

GRAPH INVERSE SEMIGROUPS, GROUPOIDS AND THEIR C^* -ALGEBRAS

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ABSTRACT. We develop a theory of graph C^* -algebras using path groupoids and inverse semigroups. Row finiteness is not assumed so that the theory applies to graphs for which there are vertices emitting a countably infinite set of edges. We show that the path groupoid is amenable, and give a groupoid proof of a recent theorem of Szymanski characterizing when a graph C^* -algebra is simple.

KEYWORDS: *Directed graphs, graph inverse semigroups, graph groupoids, graph C^* -algebras, Cuntz-Krieger algebras, amenability.*

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1. INTRODUCTION

There is now a substantial literature on the C^* -algebra associated with a (countable) directed graph. Among the motivations for this are the Bratteli diagrams for AF-algebras and especially the Cuntz-Krieger algebras, which can be interpreted in graphical terms with the matrix A , determining such an algebra, regarded as the incidence matrix of the graph. A helpful survey of the field up to 1998 has been given by A. Kumjian ([9]). In the literature surveyed by Kumjian, the graphs considered satisfy finiteness conditions on the edges coming into and going out of a vertex. Usually, *local finiteness* has been required, viz. that at each vertex v , only finitely many edges end at v , and only finitely many edges start at v . Many of these results were shown to be valid in the *row finite* case, i.e. where we only assume that finitely many edges start at v for each vertex v of the graph.

Recently, there has been interest in removing the finiteness requirements on the graph. See, for example, [1], [5], [6], [7], [8], [13], [15], [19], [20], [21], [22]. Motivation for this is to be found even in the original paper ([3]) of Cuntz in which he studied what are now called the Cuntz algebras \mathcal{O}_n . These are graph C^* -algebras, the natural graph associated with \mathcal{O}_n being a bouquet of n circles

(one vertex). When $n < \infty$, the graph is locally finite. When $n = \infty$, this is no longer the case. (Indeed up to the work of Exel and Laca ([5]), \mathcal{O}_∞ was the only known example that had been worked out of a Cuntz-Krieger algebra whose matrix was not row-finite.) Even in the case of the Cuntz algebras, there is a significant difference between the row finite/non-row finite cases. This is shown by the fact that in [3], in a number of instances, proofs given for \mathcal{O}_n with $n < \infty$ do not apply in the case of \mathcal{O}_∞ , so that different proofs were required for that case.

A representation of a directed graph \mathcal{E} is determined by certain partial isometries on a Hilbert space indexed by the vertices and edges. The C^* -algebra $C^*(\mathcal{E})$ of \mathcal{E} is the universal C^* -algebra for such representations. In the locally finite case, insight into this C^* -algebra is provided by the *path groupoid* G of \mathcal{E} . This is an r-discrete groupoid, and its units are just the infinite paths in \mathcal{E} . Then (cf. [11]) if \mathcal{E} is locally (or even row) finite and has no sinks, we have $C^*(\mathcal{E}) \cong C^*(G)$, and this groupoid connection enables one to interpret properties of $C^*(\mathcal{E})$ in dynamical terms.

When we look for this groupoid connection without assuming local finiteness, we run into a problem since the space of infinite paths in \mathcal{E} is not a locally compact Hausdorff space in any obvious way. The main objective of this paper is to show that, despite this problem with the infinite paths, there is a natural path groupoid available in complete generality. A peculiarity of this groupoid is that we have to include some *finite* paths. To determine what this groupoid is, we use an approach based on inverse semigroups.

Indeed, there is a close connection between inverse semigroups and r-discrete groupoids. This was first observed and studied by J. Renault in his monograph ([14]). Recently, the present author in his book ([12]) gave a detailed account of this connection. One of the main results of [12], Chapter 4, is a construction which associates with any (countable) inverse semigroup S a *universal* groupoid H canonically related to S such that $C^*(S) \cong C^*(H)$. The path groupoid will be a reduction of the universal groupoid for a certain inverse semigroup associated with the graph. So we have to come up with an appropriate inverse semigroup S for \mathcal{E} .

In the \mathcal{O}_n -case, the appropriate inverse semigroup is S_n which was introduced by Renault in [14], p. 141. For a general directed graph \mathcal{E} , a natural candidate for such an inverse semigroup is the inverse semigroup of path pairs (α, β) considered in [12], p. 158. As mentioned there, an unsatisfactory feature of this inverse semigroup is that, unlike the Cuntz case, it is not generated by the edges. The way out of this difficulty is suggested by the definition of a representation of \mathcal{E} referred to above. *We have to include the vertices in the inverse semigroup.*

The precise details of this are given in Section 2. The *graph inverse semigroup* of \mathcal{E} , denoted by $S_{\mathcal{E}}$, is defined as the inverse semigroup generated by the edges and vertices of \mathcal{E} subject to certain ‘‘Cuntz-like’’ relations. We show that $S = S_{\mathcal{E}}$ can be identified with an inverse semigroup of path pairs, where we permit vertices to stand as surrogates for ‘‘paths’’ of zero length. This ‘‘path-pair’’ characterization of S is useful for calculation purposes.

The universal groupoid H of S is then determined in the paper. We focus on a natural reduction of this which we will call the *graph groupoid* G . This groupoid is similar to the graph groupoid in the locally finite case and indeed is the same groupoid in that case. The main difference is that we must include in the unit

space G^0 those *finite* paths ending in a vertex $v \in V_\infty$, where V_∞ is the set of vertices emitting infinitely many edges. Sequences of infinite paths can converge to such finite paths, in particular to V_∞ -vertices. We write V_f for $V \setminus V_\infty$, the set of vertices emitting only finitely many edges.

The representations of \mathcal{E} correspond to a certain class of representations of S , and using the canonical identification of $C^*(S)$ with $C^*(H)$ in [12], Chapter 4, we show that in fact these representations correspond to the representations of G . So we obtain that $C^*(\mathcal{E}) = C^*(G)$ just as in the locally finite case.

Next we turn to the issue of the amenability of $C^*(\mathcal{E})$. A groupoid form of Cuntz’s crossed-product argument for the nuclearity (and hence amenability) of \mathcal{O}_n was given by J. Renault in his book ([14], p. 138ff.). For the case of a locally finite graph \mathcal{E} , an ingenious adaptation of this groupoid argument was given by Kumjian, Pask, Raeburn and Renault ([11]) to show that $C^*(\mathcal{E})$ is always amenable. We adapt this argument further to establish the same result in complete generality. The technical difficulty is that of dealing with the paths of finite length in G^0 .

The last section of the paper deals with the *simplicity* of $C^*(\mathcal{E})$. Kumjian, Pask, Raeburn and Renault ([11]) characterized the simplicity of $C^*(\mathcal{E})$ in terms of *condition* (K) and *cofinality* in the locally finite case. (See below.) Recently, W. Szymanski ([21], Theorem 12) characterized the simplicity of $C^*(\mathcal{E})$ for general \mathcal{E} using the approach of Exel and Laca ([5]). He showed that $C^*(\mathcal{E})$ is simple if and only if (α) and (β) hold where:

- (α) all loops in \mathcal{E} have exits;
- (β) for any $v \in V$, the smallest hereditary, saturated subset of V that contains v is equal to V .

(In (β) , a subset H of V is called *hereditary* if $w \in H$ whenever there is a path from some $v \in H$ to w , and is called *saturated* if $w \in H$ whenever $w \in V_f$ and every edge starting at w ends in H .)

We show first that for general \mathcal{E} , the condition (K) of [11] — that there are no vertices emitting exactly one loop — is equivalent to G being essentially principal (i.e. every closed, invariant subset of G^0 contains a dense subset of elements with trivial isotropy). (This result is a counterpart to the result of Kumjian, Pask and Raeburn ([10], Lemma 3.4) that, in the locally finite case (no sinks), the groupoid G is essentially free if and only if every loop in \mathcal{E} has an exit.) We then show (Theorem 5.2) that $C^*(\mathcal{E})$ is simple if (a), (b) and (c) hold, where:

- (a) \mathcal{E} has property (K);
- (b) \mathcal{E} is cofinal (i.e. given any $v \in V$ and any infinite path z , then there is a path from v to at least one of the vertices that z passes through);
- (c) if $v \in V$ and $w \in V_\infty$, then there is a path from v to w .

It is not difficult to show that (a), (b) and (c) together are equivalent to (α) and (β) together, so that Theorem 5.2 is effectively just one of the implications in Szymanski’s theorem. The point of including Theorem 5.2 in the present paper is that it gives an alternative approach to the simplicity of $C^*(\mathcal{E})$ using the groupoid G of the paper. The groupoid proof that we give is just a variant of the method used by J. Renault in his book ([14]); see also the approach of [11]) to establish the simplicity of the Cuntz C^* -algebras: using G in place of the Cuntz groupoids, Renault’s approach works for a completely general graph \mathcal{E} .

Another approach to the simplicity of graph C^* -algebras has been given by Jürgen Schweizer ([17]). Schweizer introduces the concept of a *continuous measured diagram*, in which we are given locally compact spaces D, X, Y with continuous maps $s : D \rightarrow X$, $r : D \rightarrow Y$, together with certain measures on D indexed by the elements of Y . The case where $X = Y$ gives a natural C^* -correspondence, which in turn determines a *Cuntz-Pimsner algebra* $\mathcal{O}_{\mathfrak{D}}$. The discrete case of such a diagram is just that of a *directed graph* with set of vertices X , and in that case, $\mathcal{O}_{\mathfrak{D}}$ is the same as the graph C^* -algebra. Using [18], Schweizer ([17], Corollary 5.2) proves that when X is compact and r is onto, $\mathcal{O}_{\mathfrak{D}}$ is simple if and only if the continuous measured diagram is non-periodic and minimal. The simplicity results of Schweizer and Szymanski are different. On the one hand, Schweizer's result is not restricted to the discrete (graphical) case, while on the other hand, its compactness condition on X in the graphical situation forces the vertex set to be finite. It would be interesting to relate the approach of Schweizer to the groupoid approach.

2. SOME PRELIMINARIES

Let \mathcal{E} be a directed graph. Its set of vertices is denoted by V and its set of edges by E . For each edge e , $s(e)$ is the initial vertex (source) of e and $r(e)$ the terminal vertex of e . It is assumed throughout the paper that V and E are countable sets.

The graph \mathcal{E} is called *row finite* if for each $v \in V$, the set of edges starting at v is finite. The graph is called *locally finite* if for each $v \in V$, the set of edges starting at v is finite, and the set of edges terminating at v is finite. A vertex v is called a *sink* if there are no edges starting at v . Throughout the paper we will assume that there are no sinks in \mathcal{E} (unless the contrary is explicitly specified).

A reason for this is as follows. We want to examine $C^*(\mathcal{E})$ using a "path groupoid" G , and the paths that make up the unit space of the appropriate groupoid G are such that they cannot end at a sink. So if we want to use the path groupoid to study $C^*(\mathcal{E})$, then we cannot allow sinks in \mathcal{E} . The C^* -algebra $C^*(\mathcal{E})$ still makes good sense whether there are sinks or not, and indeed, groupoid techniques can still be used using the *universal groupoid* H introduced in the paper rather than the path groupoid G . Further, as is pointed out in [1], it is technically easy to reduce to the case where there are no sinks.

The set V of vertices is the disjoint union of the sets V_f and V_∞ , where $v \in V_f$ if there are finitely many edges starting at v , while $v \in V_\infty$ if there are infinitely many edges starting at v .

The edge e with reversed orientation is denoted by e^* . So $s(e^*) = r(e)$ and $r(e^*) = s(e)$. We write E^* as the set of e^* 's.

A *finite path* is a sequence α of edges e_1, \dots, e_k where $s(e_{i+1}) = r(e_i)$ for $1 \leq i \leq k-1$. We write $\alpha = e_1 e_2 \cdots e_k$. The length $l(\alpha)$ of α is just k . Each vertex v is regarded as a finite path of length 0. We define $r(\alpha) = r(e_k)$ and $s(\alpha) = s(e_1)$. For $v \in V$, we set $r(v) = v = s(v)$. The set of finite paths in \mathcal{E} is denoted by Y . Note that $V \subset Y$. The set of infinite paths $z = z_1 z_2 \cdots$ in \mathcal{E} is denoted by Z . The length $l(z)$ of $z \in Z$ is defined to be ∞ .

Given two vertices v, w , we say that $v \geq w$ if there exists a path α such that $s(\alpha) = v, r(\alpha) = w$. If z is a path in \mathcal{E} and $v \in V$, then we write $z \geq v$ to mean that $r(z_i) \geq v$ for all i . We write $z \leq v$ if $r(z_i) \leq v$ for all i .

We now briefly discuss *inverse semigroups*. A semigroup S is called an inverse semigroup if for all $s \in S$, there exists a unique element $t \in S$ such that $sts = s, tst = t$. We write the element t as s^* . Note that $s^{**} = s$. Inverse semigroups can be identified with semigroups of partial one-to-one maps on a set that contain their inverses. Every element ss^* belongs to the set I of idempotents of S . The set I is a commutative subsemigroup of S and so is a semilattice. (In semigroup theory, the semilattice of idempotents in S is usually denoted by E , but in this paper, E stands for the set of edges in \mathcal{E} .) Homomorphic images of inverse semigroups are inverse semigroups.

A congruence ρ on a semigroup S is an equivalence relation for which $(xa, xb), (ax, bx) \in \rho$ whenever $(a, b) \in \rho$. If ρ is a congruence on S , then S/ρ is a semigroup in the natural way.

We now recall the definition ([2], pp. 40–41) of the semigroup S generated by a set X subject to the generating relations $u_\lambda = v_\lambda, \lambda \in \Lambda$. Let F_X be the semigroup of words in X with juxtaposition as product, and for $\lambda \in \Lambda$, let $u_\lambda, v_\lambda \in F_X$. Then S is just the quotient F_X/ρ where ρ is the smallest congruence on F_X containing all of the pairs (u_λ, v_λ) .

Throughout the paper, we will assume the theory of r-discrete groupoids as presented, for example, in [14] and [12].

3. THE INVERSE SEMIGROUP OF A GRAPH AND THE PATH GROUPOID

Let z be an element not in $V \cup E \cup E^*$. The *graph inverse semigroup* $S = S_{\mathcal{E}}$ of the graph \mathcal{E} is defined to be the semigroup generated by $(V \cup E \cup E^*) \cup \{z\}$ subject to the relations:

- (i) z is a zero for S ;
- (ii) $s(e)e = e = er(e), r(e)e^* = e^* = e^*s(e)$ for all $e \in E$;
- (iii) $ab = z$ if $a, b \in V \cup E \cup E^*$ and $r(a) \neq s(b)$;
- (iv) $e^*f = z$ if $e, f \in E$ and $f \neq e$.

Note that S is countable since $E \cup V$ is.

We will see in a moment that S is actually an inverse semigroup. But for the present, we describe a model T for S . The elements of T are the pairs of finite paths (α, β) with $r(\alpha) = r(\beta)$, together with a zero element z . An involution is defined on T by: $z^* = z, (\alpha, \beta)^* = (\beta, \alpha)$, and the product is determined by:

$$(3.1) \quad (\alpha, \alpha'\mu)(\alpha', \beta) = (\alpha, \beta\mu)$$

$$(3.2) \quad (\alpha, \alpha')(\alpha'\mu, \beta') = (\alpha\mu, \beta').$$

All other products in T are z . Note that for finite paths α, β , the expression $\alpha\beta$ is defined only when $r(\alpha) = s(\beta)$, and in that case, is just the finite path obtained in the obvious way by juxtaposition. Recall that each vertex v is regarded as a finite path. For any finite path μ , we specify that $s(\mu)\mu = \mu = \mu r(\mu)$. Note that the product of two pairs $(\alpha_1, \beta_1), (\alpha_2, \beta_2)$ is non-zero if and only if β_1 is an initial segment of α_2 or conversely. In this case, we say that β_1, α_2 are *comparable*. Note that if $\alpha\mu, \beta\nu$ are comparable, then so are α, β . The set I of idempotents of S is just the set of pairs (α, α) , and so can be identified with Y . The product on I of two elements $\alpha, \beta \neq z$ is given by $\alpha\beta = \alpha = \beta\alpha$ if β is an initial segment of α , and is z if α, β are not comparable.

A similar semigroup to T was suggested by the author in [12], p. 158, but there is a significant difference: in [12], the paths α, β were of finite, non-zero length, and the edges and vertices do not lie in the semigroup. However, as we will see (Proposition 3.2), the edges and vertices lie inside and generate T .

PROPOSITION 3.1. *The set T , with the above involution and product, is an inverse semigroup.*

Proof. We first have to show that the product in T is associative. If one of the terms is z , then this is obvious. So let $(\alpha_i, \beta_i) \in T$, $1 \leq i \leq 3$, $w_1 = (\alpha_1, \beta_1)((\alpha_2, \beta_2)(\alpha_3, \beta_3))$ and $w_2 = ((\alpha_1, \beta_1)(\alpha_2, \beta_2))(\alpha_3, \beta_3)$. We have to show that $w_1 = w_2$.

The cases that give $w_1 \neq z$ are as follows (for appropriate words μ, ν):

- (1) $\beta_2 = \alpha_3\mu, \beta_1 = \alpha_2\nu, w_1 = (\alpha_1, \beta_3\mu\nu)$;
- (2) $\beta_2 = \alpha_3\mu, \alpha_2 = \beta_1\nu, w_1 = (\alpha_1\nu, \beta_3)$;
- (3) $\alpha_3 = \beta_2\mu, \beta_1 = \alpha_2\mu\nu, w_1 = (\alpha_1, \beta_3\nu)$;
- (4) $\alpha_3 = \beta_2\mu, \alpha_2\mu = \beta_1\nu, w_1 = (\alpha_1\nu, \beta_3)$.

One checks directly that in each of these four cases, $w_2 = w_1$. (In case (4), one needs to consider separately the cases $\alpha_2 = \beta_1\xi$ and $\beta_1 = \alpha_2\xi$.) So if $w_1 \neq z$, then $w_1 = w_2$. Similarly, one shows that if $w_2 \neq z$, then $w_2 = w_1$. The associative law then follows.

Next we have to show that for each $t \in T$, t^* is the only element t' for which $tt't = t$, $t'tt' = t'$. If $t = z$, then this is trivial. If $t = (\alpha, \beta)$, then since $tt't = t = (\alpha, \beta)$, then $t' = (\alpha', \beta')$ for some finite paths α', β' , with β', α and β, α' comparable. If $\beta' = \alpha\mu$ with $l(\mu) > 0$, then the second component of $tt't$ would be longer than β , the second component of t , and we contradict $tt't = t$. Similarly, by considering $t'tt' = t'$ we obtain that α is not of the form $\beta'\mu$ for any μ of positive length. So $\beta' = \alpha$. Similarly, $\alpha' = \beta$ and $t' = t^*$. Obviously $tt^*t = t$, $t^*t^* = t^*$. ■

PROPOSITION 3.2. *The map $\Phi : S = S_\mathcal{E} \rightarrow T$ where*

$$\Phi(v) = (v, v), \quad \Phi(e) = (e, r(e)), \quad \Phi(e^*) = (r(e), e), \quad \Phi(z) = z,$$

is a semigroup isomorphism.

Proof. Let $W = V \cup E \cup E^* \cup \{z\}$. Then Φ extends in the obvious way to a homomorphism, also denoted $\Phi : F_W \rightarrow T$. The map $\Phi : F_W \rightarrow T$ is onto since every $t \in T$ is a product of $\Phi(w)$'s: for example, if $\alpha = \alpha_1 \cdots \alpha_k$ and $\beta = \beta_1 \cdots \beta_l$, and $t = (\alpha, \beta) \in T$, then

$$(\alpha, \beta) = \Phi(\alpha_1) \cdots \Phi(\alpha_k)\Phi(\beta_l^*) \cdots \Phi(\beta_1^*).$$

Let \sim be the congruence on F_W generated by the relations (i)–(iv) above. To prove that Φ descends to a homomorphism $\tilde{\Phi}$ from $S = F_W / \sim$ onto T , we just have to check that the elements of $\Phi(W)$ satisfy the relations in T corresponding to (i)–(iv). This is straightforward. (i) is trivially true. In (ii), for example, the equalities

$$\Phi(s(e))\Phi(e) = \Phi(e) = \Phi(e)\Phi(r(e))$$

are just

$$(s(e), s(e))(e, r(e)) = (e, r(e)) = (e, r(e))(r(e), r(e)).$$

In (iii), if $a = v \in V$ and $r(a) \neq s(b)$, then either $b = w \in V$ with $w \neq v$, or $b = e$ with $v \neq s(e)$ or $b = e^*$ with $v \neq r(e)$. Then $\Phi(a)\Phi(b)$ is either $(v, v)(w, w)$ or $(v, v)(e, r(e))$ or $(v, v)(r(e), e)$ and these are all z . The other possibilities for a are dealt with similarly. For (iv), $\Phi(e^*)\Phi(f) = (r(e), e)(f, r(f)) = z$ if $e \neq f$.

To show that $\tilde{\Phi} : S \rightarrow T$ is an isomorphism, suppose that for some $s, s' \in F_W$ we have $\Phi(s) = \Phi(s')$. We have to show that $s \sim s'$. Using the relations (i)–(iv), we can find $t, t' \in F_W$, where $s \sim t, s' \sim t'$ and, either $t = t' = z$, in which case $s \sim s'$, or, in an obvious notation, $t = \alpha\beta^*, t' = \alpha'(\beta')^*$ for paths $\alpha, \beta, \alpha', \beta'$ with $r(\alpha) = r(\beta), r(\alpha') = r(\beta')$. Then

$$(\alpha, \beta) = \Phi(t) = \Phi(s) = \Phi(s') = \Phi(t') = (\alpha', \beta')$$

and $\alpha = \alpha', \beta = \beta'$. So $t = t'$ and $s \sim s'$. ■

For the rest of the paper, we shall usually identify S with T .

As a very simple example of the inverse semigroup S of a graph \mathcal{E} , consider the case where \mathcal{E} has exactly one edge e with initial and terminal vertices v_1, v_2 respectively. (In this case, \mathcal{E} has a sink.) The elements of S are then: $v_1 = (v_1, v_1), v_2 = (v_2, v_2), (e, v_2), (v_2, e), (e, e)$ and the zero z . Then the idempotent set I of S is $\{v_1, v_2, (e, e), z\}$, and $v_1v_2 = z, v_1(e, e) = (e, e)$ and $v_2(e, e) = z$. The involution on S interchanges (e, v_2) and (v_2, e) and leaves every other element of S fixed. The other non-zero products in S are: $v_1(e, v_2) = (e, v_2) = (e, v_2)v_2, (e, v_2)(v_2, e) = (e, e), (v_2, e)(e, e) = (v_2, e), (v_2, e)(e, v_2) = v_2$ and the equalities obtained from these by applying the involution.

The next objective is to identify the universal groupoid of S for general \mathcal{E} . The universal groupoid ([12], Chapter 4) H of a countable inverse semigroup S is constructed as follows. The unit space H^0 of H is just the set of non-zero semicharacters $\chi : I \rightarrow \{0, 1\}$, where I is the commutative semigroup of idempotents of S . The topology on H^0 is just the topology of pointwise convergence on I . There is a dense subset $\bar{I} = \{\bar{\alpha} : \alpha \in I\}$ of H^0 , where $\bar{\alpha}(\beta) = 1$ if and only if $\alpha\beta = \alpha$. There is a natural right action of S on H^0 given as follows. First, an element $x \in H^0$ is in the domain D_s of $s \in S$ if $x(ss^*) = 1$. The element $x \cdot s \in H^0$ is given by: $x \cdot s(\alpha) = x(s\alpha s^*)$. The map $x \rightarrow x \cdot s$ is a homeomorphism from D_s onto D_{s^*} . The family of sets of the form $D_{\alpha, \alpha_1, \dots, \alpha_n}$, where

$$(3.3) \quad D_{\alpha, \alpha_1, \dots, \alpha_n} = D_\alpha \cap D_{\alpha_1}^c \cap \dots \cap D_{\alpha_n}^c$$

(with c standing for ‘‘complement’’ and $\alpha, \alpha_i \in I, \alpha \geq \alpha_i, 1 \leq i \leq n$) is a basis for the topology of H^0 .

The universal groupoid H is the quotient

$$\{(x, s) : x \in D_s, s \in S\} / \sim$$

where $(x, s) \sim (y, t)$ whenever $x = y$ and there exists $\alpha \in I$ such that $x(\alpha) = 1$ ($= x(ss^*) = x(tt^*)$) and $\alpha s = \alpha t$. The composable pairs are pairs of the form $((x, s), (x \cdot s, t))$, where $x \in D_s, x \cdot s \in D_t, s, t \in S$, while the product and involution on H are respectively given by $((x, s), (x \cdot s, t)) \rightarrow (x, st)$ and $(x, s) \rightarrow (x \cdot s, s^*)$. Then H is an r -discrete groupoid and the map Ψ , where $\Psi(s) = \{(x, s) : x \in D_s\} = A_s$ is an inverse semigroup isomorphism from S into the ample semigroup H^a of H . (Recall that for any r -discrete groupoid G , the ample semigroup G^a is the inverse semigroup of compact, open, Hausdorff subsets

of G .) Note that $\Psi(\alpha) = D_\alpha$ for all $\alpha \in I$. A basis for the topology of H is given by sets of the form $D_{\alpha, \alpha_1, \dots, \alpha_n} \Psi(s)$, where $\alpha, \alpha_i \in I, \alpha_i \leq \alpha, s \in S$.

We now specify the universal groupoid $H = H_\mathcal{E}$ of $S = S_\mathcal{E}$. The discussion parallels that for the Cuntz inverse semigroups $S_n, 1 \leq n \leq \infty$, given in [12], pp. 182f. We first determine the unit space H^0 of H . For $\alpha \in I$, we will write α in place of $\bar{\alpha}$. The context will make clear if α is being regarded as an element of I or as a semicharacter. Recall that Y and Z are respectively the sets of finite and infinite paths in \mathcal{E} .

PROPOSITION 3.3. *The unit space H^0 of H is the disjoint union:*

$$(3.4) \quad H^0 = Y \cup Z \cup \{z\}.$$

Then the topology for H^0 is specified by:

- (i) the singleton set $\{z\}$ is both open and closed in H^0 ;
- (ii) a neighborhood basis at $u \in Z$ is given by sets of the form $D_\alpha = \{\alpha\gamma \in Y \cup Z : \gamma \in Y \cup Z\}$, where $\alpha \in Y$ is an initial segment of u ;
- (iii) if $y \in Y_f$, then $\{y\}$ is an open set;
- (iv) if $y \in Y_\infty$, then a neighborhood basis at y is given by sets of the form D_{y, ye_1, \dots, ye_n} where e_1, \dots, e_n are edges in \mathcal{E} starting at $r(y)$.

Proof. Let $\chi \in H^0$. If $\chi(z) = 1$, then since $\alpha z = z$ for all $\alpha \in I$, we have $\chi(\alpha) = 1$, the constant non-zero semicharacter on E . Clearly $\chi = z (= \bar{z})$. The set $\{z\}$ is open in H^0 since it is just $\{\chi' \in H^0 : \chi'(z) = 1\}$. This gives (i).

Suppose then that $\chi \neq z$. Then $\chi(z) = 0$. Recall that I is identified with the set of finite paths α together with the zero z , and a product of two of these paths is $\neq z$ only when one path is an initial segment of the other, and in that case, the product is just the longer path. The set of paths α for which $\chi(\alpha) = 1$ is a subsemigroup of I and so each such path is an initial segment of any other or vice versa. There thus exists a path $\gamma \in Y \cup Z$ such that $\chi(\alpha) = 1$ if and only if γ is of the form $\alpha\mu$. Identifying χ with γ gives (3.4). Note that if $\gamma \in Y$, then $\chi = \gamma (= \bar{\gamma})$.

By the above, a basis for the topology on $Y \cup Z$ is given by sets of the form $D_{\alpha, \alpha_1, \dots, \alpha_n}$. Suppose that $u \in Z$ and that $u \in D_{\alpha, \alpha_1, \dots, \alpha_n}$. Then u starts with α but not with any α_i . So for some $n, u_1 \cdots u_n \in D_{\alpha, \alpha_1, \dots, \alpha_n}$, and so $D_{u_1 u_2 \cdots u_n} = \{u_1 \cdots u_n \gamma \in Y \cup Z : \gamma \in Y \cup Z\} \subset D_{\alpha, \alpha_1, \dots, \alpha_n}$. So the sets of the form $D_\alpha, \alpha = u_1 \cdots u_n$, form a neighborhood base at u in $Y \cup Z$. This gives (ii).

For (iii), if $y \in Y_f$ and e_1, \dots, e_n are all of the edges starting at $r(y)$, then

$$\{y\} = D_y \cap D_{ye_1}^c \cap \cdots \cap D_{ye_n}^c = D_{y, ye_1, \dots, ye_n}$$

is open in $Y \cup Z$.

Lastly, suppose that $y \in Y_\infty$ and that y is in some $D_{\alpha, \alpha_1, \dots, \alpha_n}$. Then y begins with α and the α_i 's all begin with y and are longer than y . Write $\alpha_i = ye_i \cdots$. Then $y \in D_{y, ye_1, \dots, ye_n} \subset D_{\alpha, \alpha_1, \dots, \alpha_n}$ and (iv) follows. ■

It is instructive to specify the convergent sequences in $Y \cup Z$. If $u \in Z$, then $x^n \rightarrow u$ if and only if $l(x^n) \rightarrow \infty$, and for each i , $x_i^n = u_i$ eventually. Next, let us say that a sequence $\{e^n\}$ in E is *wandering* if, for any $e \in E$, $e^n \neq e$ eventually. Let $\alpha = \alpha_1\alpha_2 \cdots \alpha_N \in Y$. Suppose that $x^n \in Y \cup Z$ and that $x^n \neq \alpha$ for any n . Then $x^n \rightarrow \alpha$ if and only if, eventually, $l(x^n) > l(\alpha)$, $x_i^n = \alpha_i$ for $1 \leq i \leq N$, and $\{x_{N+1}^n\}$ is wandering. Note that in this case, $\alpha \in Y_\infty$. It follows that each $y \in Y_\infty$ is a limit point of Z , and so we have the following proposition.

PROPOSITION 3.4. *The set Z of infinite paths is dense in $Y_\infty \cup Z$. If $\alpha \in Y$, then $\{\alpha\}$ is open in H^0 if and only if $\alpha \in Y_f$.*

THEOREM 3.5. *The universal groupoid H for S can be identified with the union of $\{z\}$ and the set of all triples of the form $(\alpha\gamma, l(\alpha) - l(\beta), \beta\gamma)$ where $\alpha, \beta \in Y$, $\gamma \in Y \cup Z$, and $\alpha\gamma, \beta\gamma \in Y \cup Z$. Multiplication on H is given by: $(x, m, y)(y, n, z) = (x, m + n, z)$ and inversion by $(x, m, y)^{-1} = (y, -m, x)$. The canonical map $\Psi : S \rightarrow H^a$ sends z to $\{z\}$ and any element (α, β) to the (compact open) set $A_{\alpha, \beta}$ of all triples of the form $(\alpha\gamma, l(\alpha) - l(\beta), \beta\gamma)$. The locally compact groupoid H is Hausdorff.*

Proof. The proof is close to that for the Cuntz semigroup S_n in [12], pp. 182–186. We can restrict consideration to $H \sim \{z\}$. Let $s = (\alpha, \beta)$, $x \in D_s$. This means that $x(ss^*) = x(\alpha) = 1$. If $\alpha' \in I$ is such that $x(\alpha') = 1$, $\alpha' \leq ss^*$, then $\alpha' = \alpha\delta \in Y$, and $\alpha's = (\alpha\delta, \beta\delta)$. Then $x = \alpha\delta\gamma$ for some γ and $(\alpha\delta\gamma, l(\alpha\delta) - l(\beta\delta), \beta\delta\gamma) = (\alpha\delta\gamma, l(\alpha) - l(\beta), \beta\delta\gamma)$. So the map $(x, s) \rightarrow (\alpha\delta\gamma, l(\alpha) - l(\beta), \beta\delta\gamma)$ is constant on the equivalence class of (x, s) in H , and so defines a map on H . The argument reverses to give that this map is a bijection. We now prove that H is Hausdorff, leaving the remaining verifications of the theorem to the reader.

Let $a = (\alpha, l(\alpha) - l(\beta), \beta)$, $b = (\alpha', l(\alpha') - l(\beta'), \beta')$ belong to H with $a \neq b$. If $A_{\alpha, \beta} \cap A_{\alpha', \beta'} = \emptyset$, then we can separate a and b using $A_{\alpha, \beta}, A_{\alpha', \beta'}$. Suppose then that $A_{\alpha, \beta} \cap A_{\alpha', \beta'} \neq \emptyset$. Then there exist γ, γ' such that

$$(\alpha\gamma, l(\alpha) - l(\beta), \beta\gamma) = (\alpha'\gamma', l(\alpha') - l(\beta'), \beta'\gamma').$$

Then $l(\alpha) - l(\beta) = l(\alpha') - l(\beta')$ and α, α' and β, β' are comparable. We can suppose that for some μ , $\alpha' = \alpha\mu$. Then $\beta' = \beta\nu$ where $l(\mu) = l(\nu)$, and $\nu = \mu$ since both are initial segments of γ . Since $a \neq b$, we have $l(\mu) > 0$. Then $A_{\alpha, \beta} \cap A_{\alpha', \beta'}^c, A_{\alpha', \beta'}$ separate a and b . So H is Hausdorff. ■

We next set $X = Y_\infty \cup Z \subset H^0$. Then (Proposition 3.3) X is a closed invariant subset of H^0 and so is a locally compact Hausdorff space in its own right. Let $G_{\mathcal{E}} = G$ be the reduction of H to X . We will call G the *path groupoid* of \mathcal{E} . (The groupoid G coincides with the usual path groupoid of a graph in the locally finite case (e.g. [11]).) Then G is a closed subgroupoid of H , and is an r -discrete groupoid in its own right with counting measures giving a left Haar system. For $\alpha, \beta \in Y$, we define $A'(\alpha, \beta) = A(\alpha, \beta) \cap G$ and $D'_\alpha = D_\alpha \cap X$.

We now relate the representations of G to those of H .

PROPOSITION 3.6. *Each representation σ of $C_c(G)$ determines a representation σ' of $C_c(H)$ by: $\sigma'(f) = \sigma(f|G)$. A representation Π of $C_c(H)$ is of the form σ' for some representation σ of $C_c(G)$ if and only if*

$$(3.5) \quad \Pi(\chi_{\{z\}}) = 0 = \Pi(\chi_{\{\alpha\}})$$

for all $\alpha \in Y_f$.

Proof. If σ is a representation of $C_c(G)$ on a Hilbert space \mathfrak{H} then $\sigma' : C_c(G) \rightarrow B(\mathfrak{H})$ is a $*$ -homomorphism. Since the homomorphism $f \rightarrow f|G$ from $C_c(H)$ into $C_c(G)$ is continuous for the I -norm, it immediately follows that σ' is I -norm continuous and so is a representation of $C_c(H)$. It is obvious from the definition that σ' satisfies (3.5) (with σ' in place of Π).

Conversely, suppose that $\Pi : C_c(H) \rightarrow B(\mathfrak{H})$ is a representation satisfying (3.5). By the proof of [12], Theorem 3.1.1, the representation Π extends to a continuous representation $\tilde{\Pi}$ of $(B_c(H), \|\cdot\|_I)$ where $B_c(H)$ is the convolution algebra of bounded Borel functions on H with compact supports. Let σ be the restriction of $\tilde{\Pi}$ to $C_c(G) \subset B_c(H)$. Then σ is a representation (*a priori* possibly degenerate) of $C_c(G)$ on \mathfrak{H} . Let $f \in C_c(H)$ and U be an open subset of H with compact closure and which contains the support of f . We have to show that $\Pi(f) = \tilde{\Pi}(f|G) (= \sigma(f|G))$. Let $\xi \in \mathfrak{H}$. As in Theorem 3.1.1 from [12], there exists a positive regular Borel measure μ on H such that for all $g \in B_c(H)$ with $\text{supp}(g) \subset U$, we have

$$(3.6) \quad \langle \tilde{\Pi}(g)\xi, \xi \rangle = \int_U g \, d\mu.$$

Now $H \setminus G$ is the reduction of H to $Y_f \cup \{z\}$ and is countable. For $a \in H \setminus G$, we have $a = ar(a)$ with $r(a) \in Y_f \cup \{z\}$. So $\Pi(\chi_{\{a\}}) = \Pi(\chi_{\{a\}})\Pi(\chi_{\{r(a)\}}) = 0$ by (3.5). From (3.6), $\mu(\{a\}) = 0$. Since $H \setminus G$ is countable, it follows that $\mu(H \setminus G) = 0$, and for $f \in C_c(G)$,

$$\langle \tilde{\Pi}(f)\xi, \xi \rangle = \int_U f|G \, d\mu = \langle \tilde{\Pi}(f|G)\xi, \xi \rangle,$$

and $\Pi(f) = \sigma(f|G)$. ■

The requirement (3.5) of the preceding proposition can be replaced by the corresponding requirement in which $\alpha \in Y_f$ is replaced by $v \in V_f$:

$$(3.7) \quad \Pi(\chi_{\{z\}}) = 0 = \Pi(\chi_{A(v,v)}) - \sum_{s(e)=v} \Pi(\chi_{A(e,e)}).$$

To show this, $\Pi(\chi_{A(v,v)}) - \sum_{s(e)=v} \Pi(\chi_{A(e,e)}) = \Pi(\chi_{\{v\}})$ and one way follows trivially since $v \in Y_f$. For the other way, we just use the fact that $\chi_{\{\alpha\}} = \chi_{\{\alpha\}} * \chi_{\{r(\alpha)\}}$ and the fact that Π is a homomorphism.

We now give the definition of a *representation* of the graph \mathcal{E} . The definition is taken from [7]. Note that in the \mathcal{O}_∞ case, condition (iii) below assumes the form ([3], p. 174) $\sum_{i=1}^r S_i S_i^* \leq 1$ for all $r \geq 1$. Further, 1 is just P_v where v is the sole vertex of the directed graph which is the infinite bouquet of circles.

DEFINITION 3.7. A representation of \mathcal{E} on a Hilbert space \mathfrak{H} is given by a family $\{P_v : v \in V\}$ of mutually orthogonal projections, and a family $\{S_e : e \in E\}$ of partial isometries such that the family of projections $S_e S_e^*$ are mutually orthogonal and

- (i) $S_e^* S_e = P_{r(e)}$ for all $e \in E$;
- (ii) if $v \in V_f$, then

$$(3.8) \quad P_v = \sum_{s(e)=v} S_e S_e^*;$$

- (iii) $P_{s(e)} S_e S_e^* = S_e S_e^*$ for all $e \in E$.

By multiplying both sides of (i) on the right by S_e^* we get

$$(3.9) \quad S_e^* = P_{r(e)} S_e^*.$$

Similarly, using (iii), we get that

$$(3.10) \quad P_{s(e)} S_e = S_e.$$

In the following, representations of graphs and inverse semigroups are assumed to be non-degenerate in the obvious sense. (Representations of locally compact groupoids are “non-degenerate” almost by definition.)

THEOREM 3.8. *There are natural one-to-one correspondences between:*

- (a) the class $\mathfrak{R}_{\mathcal{E}}$ of representations of \mathcal{E} ;
- (b) the class \mathfrak{R}_S of representations of S that vanish on z and on elements of the form $v - \sum_{s(e)=v} (e, e)$ for all $v \in V_f$;
- (c) the class \mathfrak{R}_G of representations of the locally compact groupoid G .

Proof. Let $Q = \{\{P_v\}, \{S_e\}\} \in \mathfrak{R}_{\mathcal{E}}$ and be realized on a Hilbert space \mathfrak{H} . For $\alpha = e_1 e_2 \cdots e_n \in Y$, define $S_\alpha = S_{e_1} S_{e_2} \cdots S_{e_n}$. Condition (i) in Definition 3.7 above generalizes to:

$$(3.11) \quad S_\alpha^* S_\alpha = P_{r(\alpha)}.$$

For example, if $\alpha = e_1 e_2$, then using (i), (iii) and (3.10), we have $S_\alpha^* S_\alpha = S_{e_2}^* [S_{e_1}^* S_{e_1}] S_{e_2} = S_{e_2}^* P_{s(e_2)} S_{e_2} = (S_{e_2}^* S_{e_2}) = P_{r(e_2)} = P_{r(\alpha)}$. It immediately follows from (3.10) and (3.9) that:

$$(3.12) \quad P_{s(\alpha)} S_\alpha = S_\alpha, \quad P_{r(\alpha)} S_\alpha^* = S_\alpha^*.$$

Define a $*$ -map $\Pi : S \rightarrow B(\mathfrak{H})$ by:

$$\Pi(v) = P_v, \quad \Pi(\alpha, \beta) = S_\alpha S_\beta^*, \quad \Pi(z) = 0.$$

Then Π is a $*$ -homomorphism. The proof is simple. One checks that $\Pi(st) = \Pi(s)\Pi(t)$ for the different kinds of product using (3.11) and (3.12). For example, to prove that $\Pi(\alpha, \beta\mu) = \Pi(\alpha, \alpha'\mu)\Pi(\alpha', \beta)$, one argues: $\Pi(\alpha, \alpha'\mu)\Pi(\alpha', \beta) = S_\alpha S_\mu^* S_{\alpha'}^* S_{\alpha'} S_\beta^* = S_\alpha S_\mu^* P_{r(\alpha')} S_\beta^* = S_\alpha S_\mu^* S_\beta^* = \Pi(\alpha, \beta\mu)$. Since $\Pi(z) = 0$, $S_e S_e^* = \Pi((e, e))$ and Π vanishes on elements of the form $(v - \sum_{s(e)=v} (e, e))$ for $v \in V_f$, it follows from (3.8) that $\Pi \in \mathfrak{R}_S$.

Conversely, any $\Pi' \in \mathfrak{R}_S$ determines an element of $\mathfrak{R}_\mathcal{E}$ by taking $P_v = \Pi'(v, v), S_e = \Pi'(e, r(e))$. Since $S_e S_e^* = \Pi'((e, r(e))(r(e), e)) = \Pi'(e, e)$, (ii) of Definition 3.7 follows. (i) and (iii) follow since Π' is a $*$ -homomorphism on S . This establishes the correspondence between the representations of (a) and (b).

The correspondence between the representations in (b) and (c) is implemented by using the natural bijection between representations of S and (H, Ψ) (the universal groupoid of S) established in [12], Chapter 4. We now recall what this bijection is.

Let ρ be a representation of S on a Hilbert space \mathfrak{H} . Then there is a representation Π of $C_c(H)$ determined by:

$$\Pi(\chi_{\Psi(s)}) = \rho(s)$$

for all $s \in S$ where $\Psi(s)$ is given in Theorem 3.5. Further, the map $\rho \rightarrow \Pi$ is one-to-one and onto. It remains then to show that the representations ρ of S that satisfy the conditions of (b) correspond to the representations Φ of $C_c(G)$. This follows by Proposition 3.6 (and the remark following that proposition), and so we have established the correspondence between \mathfrak{R}_S and \mathfrak{R}_G . ■

We now translate (a), (b) and (c) of the preceding theorem into C^* -algebra terms. The C^* -algebra determined by the groupoid representations of (c) is just the groupoid C^* -algebra $C^*(G)$. The C^* -algebra associated with (b) is obtained on the semigroup algebra $\ell^1(S)$ as follows. Every representation Π of S gives a bounded representation Π of $\ell^1(S)$ in the natural way, and $C^*(S)$ is just the enveloping C^* -algebra of $\ell^1(S)$ obtained by taking the biggest C^* -norm coming from such Π 's. The C^* -algebra that we want here, which we will denote by $C_0^*(S)$, is obtained in the same way only using Π 's for which $\Pi(z) = 0 = \Pi\left(v - \sum_{s(e)=v} (e, e)\right)$

for all $v \in V_f$. It is easy to show that $C_0^*(S)$ is the enveloping C^* -algebra of $\ell^1(S)/I$, where I is the closed ideal of $\ell^1(S)$ generated by elements of the form z and $v - \sum_{s(e)=v} (e, e)$, with $v \in V_f$.

We take $C^*(\mathcal{E})$ to be the universal C^* -algebra for the representations of \mathcal{E} . By the above, we then obtain the following corollary to Theorem 3.8.

COROLLARY 3.9. $C^*(\mathcal{E}) \cong C_0^*(S) \cong C^*(G)$.

In the case of \mathcal{O}_∞ , the graph \mathcal{E} is a bouquet of circles $\{e_n : n \geq 1\}$ (one vertex v). The inverse semigroup $S_\mathcal{E}$ in this case is the set of pairs (α, β) where α, β are finite words in \mathbb{P} together with the identity v and zero z . Then G is the groupoid \mathcal{O}_∞ of [12], p. 184f.

Now let \mathcal{E} be a general directed graph and $S = S_\mathcal{E}$ as before. For the purposes of the next section, we need to use a related inverse semigroup S^n with associated groupoid G^n . Here, $n \in \mathbb{P}$, $Y^n = \{y \in Y : 0 \leq l(y) \leq n\}$ and S^n is the inverse subsemigroup of S given by:

$$S^n = \{(\alpha\mu, \beta\mu) \in S : \alpha, \beta \in Y^n, l(\alpha) = l(\beta)\} \cup \{z\}.$$

It is easy to check that S^n is an inverse subsemigroup of S and that $I(S^n) = I(S)$. We now describe the universal groupoid H^n of S^n and the related groupoid G^n . The proofs of the assertions below are very similar to the corresponding proofs for S , and are left to the reader.

The universal groupoid H^n for S^n can be identified with the union of $\{z\}$ and the set of all pairs of the form $(\alpha\gamma, \beta\gamma)$, where $\alpha, \beta \in Y^n$, $l(\alpha) = l(\beta)$, $\gamma \in Y \cup Z$, and $\alpha\gamma, \beta\gamma \in Y \cup Z$. Identify $(\alpha\gamma, \beta\gamma) \in S^n$ with $(\alpha\gamma, 0, \beta\gamma) \in H$. Then H^n is a subgroupoid of H with the same unit space, and the multiplication and inversion for H^n are just those that H^n inherits as a subgroupoid of H . The canonical map $\Psi^n : S^n \rightarrow (H^n)^a$ is just the restriction of the map Ψ of Theorem 3.5 to S^n . The groupoid G^n is defined to be the reduction of H^n to X .

The principal transitive equivalence relation on a (discrete) set with k elements ($1 \leq k \leq \infty$) is denoted by T_k . (In less elaborate language, this is the equivalence relation for which any element of the set is equivalent to any other.) For the following definitions, see [14], p. 123. An r -discrete groupoid is called *elementary* if it is isomorphic to the disjoint union of a sequence of groupoid products of the form $T_k \times Q$ for some k and some locally compact space Q , and is called *AF* if its unit space is totally disconnected and it is the inductive limit of a sequence of elementary groupoids.

PROPOSITION 3.10. *The locally compact groupoid G^n is an AF-groupoid.*

Proof. Obviously, the unit space G^0 of G^n is totally disconnected. Enumerate E as e_1, e_2, \dots , and let $N \in \mathbb{P}$. Let $F = \{e_1, \dots, e_N\}$. For $0 \leq r \leq n$, let A_r be the set of all finite paths of the form $e_{p_1} \cdots e_{p_r}$ where each $e_{p_i} \in F$. We take $A_0 = s(F)$. Let B_r be the set of pairs $(\alpha\gamma, \beta\gamma) \in G^n$ where $\alpha, \beta \in A_r$ and γ is arbitrary. Clearly, B_r is a compact, open subgroupoid of G^n (being a finite union of $\Psi^n(\alpha, \beta)$'s). For each $v \in r(F)$, let

$$X^v = \{\gamma \in X : s(\gamma) = v\}.$$

Then $X^v = A(v, v)$ is a compact, open subset of X . Note that if $\alpha, \beta \in A_r$ with $r(\alpha) = v = r(\beta)$, then $(\alpha\gamma, \beta\gamma) \in B_r$ if and only if $\gamma \in X^v$. Let

$$L^N = \bigcup_{r=0}^n B_r.$$

Then L^N is also a compact, open subgroupoid of G^n . We will show that L^N is elementary.

To this end we “disjointify” the B_r 's. Let C_r be the unit space B_r^0 of B_r . The elements of C_r are of the form $\alpha\gamma$ for $\alpha \in A_r, \gamma \in X^{r(\alpha)}$. Clearly $C_n \subset C_{n-1} \subset \cdots \subset C_0 = \{x \in X : s(x) \in A_0\}$. Let $C'_n = C_n$ and, for $0 \leq r < n$, let $C'_r = C_r \setminus C_{r+1}$. Let

$$B'_r = \{(\alpha\gamma, \beta\gamma) \in B_r : \alpha, \beta \in A_r, \alpha\gamma, \beta\gamma \in C'_r\}.$$

Then L^N is the disjoint union of the compact, open subgroupoids B'_r of G^n . The fact that L^N is elementary will follow once we have shown that each B'_r is elementary. To this end, fix r , and for each $v \in r(F)$, let

$$W_v = \{(\alpha\gamma, \beta\gamma) \in B'_r : \alpha, \beta \in A_r, v = r(\alpha) = r(\beta) = s(\gamma)\}.$$

Then B'_r is the disjoint union of the compact, open subgroupoids W_v , and for each v , W_v is isomorphic to $T_{k_v} \times (X^v)'$, where $(X^v)'$ is an open subset of X^v , and k_v is the number of α 's in A_r with $r(\alpha) = v$ and such that $\alpha\gamma \in C'_r$ for some γ . So B'_r , and hence L^N , is elementary.

We now claim that G^n is the inductive limit of the L^N 's and hence is an AF-groupoid. To this end (cf. [14], pp. 122–123), G^n is a union of the increasing sequence of elementary, open subgroupoids L^N . It is easily checked that a set $V \subset G^n$ is open in the inductive limit topology if and only if it is open in G^n , so that the inductive limit topology on G^n coincides with the given topology on G^n . Further, the counting measure left Haar system on G^n is compatible with the counting measure left Haar systems on the L^N 's. So G^n is the inductive limit of the L^N 's. ■

At the S^n level, for \mathcal{O}_∞ , Cuntz observes ([3], 1.5) that \mathcal{F}^∞ is an AF C^* -algebra.

4. GRAPH GROUPOIDS ARE AMENABLE

In [3], Cuntz showed that \mathcal{O}_n ($n \geq 2$) is isomorphic to the crossed product of an AF-algebra by an automorphism, cut down by a projection. It follows ([3], [16]) that \mathcal{O}_n is therefore nuclear (=amenable). Cuntz used a separate argument to deal with the case $n = \infty$. Renault ([14], p. 138f.) showed that the Cuntz groupoids \mathcal{O}_n , $n < \infty$, are amenable as locally compact groupoids. (The amenability of \mathcal{O}_n then follows.) The groupoid case when $n = \infty$ remained unclear.

Kumjian, Pask, Raeburn and Renault ([11]) adapted Renault's original argument to establish the amenability of the groupoid $G = G_\mathcal{E}$ when \mathcal{E} is locally finite. In this section, we further adapt the argument of [11] to show that G is amenable in complete generality. In particular, this covers the groupoid case for \mathcal{O}_∞ .

As in [11], we assume initially that every vertex receives an edge. For each $v \in V$, fix an edge $e(v) \in E$ for which $r(e(v)) = v$. Next we consider the following space \mathfrak{X} of two-sided paths. The elements of \mathfrak{X} are sequences $x = \{x_i\}$ of edges, where $-\infty < i < k \leq \infty$ with k depending on x and $r(x_i) = s(x_{i+1})$ whenever that makes sense. So the sequence x is infinite to the left, and may or may not be infinite to the right. If the sequence x is finite to the right, then $r(x)$ makes sense in the obvious way, and we require that $r(x) \in V_\infty$. Define a function $h : \mathfrak{X} \rightarrow \mathbb{Z} \cup \{\infty\}$ as follows: $h(x) = \infty$ if x is infinite to the right, and otherwise, is the largest i for which x_i is defined.

For each $n \in \mathbb{Z}$, we now define a subset P_n of \mathfrak{X} by specifying that $x \in P_n$ if and only if $h(x) \geq (n-1)$ and $x_i = e(s(x_{i+1}))$ for all $i < n$. If $h(x) = n-1$, then we require x_{n-1} to be some $e(v)$. The set of x 's in P_n for which $h(x) = n-1$ can be identified with V_∞ by sending x to $r(x)$. Note that $P_n \subset P_{n-1}$.

Let $n \geq 1$. There is a natural bijection $f_{-n} : P_{-n} \rightarrow X = G^0$ defined by translating to the right by $n+1$ and chopping off what is to the left of 1. Precisely, $f_{-n}(x)_m = x_{-n+m-1}$ if $h(x) > (-n-1)$ and $f_{-n}(x) = r(x)$ when $h(x) = -n-1$. We give P_{-n} the topology for which f_n is a homeomorphism.

Note that P_{-n} is an open subset of P_{-n-1} . Indeed $f_{-n-1}(P_{-n})$ is the (open) set of elements of X of the form $e(s(x))x$ for some $x \in X$.

Let R_n be the equivalence relation on P_{-n} defined by: $(x, y) \in R_n$ if and only if $h(x) = h(y)$, and if $h(x) = -n-1$ then $r(x) = r(y)$, while if $h(x) > n$, then $x_i = y_i$ for $i > n$. Note that when $h(x) = -n-1$, then xR_ny if and only if $x = y$. Note also that if $h(x) > n$, then since $x_{n+1} = y_{n+1}$, we have $r(x_n) = s(x_{n+1}) = s(y_{n+1}) = r(y_n)$. The unit space of R_n is P_{-n} . Of course, we give R_n the product topology that it inherits as a subset of $P_{-n} \times P_{-n}$.

PROPOSITION 4.1. *The equivalence relation groupoid R_n is an AF groupoid.*

Proof. The map $(x, y) \rightarrow (f_n(x), f_n(y))$ is an isomorphism of locally compact groupoids from R_n onto G^{2n+1} and therefore is an AF groupoid by Proposition 3.10. ■

Clearly, R_n is an open subset of R_{n+1} , and the topology on R_n is the relative topology that it inherits as a subset of R_{n+1} . Also, the canonical left Haar system $\{\lambda_{n+1}(u)\}$ (counting measure on the $(R_{n+1})^u$'s) gives by restriction the left Haar system $\{\lambda_n(u)\}$ on R_n .

It follows as in [14], p. 122, that $R = \bigcup_{n \geq 1} R_n$ is an r-discrete groupoid in the inductive limit topology. (As pointed out in [11], this notion of inductive limit is more general than the r-discrete case considered by Renault inasmuch as the unit spaces of the R_n 's are increasing rather than being fixed.) It also follows that R is amenable since each R_n is AF and therefore amenable ([14], p. 123).

The argument for the amenability of G now proceeds almost exactly as in [11] (to which the reader is referred for further details). We will be content with a sketch of the argument in the present context to indicate the minor changes needed.

Let $P = \bigcup_1^\infty P_{-n}$ be the unit space of R . Then P is the inductive limit of the P_{-n} 's. The left shift map h on P , where $h(\{x_i\})_j = x_{j+1}$, is a homeomorphism on P since it maps P_{-n} homeomorphically onto P_{-n-1} , and it induces a homeomorphism σ on R , where $\sigma(u, v) = (h(u), h(v))$, which is also a groupoid homomorphism. One then considers the semi-direct product $R \times_\sigma \mathbb{Z}$. This groupoid consists ([11]) of triples $(u, v, k) \in R \times \mathbb{Z}$ with $r(u, v, k) = u, s(u, v, k) = h^{-k}v$, inverse by $(u, v, k)^{-1} = (\sigma^{-k}(v, u), -k)$ and product by:

$$(u, h^k(p), k)(p, q, l) = (u, h^k(q), k + l).$$

Then $R \times_\sigma \mathbb{Z}$ is amenable ([14], p. 96) and hence ([14], p. 92) so is its reduction G' to the closed subset P_1 of P .

We now claim that G' is isomorphic to the graph groupoid G (so that G is also amenable). Firstly, the unit space P_1 of G' can be identified with X by sending $\cdots e(s(z_1))z_1z_2\cdots$ to $z = z_1z_2\cdots$. The natural isomorphism $F : G \rightarrow G'$ is given by (cf. [11], Proposition 4.4):

$$F(xz, k, yz) = (xz, h^{-k}(yz), -k).$$

Here, $x = x_1 \cdots x_r, y = y_1 \cdots y_{r-k}$ and xz, yz on the right-hand side are abbreviations for $\cdots e(s(x_1))xz, \cdots, e(s(y_1))yz \in P_1$.

Checking that F is a groupoid isomorphism is straightforward. That F is a homeomorphism follows effectively from the way that we defined the topology of P_{-n} in terms of X . To check that F is a homomorphism, one argues ($k = l(x) - l(y), l = l(y_1) - l(a), yz = yy'z' = y_1z'$):

$$\begin{aligned} F(xz, k, yz)F(yz, l, az') &= (xz, h^{-k}(yz), -k)(yz, h^{-l}(az'), -l) \\ &= (xz, h^{-k}[h^{-l}(az')], -k - l) \\ &= (xz, h^{-(k+l)}(az'), -(k + l)) \\ &= F((xz, k, yz)(y_1z', l, az')). \end{aligned}$$

The other facts to be checked are left to the reader.

This deals with the case where every vertex v is $r(e)$ for some $e \in E$. For the general case, one considers a larger graph \mathcal{E}' obtained by adding an infinite left tail to each $v \in V$ for which $r^{-1}(\{v\}) = \emptyset$, applies the preceding result to \mathcal{E}' and performs a reduction to obtain that G is amenable. (Note that $G^0 = G(\mathcal{E})^0$ is a closed subset of $G(\mathcal{E}')^0$.) We therefore have:

THEOREM 4.2. *For any directed graph \mathcal{E} , the graph groupoid G is amenable.*

5. THE SIMPLICITY OF $C^*(\mathcal{E})$

In this section we will use the groupoid G to obtain conditions sufficient for $C^*(\mathcal{E})$ to be simple. As discussed in the Introduction, the result obtained (Theorem 5.2) is effectively due to W. Szymanski who also proved the converse. (It seems likely that the approach to the converse for the row finite case in [1] can be adapted to apply within the groupoid context of the present paper.)

For $v \in V$, a *loop based at v* is a finite path $\alpha = e_1 \cdots e_n$ such that $r(\alpha) = s(\alpha) = v$ and $r(e_i) \neq v$ for $1 \leq i < l(\alpha)$. The subsets V_0, V_1, V_2 of V are the sets of vertices v such that there are respectively no loops based at v , exactly one loop based at v and at least two distinct loops based at v . The graph \mathcal{E} is said to satisfy condition (K) ([11], Section 6) if $V_1 = \emptyset$.

Recall that a locally compact groupoid H is called *essentially principal* ([14], p. 100) if whenever F is a closed invariant subset of H^0 , then the set of u 's in F with trivial isotropy group is dense in F . It is shown in Proposition 6.3 of [11], that if \mathcal{E} is row finite and satisfies condition (K), then G is essentially principal. We now show that for general \mathcal{E} , G is essentially principal if and only if \mathcal{E} satisfies condition (K).

PROPOSITION 5.1. *The r -discrete groupoid G is essentially principal if and only if \mathcal{E} satisfies condition (K).*

Proof. Suppose that \mathcal{E} satisfies condition (K). Let F be a closed invariant subset of G . Note that any finite path $y \in Y$ has trivial isotropy. So we need only consider infinite paths. The proof of Proposition 6.3 from [11] then goes through *verbatim* to give that G is essentially principal.

Conversely, suppose that G is essentially principal and that $V_1 \neq \emptyset$. Let $v \in V_1$ and α be the loop in \mathcal{E} based at v . Let

$$C = \{z \in Z : z \geq v\}.$$

Note that $\gamma = \alpha\alpha\cdots$ belongs to C . Let $F = \overline{C}$, the closure of C in X . We claim that F is a closed invariant subset of X .

Trivially, F is closed. To prove invariance, let $\{z^i\}$ be a sequence in C and $z^i \rightarrow x \in X$. Suppose that x is equivalent to $y \in Y$. Either $l(x) = \infty$ or $l(x) < \infty$. Suppose first that $l(x) = \infty$. Then for some α' and some $z \in Z$, we have $x = \alpha'z, y = \beta z$. Then eventually, every $z^i = \alpha'w^i$, and so $\beta w^i \in C$ and $\beta w^i \rightarrow y$. So $y \in F$. If $l(x) < \infty$, then $r(x) = r(y)$, and a similar argument gives that $y \in F$. So F is invariant. Note that if $x \in F$, then $x \geq v$ for all v (so that $F \cap Z = C$).

We will contradict the assumption $V_1 \neq \emptyset$ by showing that if $x \in F$ and $s(x) = v$, then $x = \gamma$. (For then any sequence in F converging to γ will eventually coincide with γ and so eventually, the terms of the sequence will not have trivial isotropy.) We can suppose that $x \in C$ since each $x' \in F$ of finite length and with $s(x') = v$ is the limit of a sequence of such infinite paths x . For each n , there is a path y starting and finishing at v obtained by following x until we reach $r(x_n)$ and then taking a path from $r(x_n)$ to v . Since α is the only loop based at v , y is of the form $\alpha\alpha \cdots \alpha$. It follows every initial segment of x is an initial segment of γ and so $x = \gamma$. ■

The groupoid G is called *minimal* ([14], p. 35) if the only open invariant subsets of G^0 are G^0 and the empty set. Since G is amenable, we have $C^*(G) = C_{\text{red}}^*(G)$ ([14], p. 92). Assume that \mathcal{E} satisfies condition (K). Then by the above, the r -discrete groupoid G is essentially principal. So by Proposition 2.4.6 on p. 103 from [14] we have that $C^*(G)$ is simple if and only if G is minimal. We now give a groupoid proof of the simplicity result for $C^*(\mathcal{E})$ ([21], Theorem 12).

THEOREM 5.2. *The graph C^* -algebra $C^*(\mathcal{E})$ is simple if (a), (b) and (c) hold, where:*

- (a) \mathcal{E} has property (K);
- (b) \mathcal{E} is cofinal ([11]) in the sense that given $v \in V$ and $z \in Z$, there exists an n such that $v \geq r(z_n)$;
- (c) if $v \in V$ and $w \in V_\infty$, then there is a path from v to w .

Proof. Suppose that G satisfies (a), (b) and (c). By the preceding comments, we just need to show that G is minimal.

Let $U \neq \emptyset$ be an open invariant subset of X . By Proposition 3.4, $U \cap Z \neq \emptyset$. By considering a neighborhood in X of some $z \in U \cap Z$ and using (ii) of Proposition 3.3, we have that for some α , $D'_\alpha \subset U$. Let $z \in Z$. By (b), there is a path β from $r(\alpha)$ to $r(z_n)$ for some n . Then $z' = \alpha\beta z_{n+1} \cdots \in D'_\alpha \subset U$. Since z' is tail equivalent to z and U is invariant, we have $z \in U$. Next let $y \in Y_\infty$. By (c), there is a path β' from $r(\alpha)$ to $r(y)$. Then $\alpha\beta'$ is equivalent to y , and $\alpha\beta' \in U$. So $y \in U$. So $U = Z \cup Y_\infty = X$ and G is minimal. ■

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