeed, it is an easy exercise to see that a graph G has a proper k-colouring if and only if there is a homomorphism from G into the complete graph K_k. Thus we have a definition of a proper colouring in the language of homomorphisms.

The colours in the above example induce a partition of the vertex set of the graph and the homomorphism into K_n corresponds to identifying the vertices of the same colour. This can be done in general as is explained in the next section.

The reader might have asked whether between any two graphs there is a homomorphism. Proper colourings provide examples of pairs of graphs neither of which maps into the other by a homomorphism. Recall that the chromatic number $\chi(G)$ of a graph G is the least k for which G has a proper k-colouring. From the observation made in Example 2.7 it follows that if there is a homomorphism $\phi : G \longrightarrow H$, then $\chi(G) \leq \chi(H)$; in other words, a graph with chromatic number k cannot map by a homomorphism into one with a lower chromatic number. Thus the graph K_3 and any graph of chromatic number at least four containing no triangles cannot be homomorphic in either direction. The latter kind of graphs exist, for example certain Kneser graphs (defined in Section 3.4; see in particular Proposition 3.14).

2.2 Quotients

Definition 2.8 Let G be a graph and let $\mathcal{P} = \{V_1, \dots, V_k\}$ be a partition of the vertex set of G into non-empty classes. The *quotient* $G/\mathcal{P}$ of G by $\mathcal{P}$ is the graph whose vertices are the sets $V_1, \dots, V_k$ and whose edges are the pairs $[V_i, V_j]$, $i \neq j$, such that there are $u_i \in V_i$, $u_j \in V_j$ with $[u_i, u_j] \in E(G)$. The mapping $\pi_{\mathcal{P}} : V(G) \longrightarrow V(G/\mathcal{P})$ defined by $\pi_{\mathcal{P}}(u) = V_i$ such that $u \in V_i$, is the *natural* map for $\mathcal{P}$.

Quotients often provide a way of deriving the structure of an object from the structure of a larger one; factor groups are a good example. In the present context it is worth our while to investigate when the natural map $\pi_{\mathcal{P}} : V(G) \longrightarrow V(G/\mathcal{P})$ is a homomorphism. Observe that if $\pi_{\mathcal{P}}$ is a homomorphism, then it is automatically faithful.

Lemma 2.9 *A map $\phi : V(G) \longrightarrow V(H)$ is a graph homomorphism if and only if the pre-image $\phi^{-1}(I)$ of every independent subset I of $V(H)$ is an independent set.*

Corollary 2.10 *Let G be a graph and $\mathcal{P}$ a partition of the vertex set of G. Then $\pi_{\mathcal{P}}$ is a homomorphism if and only if V_i is an independent set for each i.*

Proposition 2.11 *For every homomorphism $\phi : G \longrightarrow H$ there is a partition $\mathcal{P}$ of $V(G)$ into independent sets and a monomorphism $\psi : G/\mathcal{P} \longrightarrow H$ such that $\phi = \psi \circ \pi_{\mathcal{P}}$.*

A homomorphism ϕ of a graph G into H gives rise to an equivalence relation $\equiv_\phi$, the *kernel* of ϕ, defined on V by $u \equiv_\phi v$ if and only if $\phi(u) = \phi(v)$. This, in turn, induces a partition $\mathcal{P}_\phi$ on the vertex set of G; it is this partition that works in the Corollary 2.10 and Proposition 2.11. We can then speak of the *quotient of G by ϕ* and, abusing the notation slightly, we write G/ϕ for $G/\mathcal{P}_\phi$. The language of quotients also gives us a more natural definition of a complete homomorphism.

Proposition 2.12 *A homomorphism $\phi : G \longrightarrow H$ is complete if and only if $\psi : G/\phi \longrightarrow H$ (defined in Proposition 2.11) is an isomorphism.*

One simple but useful consequence is that any homomorphism can be viewed is a sequence of identifications of pairs of vertices by complete homomorphisms.

Definition 2.13 A complete homomorphism $\phi : G \longrightarrow H$ is *elementary* if there is unique pair of (nonadjacent) vertices $u, v \in V(G)$ which are identified by ϕ. We call H an *elementary quotient* of G.

We then have the very useful observation of Lemma 2.14. It allows us to specify a homomorphism simply by saying which vertices are equivalent under the homomorphism under construction and, further, to say which vertices are identified, pair by pair.

Lemma 2.14 *Let G and H be graphs and $\phi : G \longrightarrow H$ a homomorphism. Then there is a natural number k and graphs $G = G_0, G_1, \ldots, G_k$ such that G_{i+1} is an elementary quotient of G_i when $i < k$ and $G_k \cong G/\phi$.*

One type of homomorphism which is best defined in terms of partitions although in the literature ([30, 22, 45]) it is not defined in this language, is a folding.

Definition 2.15 (i) An elementary homomorphism $\phi : G \longrightarrow H$ is a *simple fold* if the two vertices which are identified have a common neighbour.

(ii) A *folding* is a homomorphism obtained as a sequence of simple folds. If $\phi : G \longrightarrow H$ and ϕ is a folding, we say that *G folds onto H*.

The above-mentioned papers deal with minimal graphs that can fold *onto* complete graphs, a problem raised in [30] and answered in [45].

Usually, a graph cannot fold onto each of its homomorphc images. For example, a disconnected graph cannot fold onto a connected one. A simple connected example is a path of length three which cannot fold onto a triangle but can be mapped to it by an elementary homomorphism. However, we will see in the next section that some important homomorphisms *are* foldings.

2.3 Retracts

When ϕ is an endomorphism of a graph G, the image of G under ϕ is a subgraph of G and there is a natural homomorphism back from $\phi(G)$ into G, namely the inclusion map. Such relationships between G and $\phi(G)$ are natural objects to consider and lead to the idea of retracts. The concept is not proper to graph theory – retracts have their origin in topology and appear often in category theory. Hell's thesis [65] is concerned with category-theoretic aspects of graph retracts and extends the work of the Prague school ([102]). Our concerns are slightly different and so we shall not elaborate on this.

Definition 2.16 Let G and H be graphs. Then H is called a *retract* of G if there are homomorphisms $\rho : G \longrightarrow H$ and $\gamma : H \longrightarrow G$ such that $\rho \circ \gamma = id_H$. The homomorphism ρ is called a *retraction* and γ a *co-retraction.*

The composition of two retractions is again a retraction and so a retract of a retract of G is a retract of G. A co-retraction is always a faithful monomorphism and $\gamma(H)$ an induced subgraph of G. Thus retracts of G are (isomorphic to) induced subgraphs of G. We usually think of retracts as subgraphs, since an endomorphism ρ of G onto its subgraph R is a retraction whenever $\rho \restriction R = id_R$. In this case the co-retraction is naturally the inclusion map.

Observation 2.17 Since there exist homomorphisms in both directions between a graph G and any of its retracts H, it follows that G and H have the same chromatic number, odd girth and clique number. Note also that $\chi(G) = \omega(G)$ if and only if $K_{\chi(G)}$ is a retract of G.

The condition of Definition 2.16 that the composition of a retraction and its associated co-retraction be the identity can be slightly relaxed, providing a more useful characterization of retracts.

Lemma 2.18 *Let G, H be graphs. Then H is a retract of G if and only if there exist homomorphisms $\rho : G \longrightarrow H$ and $\sigma : H \longrightarrow G$ such that $\rho \circ \sigma \in \mathrm{Aut}(H)$.*

In particular, if H is a subgraph of G, then H is a retract of G is and only if there is a homomorphism $\rho : G \longrightarrow H$ whose restriction to H is an automorphism of H.

A retract of a graph G is always a quotient of G (i.e., a homomorphic image), and retractions are complete homomorphisms. In the case of connected graphs we can say more.

Proposition 2.19 *Any retraction of a connected graph is a folding.*

Proof Let $\rho : G \longrightarrow H$ be a retraction, where G is connected and H is a subgraph of G. Let $u \in V(G)$ be a vertex not in H, adjacent to some vertex v in H. Since the retraction ρ which takes G onto H must map u to a neighbour of v, the elementary homomorphism ρ_u which identifies u and $\rho(u)$ is a simple fold. The resulting graph still has H as a retract and so a folding can be constructed by repeating the argument. □

In some sense, retracts are easy to find – if we already have an endomorphism, the following lemma allows us to find a retraction.

Lemma 2.20 *Let ϕ be an endomorphism of a graph G. Then there is an n such that $R = \phi^n(G)$ is a retract of G (and ϕ^n a retraction). Further, $\phi \restriction R$ is an automorphism of R.*

Observation 2.17 shows that there are (many) subgraphs of G which are not retracts of G. Indeed, no subgraph of G with chromatic number smaller than that of G is a retract of G. On the other hand, a graph is always a retract of itself and need not have any other retracts.

2.4 Cores

Whenever one looks at substructures with some given property one is naturally tempted to ask whether there is a smallest one that has the property. This is often the case and the property of being a retract is no exception. Minimal retracts are known as cores.

Definition 2.21 A graph G is a *core* if no proper subgraph of G is a retract of G.

Natural though it is, we never work with this definition but replace it at once by an equivalent one which is technically much easier to handle. Note, however, that as it stands, Definition 2.21 applies equally well to finite and infinite graphs, but that this is not the case for its reformulated version:

Proposition 2.22 *Let G be a graph. Then G is a core if and only if every endomorphism of G is an automorphism of G.*

Proof Suppose G is a core, and let $\phi : G \longrightarrow G$ be an endomorphism. By Lemma 2.20, $\phi^n(G)$ is a retract of G for some n. Thus $\phi^n(G) = G$ by the definition of a core, and hence $\phi \in \mathrm{Aut}(G)$. The converse is trivial. □

The above proposition suggests another name for cores - *automorphic graphs.* This would be consistent with the names for similiar concepts in other branches of mathematics.

Example 2.23 The following graphs are cores. While some are easily seen to be such, others are justified later.

- K_n, $n \geq 1$;
- C_{2k+1} (the odd cycle) for $k \geq 1$;
- W_{2k+1} (the odd wheel, that is, an odd cycle with one extra vertex adjacent to all vertices of the cycle), for $k \geq 1$;
- the Petersen graph;
- any χ-critical graph, that is, a graph for which the chromatic numbers of its proper subgraphs are strictly smaller than its chromatic number.

Example 2.24 On the other hand, these are not cores.

- C_{2k} for $k > 0$;
- complete graphs with one edge removed;

- a disjoint union of cycles.

Definition 2.25 A retract H of G is called a *core of* G if it is a core.

Among all subgraphs of G which are retracts of G choose one, say H, having the smallest number of vertices. Since retracts of retracts of G are retracts of G, it follows that H is a core. Thus:

Proposition 2.26 *Every finite graph has a core.*

Proposition 2.27 *If H_0 and H_1 are cores of a graph G then they are isomorphic.*

Proof Consider such a pair of cores of G and consider the retractions ρ_0 and ρ_1 of G onto H_0 and H_1, respectively. Without loss of generality we can assume that both H_0 and H_1 are subgraphs of G. Let $\sigma_i : H_i \longrightarrow H_{1-i}$ be the restriction of ρ_{1-i} to H_i, $i = 0, 1$. Then $\sigma_{1-i}\sigma_i$ is an endomorphism of H_i, and since H_i is a core, $\sigma_{1-i}\sigma_i \in \mathrm{Aut}(H_i)$, by Proposition 2.22. This means that σ_0, σ_1 are, in fact, isomorphisms. □

Since all cores of a graph G are isomorphic, it is legitimate to speak of *the* core of a graph G; we will denote it by $G^\bullet$. Strictly speaking, $G^\bullet$ is any member of its isomorphism class and we usually take for $G^\bullet$ a subgraph of G (which is necessarily induced). This way of thinking of the core is more intuitive. For example, if G is bipartite then $G^\bullet \cong K_2$, and so any edge of G is its core.

2.5 Homomorphic equivalence

It is useful to define a relation on the class of graphs using homomorphisms.

Definition 2.28 If there is a homomorphism from a graph G to a graph H we say that G *maps to* H and we write $G \to H$.

For a simple example, note that $H \to G$ for any subgraph H of G. The relation $\to$ is clearly transitive and reflexive but is not symmetric (an even cycle maps to an odd one but not vice versa). In fact, it does not define a partial order since it is not anti-symmetric either (for example, any two bipartite graphs map one into the other even when they are not isomorphic). Thus the relation $\to$ is only a quasi-order on the class of all graphs.

Definition 2.29 We say that graphs G and H are *homomorphically equivalent* if $G \to H$ and $H \to G$. When this is the case we write $G \leftrightarrow H$.

For instance, if R is a retract of G then $G \leftrightarrow R$. Indeed, the definition of a retract provides the homomorphisms in both direction, namely the retraction $\rho : G \longrightarrow R$ and the co-retraction $\gamma : R \longrightarrow G$. This is true, in particular, of cores of graphs. Therefore, given any two graphs G, H, we have that $G \leftrightarrow H$ implies $G^\bullet \leftrightarrow H^\bullet$. However, the following much stronger result holds.

Proposition 2.30 *Let G, H be graphs such that $G \leftrightarrow H$. Then the cores of G and H are isomorphic.*

Proof Consider the cores $G^\bullet$ and $H^\bullet$ together with their retractions $\rho_G : G \longrightarrow G^\bullet$ and $\rho_H : H \longrightarrow H^\bullet$ and co-retractions $\gamma_G : G^\bullet \longrightarrow G$ and $\gamma_H : H^\bullet \longrightarrow H$. Let $\phi_G : G \longrightarrow H$ and $\phi_H : H \longrightarrow G$ be homomorphisms. Then the mapping $\psi : G^\bullet \longrightarrow G^\bullet$ defined by $\psi = \rho_G \circ \phi_H \circ \phi_G \circ \gamma_G$ is a homomorphism and hence an automorphism of $G^\bullet$. Therefore $\rho_G \circ \phi_H : H \longrightarrow G^\bullet$ is a retraction of H onto $G^\bullet$ (and $\phi_G \circ \gamma_G : G^\bullet \longrightarrow H$ a co-retraction), thus $G^\bullet$ is a core of H by Lemma 2.18. By Proposition 2.27, $G^\bullet$ is isomorphic to $H^\bullet$. □

Clearly $\leftrightarrow$ is an equivalence relation on the class of all graphs. We shall denote by $\mathcal{H}(G)$ the equivalence class of this relation containing the graph G. The following corollary follows from our discussion of retracts, cores and from the definition of homomorphic equivalence.

Corollary 2.31 *Let G be a graph. Then to within isomorphism, $G^\bullet$ is the unique graph of smallest order in $\mathcal{H}(G)$.*

The above corollary indicates why graph homomorphisms are interesting and important. The various aspects of any property which is preserved under homomorphisms can be studied on graphs best adapted for such investigations, provided they all belong to the same equivalence class (here, as elsewhere, we omit the qualifier *under homomorphisms*). For example, all the graphs in a class $\mathcal{H}(G)$ have the same odd girth, chromatic number, ultimate independence ratio, etc.

The set of equivalence classes of finite graphs can be partially ordered by homomorphisms between their elements.

Definition 2.32 Let G and H be graphs. Then $\mathcal{H}(G) \prec \mathcal{H}(H)$ if $G \to H$.

This partially ordered set is, in fact, a lattice. We will indicate what the join and the meet are in the next section. Moreover (assuming we consider only graphs having at least one edge), there is a unique equivalence class which is minimal with respect to $\prec$, namely the class $\mathcal{H}(K_2)$ consisting of all bipartite graphs. It was first noticed by Welzl [127] that the part of the lattice above the minimal class $\mathcal{H}(K_2)$ has the remarkable property of being order-dense. Although this result concerns equivalence classes of graphs we shall formulate it in terms of representatives.

Theorem 2.33 (Welzl) *Let G and H be graphs such that $G \to H$ and $H \not\to G$. Then there is a graph K such that $G \to K \to H$ and $H \not\to K \not\to G$.*

In terms of the equivalence classes this means that $\mathcal{H}(G) \prec \mathcal{H}(K) \prec \mathcal{H}(H)$, both inequalities being strict. Welzl's proof is rather long, a short one will be given in Section 5.1.

2.6 Products

We need to recall some of the various graph products that have been defined and used. We shall use the natural descriptive notation devised by Nešetřil. We will consider four products, two of them in some depth.

Definition 2.34 Let G and H be graphs. The following products of G and H are defined on the vertex set $V = V(G) \times V(H)$.

- the *cartesian* product $G \Box H$, with

$$E(G \Box H) = \{[(u,x),(v,y)] : \text{ either } u = v, [x,y] \in E(H), \text{ or } [u,v] \in E(G), x = y\};$$

- the *categorical* product $G \times H$, with

$$E(G \times H) = \{[(u,x),(v,y)] : [u,v] \in E(G), [x,y] \in E(H)\};$$

- the *strong* product $G \boxtimes H$, with

$$E(G \boxtimes H) = E(G \times H) \cup E(G \Box H);$$

- the *lexicographic* product $G[H]$, with

$$E(G[H]) = \{[(u,x),(v,y)] : \text{ either } u = v, [x,y] \in E(H), \text{ or } [u,v] \in E(G)\}.$$

All the products can be found in the literature under a variety of names. The notation in the first three products indicates exactly what the result of taking the product of K_2 by K_2 is; this defines the whole graph.

It is easy to show that all the products are associative and that all but the lexicographic product are commutative. We can then define products of k graphs $G_1, \ldots, G_k$ in the obvious manner and denote them by $\Box_{i=1}^{k} G_i$ and, analogously, for $\times$ and $\boxtimes$. The meaning of G^k (when $G_i = G$ for all i) will be specified as needed.

When a product of graphs G and H is studied, it is often useful to consider the graphs induced by $\{u\} \times V(H)$ or by $V(G) \times \{x\}$. It is easy to see that these graphs, called *fibers*, are isomorphic to G or H in the case of the cartesian, strong, and lexicographic products. Hence the next observation.

Observation 2.35 *Let G and H be graphs and let $*$ be any of the cartesian, strong or lexicographic product. Then $G \to G * H$ and $H \to G * H$. In fact, in each case G and H are induced subgraphs of the product.*

The converses are not always true. In particular, if both G and H have at least one edge, then $\omega(G \boxtimes H) = \omega(G[H]) \geq \max\{\omega(G), \omega(H)\}$, and so these products do not map into their factors. The cartesian product sometimes does have a homomorphism into its factors, but rarely (see Section 5.2).

Let X and Y be sets and $X \times Y$ their cartesian product. The *projections* of $X \times Y$ onto their factors are the mappings $\mathrm{pr}_X : X \times Y \longrightarrow X$ defined by $\mathrm{pr}_X(u,x) = u$ and $\mathrm{pr}_Y : X \times Y \longrightarrow Y$ defined by $\mathrm{pr}_Y(u,x) = x$. When G and H are graphs, and $X = V(G)$, $Y = V(H)$, we write simply pr_G and pr_H. Clearly, the projections are not homomorphisms for the cartesian, strong and lexicographic products, and they are homomorphisms for the categorical product. The latter proves the next lemma.

Lemma 2.36 *Let G and H be graphs. Then $G \times H \to G$ and $G \times H \to H$.*

The converse is not true: $K_2 \times K_3 \cong C_6$, and while the bipartite C_6 maps to K_2 and can be "wrapped around" K_3, the latter does not map into C_6

The categorical product allows us to make a lattice of homomorphism equivalence classes and, by extension, of cores. Recall that $\mathcal{H}(G)$ is the equivalence class containing G under homomorphisms.

Theorem 2.37 *The equivalence classes of graphs under homomorphisms form a lattice under the partial order $\prec$. The join of $\mathcal{H}(G)$ and $\mathcal{H}(H)$ is the equivalence class containing the disjoint union of G and H, and the meet of $\mathcal{H}(G)$ and $\mathcal{H}(H)$ is $\mathcal{H}(G \times H)$.*

Products preserve homomorphisms in the following manner. Let G, H, G', H' be graphs with $\phi : G \longrightarrow G'$ and $\psi : H \longrightarrow H'$. Define $\tau : V(G) \times V(H) \longrightarrow V(G') \times V(H')$ by $\tau(u, x) = (\phi(u), \psi(x))$. Then τ is a homomorphism for each of the products defined above. In particular, $G \Box H \leftrightarrow G^{\bullet} \Box H^{\bullet} \leftrightarrow (G \Box H)^{\bullet}$ and similarly for the other products. It is not true, however, that the product of cores of graphs is the core of the products.

Example 2.38 Consider K_2 and C_{2k+1}. Both are cores but $(K_2 \Box C_{2k+1})^{\bullet} = C_{2k+1}$. Similarly, $(K_2 \times C_{2k+1})^{\bullet} = K_2$.

3 Vertex-transitive graphs

Bijectivity is a very restrictive condition to impose on a homomorphism. Nonetheless, automorphisms play a significant role in the general study of homomorphisms, in view of some of the facts presented earlier. We have seen in Lemma 2.20 that every endomorphism $\phi : G \longrightarrow G$ acts as an automorphism on some retract of G. Also, the core graphs, which represent all classes of graphs, have the property that all of their endomorphisms are automorphisms. By themselves, these observations do not make the subject of vertex-transitive graphs a central theme in the study of homomorphisms, though they emphasize its study, for its own sake, as a fruitful venture. The practical motivation for studying vertex-transitive graphs in the context of graph homomorphisms rather comes from special classes of graphs which are vertex-transitive and play a central role in some aspects of the theory (as complete graphs do in graph colouring).

This section is therefore intended to serve two purposes. First, to present results that are specific to the context of homomorphisms of vertex-transitive graphs, such as Sabidussi's theorem, the No-Homomorphism Lemma and Welzl's theorem on cores of vertex-transitive graphs. Second, to introduce some families of graphs such as Cayley graphs, Kneser graphs and circular graphs, which are of relevance in other sections of this paper.

3.1 Cayley graphs

Let Γ be a group and S a subset of Γ that is closed under inverses and does not contain the identity[1]. The *Cayley graph* $\mathrm{Cay}(\Gamma, S)$ is the graph with Γ as its vertex set, two vertices u and v being joined by an edge if and only if $u^{-1}v \in S$. Simple examples of Cayley graphs include the cycles, which are Cayley graphs of cyclic groups, and the complete graphs K_n,

[1] Such a set is often called a *Cayley subset* of Γ.

which are Cayley graphs of any group of order n. Specific names are sometimes given to some classes of Cayley graphs; for instance, Cayley graphs of cyclic groups are often called *circulants* (the name comes from the fact that their adjacency matrices are called "circulant matrices"), and a Cayley graph $\mathrm{Cay}(F, S)$, where F is a finite field, $|F| \equiv 1 \bmod 4$, and S is the set of quadratic residues of F, is called a *Paley graph*.

Cayley graphs constitute a rich class of vertex-transitive graphs. It is well known that the left translations (that is, bijections $T_a : \Gamma \longrightarrow \Gamma$ defined by $T_a(x) = ax$) of a Cayley graph are automorphisms, thus ensuring vertex transitivity. Since a Cayley graph may also have other automorphisms, the group structure is not encoded in the graph structure.[2] In shifting the focus from automorphisms to homomorphisms, one could expect to see the relative importance of Cayley graphs diminish within the subject of graph symmetry. The topic of graph homomorphisms, however, emphasizes Cayley graphs as a central theme in the study of vertex-transitive graphs for the following reason: *up to homomorphic equivalence, Cayley graphs represent all classes of vertex-transitive graphs.*

Let G be a vertex-transitive graph, and u_0 a fixed vertex of G. Recall that $\mathrm{Aut}(G)$ denotes the automorphism group of G and $\mathrm{Stab}(u_0)$ the *stabilizer* of u_0, i.e., the subgroup of $\mathrm{Aut}(G)$ containing all automorphisms ϕ such that $\phi(u_0) = u_0$. Put

$$S = \{\sigma \in \mathrm{Aut}(G) : [u_0, \sigma(u_0)] \in E(G)\}.$$

Thus S is a union of left cosets of $\mathrm{Stab}(u_0)$.

It is easily seen that S contains the inverse of each of its elements, and that the identity does not belong to S. Thus we can define the Cayley graph $\mathrm{Cay}(\mathrm{Aut}(G), S)$. By the definition of S, for $\phi_1, \phi_2 \in \mathrm{Aut}(G)$, we have $[\phi_1, \phi_2] \in E(\mathrm{Cay}(\mathrm{Aut}(G), S))$ if and only if $[\phi_1(u_0), \phi_2(u_0)] \in E(G)$. With this characterization of adjacency in $\mathrm{Cay}(\mathrm{Aut}(G), S)$, it is easy to see that the maps ρ and γ defined below are homomorphisms between G and $\mathrm{Cay}(\mathrm{Aut}(G), S)$.

- $\rho : \mathrm{Cay}(\mathrm{Aut}(G), S) \longrightarrow G$, defined by putting $\rho(\phi) = \phi(u_0)$.
- $\gamma : G \longrightarrow \mathrm{Cay}(\mathrm{Aut}(G), S)$, defined by arbitrarily selecting $\gamma(u) = \phi_u$ such that $\phi_u(u_0) = u$ for each $u \in V(G)$.

Note that $\rho \circ \gamma(u) = u$ for any $u \in V(G)$, so ρ is a retraction. Thus, we have the following.

Theorem 3.1 (Sabidussi [108]) *Any vertex-transitive graph is a retract of some Cayley graph.* □

It is worthwhile to discuss some additional properties of the retraction ρ. Note that for any $u \in V(G)$, the set

$$\rho^{-1}(u) = \{\phi \in \mathrm{Aut}(G) : \phi(u_0) = u\}$$

is a left coset of $\mathrm{Stab}(u_0)$. In fact, ρ is a full homomorphism – if $[u, v]$ is an edge of G, then $\rho^{-1}(u) \cup \rho^{-1}(v)$ induces a complete bipartite graph; this accounts for the freedom encountered in defining γ. But we have more: $\mathrm{Cay}(\mathrm{Aut}(G), S)$ is isomorphic to the lexicographic product

[2]Nonetheless, a result of Babai and Godsil provides evidence that most Cayley graphs have a regular automorphism group, see [7].

$G[\overline{K}_k]$ with $k = |\mathrm{Stab}(u_0)|$. Thus the structure of $\mathrm{Cay}(\mathrm{Aut}(G), S)$ is independent of the choice of u_0. A graph isomorphic to $G[\overline{K}_k]$ for some k is called a *multiple* of G.

With this in mind, we give an alternative statement of Theorem 3.1 which is closer to its original spirit.

Theorem 3.1 (Multiples version) *Let G be a vertex-transitive graph. Then some multiple of G is a Cayley graph.* □

In our approach, a Cayley graph on $\mathrm{Aut}(G)$ is defined first, then the structure of G is recovered by retracting $\mathrm{Cay}(\mathrm{Aut}(G), S)$ onto it. It is possible to interchange the order of these steps; this gives rise to the notion of a Cayley coset graph. Let Γ be a group, Γ_0 a subgroup of Γ and S a subset of Γ that is closed under inverses and does not contain the identity. The *Cayley coset graph* $\mathrm{Ccg}(\Gamma, S, \Gamma_0)$ is defined by putting

$$\begin{aligned} V(\mathrm{Ccg}(\Gamma, S, \Gamma_0)) &= \{a\Gamma_0 : a \in \Gamma\}, \\ E(\mathrm{Ccg}(\Gamma, S, \Gamma_0)) &= \{[a\Gamma_0, b\Gamma_0] : a\Gamma_0 \neq b\Gamma_0 \text{ and } a\Gamma_0 \cap b\Gamma_0 S \neq \emptyset\}. \end{aligned}$$

The condition $a\Gamma_0 \cap b\Gamma_0 S \neq \emptyset$ means that there exist $c \in a\Gamma_0$, $d \in b\Gamma_0$ such that $d^{-1}c \in S$. In particular, our retraction ρ amounts to collapsing $\mathrm{Cay}(\mathrm{Aut}(G), S)$ onto the coset graph $\mathrm{Ccg}(\mathrm{Aut}(G), S, \mathrm{Stab}(u_0))$. Thus, Theorem 3.1 also admits the following formulation.

Theorem 3.1 (Cayley coset graphs version) *Any vertex-transitive graph is isomorphic to a Cayley coset graph.* □

In general, the natural map from a Cayley graph $\mathrm{Cay}(\Gamma, S)$ to a Cayley coset graph $\mathrm{Ccg}(\Gamma, S, \Gamma_0)$ is a homomorphism if and only if $\Gamma_0 \cap S = \emptyset$. However, an element of $\Gamma_0 \cap S$ does not contribute to the structure of $\mathrm{Ccg}(\Gamma, S, \Gamma_0)$ in any way, so the condition $\Gamma_0 \cap S = \emptyset$ is not a severe restriction on a Cayley coset graph.

3.2 Indepedemce ratio and the No-Homomorphism Lemma

Other than trial and error, there seems to be no sure-fire way to determine whether there exists a homomorphism from a given graph to another. Parameters such as the odd girth and the chromatic number provide some restrictions, but these are far from being exhaustive. The basic idea behind the No-Homomorphism Lemma is to use symmetry to find new restrictions for the existence of homomorphisms between graphs. Only the simplest version is given here, but it is clear from the proof that many generalizations and variations are possible.

We write $\alpha(G)$ for the *independence number* of a graph G, that is, the maximum number of vertices in an independent set of G.

Definition 3.2 The *independence ratio* of a graph G is $i(G) = \alpha(G)/|V(G)|$.

Lemma 3.3 (No-Homomorphism Lemma, Albertson and Collins [5]) *Let G, H be graphs such that H is vertex-transitive and $G \to H$. Then*

$$i(G) \geq i(H).$$

Proof Let $\mathcal{S}(H)$ denote the family of independent sets of size $\alpha(H)$ in H. By symmetry, every vertex of H is in the same number, say m, of members of $\mathcal{S}(H)$. We then have

$$\alpha(H) \cdot |\mathcal{S}(H)| = m \cdot |H|, \tag{3.1}$$

since each expression counts the number of inclusions $u \in I$, with $u \in V(H)$ and $I \in \mathcal{S}(H)$.

Let $\phi : G \longrightarrow H$ be a homomorphism. Then, for each $I \in \mathcal{S}(H)$, we have $|\phi^{-1}(I)| \leq \alpha(G)$. Summing this inequality for all members of $\mathcal{S}(H)$, we get

$$\sum_{I \in \mathcal{S}(H)} |\phi^{-1}(I)| \leq \alpha(G) \cdot |\mathcal{S}(H)|. \tag{3.2}$$

However, each $u \in V(G)$ contributes exactly m to the sum $\sum_{I \in \mathcal{S}(H)} |\phi^{-1}(I)|$, since $\phi(u)$ belongs to exactly m members of $\mathcal{S}(H)$. Thus,

$$\sum_{I \in \mathcal{S}(H)} |\phi^{-1}(I)| = m \cdot |G|. \tag{3.3}$$

Combining (3.1), (3.2), (3.3), we get

$$i(G) = \alpha(G)/|G| \geq m/|\mathcal{S}(H)| = \alpha(H)/|H| = i(H). \tag{3.4}$$

□

The name "No-Homomorphism Lemma" is derived from the contrapositive: *if H is vertex-transitive and $i(G) < i(H)$, then $G \not\to H$.* One trivial consequence of this result is the well-known fact that the chromatic number of a graph G is at least $|G|/\alpha(G)$. In terms of Lemma 3.3, we may rephrase this by saying that a necessary condition for G to admit a homomorphism into K_n is that $i(G) \geq 1/n$.

Example 3.4 Consider the odd cycles. We have $\alpha(C_{2k+1}) = k$, and the No-Homomorphism Lemma states that if $C_{2k+1} \to C_{2k'+1}$, then $k/(2k+1) \geq k'/(2k'+1)$, i.e., $k \geq k'$.

In fact, there exists a homomorphism from C_{2k+1} to $C_{2k'+1}$ if and only if $k \geq k'$, but sufficiency cannot be deduced from Lemma 3.3.

Example 3.5 Let $G = \mathrm{Cay}(\mathbb{Z}_8, \{\pm 1, \pm 2\})$ and $H = \mathrm{Cay}(\mathbb{Z}_7, \{\pm 1, \pm 2\})$. Then $\alpha(G) = \alpha(H) = 2$. So,

$$i(G) = 2/8 < 2/7 = i(H).$$

Since H is vertex-transitive, this implies that there exists no homomorphism from G to H.

In fact, there exists no homomorphism from H to G either for the graphs in the above example, though this cannot be deduced from the No-Homomorphism Lemma. A direct verification of this statement could be messy. However, if we define the parameter $\pi(G)$ as the maximum number of vertices in an induced subgraph of G that does not contain a triangle, we get $\pi(G) = 5$, $\pi(H) = 4$, so $\pi(H)/|H| < \pi(G)/|G|$. The reader might want to check how the proof of the No-Homomorphism Lemma can be adapted to deduce from this that $H \not\to G$.

The critical situation in the No-Homomorphism Lemma is the case of homomorphisms between graphs with the same independence ratio. The next result shows how the behaviour of homomorphisms is restricted in that case.

Lemma 3.6 *Let G, H be graphs such that H is vertex-transitive, $i(G) = i(H)$ and $G \to H$. Then for any independent set I of cardinality $\alpha(H)$ in H and any homomorphism $\phi : G \longrightarrow H$ we have*

$$|\phi^{-1}(I)| = \alpha(G).$$

Proof With the notation of the proof of Lemma 3.3, our hypotheses imply that equality holds in (3.4). Thus, $m \cdot |G| = \alpha(G) \cdot |\mathcal{S}(H)|$. By (3.3), we then have $\sum_{I \in \mathcal{S}(H)} |\phi^{-1}(I)| = \alpha(G) \cdot |\mathcal{S}(H)|$. Since $|\phi^{-1}(I)| \leq \alpha(G)$ for all $I \in \mathcal{S}(H)$, we must have $|\phi^{-1}(I)| = \alpha(G)$ for all $I \in \mathcal{S}(H)$. □

In particular, the situation described in Lemma 3.6 arises in the case of homomorphically equivalent graphs. If G and H are homomorphically equivalent vertex-transitive graphs, then the No-Homomorphism Lemma states that $i(G) \geq i(H)$ and $i(H) \geq i(G)$. Thus, $i(G) = i(H)$, and Lemma 3.6 says that any homomorphism between G and H induces a map between $\mathcal{S}(H)$ and $\mathcal{S}(G)$. Sometimes this information helps to characterize such homomorphisms (see Proposition 3.15).

3.3 Cores of vertex-transitive graphs

Theorem 3.1 tells us that any vertex-transitive graph is a retract of a Cayley graph. As retractions often provide a way of simplifying the structure of an object, it would be worthwhile to investigate the retracts *of* vertex-transitive graphs as well. However, retractions do not generally preserve vertex transitivity; for instance, any even cycle retracts onto a path of the same diameter. With respect to symmetry, the most interesting retract of a vertex-transitive graph is its core. Recall that core graphs have the important property that all of their endomorphisms are automorphisms, and this is the basis of many useful properties of cores of vertex-transitive graphs, starting with the following (also independently proved by MacGillivray):

Theorem 3.7 (Welzl [128]) *Let G be a vertex-transitive graph. Then its core $G^\bullet$ is vertex-transitive.*

Proof Fix a retraction $\rho : G \longrightarrow G^\bullet$ with co-retraction $\gamma : G^\bullet \longrightarrow G$. For any $u, v \in V(G^\bullet)$, there exists an automorphism ϕ of G mapping $\gamma(u)$ to $\gamma(v)$. We then have $\rho \circ \phi \circ \gamma(u) = v$ (since $\rho \circ \gamma$ is the identity map on $G^\bullet$). But since $G^\bullet$ is a core, the map $\rho \circ \phi \circ \gamma : G^\bullet \longrightarrow G^\bullet$ is an automorphism of $G^\bullet$. Thus, $\mathrm{Aut}(G^\bullet)$ acts transitively on $G^\bullet$. □

Along the same lines, it is easy to show that the core of an edge-transitive graph is again edge-transitive and the same idea applies to many other types of symmetry.

Theorem 3.7 allows us to apply the results of the preceding section. A vertex-transitive graph and its core are homomorphically equivalent vertex-transitive graphs, so Lemma 3.3 states that they have the same independence ratio. Since this ratio is defined with integer parameters on graphs, we deduce a sufficient condition for a graph to be a core.

Corollary 3.8 *Let G be a vertex-transitive graph. If $\alpha(G)$ and $|G|$ are relatively prime, then G is a core.*

In particular, this occurs for vertex-transitive graphs on a prime number of vertices. It is known that such a graph is isomorphic to a circulant $\mathrm{Cay}(\mathbb{Z}_p, S)$, whose automorphism group is generated by the left translations and the automorphisms of $\mathbb{Z}_p$ that preserve the connection set S (see Turner [122]). By the preceding result, these graphs are also cores, and thus have no endomorphisms other than their automorphisms.

In general, the result of Lemma 3.6 cannot be formulated for vertices rather than maximum independent sets. For instance, consider the homomorphisms between the 6-cycle and the 4-cycle. Note that the cardinalities of inverse images of individual vertices vary, while the sum over maximal independent sets is constant. The next result show that the situation is different in the case of cores.

Theorem 3.9 *Let ϕ be a homomorphism of a vertex-transitive graph G onto $G^\bullet$. Then all inverse images $\phi^{-1}(u), u \in V(G^\bullet)$, have the same cardinality, namely $|G|/|G^\bullet|$.*

Proof Fix a homomorphism $\gamma : G^\bullet \longrightarrow G$ and a vertex u of $G^\bullet$. Put

$$X = \{(v, \psi) : v \in V(G^\bullet), \psi \in \mathrm{Aut}(G) \text{ and } \phi \circ \psi \circ \gamma(v) = u\}.$$

The cardinality of X can be evaluated in two ways: First, for any $\psi \in \mathrm{Aut}(G)$, the map $\phi\circ\psi\circ\gamma : G^\bullet \longrightarrow G^\bullet$ is an automorphism of $G^\bullet$, so it must be bijective. Therefore there exists a unique $v \in V(G^\bullet)$ such that $\phi\circ\psi\circ\gamma(v) = u$, i.e., $(v, \psi) \in X$. Thus $|X| = |\mathrm{Aut}(G)|$. Second, for any $v \in V(G^\bullet)$ and $x \in \phi^{-1}(u)$, the set of automorphisms of G mapping $\gamma(v)$ to x is a left-coset of $\mathrm{Stab}(\gamma(v))$, and has cardinality $|\mathrm{Aut}(G)|/|G|$. Thus $|X| = |G^\bullet| \cdot |\phi^{-1}(u)| \cdot (|\mathrm{Aut}(G)|/|G|)$. Combining these two expressions for $|X|$, we have $|\phi^{-1}(u)| = |G|/|G^\bullet|$. □

This last result is an analogue for cores of vertex-transitive graphs of Lagrange's theorem: the order of $G^\bullet$ divides that of G.

Any fixed retraction ρ and co-retraction γ between a graph G and its core $G^\bullet$ induce a map between $\mathrm{Aut}(G)$ and $\mathrm{Aut}(G^\bullet)$, defined by mapping $\phi \in \mathrm{Aut}(G)$ to $\rho \circ \phi \circ \gamma \in \mathrm{Aut}(G^\bullet)$. Some results of this section are applications of this fact, but notice that this induced mapping between groups is devoid of algebraic significance. A case in point is the fact that the core of a Cayley graph is not necessarily a Cayley graph. The Petersen graph is a well-known example of a vertex-transitive graph which is not a Cayley graph. It will be shown in the next section that the Petersen graph is a core. However, by Theorem 3.1, the Petersen graph is a retract of a Cayley graph, so it is the core of a Cayley graph.

3.4 Kneser graphs

Let r, s be integers such that $1 \leq r < s/2$. The *Kneser graph* $K(r, s)$ is the graph whose vertices are the r-subsets of $[s]$, two vertices being joined by an edge if and only if they are disjoint. Kneser graphs are a combinatorial structure which arises naturally in different contexts. The graphs $K(r, 2r + 1), r \geq 1$ are also called *odd graphs*; amongst them we find $K(2, 5)$, which is the Petersen graph. Also, $K(1, s)$ is the complete graph K_s. We will see below a connection between Kneser graphs and the Erdős-Ko-Rado theorem, and other sections of this paper will show their relation with the lexicographic product of graphs and some parameters such as the fractional chromatic number.

It will also be shown that Kneser graphs provide examples of graphs with arbitrarily large odd girth and chromatic number.

The name 'Kneser graph' is derived from Kneser's conjecture, which is the following statement.

> *Let r, s be integers such that $2 \leq r < s/2$. If the r-subsets of a s-set are partitioned into $s - 2r + 1$ families, then one family contains two disjoint sets.*

Note that a partition of the r-subsets of $[s]$ into families of pairwise intersecting subsets amounts to a proper vertex colouring of the Kneser graph $K(r, s)$. Thus Kneser's conjecture says that $\chi(K(r, s)) \geq s - 2r + 2$. On the other hand, it is easily seen that $s - 2r + 2$ colours are sufficient: For $k = 0, \dots, s - 2r$, assign the colour k to the set

$$I_k = \{A \in V(K(r, s)) : \min A = k\},$$

and assign the colour $s - 2r + 1$ to the set

$$I_{s-2r+1} = \{A \in V(K(r, s)) : \min A \geq s - 2r + 1\}.$$

This graph-theoretic interpretation of Kneser's conjecture is due to Lovász, who gave the first proof of it.

Theorem 3.10 (Lovász [91]) *The chromatic number of $K(r, s)$ is $s - 2r + 2$.*

Lovász used a surprising approach; simplifications and generalizations were later made by Bárány [14] and Walker [126], but the crux remains the basic idea of Lovász, which is, to transpose the problem to some topological space associated with the graph, and then use some powerful tools of algebraic topology, notably the Borsuk-Ulam antipodal theorem. It is somewhat unsettling that there is as yet no genuine graph-theoretic argument that explains why $K(r, s)$ cannot be coloured with $s - 2r + 1$ colours; on the other hand, Lovász's result gives an interesting topological flavour to the problem of determining whether there exists a homomorphism between two given graphs.

The independence number of Kneser graphs is related to the following classical inequality.

Theorem 3.11 (Erdős, Ko, Rado [42]) *Let r, s be integers such that $1 \leq r < s/2$, and $\mathcal{F}$ a family of pairwise intersecting r-subsets of $[s]$. Then*

$$|\mathcal{F}| \leq \binom{s-1}{r-1}.$$

As the families of pairwise intersecting r-subsets of $[s]$ are precisely the independent sets of $K(r, s)$, this result gives a bound for $\alpha(K(r, s))$. This bound is easily seen to be sharp, as it coincides with the cardinality of each set

$$J_k = \{A \in V(K(r, s)) : k \in A\}, \quad k = 0, \dots, s - 1.$$

One other combinatorial inequality provides further information on the structure of Kneser graphs:

Theorem 3.12 (Hilton, Milner [81]) *Let r, s be integers such that $1 \leq r < s/2$, and $\mathcal{F}$ a family of pairwise intersecting r-subsets of $[s]$. If $\bigcap \mathcal{F} = \emptyset$, then*

$$|\mathcal{F}| \leq \binom{s-1}{r-1} - \binom{s-r-1}{r-1} + 1.$$

This result shows that the only independent sets of maximal cardinality in $K(r,s)$ are the sets $J_0, \dots, J_{s-1}$ defined above. We see from this that the automorphism group of $K(r,s)$ is isomorphic to the symmetric group S_s: Any permutation of the base set $[s]$ induces an automorphism of $K(r,s)$ in a natural way. Conversely, any automorphism of $K(r,s)$ must interchange the sets $J_1, \dots, J_s$ (since these are the only independent sets of maximal size) thus induces a permutation of $[s]$. With the help of Lemma 3.6, it is possible to complete the characterization of endomorphisms of Kneser graphs.

Proposition 3.13 *All Kneser graphs are cores.*

Proof Let $\phi : K(r,s) \longrightarrow K(r,s)$ be a homomorphism. Then for $k = 0, \dots, s-1$, $|\phi^{-1}(J_k)| = \alpha(K(r,s))$, so there exists an index $\psi(k) \in [s]$ such that $\phi^{-1}(J_k) = J_{\psi(k)}$. By definition of the sets $J_k, k = 0, \dots, s-1$, this means that for any $A \in V(K(r,s))$, we have $k \in \phi(A)$ if and only if $\psi(k) \in A$. Therefore, ψ must be bijective, ϕ is the automorphism induced by $\psi^{-1} : [s] \longrightarrow [s]$, and $K(r,s)$ is a core. □

The Kneser graphs with a given chromatic number k are the graphs $K(n, 2n-2+k)$, $n \geq 1$, and the odd girth of these can be made arbitrarily large. Thus, we get a constructive proof of the well-known result that there exist graphs with arbitrarily large odd girth and chromatic number (see [41]). The following statement provides specific examples for any given odd girth and chromatic number.

Proposition 3.14 *Given integers k, n such that $n \geq 3$, let $r = k(n-2)$, $s = (2k+1)(n-2)$. Then $K(r,s)$ has chromatic number n and odd girth $2k+1$.*

Proof By Theorem 3.10, $\chi(K(r,s)) = n$. Also, since Kneser graphs are vertex-transitive, Lemma 3.3 states that $C_{2i+1} \to K(r,s)$ only if $i/(2i+1) \geq k/(2k+1)$, that is, $i \geq k$. Thus, the odd girth of $K(r,s)$ is at least $2k+1$. The interested reader may want to verify that $K(r,s)$ indeed contains a $(2k+1)$-cycle, but this also follows from our discussion of circular graphs in the next section. □

At this point it is worthwhile to reflect upon the condition $r < s/2$ that we imposed in the definition of Kneser graphs. If we allow $r = s/2$, the resulting graph $K(r,2r)$ is a perfect matching. The formulas above still provide the correct chromatic number and independence number, but the subsequent results on the structure of independent sets and automorphisms groups are no longer valid. If $s/2 < r \leq s$, the resulting graph $K(r,s)$ has no edge, and the formulas for the chromatic number and the independence number are wrong in most cases. However, for some applications of Kneser graphs such as subset colourings and the fractional chromatic number, it is convenient to consider these marginal graphs as Kneser graphs. The context will make clear what restrictions on r and s are to be imposed.

The remainder of this section is devoted to homomorphisms between Kneser graphs. A question raised by Hell asks for which integers r, s, r', s' we have $K(r,s) \to K(r',s')$ (see Godsil [50]). Lovász's theorem and the No-Homomorphism Lemma provide some restrictions. If $K(r,s) \to K(r',s')$, then

$$\chi(K(r,s)) = s - 2r + 2 \leq s' - 2r' + 2 = \chi(K(r',s'))$$

and

$$i(K(r,s)) = r/s \geq r'/s' = i(K(r',s')).$$

Also, the proof of Proposition 3.13 can easily be adapted to the following characterization of homomorphisms between Kneser graphs with the same independence ratio.

Proposition 3.15 (Stahl [113]) *Let r, s be relatively prime numbers such that $1 \leq r < s/2$. Then $K(mr, ms) \to K(nr, ns)$ for integers m, n if and only if n is a multiple of m.*

We now turn our attention to a wide list of homomorphisms between Kneser graphs. Note that $K(r,s)$ is an induced subgraph of $K(r, s+1)$. Also, it is easily seen that there exists a homomorphism from $K(r+1, s+1)$ to $K(r,s)$ defined by removing the maximal element from each $A \in V(K(r+1, s+1))$. A refinement of this idea is used in the proof of the following result:

Proposition 3.16 (Stahl [113]) *For any two integers such that $1 \leq r < s/2$ we have $K(r+1, s+2) \to K(r,s)$.*

Proof Define a map ϕ from $K(r+1, s+2)$ to $K(r,s)$ by

$$\phi(A) = \begin{cases} A \setminus \{\max(A)\} & \text{if } |A \cap \{s, s+1\}| \leq 1 \\ (A \setminus \{s, s+1\}) \cup \{\max(\overline{A})\} & \text{if } \{s, s+1\} \subseteq A \end{cases}$$

(where $\overline{A}$ denotes the complement of A). We show that ϕ is a homomorphism. Take $A, B \in V(K(r+1, s+2))$ such that $[A, B] \in E(K(r+1, s+2))$. Then $A \cap B = \emptyset$, so A and B cannot both contain $\{s, s+1\}$. We may therefore assume that $|A \cap \{s, s+1\}| \leq 1$. We then have $\phi(A) \subseteq A$. If $|B \cap \{s, s+1\}| \leq 1$, then $\phi(B) \subseteq B$, thus $\phi(A) \cap \phi(B) = \emptyset$ and $[\phi(A), \phi(B)] \in E(K(r,s))$. Otherwise, we have $\phi(B) \setminus B = \{k\}$, where $k \leq s-1$ and $\{k+1, \ldots, s+1\} \subseteq B$. Since $A \cap B = \emptyset$, we have $\max(A) \leq k$, thus $k \notin \phi(A)$. But $\phi(B) \setminus \{k\} \subseteq B$, so $\phi(A) \cap \phi(B) = \emptyset$ and $[\phi(A), \phi(B)] \in E(K(r,s))$. □

Note that the graphs $K(r, k-2+2r), r \in \mathbb{N}$, are precisely the Kneser graphs with chromatic number k. By the preceding result, these are linearly ordered by the relation $\to$. Combining this with inclusions, we get the following.

Corollary 3.17 *Let r, r', s, s' be integers such that $r \geq r'$, $1 \leq r < s/2$ and $1 \leq r' < s'/2$. Then $K(r,s) \to K(r', s')$ if and only if $s - 2r + 2 \leq s' - 2r' + 2$.* □

It remains to investigate homomorphisms from $K(r,s)$ to $K(r', s')$, with $r < r'$. The next result allows us to make use of the homomorphisms we have already defined.

Proposition 3.18 *Let G be a graph, and let r, r', s, s' be integers such that $1 \leq r < s/2$ and $1 \leq r' < s'/2$. If $G \to K(r,s)$ and $G \to K(r', s')$ then $G \to K(r+r', s+s')$.*

Proof Let $\phi_1 : G \longrightarrow K(r,s)$ and $\phi_2 : G \longrightarrow K(r', s')$ be homomorphisms. Define $\phi : G \longrightarrow K(r+r', s+s')$ by putting $\phi(u) = \phi_1(u) \cup \{i+s : i \in \phi_2(u)\}$. Then ϕ is a homomorphism. □

By Theorem 3.16, $K(r,s) \to K(r-k, s-2k)$ for all $k \in \{1,\dots,r-1\}$, so this last result shows that $K(r,s) \to K(nr-k, ns-2k)$ for all $n \in \mathbb{N}$ and $k \in \{1,\dots,r-1\}$. This is where the question stands today. Stahl [113] conjectures that $K(r,s) \to K(r',s')$ if and only if $s' \geq ns-2k$, where $r' = nr-k$. This conjecture is known to be true for low values of r, and for all odd graphs $K(r, 2r+1)$ (see Stahl [114]).

3.5 Circular graphs

This section presents a class of graphs that behaves with respect to $\to$ like the set of rationals with respect to its usual order. Let r,s be integers such that $1 \leq r \leq s/2$. We define the *circular graph* G_s^r as the Cayley graph $\mathrm{Cay}(\mathbb{Z}_s, \{r, r+1, \dots, s-r\})$. These graphs play an important role in the definition of the circular chromatic number (see Section 4) and some essential properties of this parameter are derived from the results presented here. It is worthwhile to warn the reader about some notation used in the proofs. Formally, the elements of $\mathbb{Z}_s$ are congruence classes modulo s, but are usually written as integers from $[s]$. This convention becomes inconvenient and confusing when defining maps between groups $\mathbb{Z}_s$ and $\mathbb{Z}_{s'}$. Consequently some proofs will be given in more detail than usual in order to avoid any possible misinterpretation.

Lemma 3.19 *Let r,s be positive integers such that $r \leq s/2$. Then $\alpha(G_s^r) = r$.*

Proof Obviously, $\alpha(G_s^r) \geq r$, since $\{0,1,\dots,r-1\}$ is an independent set. Let I be an independent set of G_s^r and u an element of I. Then $I \subseteq \{u-(r-1),\dots,u,\dots,u+r-1\}$. However, I can contain only one vertex of each edge $[u-i, u-i+r]$, so $|I| \leq r$. □

This simple result shows that the independence ratio of G_s^r is r/s, and allows us to use the No-Homomorphism Lemma to investigate the existence of homomorphisms between circular graphs. Also note that the Erdős-Ko-Rado inequality (Theorem 3.11) can be deduced from Lemma 3.19 (see [120]). Define $\phi : G_s^r \longrightarrow K(r,s)$ by putting $\phi(u) = \{u, u+1, \dots, u+r-1\}$ (we identify a congruence class in $\mathbb{Z}_s$ with its unique representative in $[s]$). Clearly, ϕ is a homomorphism, and with Lemma 3.3, the independence ratio of G_s^r provides an upper bound for $\alpha(K(r,s))$ which coincides with the Erdős-Ko-Rado bound.

Lemma 3.20 (Bondy, Hell [18]) *Let r,s,k be integers such that $r \leq s/2$. Then $G_s^r \leftrightarrow G_{ks}^{kr}$.*

Proof Let ϕ be the unique group homomorphism mapping the generator 1 of $\mathbb{Z}_s$ to the element k of $\mathbb{Z}_{ks}$. It is easily seen that ϕ is a homomorphism from G_s^r to G_{ks}^{kr}. We can also define a homomorphism $\psi : G_{ks}^{kr} \longrightarrow G_s^r$ by putting $\psi(v) = u$ if and only if $v \in \{\phi(u), \phi(u)+1, \dots, \phi(u)+k-1\}$. □

Proposition 3.21 (Bondy, Hell [18]) *Let r,s,m,n be integers such that $r \leq s/2$ and $m \leq n/2$. Then $G_s^r \to G_n^m$ if and only if $r/s \geq m/n$.*

Proof By Lemma 3.3 and Lemma 3.19, if there exists a homomorphism from G_s^r to G_n^m, then $r/s \geq m/n$. Conversely, suppose that $r/s \geq m/n$. By Lemma 3.20, $G_s^r \leftrightarrow G_{ns}^{nr}$

and $G_n^m \leftrightarrow G_{sn}^{sm}$. But $nr \geq sm$, so G_{ns}^{nr} is a subgraph of G_{sn}^{sm}. Therefore, there exists a homomorphism from G_s^r to G_n^m. □

This last result shows that with respect to the relation $\rightarrow$, the class of circular graphs behaves like the rationals in $]0, 1/2]$ with respect to the relation $\geq$. This is an example of a family of graphs which is 'dense' in the sense of Theorem 2.33: between any two rationals $r/s < r'/s'$ in $]0, 1/2]$, there exists a rational r''/s'' which is strictly in between them; we then have $G_{s'}^{r'} \rightarrow G_{s''}^{r''} \rightarrow G_s^r$ but no two of the three graphs are homomorphically equivalent. Recall that all circular graphs are Cayley graphs and so are vertex-transitive. It is natural to ask whether Welzl's density theorem 2.33 can be restricted to vertex-transitive graphs. This question was raised by Welzl [128]. Albertson and Booth [4] have shown that every vertex-transitive graph is an endpoint of an interval containing no other vertex-transitive graph. They also proved a special case of the following theorem.

Theorem 3.22 (Tardif [118]) *If G and H are vertex-transitive graphs such that G maps strictly into H then there is a vertex-transitive graph K such that $G \rightarrow K \rightarrow H$ and no two of the three graphs are homomorphically equivalent.*

By Proposition 3.21, $G_s^r \not\rightarrow G_{s'}^{r'}$ for any integers r', s' such that $r/s < r'/s'$. We conclude this section with a result showing that G_s^r is critical with respect to this property. Recall that $G - u$ is the graph obtained from G by the removal of the vertex u and all edges incident with it.

Proposition 3.23 (Bondy, Hell [18]) *Let r, s be relatively prime integers such that $r < s/2$. Then there exists integers m, n such that $r/s < m/n$, and for any $u \in V(G_s^r)$, $G_s^r - u \rightarrow G_n^m$.*

Proof Since r and s are relatively prime, there exist integers $a < s$ and $b < r$ such that $ar = bs + 1$. Put $m = r - b$ and $n = s - a$. Then, $r/s < m/n$. We show that for any $u \in V(G_s^r)$ there exists a homomorphism from $G_s^r - u$ to G_n^m.

By vertex transitivity, it suffices to consider the case $u = 0$. Also, by Lemma 3.20, it suffices to find a homomorphism from $G_s^r - \{0\}$ to G_{rn}^{rm}. To do this, we identify the vertices of $G_s^r - \{0\}$ with their representatives $1, \ldots, s-1$ in $\mathbb{N}$, and the vertices of G_{rn}^{rm} with their representatives $0, \ldots, rn-1$ in $\mathbb{N}$ (to recover multiplication in $\mathbb{N}$). We then define $\phi : G_s^r - \{0\} \longrightarrow G_{rn}^{rm}$ by putting $\phi(i) = im$. This map is well defined since $1 \leq i \leq s-1$ implies $0 \leq im \leq rn-1$. Take $1 \leq i < j \leq s-1$ such that $[i, j]$ is an edge of $G_s^r - \{0\}$. Then $j - i \in \{r, \ldots, s-r\}$, so $\phi(j) - \phi(i) = (j-i)m \in \{rm, \ldots, r(n-m)\}$ (since $(s-r)m \leq r(n-m)$). Thus, $[\phi(i), \phi(j)]$ is an edge of G_{rn}^{rm}, and ϕ is a homomorphism. □

4 Graph colourings and variations

One of the sources of the subject of graph homomorphisms is the theory of graph colourings. We have seen (Example 2.7) that a proper colouring of a graph G with k colours can be viewed as a homomorphism from G into K_k. In this section we will explore the homomorphism approach to graph colourings, beginning with definitions phrased in the language of homomorphisms. We begin with proper colourings and the chromatic number. We will be brief most of the time since there are recent publications surveying much of the literature on graph colourings; see [82, 121, 142].

A k-colouring of a graph G is a homomorphism into the complete graph K_k. Such a homomorphism is often wasteful in that not all the vertices (or edges) of K_k are images of vertices (or edges) of G. It is, therefore, of interest to restrict our attention only to colourings that use all of the target graph.

Definition 4.1 Given a natural number k, a *complete k-colouring* of a graph G is a complete homomorphism $\phi : G \longrightarrow K_k$.

That is, a k-colouring of G is complete if it is proper and every pair of colours appears on the endpoints of some edge of G. The definition can be rephrased by saying that a complete colouring of a graph G is a partition $\mathcal{P}$ of $V(G)$ into independent sets such that $G/\mathcal{P}$ is a complete graph. When the partition has k classes, we speak of a *complete k-colouring*.

We will prove (Proposition 4.12) that if there is a complete homomorphism from G onto K_r and onto K_t then there also is one onto K_s for any s between r and t. Since clearly r is bounded below (by 1) and t is bounded above (by $|G|$), it is reasonable to ask for tight bounds. These have been studied extensively, especially the former, and we consider them briefly in the next two sections.

4.1 Chromatic number

There is a large number of papers dealing with the minimum value of the k for which there is a complete k-colouring of a graph, most not concerned with homomorphisms. We will only mention, in the language of homomorphisms, a small number of very basic results.

Lemma 4.2 *Let G and H be graphs and assume that $G \to H$. Then G has a proper k-colouring whenever H does.*

Proof Let $\phi : G \longrightarrow H$ and $\psi : H \longrightarrow K_k$ be homomorphisms. Then the composition of ψ and ϕ is a k-colouring of G. $\square$

The chromatic number of a graph is usually defined as the least k so that the graph has a proper k-colouring. We do the same thing in the language of homomorphisms.

Definition 4.3 Let G be a graph. The *chromatic number* of G, denoted by $\chi(G)$, is the least n such that there is a homomorphism from G onto K_n.

We can immediately observe the following.

Lemma 4.4 *Let G be a graph with $\chi(G) = n$. Then every homomorphism $G \longrightarrow K_n$ is complete.*

Lemma 4.2 then has a few corollaries.

Corollary 4.5 *If $G \to H$ then $\chi(G) \le \chi(H)$.*

Corollary 4.6 *If H is the quotient of G by an elementary homomorphism then $\chi(G) \leq \chi(H) \leq \chi(G) + 1$.*

Proof Let $\phi : G \longrightarrow H$ be an elementary homomorphism and let u, v be the only vertices of G such that $\phi(u) = \phi(v)$. Let $\psi : G \longrightarrow K_n$ (recall that $V(K_n) = [n]$). By Lemma 4.5 we only need to prove the upper bound. Define a complete homomorphism $\psi' : H \longrightarrow K_{n+1}$ by $\psi'(z) = \psi(\phi^{-1}(z))$ for $z \neq \{u, v\}$ (this is well defined since ϕ is elementary) and $\psi'(u) = \psi'(v) = n$. □

Corollary 4.7 *The chromatic number of a graph and its core are the same.*

This corollary provides a sufficient condition for a graph to be a core. If we define a graph G to be χ-*critical* whenever each of its proper induced subgraphs has chromatic number strictly smaller than $\chi(G)$, we can immediately deduce that all χ-critical graphs are cores.

Another way to define the chromatic number of a graph is described by X.Zhu in [138]. Let I be an interval of length 1 and $r \geq 1$ any real number. Let $I(r)$ be the graph whose vertices are the open intervals in I of length $1/r$, two of which are joined by an edge if and only if their intersection is empty (the non-intersection graph of these intervals).

Theorem 4.8 (X.Zhu [138]) *For any graph G, $\chi(G) = \inf\{r : G \to I(r)\}$.*

Proof Let $r \geq 1$ be a real number. Define $\phi : I(r) \longrightarrow K_{\lfloor r \rfloor}$ by $\phi((a, a + 1/r)) = i$ if and only if $i/r \leq a < (i + 1)/r$. This is a homomorphism. Since I clearly contains $\lfloor r \rfloor$ pairwise disjoint open intervals of length $1/r$, $I(r) \leftrightarrow K_{\lfloor r \rfloor}$. □

This way of defining a parameter related to graph homomorphism is often useful and will appear throughout the sections on colourings.

4.2 Achromatic number

The maximum k for which a graph has a complete k colouring is a more recent idea than the chromatic number. It stems partly from Proposition 4.12.

Definition 4.9 The *achromatic number* of a graph G, denoted by $\mathrm{achr}(G)$, is the largest s such that there is a complete homomorphism from G onto K_s.

For example, $\mathrm{achr}(K_n) = n$, $\mathrm{achr}(P_7) = 3$, $\mathrm{achr}(P_8) = 4$, $\mathrm{achr}(P_{11}) = 5$, etc. The next lemma follows easily from the definition.

Lemma 4.10 ([59]) *If $\phi : G \longrightarrow H$ is complete then $\mathrm{achr}(G) \geq \mathrm{achr}(H)$.*

As with the chromatic number, the achromatic number of a quotient of a graph G by an elementary homomorphism cannot differ much from that of G.

Proposition 4.11 ([59]) *If there is an elementary homomorphism $\phi : G \longrightarrow H$, then*

$$\mathrm{achr}(G) - 2 \leq \mathrm{achr}(H) \leq \mathrm{achr}(G).$$

The lower bound is attained by graphs constructed from complete bipartite graphs with at least three vertices in each class of the bipartition by removing two non-adjacent edges. The elementary homomorphism identifying the endpoints of the removed edges in one of the classes of the bipartition is onto a complete bipartite graph whose achromatic number is 2 while the achromatic number of the original graph is 4.

The next result, known as the Interpolation Theorem for complete homomorphisms, is due to Harary, Hedetniemi and Prins [60].

Proposition 4.12 *Let G be a graph. There is a complete homomorphism from G onto K_s for each s such that $\chi(G) \leq s \leq \mathrm{achr}(G)$.*

Proof We give the proof in terms of elementary quotients. By definition of the achromatic number there is a quotient G/ϕ which is the complete graph on $\mathrm{achr}(G)$ vertices and, by Lemma 2.14 there are graphs $G_0, \ldots, G_k$ such that $G_0 = G$, $G_k \cong K_{\mathrm{achr}(G)}$, and G_{i+1} is an elementary quotient of G_i, $i = 0, \ldots, k-1$. Now $G \to G_i$ and $G_i \to K_{s_i}$ for each i, with $s_i = \chi(G_i)$. Since, by Lemma 4.6, the sequence $s_0, s_1, \ldots, s_k = s$ is non-decreasing and $s_i \leq s_{i-1} + 1$, there is, for each $\chi(G) \leq s \leq \mathrm{achr}(G)$, a least i_s such that $\chi(G_{i_s}) = s$. Thus, the composition of the natural map from G to G_{i_s} with any s-colouring of G_{i_s} is a complete homomorphism from G to K_s. □

The achromatic number has been studied by many. For example, in their work on graphs with high achromatic number, Hell and Miller ([69]) use quotients obtained by defining u and v to be equivalent if and only if u and v have the same neighbourhood. Graphs in which distinct vertices have distinct neighbourhoods are called *irreducible*. The important theorem then says than the number of irreducible graphs with a given achromatic number is finite.

Theorem 4.13 ([69]) *For every k there is a K such that $|V(G)| \leq K$ for any irreducible graph G with $\mathrm{achr}(G) = k$.*

In [68] Hell and Miller also study the achromatic number of paths and cycles by exploring the relationship between eulerian trails and complete homomorphisms.

Lemma 4.14 ([68]) *Let G be a graph, n an integer.*

(1) *There is a complete homomorphism from P_n onto G if and only if G is the underlying graph of some multigraph G' with $n-1$ edges and an eulerian trail.*

(2) *There is a complete homomorphism from C_n onto G if and only if G is the underlying graph of some multigraph G' with n edges and an eulerian tour.*

Paths, of course, are much easier to analyze than general graphs since, as the lemma indicates, they only need to be "wrapped around" a complete graph in order to get a lower bound on their achromatic number. It is an indication of the difficulty of the general problem that, except in a few special cases, the achromatic number is not known even for trees. For a recent survey of results on the achromatic number see [82].

Hedrlín, Hell and Ko consider another kind of interpolation in [63] by defining the class $\mathcal{H}_{n+\frac{1}{2}}$ of graphs obtained form K_n by adding two new vertices, each adjacent to some but not all the vertices of K_n, each of which is adjacent to at least one of the two new vertices. Note that the graphs in $\mathcal{H}_{n+\frac{1}{2}}$ are homomorphically equivalent to K_n.

Theorem 4.15 (Hedrlín, Hell, Ko) *For every graph G for which there are complete homomorphisms onto K_n and K_{n+1} there is a graph $K \in \mathcal{H}_{n+\frac{1}{2}}$ and a complete homomorphism of G onto K.*

4.3 Kneser colourings

¿From the point of view of homomorphisms, a proper colouring is a homomorphism of a graph into a complete graph. The minimum and the maximum order of the complete graphs for which there is a surjective homomorphism from G define numerical parameters which give us information about the graphs by implying the existence of homomorphisms. It is then natural to realize that any homomorphism could be called a 'colouring' if we do not restrict the image to complete graphs. This point of view is adopted by Hell and Nešetřil in [72] (see also [56]). This, however, destroys the very desirable property of having a link between homomorphisms and numerical parameters. In the case of the chromatic and the achromatic numbers, this link is a consequence of the structure of the images allowed (complete graphs), and of our knowledge of homomorphisms between the members of this family. Many parameters that have been studied in graphs can be presented within the same framework. Start with a given family $\mathcal{F}$ of graphs. For a graph G, the homomorphisms from G to members of $\mathcal{F}$ are called 'colourings' by $\mathcal{F}$. Then if some numerical parameter π can be naturally associated to all members of $\mathcal{F}$, we may expand the definition of this parameter to the class of all graphs, by putting

$$\pi(G) = \inf\{\pi(H) : H \in \mathcal{F} \text{ and } G \to H\} \tag{4.1}$$

and hope that the infimum will turn out to be minimum. In some cases this happens naturally, as it does for the chromatic number. The parameters defined in this manner share some fundamental properties, the principal one being that $\pi(G) \leq \pi(H)$ whenever $G \to H$.

Recall that the Kneser graph $K(r,s)$, $r < s/2$, is the non-intersection graph of the r-element subsets of $[s]$.

Definition 4.16 Let r, s be positive integers, $r < s/2$. A *Kneser (r,s)-colouring* of a graph G is a homomorphism from G to the Kneser graph $K(r,s)$.

This concept was introduced by Stahl [113]. The "normal" definition is clearly the following. An *(r,s)-subset colouring*, $r < s/2$, of a graph G is a map ϕ from the vertex set of G to the r-subsets of the set $\{1,\ldots,s\}$ such that adjacent vertices are mapped to disjoint sets.

Notice that we have $K(r,s) \to K(r,s+1)$ for all $s \in \mathbb{N}$. This allows a definition of the r-chromatic number.

Definition 4.17 Given a natural number r, the *r-chromatic number* $\chi_r(G)$ of a graph G is the least s such that $G \to K(r,s)$.

In particular, a Kneser $(1,s)$-colouring is a homomorphism into a complete graph on s vertices and so $\chi_1(G) = \chi(G)$ for any graph G. The chromatic number $\chi(G)$ of a graph G is thus generalized to a sequence of numerical parameters $\{\chi_r(G)\}_{r\in\mathbb{N}}$. Such a sequence tells us more about the graph than the chromatic number alone. For example, there is an infinite sequence of Kneser graphs with a given chromatic number but the sequence of r-chromatic numbers determines a Kneser graph uniquely.

Observe that for a graph G, $\chi_r(G) = r$ if and only if G has no edges, and that $\chi_r(G) = 2r$ if and only if G is bipartite with at least one edge. From the No-Homomorphism Lemma 3.3 we have that if there is a homomorphism from G into $K(r, s)$, then $\alpha(G)/|G| \geq r/2$ and so $\chi_r(G) \geq r|G|/\alpha(G)$.

Obviously, homomorphisms between Kneser graphs can be translated as relationships between members of the sequence of r-chromatic numbers. For instance, from Propositions 3.16 and 3.18 we have that $K(r + 1, s + 2) \to K(r, s)$ and $\chi_{r+1}(G) \geq \chi_r(G) + 2$, as well as $\chi_{r+s}(G) \leq \chi_r(G) + \chi_s(G)$, see [113].

The latter is related to homomorphisms between categorical products of Kneser graphs (see Section 5). Some additional properties of the r-chromatic numbers are linked to the fractional chromatic number defined below. Also, the r-chromatic numbers of graphs are related to the chromatic number of lexicographic products of graphs (see Section 5.3).

4.4 Circular colourings

The next family of graphs does not immediately provide a minimum in the equation (4.1) defining π, but it can be shown that the infimum is, in fact, a minimum.

Let $1 \leq r \leq s/2$. Recall that G_s^r is the Cayley graph $\mathrm{Cay}(\mathbb{Z}_s, \{r, r + 1, \ldots, s - r\})$.

Definition 4.18 Let $1 \leq r \leq s/2$. A *circular* (r, s)*-colouring* of a graph G is a homomorphism $\phi : G \longrightarrow G_s^r$

This concept was introduced by Vince [125], without the name, with the next (equivalent) definition.

Definition 4.19 A *circular* (r, s)*-colouring* of a graph G is a map $\phi : V(G) \to \{0, \ldots s - 1\}$ such that

$$[u, v] \in E(G) \Rightarrow r \leq |\phi(u) - \phi(v)| \leq s - r.$$

Our presentation follows the combinatorial approach of Bondy and Hell [18] rather than that of Vince who used continuous methods. The name “circular” is suggested by the circulant target graphs of the homomorphisms.

The *circular chromatic number*[3] $\chi_c(G)$ of a graph G is defined by putting

$$\chi_c(G) = \inf\{s/r : G \to G_s^r\}.$$

The range of this new parameter is not restricted to the integers, so we have to define it as an infimum instead of a minimum. ¿From Proposition 3.21 we have that $G \to G_s^r$ if and only if $s/r \geq \chi_c(G)$. Further, since by Proposition 3.23 we only need to consider surjective homomorphisms, these results imply that the infimum is actually a minimum.

Theorem 4.20 (Bondy and Hell [18]) *The circular chromatic number of a graph G is*

$$\chi_c(G) = \min\{s/r : G \to G_s^r \text{ and } s \leq |G|\}.$$

[3] The name *star chromatic number* usually given to this parameter only relates to the current notation $\chi*$. The name *circular* is not only justified by the homomorphisms into circulants but also by the characterization of the parameter due to X.Zhu given later.

Thus, the circular chromatic number of a finite graph G is always rational, and completely determines the pairs of integers r, s such that G has a circular (r,s)-colouring: $G \to G^r_s$ if and only if $s/r \geq \chi_c(G)$.

Also note that since G^s_1 is the complete graph K_s, we have the inequalities

$$\chi(G) - 1 < \chi_c(G) \leq \chi(G).$$

The following characterization of circular colourings and of the circular chromatic number is due to X.Zhu [138]. Let C be a circle in $\mathbb{R}^2$ of length 1, and let $r \geq 1$ be any real number. Let $C(r)$ be the non-intersection graph of the open arcs on C of length $1/r$.

Theorem 4.21 *For any graph G, $\chi_c(G) = \inf\{r : G \to C(r)\}$.*

¿From the above theorem together with Theorem 4.8 we can obtain a sufficient condition for graphs for to have the same the chromatic and the circular chromatic number.

Theorem 4.22 *If G has a vertex v which is adjacent to every other vertex of G, then $\chi_c(G) = \chi(G) = n$.*

Proof A vertex adjacent to all other vertices in G must map to an interval I which intersects no other. Thus an r-circle colouring exists if and only if an r-interval colouring exists since the circle colouring corresponds to an interval one by "cutting" the circle at one of the endpoints of I (the other direction is trivial). □

By refining somewhat the proof of the theorem we obtain a slightly more general result.

Corollary 4.23 *Suppose that G has chromatic number s and a vertex whose neighbours induce a subgraph of chromatic number $s-1$. Then $\chi_c(G) = \chi(G)$. In particular, if $\chi(G) = s$ and G contains (a copy of) K_s then $\chi_c(G) = \chi(G)$.*

The corollary can, of course, be obtained simply by considering what the circular graph which realizes the circular chromatic number must look like in this case.

Abbott and B.Zhou [1], and, independently, G.Gao, Mendelsohn and H.Zhou [47], have found a more general sufficient condition for the circular chromatic number to be equal to the chromatic number, again by using r-circle colourings.

Theorem 4.24 *If G is a graph whose complement is disconnected, then $\chi_c(G) = \chi(G)$.*

The circular chromatic number has been studied by Gao, Hahn, Hell, H.Zhou, X.Zhu [18, 26, 46, 48, 115, 138] and others. We shall not list here the many results obtained for particular classes of graphs, for more see the survey [142]. Deuber and X.Zhu [34] have generalized the circular chromatic number further (incidentally again justifying the change of name). They start with a graph G and a weight function which assigns to each vertex of G a non-negative real weight $w(u)$. Then they consider a circle of circumference t in the plane and a mapping ϕ assigning to each vertex u of a graph G an arc of length $w(u)$ on the circle in such a way that the arcs assigned to adjacent vertices do not intersect. The *weighted circular chromatic number* $\chi_c(G, w)$ is then the infimum of the t for which such a ϕ exists.

Clearly $\chi_c(G,1) = \chi_c(G)$. Other weighted colourings are considered as well, but the methods do not use homomorphism.

The interest of the circular chromatic number comes in part from the bound it provides on the ultimate independence ratio discussed in Section 4.7. The fractional chromatic number considered next also gives a bound on the ultimate independence ratio.

4.5 Fractional chromatic number

Definition 4.25 Let $\mathcal{I}(G)$ denote the set of all independent sets of a graph G. A *fractional colouring* of G is a weight function $\mu : \mathcal{I}(G) \to [0,1]$ such that the constraints

$$\sum_{u \in I \in \mathcal{I}(G)} \mu(I) \geq 1$$

are satisfied for all $u \in V(G)$.

While the concepts of both subset colouring and circular colouring remain close to our intuitive notion of colouring, that is, the affectation of colours to vertices, the idea of a fractional colouring is somewhat different. It can be motivated by our definitions of colourings through homomorphisms: we assign vertices to colours rather than the other way around. Thus, the usual vertex colourings are constructed by selecting disjoint independent set which are assigned to different colours. The corresponding weight function has value 1 for all selected independent sets, and 0 for all other independent sets. With fractional colourings, we allow independent sets to be 'partially' selected. Examples of such fractional colourings can be found by looking at the Kneser (r,s) colourings of a graph. If $\phi : G \longrightarrow K(r,s)$ is a homomorphism, we may define a fractional colouring $\mu : \mathcal{I}(G) \to [0,1]$ by putting $\mu(I) = k_I/r$, where k_I is the number of indices $i \in [s]$ such that $I = \{u \in V(G) : i \in \phi(u)\}$.

The *fractional chromatic number* $\chi_f(G)$ of a graph G is defined by putting

$$\chi_f(G) = \inf\{\sum_{I \in \mathcal{I}(G)} \mu(I) : \mu \text{ is a fractional colouring of } G\}.$$

Thus, the fractional chromatic number of a graph is the solution of a linear program with $|\mathcal{I}(G)|$ variables and $|G|$ constraints. Therefore, the infimum is always attained and is truly a 'fraction'. Note that this definition of the fractional chromatic number does not seem to relate to homomorphisms. However, we can express the fractional chromatic number of a graph in terms of graph homomorphisms in at least two ways, the second of which uses the independence number $\alpha(G)$ of the graph G.

Proposition 4.26 *Let G be a graph. Then*

$$\chi_f(G) = \inf\{s/r : G \to K(r,s)\} = \sup\{|H|/\alpha(H) : H \to G\}.$$

Some relations between the fractional chromatic number, the circular chromatic number and the subset chromatic numbers of a graph can be derived from the first of these equalities. First of all we easily have $\chi_f(G) \leq \chi_r(G)/r$ for all $r \in \mathbb{N}$ by the definition of these parameters (in fact, $\chi_f(G) = \lim_{r\to\infty} \chi_r(G)/r$). We also have $G_r^s \to K(r,s)$, so that $\chi_c(G) \leq \chi_f(G)$.

Unlike the circular chromatic number, the fractional chromatic number of a graph can differ arbitrarily from its chromatic number, the simplest example being that of the Kneser graphs themselves. Finally, note that the last characterization of the fractional chromatic number strays from our intended mold. This may be explained by the dual version of the linear program defining the fractional chromatic number of a graph, which can be expressed as follows.

$$\begin{gathered}\max \textstyle\sum_{u \in V(G)} \nu(u) \\ \text{subject to } \textstyle\sum_{u \in I} \nu(u) \leq 1 \text{ for all } I \in \mathcal{I}(G).\end{gathered}$$

Observe also that the $0-1$ weight functions ν on $V(G)$ that satisfy the above constraints are obtained by assigning the value 1 to all vertices of some complete subgraph of G and 0 to all the other vertices. Hence the above expression can be thought of as a *fractional clique number* of G. We can then note that while the chromatic number and the clique number of a graph can differ arbitrarily, the fractional versions always coincide.

¿From the primal and the dual formulations of the fractional chromatic number we can prove something for vertex-transitive graphs.

Theorem 4.27 *If G is a vertex-transitive graph then $\chi_f(G) = |G|/\alpha(G)$.*

The No-Homomorphism Lemma 3.3 can be viewed as a consequence of this and Proposition 4.26.

An alternate description of the fractional chromatic number is given by X.Zhu in [48] and [34]. Let I be an interval of length 1 and $r \geq 1$ any real number. Let $M(r)$ be the non-intersection graph of the measurable subsets of I of measure $1/r$.

Theorem 4.28 (X.Zhu [48]) *For any graph G, $\chi_f(G) = \inf\{r : G \to M(r)\}$.*

4.6 Chromatic difference sequence

There is but a small step from the chromatic number to asking *How many vertices of a graph can be coloured with a given number of colours?* More precisely, following Albertson and Berman [3], we define, for a given graph G and a positive integer k, $\alpha_k(G)$ to be the maximum number of vertices in a k-colourable induced subgraph of G. We also define the differences of successive α_k's by setting $\beta_k(G) = \alpha_k(G) - \alpha_{k-1}(G)$, for $k > 0$ and $\alpha_0(G) = 0$. Thus $\sum_{i=1}^{k} \beta_i(G) = \alpha_k(G)$.

Definition 4.29 Let G be a graph and let $n = |G|$. The *chromatic difference sequence* of G is the sequence $(\beta_1(G), \ldots, \beta_{\chi(G)}(G))$. The *normalized* chromatic difference sequence of G is the sequence $(\beta_1(G)/n, \ldots, \beta_{\chi(G)}(G)/n)$.

We will say that a sequence $a_1, \ldots, a_n$ *dominates* the sequence $b_1, \ldots, b_n$, if $\sum_{i=1}^{k} a_i \geq \sum_{i=1}^{k} b_i$ for all k, with equality when $k = n$ (this can be extended to sequences of different lengths, see [132]).

Chromatic difference sequences have been used to prove that graphs do not have homomorphisms into other graphs. The No-Homomorphism Lemma 3.3 was first proved in this context in the following form.

Lemma 4.30 ([5]) *If there is a homomorphism from a graph G to a graph H and H is vertex-transitive, then the normalized chromatic sequence of G dominates that of H.*

It was then used to show that for the Petersen graph P, $P^r \not\to P^s$ for any $r > s$, where P^n is the n-th cartesian power of P. Others, notably H. Zhou in his thesis [135] and subsequent papers, considered the behaviour of the chromatic difference sequences under homomorphisms and in relation to graph products.

Theorem 4.31 *The normalized chromatic difference sequence of G^k is equal to that of G for any Cayley graph G of an abelian group, with the cartesian product.*

In general, H.Zhou has shown the following.

Theorem 4.32 ([132]) *Let G and H be graphs. Then:*

(1) *The normalized chromatic difference sequence of $G \times H$ dominates those of G and H.*

(2) *The normalized chromatic difference sequence of $G \Box H$ is dominated by those of G and H, and so the normalized chromatic difference sequence of the cartesian power G^k is dominated by that of G.*

Albertson and Collins [5] conjectured that the chromatic difference sequence of a vertex-transitive graph was always non-increasing. In a similar direction, H.Zhou asked if a Cayley graph G has a non-increasing *achievable* chromatic difference sequence (that is, one with colour classes of sizes β_k for $k = 1, \ldots, \chi(G)$). Tardif [119] constructed a circulant whose chromatic difference sequence is not monotonic.

Example 4.33 The chromatic difference sequence of $(C_7[C_5]) \times K_6$ is $(36, 36, 36, 36, 31, 35)$.

While the chromatic difference sequence does not lead to a numerical parameter, a concept whose roots are in the study of the chromatic difference sequence does. It is the ultimate independence ratio of a graph considered next.

4.7 Ultimate independence ratio

Recall that the independence ratio of a graph G is $i(G) = \alpha(G)/|G|$ (Definition 3.2). Since in this section we will be concerned with the cartesian product, we will have $G^1 = G$ and $G^k = G^{k-1} \Box G$.

We begin by asking about $\alpha(G \Box H)$. Clearly, $\alpha(G \Box H) \leq |G|\alpha(H)$ since each fiber $F_u = \{u\} \times V(H)$ of $G \Box H$ can contribute at most $\alpha(G)$ vertices to a maximum independent set of $G \Box H$. This is achievable if and only if we can choose, in each fiber F_u, a maximum independent set $\{u\} \times I_u$ of size $\alpha(G)$ so that $I_u \cap I_v = \emptyset$ whenever $[u, v]$ is an edge of G. This can be expressed in a setting using homomorphisms.

Let G be a graph and define the *maximum independent set graph* of G, $\mathrm{Ind}(G)$, to be the non-intersection graph the maximum independent sets of G.

Definition 4.34 An *independent set cover* of a graph G by a graph H is a homomorphism $\iota : G \longrightarrow \mathrm{Ind}(H)$. When G has an independent set cover by G, we say simply that G *has an independent set cover*.

For example, $\mathrm{Ind}(C_{2k+1})$ is the odd cycle C_{2k+1}; the maximum independent set graph of the Petersen graph consists of five isolated vertices. Thus each odd cycle has an independent set cover but the Petersen graph does not.

The following results are direct consequences of the above discussion.

Lemma 4.35 *Let G and H be graphs. Then*

$$i(G \Box H) \leq \min\{i(G), i(H)\},$$

with equality if and only if $G \to \mathrm{Ind}(H)$ or $H \to \mathrm{Ind}(G)$, depending on whether $i(H) \leq i(G)$ or $i(G) \leq i(H)$.

In particular this says that a graph G has an independent set cover if and only if $i(G^2) = i(G)$. In the language of Section 5.2 this can be restated by saying that the graph $\mathrm{Hom}(G, \mathrm{Ind}(G))$ (see Definition 5.18) is non-empty. In fact we have the stronger statement that $i(G^2) = i(G)$ if and only if the graphs $\mathrm{Hom}(G, \mathrm{Ind}(G))$ and $\mathrm{Ind}(G^2)$ are isomorphic.[4]

Given a graph G, consider the powers G^k. By Lemma 4.35 we have that $i(G^k) \geq i(G^{k+1}) > 0$ for all k. Hence the sequence $i(G^k)$, $k = 1, 2, \ldots$, has a limit ≥ 0. In fact the sequence is bounded away from zero since by Lemma 5.13, $\chi(G^k) = \chi(G)$ for all k, so that $i(G^k) \geq 1/\chi(G^k) = 1/\chi(G)$. Hell, Yu and H. Zhou introduced the concept of ultimate independence ratio in [75].

Definition 4.36 The *ultimate independence ratio* of a graph G, denoted by $I(G)$, is

$$I(G) = \lim_{k \longrightarrow \infty} i(G^k).$$

It is not known if the limit is always rational.

There exist graphs for which $I(G)$ is strictly less than $i(G^k)$ for any k. For example, if G is the Petersen graph, then $I(G) = 1/3$ whereas $i(G^k) = \alpha(G^k)/10^k$ (two proper fractions with relatively prime denominators). At the other extreme are graphs whose ultimate independence ratio is already equal to $i(G)$. A class of such graphs is provided by the following result of Hell, Yu and H. Zhou [75] which is based on Lemma 4.35.

Theorem 4.37 *If $G^2 \to G \to \mathrm{Ind}(G)$ then $I(G) = i(G)$.*

Thus the powers of a graph satisfying $G^2 \to G$ and having an independent set cover all have the same independence ratio. Conversely, if G is any graph for which $I(G) = i(G)$, then in particular $i(G^2) = i(G)$, hence by the remark after Lemma 4.35 we have that the condition $G \to \mathrm{Ind}(G)$ in Theorem 4.37 is necessary. We do not know whether the same is true for the condition $G^2 \to G$. In general, Theorem 4.37 is false without it (see the end of this section).

We can use Theorem 4.37 to prove the following result.

Theorem 4.38 *Let G be a Cayley graph $\mathrm{Cay}(\Gamma, S)$ on an abelian group Γ. Then $I(G) = i(G)$.*

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[4] This was noticed independently by B. Larose, F. Laviolette and G. Sabidussi.

Proof For an abelian group Γ the mapping $\phi : G^2 \longrightarrow G$ defined by $\phi(g,h) = gh$ is a homomorphism. Let I be any maximum independent set of G and for any $g \in \Gamma$ set $I_g = gI$. Since left translations are automorphisms, I_g is a maximum independent set. If now gh is an edge of G, $I_g \cap I_h = \emptyset$ since otherwise $gk = hk'$ for some $k, k' \in I$ and so $k^{-1}k' = h^{-1}g \in S$, contradicting the independence of I. Thus the map $g \mapsto I_g$ is a homomorphism $G \to \mathrm{Ind}(G)$. □

The above theorem can be generalized to normal Cayley graphs, see Section 5. For arbitrary Cayley graphs very little is known, and results even for Cayley graphs of dihedral groups are nonexistent. We do know, however, that the tempting conjecture that $I(G) = i(G)$ for Cayley graphs fails for infinitely many Cayley graphs, see [57].

Corollary 4.39 *If G is a circulant, then $I(G) = i(G)$. In particular, $I(C_{2k}) = 1/2$ and $I(C_{2k+1}) = k/(2k+1)$ for $k > 0$.*

With the ultimate independence ratio we can prove an analogue of the No-Homomorphism Lemma 3.3. It does not require that the target graph H be transitive, but we pay a price by having to deal with limits. The proof uses multiples of graphs mentioned in Section 3.1.

Theorem 4.40 (Hahn, Hell, Poljak [55]) *If $G \to H$ then $I(H) \leq I(G)$. In particular, if H is a retract of G, then $I(H) = I(G)$.*

This gives, as corollaries, that $I(G) \leq 1/\omega(G)$ ($\omega(G)$ is the size of a largest clique in G) and that $I(G) = 1/\chi(G)$ for perfect graphs.

Homomorphisms also allow us to prove:

Theorem 4.41 (Hahn, Hell, Poljak [55]) *For any graph G,*

$$\frac{1}{\chi(G)} \leq I(G) \leq \frac{1}{\chi_f(G)}.$$

X.Zhu gives more bounds related to homomorphisms. The first was also proved independently by Favaron[44].

Theorem 4.42 (X.Zhu [137]) *For any graph G, the sequence $1/\chi_f(G^k)$, $k = 1, 2, \ldots$, is non-increasing and*

$$I(G) = \lim_{k\to\infty} \frac{1}{\chi_f(G^k)}.$$

Theorem 4.43 (X.Zhu [137]) *For any graph G, $I(G) \geq 1/\chi_c(G)$.*

In general, the problem of determining the ultimate independence ratio of a graph is open. The conjectures which would equate $I(G)$ to one of the bounds have proven wrong (for example, X.Zhu constructs a graph with $1/\chi(G) < I(G) < 1/\chi_f(G)$ in [137]). The simplest case of interest – because of the methods which must be invented – is that of the odd wheels in general and W_5 in particular. From solving the linear program for the fractional chromatic number of the square of W_5 we get $1/4 \leq I(W_5) \leq 11/41$. The working hypothesis is that $1/4$ is the value for any odd wheel.

As an application of these remarks and in order to illustrate the usefulness of Theorem 4.40, let us show that in general Theorem 4.37 fails if we only assume that $G \to \mathrm{Ind}(G)$. In other words, it is not true that if $i(G^2) = i(G)$, then all the higher powers of G also have the same independence ratio. Let W_5' be a 5-wheel with a "double hub", i.e. a pentagon with two additional vertices that are joined to all vertices of the pentagon but not to each other. Clearly the wheel W_5 is a retract of W_5'. Hence by Theorem 4.40,

$$I(W_5') = I(W_5) \leq 11/41 < 2/7 = i(W_5').$$

On the other hand, it is easy to check that $\mathrm{Ind}(W_5') = W_5$, and hence the condition $W_5' \to \mathrm{Ind}(W_5')$ is satisfied.

5 Graph products

The main constructions known as 'graph products' have already been introduced in Section 2.6. In this section, we take a closer look at some of the themes linking graph homomorphisms and products. The principal motivation of this subject is the investigation of the behaviour of some parameters, such as the chromatic number, the clique number or the independence number, under some product. In terms of homomorphisms, these questions often amount to finding criteria for the existence of a homomorphism from a product of graphs to a given graph, or from a given graph to a product of graphs.

5.1 The categorical product and Hedetniemi's conjecture

Recall that the categorical product $G \times H$ of two graphs G and H is the graph with vertex set $V(G) \times V(H)$, where two vertices (u_1, u_2), (v_1, v_2) are adjacent if and only if u_1, v_1 are adjacent in G, and u_2, v_2 are adjacent in H. This is the product with the universal property in the category of graphs. Indeed, both projections pr_G and pr_H of $G \times H$ onto its factors are homomorphisms, and a graph K admits homomorphisms $\phi_1 : K \longrightarrow G$ and $\phi_2 : K \longrightarrow H$ if and only if it admits a homomorphism $\phi : K \longrightarrow G \times H$ such that $\phi_1 = \mathrm{pr}_1 \circ \phi$ and $\phi_2 = \mathrm{pr}_2 \circ \phi$. This homomorphism ϕ is defined by putting $\phi(u) = (\phi_1(u), \phi_2(u))$. In general, there need not exist a homomorphism from G to $G \times H$; such a homomorphism exists if and only if $G \to H$. However, two homomorphisms $\phi_1 : G \longrightarrow G'$ and $\phi_2 : H \longrightarrow H'$ can be used coordinatewise to define a homomorphism $\phi : G \times H \longrightarrow G' \times H'$. In terms of the quasi-order $\to$, the universal property of the categorical product of graphs says that for graphs G, H, K, we have $K \to G \times H$ if and only if both $K \to G$ and $K \to H$ hold. Thus, the class $\mathcal{H}(G \times H)$ is the infimum of the classes $\mathcal{H}(G)$ and $\mathcal{H}(H)$ with respect to the partial order $\prec$.

On a more concrete level, note that both the odd girth and the clique number of a graph are defined by homomorphisms from odd cycles or complete graphs into this graph. With this in mind, the following facts are direct applications of the universal property of the categorical product.

Observation 5.1 *Let G, H be graphs. Then the odd girth of $G \times H$ is the maximum of the odd girths of G and H, and the clique number of $G \times H$ is the minimum of the clique numbers of G and H.*

It is very enlightening to be able to express the value of some parameters in a product as a function of the same parameters in the factors in this fashion. We may wonder if we might be able to do the same with other parameters. In particular, note that the chromatic number of $G \times H$ is defined by homomorphisms *from* $G \times H$ *to* complete graphs, and its characterization does not fall within the framework of the defining property of the categorical product. Thus, the colourings of $G \times H$ are much less predictable. Some 'canonical' colourings can be defined by projecting $G \times H$ onto one of its factors and then colouring this factor, but other colourings also exist. However, we do not know any instances where less than $\min\{\chi(G), \chi(H)\}$ colours are used. It is believed that such colourings are impossible, and this is the essence of the following.

Conjecture 5.2 (Product conjecture, Hedetniemi [62]) Let G, H be graphs. Then

$$\chi(G \times H) = \min\{\chi(G), \chi(H)\}. \tag{5.1}$$

It is easy to see that the identity (5.1) holds for $\min\{\chi(G), \chi(H)\} \leq 3$. In terms of homomorphisms, Conjecture 5.2 admits many equivalent formulations, which take into account the obvious inequality $\chi(G \times H) \leq \min\{\chi(G), \chi(H)\}$. Each of the following statements is equivalent to the product conjecture.

- Let n be a positive integer. If for two graphs G, H we have $G \not\to K_n$ and $H \not\to K_n$, then $G \times H \not\to K_n$.
- Let G, H be m-chromatic graphs. Then there exists an m-chromatic graph K such that $K \to G$ and $K \to H$.

The first of these statements amounts to saying that K_n is *multiplicative* (see Definition 5.9 below). Note that in the second statement, no restriction is made on the graph K. Of course, if the product conjecture is true, we can put $K = G \times H$. However, it is sometimes convenient to consider other possible choices for K. For instance, we can try to find a graph K that is χ-critical. In particular, the only 3-colourable graphs that are χ-critical are the graphs K_1, K_2 and the odd cycles. This simple family of graphs accounts for the validity of the product conjecture for $\min\{\chi(G), \chi(H)\} \leq 3$.

The next step towards a proof of the product conjecture would be to settle the case $\min\{\chi(G), \chi(H)\} = 4$. However the family of 4-critical graphs is much more complex than the family of odd cycles, and there is no hope of generalizing the arguments that worked before. Nonetheless, progress has indeed been made with a new approach due to El-Zahar and Sauer, and it is now known that the identity (5.1) holds for $\min\{\chi(G), \chi(H)\} \leq 4$.

For a graph G and an integer n, put

$$\mathcal{F} = \{H : G \times H \text{ is } n\text{-colourable}\}.$$

Then, $H_1 \to H_2$ and $H_2 \in \mathcal{F}$ implies $H_1 \in \mathcal{F}$. Thus, $\mathcal{F}$ is an ideal with respect to the quasi-order $\to$. If G is n-colourable, then $\mathcal{F}$ contains all graphs; the interesting situation arises when $\chi(G) > n$. Then, we find a counterexample to the product conjecture if there exists a graph $H \in \mathcal{F}$ such that $\chi(H) > n$. The following construction, due to El-Zahar and Sauer, allows us to find a maximal element in $\mathcal{F}$.

Definition 5.3 For a graph G and an integer n, we define the *n-colouring graph* $\mathcal{C}_n(G)$ of G by putting

$$\begin{aligned} V(\mathcal{C}_n(G)) &= \{f : V(G) \longrightarrow [n]\}, \\ E(\mathcal{C}_n(G)) &= \{[f, g] : \text{ for all } (u, v) \in V(G)^2, [u, v] \in E(G), f(u) \neq g(v)\}. \end{aligned}$$

The vertices of $\mathcal{C}_n(G)$ are arbitrary functions, not proper colourings in general. Thus, $|\mathcal{C}_n(G)| = n^{|G|}$, and the structure of G only determines the structure of $E(\mathcal{C}_n(G))$. By the definition of $E(\mathcal{C}_n(G))$, a function f is adjacent to itself (i.e., $[f, f]$ is a loop) if and only if it is a proper colouring of G.[5] Thus, $\mathcal{C}_n(G)$ has no loops if and only if $\chi(G) > n$.

For $H \in \mathcal{F}$ and a proper n-colouring ϕ of $G \times H$, we define a map $\psi : H \longrightarrow \mathcal{C}_n(G)$ by putting $\psi(x) = f_x$, where $f_x : V(G) \longrightarrow \{0, \ldots, n-1\}$ is the function defined by $f_x(u) = \phi(u, x)$. By the definition of $\mathcal{C}_n(G)$, it is straightforward to verify that ψ is a homomorphism. Conversely, we can define a proper n-colouring ϕ of $G \times \mathcal{C}_n(G)$ by putting $\phi(u, f) = f(u)$. Thus, $\mathcal{C}_n(G) \in \mathcal{F}$, and $\mathcal{F} = \{H : H \to \mathcal{C}_n(G)\}$.

Now assume that $\chi(G) > n$. If $\chi(\mathcal{C}_n(G)) \leq n$, then $\chi(H) \leq n$ for all $H \in \mathcal{F}$, and there does not exist any graph H such that $\chi(H) > n$ and $\chi(G \times H) \leq n$. On the other hand, if $\chi(\mathcal{C}_n(G)) > n$, we find a counterexample to the product conjecture since $\chi(G \times \mathcal{C}_n(G)) \leq n$. Therefore, the product conjecture is equivalent to the statement that for any graph G and for any $n < \chi(G)$, we have $\chi(\mathcal{C}_n(G)) \leq n$. The main results of El-Zahar and Sauer are the following.

Theorem 5.4 (El-Zahar, Sauer [40]) *Let G be a graph such that $\chi(G) \geq 4$. Then $\chi(\mathcal{C}_3(G)) = 3$.*

Corollary 5.5 *If $\chi(G) \geq 4$ and $\chi(H) \geq 4$, then $\chi(G \times H) \geq 4$.*

The proof of Theorem 5.4 is intricate and, once again, it is not clear how it can be generalized to higher values of n. However, El-Zahar and Sauer's approach provides a fruitful insight into colourings of a categorical product of graphs. For instance, for a graph G and $n = \chi(G) - 1$, we at least know that $\chi(\mathcal{C}_n(G))$ is bounded, so there exists an integer $m \geq \chi(G)$ such that if $\chi(H) \geq m$, then $\chi(G \times H) = \chi(G)$. The situation is different if we do not fix a graph G and consider the function $a : \mathbb{N} \longrightarrow \mathbb{N}$ defined by putting

$$a(m) = \min\{\chi(G \times H) : \chi(G) = \chi(H) = m\}.$$

Though a is non decreasing, it is not even clear that a is not bounded. Poljak and Rödl [101] obtained the surprising result that *if* a is bounded, then $a(m) \leq 16$ for all $m \in \mathbb{N}$ (Poljak [100] later improved this bound to 9).

The next three results show how our knowledge of the general structure of the graph $\mathcal{C}_n(G)$ helps to solve further instances of the product conjecture.

Lemma 5.6 (El-Zahar, Sauer [40]) *Let G be a connected graph and n an integer such that $n < \chi(G)$. Then $\mathcal{C}_n(G)$ contains a unique complete subgraph of cardinality n, namely, the subgraph induced by the constant functions.*

Proof Let $f_1, \ldots, f_n$ denote the vertices of a complete subgraph in $\mathcal{C}_n(G)$. Suppose that G contains a vertex u such that the values $f_1(u), \ldots, f_n(u)$ are all distinct. Then for any neighbour v of u and for any $i \in \{1, \ldots, n\}$, we have $f_i(v) \neq f_j(u)$ for all $j \neq i$. Thus, $f_i(v) = f_i(u)$ for all $i \in \{1, \ldots, n\}$, and the values $f_1(v), \ldots, f_n(v)$ are all distinct. Since G is

[5] Technically, we do not allow loops in this paper, so we should always specify that $\chi(G) > n$. However, the validity of the approach remains unaltered even if we disregard the possible presence of loops.

connected, we can repeat this argument to show that $f_i(u) = f_i(v)$ for all $u, v \in V(G)$ and $i \in \{1, \ldots, n\}$. Therefore, $\{f_1, \ldots, f_n\}$ are the constant functions.

It remains to consider the case where for all $u \in V(G)$, the sequence $f_1(u), \ldots, f_n(u)$ contains at least one repetition. We show that if this were the case, then G would be n-colourable. Define a map $\phi : G \longrightarrow \{0, \ldots, n-1\}$ by selecting, for each $u \in V(G)$, a value $\phi(u)$ that appears at least twice in the sequence $f_1(u), \ldots, f_n(u)$. It is easy to verify that ϕ is a proper colouring of G. Since $\chi(G) > n$, such a colouring is impossible. Therefore, $\mathcal{C}_n(G)$ contains a unique complete subgraph of cardinality n. $\square$

Theorem 5.7 (Burr, Erdős, Lovász [23]) *Let H be a graph such that each vertex of H is contained in a complete subgraph of cardinality n, and $\chi(H) > n$. Then for any graph G such that $\chi(G) > n$ we have $\chi(G \times H) > n$.*

Proof We may assume that G is connected. Suppose that $\chi(G \times H) \leq n$. Then $H \to \mathcal{C}_n(G)$, and any homomorphism $\phi : H \longrightarrow \mathcal{C}_n(G)$ must map all vertices of H to the unique n-clique of $\mathcal{C}_n(G)$. This implies that H is n-colourable, which contradicts our hypothesis. $\square$

Theorem 5.8 (Duffus, Sands, Woodrow [38]) *Let G, H be two connected graphs, both containing a complete subgraph of cardinality n, and such that $\chi(G), \chi(H) > n$. Then $\chi(G \times H) > n$.*

Proof Suppose that $\chi(G \times H) \leq n$. Then there exists a homomorphism $\phi : H \longrightarrow \mathcal{C}_n(G)$, and since H is connected, ϕ must map all vertices of H into the same connected component of $\mathcal{C}_n(G)$, namely, the component C that contains the constant functions. It remains to show that if G contains a complete subgraph of cardinality n, then C is n-colourable.

Let $u_1, \ldots, u_n$ denote the vertices of a complete subgraph of G. Suppose that C contains a function f such that the values $f(u_1), \ldots, f(u_n)$ are all distinct. For any neighbour g of f, we have $f(u_i) \neq g(u_j)$ for all $i \neq j$, so $f(u_i) = g(u_i)$ for $i = 1, \ldots, n$. Thus, the values $g(u_1), \ldots, g(u_n)$ are all distinct, and since C is connected, any function h in C has the property the values $h(u_1), \ldots, h(u_n)$ are all distinct. However, this is impossible since C contains the constant functions.

Therefore, for any function f in C, the values $f(u_1), \ldots, f(u_n)$ are not all distinct. We can then define a map $\phi : C \longrightarrow \{0, \ldots, n-1\}$ by selecting, for each $f \in V(C)$, a value $\phi(f)$ that appears at least twice in the sequence $f(u_1), \ldots, f(u_n)$. It is easy to verify that ϕ is a proper n-colouring of C. $\square$

Note that since the categorical product of graphs is commutative, we have $H \to \mathcal{C}_n(G)$ if and only if $G \to \mathcal{C}_n(H)$. This accounts for the similarity of the proofs of Lemma 5.6 and Theorem 5.8. The reader may have noticed that connectedness plays an important role in these proofs. Of course, if the product conjecture is false, then there exist two *connected* graphs G and H such that $\chi(G \times H) < \min\{\chi(G), \chi(H)\}$. Also, it is an easy exercise to verify that if $C_1, \ldots, C_m$ are the connected components of G, then $\mathcal{C}_n(G)$ is isomorphic to $\times_{k=1}^{m} \mathcal{C}_n(C_k)$. Hence, everything seems to indicate that difficulties related to connectedness are avoidable. However, a closer inspection of the order-theoretic properties of the construction $\mathcal{C}_n$ with respect to the quasi-order $\rightarrow$ reveals that connectedness is indeed an important issue.[6]

Given a homomorphism $\phi : G \longrightarrow H$ one can define a homomorphism $\psi : \mathcal{C}_n(H) \longrightarrow \mathcal{C}_n(G)$ by putting $\psi(f) = f'$, where $f'(u) = f(\phi(u))$. Thus, the construction $\mathcal{C}_n$ 'reverses arrows' in the sense that $G \rightarrow H$ implies $\mathcal{C}_n(H) \rightarrow \mathcal{C}_n(G)$. Suppose that neither G nor $\mathcal{C}_n(G)$ is n-colourable. Then we may iterate the construction and consider $\mathcal{C}_n(\mathcal{C}_n(G))$. Since $\mathcal{C}_n(G) \times G$ is n-colourable, we have $G \rightarrow \mathcal{C}_n(\mathcal{C}_n(G))$. Therefore, $\chi(\mathcal{C}_n(\mathcal{C}_n(G))) \geq \chi(G)$, and $\mathcal{C}_n(\mathcal{C}_n(G))$ is not n-colourable. Again, $\mathcal{C}_n(\mathcal{C}_n(G)) \times \mathcal{C}_n(G)$ is n-colourable, and $\mathcal{C}_n(G) \rightarrow \mathcal{C}_n(\mathcal{C}_n(\mathcal{C}_n(G)))$. However, since the construction $\mathcal{C}_n$ reverses arrows and $G \rightarrow \mathcal{C}_n(\mathcal{C}_n(G))$, we also have $\mathcal{C}_n(\mathcal{C}_n(\mathcal{C}_n(G))) \rightarrow \mathcal{C}_n(G)$, that is, $\mathcal{C}_n(\mathcal{C}_n(\mathcal{C}_n(G)))$ and $\mathcal{C}_n(G)$ are homomorphically equivalent. Let H_1, H_2 denote the cores of $\mathcal{C}_n(G)$ and $\mathcal{C}_n(\mathcal{C}_n(G))$, respectively. We then have $H_1 \leftrightarrow \mathcal{C}_n(H_2)$ and $H_2 \leftrightarrow \mathcal{C}_n(H_1)$.

Hence, if the product conjecture is false, then there exist an integer n and two core graphs G and H such that $G \leftrightarrow \mathcal{C}_n(H)$ and $H \leftrightarrow \mathcal{C}_n(G)$.[7] In particular, G and H must contain complete subgraphs of cardinality n. Thus, if G and H were connected, we would have a contradiction to Corollary 5.8. So, we have to assume that G and H are disconnected.

An essential element of the proofs of Lemma 5.6 and Corollary 5.8 is the recognition of 'invariants' of a connected component of a n-colouring graph. In the proof of Theorem 5.4, the role of these invariants is played by a parameter defined as the 'parity' of a 3-colouring of an odd cycle. It seems that more invariants of this type need to be discovered if any progress is to be made with this approach to the product conjecture. However, the product conjecture admits interesting variations, and the methods of El-Zahar and Sauer have been applied successfully to some of these.

Definition 5.9 A graph K is called *multiplicative* if for any graphs G and H such that $G \not\rightarrow K$ and $H \not\rightarrow K$, we have $G \times H \not\rightarrow K$.

With this terminology, the product conjecture states that all complete graphs are multiplicative, and falls within the general problem of characterizing all multiplicative graphs. In this respect, a circular chromatic number analogue of Hedetniemi's conjecture has been formulated by X.Zhu in [138]: *all circular graphs are multiplicative.*

Conjecture 5.10 Let G and H be graphs. Then $\chi_c(G \times H) = \min\{\chi_c(G), \chi_c(H)\}$.

Note however that there exist graphs which are not multiplicative. For instance, if G, H are graphs such that $G \not\rightarrow H$ and $H \not\rightarrow G$, then $K = G \times H$ is not multiplicative.

By analogy with the definition of $\mathcal{C}_n(G)$, we define, for any graph K, the *K-colouring graph* $\mathcal{C}_K(G)$ of a graph G. The vertices of $\mathcal{C}_K(G)$ are all functions $f : V(G) \longrightarrow V(K)$, and two functions f, g are joined by an edge if and only if for all $[u, v] \in E(G)$, we have $[f(u), g(v)] \in E(K)$. In particular, $\mathcal{C}_{K_n}(G) = \mathcal{C}_n(G)$. Note that a function $f \in V(\mathcal{C}_K(G))$ is adjacent to itself if and only if it is a homomorphism form G to K. Hence, $\mathcal{C}_K(G)$ is a genuine (irreflexive) graph if and only if G has *no* "K-colourings"; in this respect, the name "K-colouring graph" is somewhat misleading[8].

[6] These order-theoretic properties are investigated by Duffus and Sauer [39].

[7] If we allow loops, this relationship holds between the complete graph K_n and the one-vertex graph with a loop. If the product conjecture is true, this is the only instance.

[8] Also note that the categorical product is sometimes called the tensor product, although it is not a tensor product from the point of view of category theory.

The essential properties of K-colouring graphs are summarized in the following statement.

Lemma 5.11 *Let G, H, K be graphs. Then $G \times H \to K$ if and only if $H \to \mathcal{C}_K(G)$.*

Proof Let $\phi : G \times H \longrightarrow K$ be a homomorphism. We define a map $\psi : V(H) \longrightarrow V(\mathcal{C}_K(G))$ by putting $\psi(x) = f_x$, where $f_x(u) = \phi(u, x)$. By the definition of $\mathcal{C}_K(G)$, it is straightforward to verify that ψ is a homomorphism.

Conversely, for a homomorphism $\psi : H \longrightarrow \mathcal{C}_K(G)$, we can define a homomorphism $\phi : G \times H \longrightarrow K$ by $\phi(u, x) = f_x(u)$, where $f_x = \psi(x)$. □

In particular, the relation $K \to \mathcal{C}_K(G)$ holds for any graphs G and K. Also, a graph K is multiplicative if and only if for every graph G such that $G \not\to K$, we have $\mathcal{C}_K(G) \to K$. In this connection, Theorem 5.4 has been generalized to the following.

Theorem 5.12 (Häggkvist, Hell, Miller, Neumann-Lara, [54]) *All odd cycles are multiplicative.*

In addition, the categorical product and the concept of multiplicative graphs is also investigated in the context of directed graphs. There, the approach of El-Zahar and Sauer has some noteworthy competition, namely the concept of 'complete sets of obstructions', which generalizes the role of the odd cycles in the characterization of bipartite graphs (see [54]). In another direction, infinite graphs and infinite chromatic numbers have also been considered with respect to the product conjecture. The categorical product of two graphs with countably infinite chromatic number has countably infinite chromatic number, but Hajnal [58] has shown that the product conjecture is false for higher cardinalities.

We conclude this section with an elegant proof of Welzl's density theorem 2.33 that uses the concepts presented above. This proof (to our knowledge unpublished) is due to M. Perles [99] and is very different form Welzl's original approach.

Proof of Theorem 2.33 Let G, H be graphs with at least one edge, and such that $G \to H$ and $H \not\to G$. Note that H need not be connected. However, we may assume that every connected component of H is nonbipartite. Let m denote the maximum of the odd girths of the connected components of H, and put $n = \chi(\mathcal{C}_G(H))$ (since $H \not\to G$, $\mathcal{C}_G(H)$ has no loops and hence its chromatic number is well defined). It is well known that there exists a graph L with odd girth greater than m and chromatic number greater than n (for instance, some Kneser graphs satisfy this property, see Proposition 3.14). We then have $L \not\to \mathcal{C}_G(H)$, and $H' \not\to L$ for every connected component H' of H. Let K be the disjoint union of the graphs G and $H \times L$. Obviously, we have $G \to K \to H$; we show that $H \not\to K \not\to G$. Suppose that $K \to G$. Then $H \times L \to G$, thus $L \to \mathcal{C}_G(H)$ by Lemma 5.11; this contradicts our choice of L. Similarly, suppose that $H \to K = G \cup H \times L$. Since $H \not\to G$, we must have $H' \to H \times L$ for some connected component H' of H. We then have $H' \to l$, which again contradicts our choice of L. Therefore, we have $G \to K \to H$ and $H \not\to K \not\to G$; that is, K is homomorphically "strictly in between" G and H, and this concludes the proof. □

5.2 The cartesian product and normal Cayley graphs

Recall that the cartesian product $G \Box H$ of two graphs G and H has vertex set $V(G) \times V(H)$, and two vertices $(u_1, v_1), (u_2, v_2)$ are joined by an edge if either $u_1 = u_2$ and $[v_1, v_2] \in E(H)$, or $[u_1, u_2] \in E(G)$ and $v_1 = v_2$. Hence the projections of $G \Box H$ onto its factors are not homomorphisms (except in the case of factors with no edges). However, for any $(u, v) \in V(G) \times V(H)$, the fibers induced by the sets $\{u\} \times V(H)$ and $V(G) \times \{v\}$ are isomorphic copies of H and G, respectively. This shows that both relations $G \rightarrow G \Box H$ and $H \rightarrow G \Box H$ hold. Thus, in shifting the focus from the categorical product to the cartesian product, we find that the respective roles of the product and the factors are reversed, in the sense that the factors admit natural maps into the product instead of the opposite. In particular, the inequality relating the respective chromatic numbers of the product now becomes $\chi(G \Box H) \geq \max\{\chi(G), \chi(H)\}$. However, in the case of the cartesian product, the analogue of the product conjecture admits a simple proof.

Lemma 5.13 ([107]) *For graphs G and H, $\chi(G \Box H) = \max\{\chi(G), \chi(H)\}$.*

Proof Recall our convention that for any $n \in \mathbb{N}$, the graph K_n has as its vertices the integers $\{0, 1, \ldots, n-1\}$. Let $m = \max\{\chi(G), \chi(H)\}$. Given homomorphisms $\phi : G \longrightarrow K_{\chi(G)}$ and $\psi : H \longrightarrow K_{\chi(H)}$, we define $\nu : G \Box H \longrightarrow K_m$ by $\nu(u, x) = \phi(u) + \psi(x) \bmod m$. This is a homomorphism and so $\chi(G \Box H) \leq m$. Since both G and H map into $G \Box H$, we also have the inverse inequality. $\Box$

¿From the point of view of homomorphisms, Lemma 5.13 admits the following interpretation. For any graphs G, H such that $G \rightarrow K_n$ and $H \rightarrow K_n$, we have $G \Box H \rightarrow K_n$. By analogy with the concept of multiplicative graphs (Definition 5.9), this property suggests the following problem.

> *Characterize the graphs K with the property that for any graphs G, H such that $G \rightarrow K$ and $H \rightarrow K$, we have $G \Box H \rightarrow K$.*

In particular, for any graph K, we have $K \rightarrow K$, so this property implies $K \Box K \rightarrow K$. Conversely, any pair of homomorphisms $\phi_1 : G \longrightarrow K$ and $\phi_2 : H \longrightarrow K$ can be used coordinatewise to define a homomorphism $\phi : G \Box H \longrightarrow K \Box K$. Therefore, a graph K has the property described above if and only if $K \Box K \rightarrow K$.

Definition 5.14 A graph K is called *hom-idempotent* if $K \Box K \leftrightarrow K$.

As any graph K satisfies $K \rightarrow K \Box K$, a graph K is hom-idempotent if and only if $K \Box K \rightarrow K$. This property seems very restrictive, since the projections are not homomorphisms. The complete graphs are hom-idempotent, but a close inspection of the proof of Lemma 5.13 hints at a construction of other examples.

Definition 5.15 A *normal Cayley graph* is a Cayley graph $\mathrm{Cay}(\Gamma, S)$ such that $x^{-1}sx \in S$ for any $x \in \Gamma$ and $s \in S$.

Note that the condition $x^{-1}sx \in S$ is satisfied whenever the group Γ is abelian. Thus, all Cayley graphs of abelian groups, and in particular all circulants, are normal Cayley graphs. The defining property of normal Cayley graphs has the following interpretation.

Lemma 5.16 *A Cayley graph* $\mathrm{Cay}(\Gamma, S)$ *is normal if and only if both the left translations and the right translations of* Γ *are automorphisms of* $\mathrm{Cay}(\Gamma, S)$.

Proof Since the left translations are always automorphisms of Cayley graphs, it suffices to consider the right translations. Suppose that $\mathrm{Cay}(\Gamma, S)$ is normal. For $w \in \Gamma$, let $T : \Gamma \longrightarrow \Gamma$ be the right translation $T(u) = uw$. For $[u, v] \in E(\mathrm{Cay}(\Gamma, S))$, we have $u^{-1}v \in S$, so $T(u)^{-1}T(v) = w^{-1}u^{-1}vw \in S$ and $[T(u), T(v)] \in E(\mathrm{Cay}(\Gamma, S))$. Therefore, T is an automorphism of $\mathrm{Cay}(\Gamma, S)$. Conversely, suppose that all right translations of a Cayley graph $\mathrm{Cay}(\Gamma, S)$ are automorphisms. Then, for $s \in S$ and $w \in \Gamma$, we have $[1_\Gamma, s] \in E(\mathrm{Cay}(\Gamma, S))$, so $[w, sw] \in E(\mathrm{Cay}(\Gamma, S))$, that is, $w^{-1}sw \in S$. □

Corollary 5.17 (Hahn, Hell, Poljak [55]) *Let* $\mathrm{Cay}(\Gamma, S)$ *be a normal Cayley graph. Then the map* $\phi : \mathrm{Cay}(\Gamma, S) \Box \mathrm{Cay}(\Gamma, S) \longrightarrow \mathrm{Cay}(\Gamma, S)$ *defined by putting* $\phi(x, y) = xy$ *is a homomorphism. Therefore,* $\mathrm{Cay}(\Gamma, S)$ *is hom-idempotent.*

Proof The map ϕ acts as a right translation on each fiber of the first factor and as a left translation on each fiber of the second factor. Thus, by Lemma 5.16, ϕ is a homomorphism. □

Two classes of circulants appear respectively in the definition of the chromatic number and the circular chromatic number, namely the complete graphs and the circular graphs. By Corollary 5.17, these are hom-idempotent graphs, and this accounts for the identities $\chi(G \Box H) = \max\{\chi(G), \chi(H)\}$ and $\chi_c(G \Box H) = \max\{\chi_c(G), \chi_c(H)\}$. In contrast, the fractional chromatic number is defined using homomorphisms into Kneser graphs, which are not hom-idempotent. It follows that in general, $\chi_f(G \Box H)$ cannot be expressed as a function of $\chi_f(G)$ and $\chi_f(H)$. For instance, we have $\chi_f(C_5) = \chi_f(K(2, 5)) = 5/2$. However, since C_5 is hom-idempotent, we have $\chi_f(C_5 \Box C_5) = \chi_f(C_5) = 5/2$, while $\chi_f(K(2, 5) \Box K(2, 5)) = 50/17$, as shown by Albertson and Collins [5].

These examples show how hom-idempotent graphs are related to parameters which are well behaved with respect to the cartesian product. The characterization of hom-idempotent graphs is closely related to the definition of the cartesian product of graphs as a tensor product in the category of graphs.

Definition 5.18 Let G and K be graphs. We define the *homomorphism graph* $\mathrm{Hom}(G, K)$ by putting

$$\begin{aligned} V(\mathrm{Hom}(G, K)) &= \{\phi : G \longrightarrow K : \ \phi \text{ is a homomorphism}\}, \\ E(\mathrm{Hom}(G, K)) &= \{[\phi, \psi] : \ \text{for all } u \in V(G), [\phi(u), \psi(u)] \in E(K)\}. \end{aligned}$$

It is interesting to compare this definition to that of the graph $\mathcal{C}_K(G)$ defined in the preceding section. Note that $\mathcal{C}_K(G)$ has $|K|^{|G|}$ vertices regardless of the structure of G and K, whereas the vertices of $\mathrm{Hom}(G, K)$ are homomorphisms, and hence their number depends heavily on the structure of both G and K. In particular, the number of vertices of $\mathrm{Hom}(G, K_n)$ is the value $p_G(n)$, where p_G is the chromatic polynomial of G. The relative importance of structure is reversed when we consider the edge sets of $\mathcal{C}_K(G)$ and $\mathrm{Hom}(G, K)$. The definition of $E(\mathcal{C}_K(G))$ uses both definitions of $E(G)$ and $E(K)$, while the definition of

$E(\mathrm{Hom}(G, K))$ omits that of $E(G)$. In fact, $\mathrm{Hom}(G, K)$ can be viewed as a subgraph of the categorical product of $|G|$ copies of K. This brings forth the last point of comparison between these graphs. The relation $\mathrm{Hom}(G, K) \to K$ trivially holds in all cases, as for any $u \in V(G)$, the *evaluation map* $\epsilon_u : \mathrm{Hom}(G, K) \to K$ defined by $\epsilon_u(\phi) = \phi(u)$ is a homomorphism. However, to decide in which cases the relation $\mathcal{C}_K(G) \to K$ holds, amounts to characterizing multiplicative graphs. Note, however, that the relation $K \to \mathcal{C}_K(G)$ always holds.

Any homomorphism $\phi : G \square H \longrightarrow K$ acts as a homomorphism on each of the fibers of H. It is the kind of interplay needed between these that dictates the definition of $\mathrm{Hom}(H, K)$. Formally, this means that the homomorphism graphs can be used in a characterization of the cartesian product of graphs as a tensor product. The essential points of this characterization are summarized in the following technical statement.

Proposition 5.19 *Let G, H, K be graphs.*

(i) *Let $\phi : G \square H \longrightarrow K$ be a homomorphism. Then the map $\psi : G \longrightarrow \mathrm{Hom}(H, K)$ defined by setting $\psi(u) = \phi_u$ is a homomorphism, where $\phi_u : H \longrightarrow K$ is defined by $\phi_u(v) = \phi(u, v)$.*

(ii) *Let $\psi : G \longrightarrow \mathrm{Hom}(H, K)$ be a homomorphism. Then the map $\phi : G \square H \longrightarrow K$ defined by $\phi(u, v) = \phi_u(v)$ is a homomorphism, where $\phi_u = \psi(u)$.*

We omit the proof which is a straightforward application of the definition. Our main interest is the hom-idempotent graphs, that is, the situation where $G = H = K$. Proposition 5.19 states that for a graph K, we have $K \square K \to K$ if and only if $K \to \mathrm{Hom}(K, K)$. Since the relation $\mathrm{Hom}(K, K) \to K$ always holds, we have the following.

Corollary 5.20 *A graph K is hom-idempotent if and only if $K \leftrightarrow \mathrm{Hom}(K, K)$.*

We can make this characterization more precise by restricting it to core graphs.

Proposition 5.21 *Let K be a core. Then $\mathrm{Hom}(K, K)$ is a normal Cayley graph.*

Proof If K is a core, then $V(\mathrm{Hom}(K, K)) = \mathrm{Aut}(K)$. Two automorphisms ϕ, ψ of K are adjacent if and only for all $u \in V(K)$, we have $[\phi(u), \psi(u)] \in E(K)$. Applying ϕ^{-1} yields $[u, \phi^{-1} \circ \psi(u)] \in E(K)$ for all $u \in V(K)$. Therefore, $\mathrm{Hom}(K, K)$ is the Cayley graph $\mathrm{Cay}(\mathrm{Aut}(K), S_K)$, where

$$S_K = \{\sigma \in \mathrm{Aut}(K) : [u, \sigma(u)] \in \mathrm{Aut}(K) \text{ for all } u \in V(K)\}.$$

The elements of S_K are called the *shifts* of K. It remains to show that for any $\sigma \in S_K$ and any $\phi \in \mathrm{Aut}(K)$, we have $\phi^{-1} \circ \sigma \circ \phi \in S_K$. For any $u \in V(K)$, we have $[\phi(u), \sigma \circ \phi(u)] \in E(K)$ by the definition of shifts. Applying ϕ^{-1} yields $[u, \phi^{-1} \circ \sigma \circ \phi(u)] \in E(K)$. Thus, $\phi^{-1} \circ \sigma \circ \phi \in S_K$, and $\mathrm{Hom}(K, K) = \mathrm{Cay}(\mathrm{Aut}(K), S_K)$ is a normal Cayley graph. □

Since homomorphisms can be used coordinatewise, hom-idempotency of a graph K is equivalent to hom-idempotency of its core. Therefore, combining the previous results with Corollary 5.17, we get the following.

Theorem 5.22 (Larose, Laviolette, Tardif [90]) *A graph is hom-idempotent if and only if it is homomorphically equivalent to a normal Cayley graph.* □

Thus, up to homomorphic equivalence, the examples described in Corollary 5.17 represent all hom-idempotent graphs. This is surprising in view of the fact that even for a core graph K, the homomorphisms $\phi : K\Box K \longrightarrow K$ are usually not group operations. For instance, any latin square on the vertex set of K_n defines a homomorphism $\phi : K_n\Box K_n \longrightarrow K_n$, and only a few of these are group operations. Of course, some always are, and K_n is indeed a normal Cayley graph. However, it is conceivable that there exists a hom-idempotent core K such that none of the homomorphisms $\phi : K\Box K \longrightarrow K$ are group operations, though no such example is known.

Since the fibers are isomorphic copies of the factors in a cartesian product of graph, we have for any graph K and any $n \leq m$ that $K^n \to K^m$ (where K^n is short for $\Box_{i=1}^n K$). In fact, it is quite common that the relation $\to$ defines a strict order on the cartesian powers of a graph. At the other extreme lie the hom-idempotent graphs, which have the property that all of their cartesian powers are homomorphically equivalent. Between these two extremes, there exist graphs which satisfy some nontrivial relations $K^m \to K^n$ but not all of them. The next result shows that normal Cayley graphs also help to characterize all of these graphs.

Theorem 5.23 (Larose, Laviolette, Tardif [90]) *Let K be a graph and $m > n$ integers. Then $K^m \to K^n$ if and only if there exist normal Cayley graphs* $\mathrm{Cay}(\Gamma_1, S_1), \ldots, \mathrm{Cay}(\Gamma_n, S_n)$ *such that* $\mathrm{Cay}(\Gamma_k, S_k) \to K$ *for $k = 1, \ldots, n$, and* $K \to \Box_{k=1}^n \mathrm{Cay}(\Gamma_k, S_k)$.

We have seen in Example 3.5 that $G = \mathrm{Cay}(\mathbb{Z}_8, \{\pm 1, \pm 2\})$ and $H = \mathrm{Cay}(\mathbb{Z}_7, \{\pm 1, \pm 2\})$ are two circulants such that $G \not\to H$ and $H \not\to G$. Both of these are normal Cayley graphs. Let K be the graph obtained from the disjoint union of these two graphs by adding an edge between the element 0 of $\mathbb{Z}_8$ and the element 0 of $\mathbb{Z}_7$. Then we have $G \to K$ and $H \to K$. Also, a homomorphism $\phi : K \longrightarrow G\Box H$ can be defined by putting $\phi(u) = (u, 0)$ if $u \in \mathbb{Z}_8$ and $\phi(u) = (1, u)$ if $u \in \mathbb{Z}_7$. Therefore, we also have $K \to G\Box H$.

Theorem 5.23 then states that for any $m > 2$, we have $K^m \to K^2$. A homomorphism from K^m to K^2 can be constructed as follows. First, use $\phi : K \longrightarrow G\Box H$ coordinatewise to define a homomorphism from K^m to $(G\Box H)^m$. Then, note that $G\Box H$ is a normal Cayley graph (the cartesian product of normal Cayley graphs always is), so the multiplication of the coordinates is a homomorphism from $(G\Box H)^m$ to $G\Box H$. Finally, use any homomorphisms $\psi_1 : G \longrightarrow K$ and $\psi_2 : H \longrightarrow K$ coordinatewise to define a homomorphism from $G\Box H$ to K^2.

However, there is no homomorphism from K^2 to K. Indeed, we have $K^2 \leftrightarrow G\Box H$, and every edge of $G\Box H$ is contained in a triangle. Hence, no edge of $G\Box H$ can be mapped by homomorphism to the edge of K joining the element 0 of $\mathbb{Z}_8$ and the element 0 of $\mathbb{Z}_7$. Therefore, the image of a homomorphism from $G\Box H$ to K would either be contained in G or in H. However, this is impossible since $G \not\to H$ and $H \not\to G$.

We now turn our attention to homomorphisms *into* a cartesian product of graphs. Let $\phi : G \longrightarrow \Box_{k=1}^n G_k$ be a homomorphism. Then the maps $\mathrm{pr}_k \circ \phi : G \longrightarrow G_k, k = 1, \ldots, n$, need not be homomorphisms. However, for any k, the partition

$$\mathcal{P}_k = \{(pr_k \circ \phi)^{-1}(u) : u \in V(G_k)\}$$

of $V(G)$ has the property that $G/\mathcal{P}_k \to G_k$, and the least common refinement of $\mathcal{P}_1, \ldots, \mathcal{P}_n$ is the partition

$$\mathcal{P} = \{\phi^{-1}(u) : u \in V(\Box_{k=1}^n G_k)\}$$

of $V(G)$. Hence, the essential information concerning ϕ is encoded in the family $\mathcal{P}_1, \ldots, \mathcal{P}_n$ of partitions of $V(G)$. This representation is still cumbersome, but we will be able to refine it by using the structure of the cartesian product of graphs.

Definition 5.24

- Let $\phi : G \longrightarrow \Box_{k=1}^{n} G_k$ be a homomorphism. The *edge-labeling* induced by ϕ is the map

$$\ell_\phi : E(G) \longrightarrow \{1, \ldots, n\},$$

  where $\ell_\phi([u,v])$ is the unique index k such that $\mathrm{pr}_k \circ \phi(u) \neq \mathrm{pr}_k \circ \phi(v)$.

- Let $\ell : E(G) \longrightarrow \{1, \ldots, n\}$ be a map. Then for $k \in \{1, \ldots, n\}$, let $G/\ell^{-1}(k)$ denote the quotient $G/\mathcal{Q}_k$, where $\mathcal{Q}_k$ is the partition of $V(G)$ whose cells are the (vertex-sets of) connected components of $G - \ell^{-1}(k)$.

Let $\phi : G \longrightarrow \Box_{k=1}^{n} G_k$ be a homomorphism with induced edge labeling ℓ_ϕ. Then $\mathrm{pr}_k \circ \phi$ is constant on each connected component of $G - \ell_\phi^{-1}(k)$, and there exists a homomorphism $\phi_k : G/\ell_\phi^{-1}(k) \longrightarrow G_k$ (not necessarily injective) such that $\mathrm{pr}_k \circ \phi = \phi_k \circ \pi_k$ (where $\pi_k : G \longrightarrow G/\ell_\phi^{-1}(k)$ is the natural map) for $k = 1, \ldots, n$. In addition, we can use the natural maps $\pi_k : G \longrightarrow G/\ell_\phi^{-1}(k)$ coordinatewise to define a homomorphism $\pi : G \longrightarrow \Box_{k=1}^{n} G/\ell_\phi^{-1}(k)$. To some extent, this shows how some essential information concerning homomorphisms of G into a cartesian product is encoded in maps from $E(G)$ to $\{1, \ldots, n\}$ for some integer n, in other words, in edge partitions. The following is a characterization of the maps which give rise to a homomorphism in this fashion.

Observation 5.25 *Let G be a graph, n an integer, and $\ell : E(G) \longrightarrow \{1, \ldots, n\}$ a map. Then the following conditions are equivalent.*

(i) *There exists a homomorphism $\phi : G \longrightarrow \Box_{k=1}^{n} G_k$ such that $\ell_\phi = \ell$.*

(ii) *The natural map $\pi : G \longrightarrow \Box_{k=1}^{n} G/\ell^{-1}(k)$ is a homomorphism.*

(iii) *For any edge $e = [u,v]$ of G and for any path P from u to v, there exists an edge e' of P such that $\ell(e') = \ell(e)$.*

Therefore, the problem of finding a homomorphism from a graph to a cartesian product of graphs often amounts to finding a suitable edge partition. In that vein, we find the long standing problem of characterizing the induced subgraphs of cubes (a *cube* is a cartesian product of copies of K_2). This problem was brought forth by Shapiro [112] along with the first definition of the cartesian product of graphs, and has since been studied by many (see [27, 31]). As is to be expected, the problems concerning homomorphisms into cartesian product of graphs turn out to be as intricate as many other problems concerning graph homomorphisms. However, a recent wave of interest in some particular homomorphisms called isometric embeddings has given rise to interesting results concerning embeddings into cartesian products. These and other aspects of isometric embeddings are investigated in Section 5.4.

5.3 The strong product and the lexicographic product

The strong product $G \boxtimes H$ and the lexicographic product $G[H]$ bear a close resemblance to each other. The edges of $G \boxtimes H$ are the pairs $[(u_1, u_2), (v_1, v_2)]$ such that u_i and v_i are adjacent or equal for $i = 1, 2$. All these edges are contained in $G[H]$, in addition to all other edges $[(u_1, u_2), (v_1, v_2)]$ such that $[u_1, v_1] \in E(G)$. In particular, $G \boxtimes K_n = G[K_n]$ for any graph G and any integer n. This identity summarizes the relation between the colouring problems between both of these products. The problem of finding the chromatic number of a lexicographic product of graphs reduces to that of finding the chromatic number of the graphs $G[K_n]$, which in turn provides bounds on the chromatic number of a strong product of graphs.

In section 4.3, we defined the n-chromatic number of a graph G as the least integer $m = \chi_n(G)$ such that G admits a Kneser (n, m)-colouring, that is, a homomorphism $\phi : G \longrightarrow K(n, m)$.

Lemma 5.26 *For any graph G, and any integer n, $\chi(G[K_n]) = \chi_n(G)$.*

Proof Let $\phi : \chi(G[K_n]) \longrightarrow [m]$ be a colouring. Then, we find a (n, m)-subset colouring of G by assigning to each vertex u the set of colours used on the fiber $\{(u, v) : v \in K_n\}$. Conversely, if $\phi : G \longrightarrow K(n, m)$ is a homomorphism, then we can m-colour $G[K_n]$ by colouring the fiber $\{(u, v) : v \in K_n\}$ with colours in $\phi(u)$ for each $u \in V(G)$. □

Theorem 5.27 (Stahl [113]) *For any graphs G and H, we have $\chi(G[H]) = \chi_n(G)$, where $n = \chi(H)$.*

Proof By Lemma 5.26, we have $\chi(G[H]) \leq \chi_n(G)$ since $G[H] \rightarrow G[K_n]$. On the other hand, for any m-colouring $\phi : G[H] \longrightarrow [m]$, each fiber $\{(u, v) : v \in V(H)\}$ uses at least $n = \chi(H)$ colours, and the fibers corresponding to adjacent vertices of G are coloured with disjoint sets of colours. Thus, we can define a Kneser (n, m)-colouring of G by independently selecting, for each u in $V(G)$, a n-subset of the colours used for its fiber. □

Thus, the chromatic number of $G[H]$ does not depend on the structure of H, only on its chromatic number. A similar phenomenon occurs when we consider the circular chromatic number of a lexicographic product of graphs.

Theorem 5.28 (Zhu [138]) *For any graphs G and H, we have $\chi_c(G[H]) = \chi_c(G[K_n])$, where $n = \chi(H)$.*

In the case of the strong product, Lemma 5.26 can be used to find lower bounds for the chromatic number.

Lemma 5.29 *For any graphs G and H,*

$$\chi(G \boxtimes H) \geq \max\{\chi_{\omega(H)}(G), \chi_{\omega(G)}(H)\}.$$

Proof Since $G[K_{\omega(H)}] \simeq G \boxtimes K_{\omega(H)} \rightarrow G \boxtimes H$ and $H[K_{\omega(G)}] \simeq H \boxtimes K_{\omega(G)} \rightarrow H \boxtimes G \simeq G \boxtimes H$, the result follows from Lemma 5.26. □

It is also possible to express bounds for the chromatic number of a strong product of graphs in terms of the chromatic numbers and clique numbers of the factors alone. If G has at least one edge, then for any n, $\chi_n(G) \geq 2n$. For $m = \chi_n(G)$, we have $G \to K(n, m)$. By Theorem 3.10, $\chi(K(n, m)) = m - 2n + 2$, and $\chi(G) + 2n - 2 \leq m = \chi_n(G)$. Therefore, we have the following.

Theorem 5.30 (Klavžar, Milutinović [85]) *If G and H are graphs with at least one edge, then*

$$\chi(G \boxtimes H) \geq \max\{\chi(G) + 2\omega(H) - 2, \chi(H) + 2\omega(G) - 2\}. \qquad \square$$

Note that this bound is obtained by considering subgraphs of the product which are isomorphic to lexicographic products. Vesztergombi [124] proposed a different approach which takes into account the full structure of the strong product of graphs. For a graph G and an integer m, define the graph $\mathcal{C}_m^{\boxtimes}(G)$ by putting

$$\begin{aligned} V(\mathcal{C}_m^{\boxtimes}(G)) &= \{\phi : G \longrightarrow [m] : \phi \text{ is a proper colouring}\}, \\ E(\mathcal{C}_m^{\boxtimes}(G)) &= \{[\phi, \psi] : \phi(u) \neq \psi(u) \text{ for all } u \in V(G) \text{ and} \\ &\qquad\qquad \phi(u) \neq \psi(v) \text{ for all } [u, v] \in E(G)\}. \end{aligned}$$

This construction bears the same relation to the colourings of a strong product of graphs as do the n-colouring graphs of Definition 5.3 to the colourings of a categorical product of graphs.[9] Namely, the following holds.

Theorem 5.31 (Vesztergombi [124]) *For two graphs G and H, $G \boxtimes H$ is m-colourable if and only if $H \to \mathcal{C}_m^{\boxtimes}(G)$.*

It is clear that the chromatic number of a strong product of graphs cannot be expressed as a function of the chromatic numbers of the factors. Thus, there is no strong product-analogue of the product conjecture. Nonetheless, it would be interesting to know more about the general structure of $\mathcal{C}_m^{\boxtimes}(G)^\bullet$. In particular, Vesztergombi [124] has shown that $\mathcal{C}_m^{\boxtimes}(K_n)^\bullet = K(n, m)$, and $\mathcal{C}_m^{\boxtimes}(C_5)^\bullet = C_5$.

We now turn our attention to retracts of strong products and lexicographic products of graphs. These are more manageable than the retracts of products we have investigated so far. We will be particularly interested in finding out when the product of two cores is a core. Since homomorphisms can be used coordinatewise, it is of course necessary that both factors be cores. However, this condition is not sufficient for either of the products. Consider the graphs $K(3, 9)$ and K_3. Both are cores; however, their product $K(3, 9) \boxtimes K_3 \simeq K(3, 9)[K_3]$ is a 9-chromatic graph with a 9-clique, so its core is K_9.

Proposition 5.32 *For any connected graphs G, H, the core of $G[H]$ is $G'[H^\bullet]$, where G' is a subgraph of G which is itself a core.*

Proof Clearly, $G[H] \leftrightarrow G[H^\bullet]$. Let $\rho : G[H^\bullet] \longrightarrow G[H^\bullet]^\bullet$ be a retraction, and identify $G[H^\bullet]^\bullet$ with a subgraph of $G[H^\bullet]$. Then for any $u \in V(G)$ and $[v, w] \in E(H^\bullet)$ such that $(u, v) \in V(G[H^\bullet]^\bullet)$, $\rho(u, w)$ must be adjacent to (u, v) and cannot be adjacent to (u, w).

[9] Note that the definition of $\mathcal{C}_m^{\boxtimes}(G)$ predates that of $\mathcal{C}_m(G)$.

Since $H^\bullet$ is connected, it follows that ρ maps the fiber $F_u = \{(u,v) : v \in V(H^\bullet)\}$ into itself. Since $H^\bullet$ is a core, this means that $G[H^\bullet]^\bullet = G'[H^\bullet]$, where G' is a subgraph of G. $\square$

However note that not all retracts of $G[H]$ are lexicographic products, since retractions can be performed independently on the fibers of H. The situation is different in the case of the strong product, as the following result shows.

Theorem 5.33 (Imrich, Klavžar [84]) *Let G, H be connected graphs, and R a retract of $G \boxtimes H$. Then there exist subgraphs G' of G and H' of H such that R is isomorphic to $G' \boxtimes H'$.*

In particular, the core of $G \boxtimes H$ is a strong product of subgraphs of G and H. The example of $K(3,9)[K_3]$ shows that neither Proposition 5.32 nor Theorem 5.33 can be refined to the statement that these subgraphs are retracts of the respective factors. The reason lies in the fact that the presence of triangles in the factors allows some interplay between the factors. This is made clear by the following results.

Proposition 5.34 *Let G and H be connected graphs such that G does not contain any triangles. Then $G[H]^\bullet = G^\bullet[H^\bullet]$.*

Proof By Proposition 5.32, $G[H]^\bullet = G'[H^\bullet]$, where G' is a core, and it remains to show that it is the core of G. Let $\rho : G[H^\bullet] \longrightarrow G'[H^\bullet]$ be a retraction. Put

$$S = \{u \in V(G) : \rho(F_u) = F_{\psi(u)} \text{ for some } \psi(u) \in V(H^\bullet)\}.$$

We show that $S = V(G)$. Clearly, S is not empty since $V(G') \subseteq S$. If u' is adjacent to some $u \in S$, then $\rho(F_{u'})$ cannot intersect $F_{\psi(u)}$ since every vertex of $F_{u'}$ is adjacent to all vertices of F_u. Also, since $H^\bullet$ is connected, $\mathrm{pr}_G(F_{u'})$ induces a connected subgraph of G, whose vertices are all adjacent to $\psi(u)$. However, G does not contain any triangles, so $\mathrm{pr}_G(F_{u'})$ consists of a single vertex $\psi(u')$. Then, ρ must map $F_{u'}$ to $F_{\psi(u')}$ bijectively, since all fibers are isomorphic to the core $H^\bullet$. Thus, $u' \in S$. G being connected, this implies that $S = V(G)$. Therefore, $\psi : V(G) \longrightarrow G'$ is a well-defined retraction, and $G' = G^\bullet$. $\square$

Theorem 5.35 (Imrich, Klavžar [84]) *Let G, H be connected triangle-free graphs, and R a retract of $G \boxtimes H$. Then there exist retracts G' of G and H' of H such that R is isomorphic to $G' \boxtimes H'$. In particular, $(G \boxtimes H)^\bullet = G^\bullet \boxtimes H^\bullet$.*

In the case of these two products, retracts and cores can be expressed as products of subgraphs or retracts. The situation is different with other products. In particular, the identity $(G \times G)^\bullet = G^\bullet$ holds for all graphs, and $G^\bullet$ cannot be expressed as a categorical product of subgraphs of G. Also, the hom-idempotent graphs are the graphs G that satisfy $(G \Box G)^\bullet = G^\bullet$, and $G^\bullet$ is a cartesian product of subgraphs of G. In general, it is known (see Tardif [117]) that if $(\Box_{k=1}^n G_k)^\bullet$ is vertex-transitive, then it is a cartesian product of subgraphs of the factors. However, it is not known if the same holds for all cores of connected cartesian products of graphs.

5.4 Isometric embeddings and retracts

The *distance* $d_G(u, v)$ between two vertices u, v of a connected graph G is the length of a shortest path joining them. Such a path is called a *uv-geodesic*. The function d_G itself is often called the 'geodesic metric' or 'shortest path metric' of G. It is indeed a metric in the usual sense, as it satisfies the axioms $d_G(u, v) = 0$ if and only if $u = v$, $d_G(u, v) = d_G(v, u)$ and the triangle inequality $d_G(u, w) \leq d_G(u, v) + d_G(v, w)$.

The interplay between graph homomorphisms and the shortest path metric is illustrated by the following observation. Let G, H be connected graphs, and $\phi : G \longrightarrow H$ a homomorphism. Then, for any $u, v \in V(G)$ and any uv-geodesic P, $\phi(P)$ is a $\phi(u)\phi(v)$-trail, and we see from this that $d_H(\phi(u), \phi(v)) \leq d_G(u, v)$. This motivates the following definitions.

Definition 5.36 Let G, H be connected graphs, and $\phi : V(G) \longrightarrow V(H)$ a map.

(i) ϕ is a *contraction* if $d_H(\phi(u), \phi(v)) \leq d_G(u, v)$ for any two vertices u, v of G.

(ii) ϕ is an *isometric embedding* if $d_H(\phi(u), \phi(v)) = d_G(u, v)$ for any two vertices u, v of G.

Any homomorphism between connected graphs is a contraction, but the converse fails since contractions can map adjacent vertices to the same vertex. However, isometric embeddings are homomorphisms. The term 'embedding' is normally reserved for injective maps and indeed, it is easy to see that any isometric embedding is a full monomorphism. The following result shows that some important homomorphisms that we have already encountered are in fact isometric embeddings.

Lemma 5.37 *Let R be a retract of a connected graph G. Then any co-retraction $\gamma : R \longrightarrow G$ is an isometric embedding.*

Proof Let $\rho : G \longrightarrow R$ be a retraction such that $\rho \circ \gamma = \mathrm{id}_R$. Then for any $u, v \in V(R)$, we have

$$d_R(u, v) \geq d_G(\gamma(u), \gamma(v)) \geq d_R(\rho \circ \gamma(u), \rho \circ \gamma(v)) = d_R(u, v),$$

hence γ is an isometric embedding. □

Therefore, metric considerations turn out to play an important role in the characterization of retracts of graphs, though we have managed to avoid the subject up to this point. Part of the reasons for avoiding it lie in the fact that when considering discrete metric spaces, the natural morphisms are the contractions rather than graph homomorphisms.

Isometric embeddings into products of graphs often provide an insight into the metric structure of a graph; this has become a flourishing topic in view of some recent applications of graph theory to computer science. Szamkołowicz [116] has shown that there exist precisely two products of graphs in which the distance can be expressed as a reasonably 'nice' function of the coordinate distances. These are the cartesian product, where the distance is the sum of coordinate distances, and the strong product, where the distance is the maximum of coordinate distances. Each of these products gives rise to an interesting theory regarding isometric embeddings and retracts. We shall present both in some detail.

Recall that Observation 5.25 shows how the analysis of homomorphisms into cartesian products is reduced to the study of edge labelings and natural maps into products of the quotients introduced in Definition 5.24

Theorem 5.38 (Graham, Winkler [52]) *Let G be a graph, n an integer, and $\ell : E(G) \longrightarrow \{1, \ldots, n\}$ a map. Then the following conditions are equivalent.*

(i) *There exists an isometric embedding $\phi : G \longrightarrow \square_{k=1}^{n} G_k$ such that $\ell_\phi = \ell$.*

(ii) *The natural map $\pi : G \longrightarrow \square_{k=1}^{n} G/\ell^{-1}(k)$ is an isometric embedding.*

(iii) *For all $u, v \in V(G)$, for all $k \in \{1, \ldots, n\}$ and for all uv-geodesic P, we have*

$$|E(P) \cap \ell^{-1}(k)| = \min\{|E(Q) \cap \ell^{-1}(k)| : Q \text{ is a } uv\text{-path }\}.$$

(iv) *If $[u, v], [u', v']$ are edges of G such that $\ell^{-1}([u, v]) \neq \ell^{-1}([u', v'])$, then*

$$d_G(u, u') - d_G(u, v') = d_G(v, u') - d_G(v, v').$$

In particular, condition (iv) suggests the following. Define a relation θ on the edge set of a connected graph G by putting

$$[u, v]\theta[u', v'] \equiv d_G(u, u') + d_G(v, v') \neq d_G(u, v') + d_G(v, u').$$

This relation was introduced by Djoković for the purpose of characterizing the graphs which admit an isometric embedding in a cube. Note that θ is well defined, reflexive and symmetric. Let $\hat{\theta}$ denote the transitive closure of θ. Then $\hat{\theta}$ is an equivalence relation and partitions $E(G)$ into classes $E_1, \ldots, E_n$. Define $\ell : E(G) \longrightarrow \{1, \ldots, n\}$ by putting $\ell(e) = k$ if $e \in E_k$. Then the natural map $\pi : G \longrightarrow \square_{k=1}^{n} G/\ell^{-1}(k)$ is an isometric embedding. This map is optimal in the sense that if $\phi : G \longrightarrow \square_{k=1}^{m} G_k$ is an isometric embedding that is 'irredundant' (that is, $\mathrm{pr}_k \circ \phi(G) = G_k$ for all k and none of the factors G_k is isomorphic to K_1), then ϕ can be refined to π.

Thus, isometric embeddings into cartesian products of graphs admit an efficient characterization, contrary to homomorphisms in general. A similar treatment applies to isomorphisms into cartesian products, which can be viewed as a particular kind of isometric embeddings (see [83]). However, we shall not go into this subject and turn our attention to the retracts of a cartesian product of graphs.

Recall that the cubes are the cartesian powers of K_2. These are the simplest instances of cartesian product of graphs. No general characterization of the retracts of cartesian products of graphs is known, but such a characterization exists for simple structures such as cubes. We define the *diameter* of a graph G as the maximum distance between any two vertices of G. For instance, the cube $Q_n = \square_{k=1}^{n} K_2$ has diameter n. Also, the *covering graph* (or *unoriented Hasse diagram*) of a partially ordered set $(P, \leq)$ is the graph G which has P as its vertex set, and whose edges are the pairs $[p, q]$ such that p *covers* q, that is, $p < q$ and no element r of P satisfies $p < r < q$.

Theorem 5.39 (Duffus, Rival [37]) *The diameter-preserving retracts of cubes are the covering graphs of distributive lattices.*

This result may come as a surprise, since it is not clear how lattice operations can be related to a graph structure, let alone to graph retractions. The following concept helps to clarify these matters. A *median* of three vertices of a connected graph is a vertex that lies simultaneously on geodesics between any two of them. It often happens that three given vertices of a graph have no median, or more than one. A graph G is called a *median graph* if any triple of (not necessarily distinct) vertices of G has a unique median. For instance, trees are simple examples of median graphs. Other examples include the cubes and the covering graphs of distributive lattices. In fact, if G is the covering graph of a distributive lattice L, then the median of any three vertices u, v, w of G coincides with their order-theoretic median $(u \wedge v) \vee (u \wedge w) \vee (v \wedge w)$ in L.

Let R be a retract of a median graph G. By Lemma 5.37, any co-retraction $\gamma : R \longrightarrow G$ is an isometric embedding. Let $\rho : G \longrightarrow R$ be a retraction such that $\rho \circ \gamma = \mathrm{id}_R$. Let u, v, w be vertices of R, and $x \in V(G)$ the median of $\gamma(u)$, $\gamma(v)$ and $\gamma(w)$. Then $\rho(x)$ is a median of u, v and w, which is easily seen to be unique since γ is an isometric embedding. Thus, R is a median graph, and we have shown that the class of median graphs is closed under taking retracts. In fact, Theorem 5.39 generalizes to the following characterization.

Theorem 5.40 (Bandelt [10]) *The median graphs are the retracts of cubes.*

The reader may have noticed that the essential feature of retractions that is used in this characterization is the fact that they are contractions, not that they are edge-preserving. Indeed, we can define a *weak retract* of a graph G as a graph R such that there exist contractions $\rho : G \longrightarrow R$ and $\gamma : R \longrightarrow G$ satisfying $\rho \circ \gamma = \mathrm{id}_R$. The metric properties defining median graphs can then be relaxed to a definition of *quasi-median graphs*, which are the weak retracts of Hamming graphs (that is, cartesian products of complete graphs, see Chung, Graham, Saks [28] and Wilkeit [129]).

In view of these observations, the median graphs can also be characterized as the weak retracts of cubes. Thus, Theorem 5.40 not only provides a characterization of the retracts of cubes, but also shows that no other graphs are found if we allow contractions to serve as retractions. This is a bit surprising in view of the freedom gained when considering contractions rather than homomorphisms. However, note that median graphs are bipartite, and in the context of bipartite graphs, contractions and homomorphisms bear a special relationship to each other, as is shown by the following result.

Proposition 5.41 *Let $\phi : G \longrightarrow H$ be a contraction, where G and H are bipartite graphs. Then there exists a bipartite graph G' and a homomorphism $\psi : G \longrightarrow H \Box G'$ such that $\phi = \mathrm{pr}_H \circ \psi$.*

Proof Put $F = \{[u, v] \in E(G) : \phi(u) = \phi(v)\}$, and let $\mathcal{P}$ be the partition of $V(G)$ whose cells are the connected components of $G - F$. Put $G' = G/\mathcal{P}$, and let the map $\psi : G \longrightarrow H \Box G'$ be defined by $\psi(u) = (\phi(u), \pi(u))$, where $\pi : G \longrightarrow G'$ is the natural map. We then have $\phi = \mathrm{pr}_H \circ \psi$.

We show that ψ is a homomorphism. Let $[u, v]$ be an edge of G. If $\phi(u) \neq \phi(v)$, then $[\phi(u), \phi(v)] \in E(H)$ and $\pi(u) = \pi(v)$, since $[u, v] \notin F$. Thus, $[\psi(u), \psi(v)] \in E(H \Box G')$. Now suppose that $\phi(u) = \phi(v)$. Since $[u, v] \in E(G)$ and G is bipartite, any uv-path P has odd length. If $E(P) \cap F$ were empty, then $\phi(P)$ would be an odd closed trail in H, which is impossible since H is bipartite. Therefore, u and v belong to different connected components of $G - F$, and $[\pi(u), \pi(v)] \in E(G')$. Thus, $[\psi(u), \psi(v)] \in E(H \Box H')$.

Along the the same lines, it is easy to show that any cycle of G' must have even length. Thus, G' is also bipartite. $\Box$

We now turn our attention to isometric embeddings into strong products of paths. We begin with a pleasing result that is both general and simple.

Proposition 5.42 (Quilliot [104], Nowakowski, Rival [97]) *Every connected graph admits an isometric embedding into a strong product of paths.*

Proof Let G be a connected graph with diameter n, and $P = x_0, x_1, \dots, x_n$ a path. For any $u \in V(G)$, we can define a map $\phi_u : G \longrightarrow P$ by putting $\phi_u(v) = x_k$, where $k = d_G(u, v)$. By the triangle inequality, ϕ_u is a contraction. We can then use the maps $\phi_u, u \in V(G)$, coordinatewise to define a contraction $\phi : G \longrightarrow \boxtimes_{u \in V(G)} P$. For any $v, w \in V(G)$, we then have

$$d_{\boxtimes_{u \in V(G)} P}(\phi(u), \phi(v)) = \max\{|d_G(u, v) - d_G(u, w)| : u \in V(G)\}.$$

This maximum is attained by putting $u = v$ (or $u = w$). Thus,

$$d_{\boxtimes_{u \in V(G)} P}(\phi(u), \phi(v)) = d_G(v, w),$$

and ϕ is an isometric embedding. □

As was the case for the cubes, a characterization of the weak retracts of strong products of paths is heavily based on some special metric properties. We use the following concepts.

Definition 5.43 Let G be a connected graph.

(i) For $u \in V(G)$ and an integer r, the *ball* of center u and radius r is the set

$$B(u, r) = \{v \in V(G) : d_G(u, v) \le r\}.$$

(ii) G is a *Helly graph* if the family of its balls has the Helly property, that is, for any family $B(u_1, r_1), \dots, B(u_n, r_n)$ of pairwise intersecting balls, we have $\bigcap_{k=1}^{n} B(u_k, r_k) \ne \emptyset$.

In particular, all trees are Helly graphs, since their balls are subtrees, and these satisfy the Helly property. The balls of a strong product of graphs are products of balls, and a strong product of Helly graphs is a Helly graph. In particular, the strong products of paths are Helly graphs. Also, weak retractions preserve intersecting balls, so the weak retracts of Helly graphs are Helly graphs. All in all, the situation is very similar to the case of cubes and median graphs, and indeed, we get the following:

Proposition 5.44 (Quilliot [104], Nowakowski, Rival [98]) *The Helly graphs are the weak retracts of strong product of paths.*

The proof of Proposition 5.44 is based on the following property.

Theorem 5.45 (Extension Theorem, Quilliot [103]) *Let G be a connected graph and H a Helly graph. Let S be a subset of $V(G)$ and $\phi_0 : S \longrightarrow V(H)$ a map such that $d_H(\phi_0(u), \phi_0(v)) \le d_G(u, v)$ for all $u, v \in S$. Then there exists a contraction $\phi : G \longrightarrow H$ whose restriction to S is ϕ_0.*

Proof We construct ϕ by gradually expanding the domain S of ϕ_0. If $S = V(G)$, we are done. Otherwise, there exists $u \in V(G)\backslash S$. Let $S' = S \cup \{u\}$ and define $\phi_0' : S' \longrightarrow H$ as follows. Put $\phi_0'(v) = \phi_0(v)$ if $v \in S$, and notice that the balls $B(\phi_0(v), d_G(u,v)), v \in S$, are pairwise intersecting. Indeed, the condition $B(\phi_0(v_1), d_G(u,v_1)) \cap B(\phi_0(v_2), d_G(u,v_2)) \neq \emptyset$ is equivalent to the condition $d_H(\phi_0(v_1), \phi_0(v_2)) \leq d_G(u,v_1) + d_G(u,v_2)$, and the latter is fulfilled since ϕ_0 does not increase distances. Since H is a Helly graph, there exists $w \in \bigcap_{v\in S} B(\phi_0(v), d_G(u,v))$; put $\phi_0'(v) = w$. Then, S' and ϕ_0' still satisfy the conditions of the theorem, so we can iterate our construction. Eventually, we get $S = V(G)$. $\square$

We see that Proposition 5.42 and Theorem 5.45 imply Proposition 5.44. If G is a Helly graph, and $\psi : G \longrightarrow \boxtimes_{k=1}^{n} P$ an isometric embedding, then it is a full monomorphism, and we can define a partial inverse $\phi_0 : S = \phi(V(G)) \longrightarrow V(G)$ such that $\phi_0 \circ \phi(u) = u$ for all $u \in V(G)$. Since G is a Helly graph, ϕ_0 extends to a contraction $\psi : \boxtimes_{k=1}^{n} P \longrightarrow G$, so G is a weak retract of $\boxtimes_{k=1}^{n} P$.

These results on Helly graphs show how the context of contractions and weak retractions strays from the subject of homomorphisms. By contrast, Theorem 5.33 states that the retracts of strong products of paths are strong products of (possibly shorter) paths. However, this topic also brought us close to one of the earliest results concerning the retracts of graphs.

Theorem 5.46 (Hell [65]) *Let T be a tree, G a bipartite graph, and $\gamma : T \longrightarrow G$ an isometric embedding. Then T is a retract of G.*

Proof Since any tree is a Helly graph, the partial inverse $\rho_0 : \gamma(V(T)) \longrightarrow T$ of γ extends to a contraction $\rho : G \longrightarrow T$. Both G and T are bipartite, so by Proposition 5.41, there exists a bipartite graph H and a homomorphism $\phi : G \longrightarrow T \Box H$ such that $\rho = \mathrm{pr}_T \circ \phi$. We can then properly 2-colour H to obtain a homomorphism $\psi : G \longrightarrow T \Box K_2$. Since $\rho \circ \gamma = \mathrm{id}_T$, the vertices of $\psi \circ \gamma(T)$ have all the same second coordinate, say 0. We then define a map $\rho' : T \Box K_2 \longrightarrow T$ as follows. Fix a vertex $u \in T$ with a neighbour v, and define $\sigma : T \longrightarrow T$ by taking $\sigma(w)$ as the neighbour of w on the unique uw-geodesic, if $u \neq w$, and putting $\sigma(u) = v$. The map $\rho' : T \Box K_2 \longrightarrow T$ is then defined by putting $\rho'((w,i)) = w$ if $i = 0$, and $\rho'((w,i)) = \sigma(w)$ if $i = 1$. Thus, $\rho' \circ \psi : G \longrightarrow T$ is a homomorphism such that $\rho' \circ \psi \circ \gamma = \mathrm{id}_T$, so T is a retract of G. $\square$

A graph G is called an *absolute retract* of the bipartite graphs if it is a retract of any bipartite graph into which it admits an isometric embedding. For instance, Theorem 5.46 states that all trees are absolute retracts of the bipartite graphs. By analogy, Theorem 5.45 is a characterization of the Helly graphs as the ‘absolute weak retracts’ of the connected graphs. In fact, the characterization of the absolute retracts of the bipartite graphs is similar to our presentation of Helly graphs (see [11, 66]). Note that a categorical product of paths is not connected; however, the connected components of categorical products of paths play the same role in characterizing the absolute retracts of the bipartite graphs as do the strong products of paths in that of the ‘absolute weak retracts’ of the connected graphs: Every bipartite graph is an isometric subgraph of a connected component of a categorical product of paths, and the absolute retracts of the bipartite graphs are precisely the retracts of connected components of categorical products of paths.

6 Remarks

We have concentrated on finite graphs two reasons. The first is that when an infinite graph has finite homomorphic images, compactness comes to play. For example, if H is finite, then an arbitrary graph G admits a homomorphism to H if and only if all of its finite subgraphs do (see [21, 51]).

The second reason for not considering infinite graphs in this paper is the work Bauslaugh ([15, 16, 17]) and the fact that cores are fundamental to the study of homomorphisms. It turns out that not only does Proposition 2.22 not generalize to infinite graphs, but many problems related to cores and homomorphisms are undecidable. We survey some of this work in [56]. Bauslaugh's work treats directed graphs, but the cores present a problem even for undirected ones. For an example, consider the ray whose vertices are the positive integers and each of whose edges connects two consecutive integers. For each $k > 1$, let G_k be a distinct copy of the complete graph on k vertices $\{v_1, \dots, v_k\}$. Construct a graph H by identifying k with v_k for $k \in \mathbb{N}$. We claim this graph has no core in the sense that there is no minimal retract. Indeed, any retraction can map G_i to G_j only if $i < j$ and so the retracts of H are the infinite components of $H - k$, $k \in \mathbb{N}$. Clearly there is no minimal one.

We have also skipped some other interesting work. For example, the concept of a *rigid* graph, studied by Hell, Nešetřil, Babai and others (see, for example, [2, 64, 29, 8, 9], and also [102] for many more references), is a natural one: consider a graph whose only endomorphism is the identity. Brewster and MacGillivray [20] have looked at graphs with the property that every homomorphic image is a subgraph. Also, all the work, some quite recent by Hell, MacGillivray, Bang-Jensen ([12, 13]) and others, on homomorphisms of directed graphs is omitted, as is the large body of results on the complexity of what has become known as *H-colourings*, that is, problems of the existence of homomorphisms into given graphs H. Both the complexity and the undecidability considerations, as well as the related topics of homomorphisms into directed graphs are treated in [56].

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