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Graph polynomials
and the discrete Fourier transform

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Merton College

Oxford

A thesis submitted for the degree of

Doctor of Philosophy

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The main impetus for this thesis is to obtain new identities about colourings and flows in graphs. Chapter 2 contains two smaller results about the Tutte polynomial which have already been accepted for publication [26, 27]. The thesis then examines the different polynomials introduced by Matiyasevich [43, 45] and Alon and Tarsi [6]. Chapter 3 is devoted to setting up the necessary machinery for a unifying treatment of these polynomials and introduces the relevant notions of discrete Fourier analysis. The groundwork carried out in this chapter leads to an assortment of identities, often just for cubic graphs, involving the number of proper edge colourings and nowhere-zero flows. The main results of Chapter 4 are new properties of the coefficients of the Matiyasevich polynomial. Chapter 5 contains some curious correlation identities, which emerge from a development of the ideas of Matiyasevich [45]. Some of these identities have been re-proved by methods which do not rely on the discrete Fourier transform [69]. Chapter 5 concludes with an extension of these ideas to face colourings of cubic graphs embedded in surfaces. Chapter 6 focuses on cubic graphs and the enumeration of nowhere-zero flows and proper edge colourings. In its final section, identities are derived which relate proper edge 3-colourings to colourings of the edges with a larger number of colours. The thesis closes with a short outline of possible future extensions.

Contents

This thesis is dedicated to my parents.

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Chapter 1

Introduction

In this chapter we outline the content of this thesis and give an account of a theorem of Alon and Tarsi in [7] and a theorem of Matiyasevich in [45] which, together, motivated the work set out in Chapters 3-6. All definitions and notation are introduced as they are needed, although the basic terms of graph theory are assumed. We refer the reader to [18] and [14] for more information about the notions from graph theory encountered in this thesis.

In Chapter 2 we consider the evaluation of some specialisations of the Tutte polynomial modulo a fixed integer. (See for example [68] for the significance of this polynomial in many areas of combinatorics and physics.) With minor modifications, §2.2 is a paper cowritten with Dominic Welsh which has appeared in [26] and §2.3 is a paper which has appeared in [27]. The main import of Annan's results in [8, §3] is that in all but a few unresolved cases the evaluation of the Tutte polynomial at a point (x, y) modulo any integer $k \geq 3$ is not possible in random polynomial time (provided $\mathcal{RP} \neq \mathcal{NP}$) if its evaluation at some congruent point in $\mathbb{Z}$ is not possible in polynomial time (provided $\#\mathcal{P} \neq \mathcal{FP}$). In §2.3 we find some exceptions to the converse statement which are counterexamples to Annan's Conjecture 3.6.3 in [8].

The extensive groundwork carried out in Chapter 3 is a result of the synthesis of the methods of Alon and Tarsi [6, 7] and Matiyasevich [43, 45], and in the use of *half-edges* and *weights* is comparable to Jaeger's [36] use of *half-edges*, *transitions* and *weight functions*. In §§10-12 of Chapter 3 definitions and theorems from discrete Fourier analysis are tailored for our purposes. Taking the discrete Fourier transform of our weight functions gives many of the results set out in Chapters 4-6.

In Chapter 4 we obtain some new results about the *Matiyasevich polynomial*, the subject of Matiyasevich's paper [43] and given this name in for example [21]. Matiyasevich in [43] considers this polynomial as a special case of the *graph polynomial*, which is the subject of Matiyasevich's earlier paper [42] as well as Alon and Tarsi's papers [6, 7, 62].

We define these polynomials in the second part of this introduction.

In Chapter 5 we derive some new results analogous to Matiyasevich's probabilistic restatements of the Four Colour Theorem [45] and in particular provide comparable probabilistic restatements of Alon and Tarsi's interpretation of the coefficients and the ℓ_2-norm of the graph polynomial in [6, 7].

In Chapter 6 we use the discrete Fourier transform to generate a number of expressions for the number of nowhere-zero k-flows and the number of proper edge k-colourings of a cubic graph, which appear to be new and for which it would be interesting to have a simpler direct proof.

Finally in Chapter 7 we highlight a few of the questions and open problems which arise in the course of Chapters 2-6.

We conclude this introduction with a definition of the graph polynomial and the Matiyasevich polynomial, together with a presentation of a theorem of Alon and Tarsi [6] and a theorem of Matiyasevich [45]. The reader may wish to refer back to these before embarking on Chapter 3, as motivation for the unification and generalisation to be found in this chapter of the methodology of the authors just cited.

Suppose that $G = (V, E)$ is a loopless graph with a fixed orientation. If $\{v_0, v_1\} \in E$ then we write $v_0 < v_1$ if v_0 is directed towards v_1. The *graph polynomial* of G, first studied by Petersen [53], is defined by

$$f(G; (x_v)) = \prod_{\{v_0,v_1\}\in E,\, v_0<v_1} (x_{v_1} - x_{v_0}),$$

where (x_v) denotes a vector of $|V|$ commuting indeterminates indexed by the vertex set of G.

For $k \in \mathbb{N}$, Alon and Tarsi [7] take the graph polynomial $f(G)$ modulo the ideal $(x_v^k - 1)$ of $\mathbb{C}[(x_v)]$ generated by the polynomials $\{x_v^k - 1 : v \in V\}$ to obtain the following polynomial:

$$f_k(G; (x_v)) = \prod_{\{v_0,v_1\}\in E,\, v_0<v_1} (x_{v_1} - x_{v_0}) \mod (x_v^k - 1).$$

Let Z_k be the cyclic group of integers modulo k whose elements have been identified with the first k nonnegative integers $\{0, 1, 2, \ldots, k-1\}$. For a polynomial $p \in \mathbb{C}[(x_v)]/(x_v^k - 1)$ given by

$$p((x_v)) = \sum_{\lambda:V\to Z_k} c_\lambda \prod_{v\in V} x_v^{\lambda(v)},$$

the ℓ_2-*norm* of p is defined by

$$\|p\|_2^2 = \sum_{\lambda:V\to Z_k} |c_\lambda|^2.$$

Alon and Tarsi [7] apply a theorem from the theory of discrete Fourier analysis known as Parseval's formula (recorded in §3.10 of this thesis) to obtain the following expression for the ℓ_2-norm of $f_k(G)$:

Theorem [7] *For any graph $G = (V, E)$ and $k \in \mathbb{N}$,*

$$\|f_k(G)\|_2^2 = 4^{|E|}k^{-|V|} \sum_{\lambda:V\to Z_k} \prod_{\{v_0,v_1\}\in E,\, v_0<v_1} \sin^2[\frac{\pi(\lambda(v_1)-\lambda(v_0))}{k}].$$

In particular

$$\|f_3(G)\|_2^2 = 3^{|E|-|V|}P(G;3),$$

where $P(G;3)$ is the number of proper vertex 3-colourings of G.

We now turn to Matiyasevich's theorem from [45].

Suppose that $G = (V, E)$ is a bridgeless planar cubic graph embedded in the plane with planar dual G^*. The *Matiyasevich polynomial* [43, 45, 21] of G is defined by

$$\prod_{e_0<e_1<e_2} (x_{e_0} - x_{e_1})(x_{e_1} - x_{e_2})(x_{e_2} - x_{e_0}) \mod (x_e^3 - 1),$$

where the product is over all triples of edges $\{e_0, e_1, e_2\}$ incident with a common vertex of G taken in a clockwise order $e_0 < e_1 < e_2$ in the embedding of G in the plane and $(x_e^3 - 1)$ denotes the ideal generated by the polynomials $\{x_e^3 - 1 : e \in E\}$.

In [43, 44] Matiyasevich has a number of results about the coefficients of this polynomial, and in [45, 46] he gives some of them probabilistic interpretations, one of which we will describe here. First we need to give some preparatory definitions.

Each edge e of G is divided into two *half-edges* (the two *halves* of e), one of each incident with either end of e. Each vertex is incident with three half-edges since G is cubic. The set of half-edges of G is denoted by H and a *proper half-edge colouring* of G is a colouring function $\mu : H \to \{0, 1, 2\}$ with the property that each vertex of G is incident with half-edges of distinct colours.

Given a plane embedding of G, a vertex v is *clockwise* in a given proper half-edge colouring of G if the colours $0, 1, 2$ appear in a clockwise order around v, and *anticlockwise* otherwise. A proper half-edge colouring is *even* or *odd* according as the parity of the number of anticlockwise vertices is even or odd. Any given proper half-edge colouring of G *induces* a (not necessarily proper) edge 3-colouring of G by adding together the colours on the two halves of an edge modulo 3.

We now choose two proper half-edge colourings μ, μ' uniformly at random (u.a.r.) from the set of $6^{|V|}$ proper half-edge colourings of G. Define "Equivalent" to be the event that μ, μ' induce the same edge colouring and "Same Parity" to be the event that the number of anticlockwise vertices in each of μ, μ' has the same parity.

Matiyasevich shows that the Four Colour Theorem is equivalent to the following statement: given that two proper half-edge colourings of a plane bridgeless cubic graph chosen u.a.r. have the same parity, the probability that they induce the same colouring is increased. More specifically, he proves the following:

Theorem [45] *Let $G = (V, E)$ be a plane cubic graph. Then*

$$\Pr(\text{Equivalent} \mid \text{Same Parity}) - \Pr(\text{Equivalent}) = \frac{1}{4}\left(\frac{1}{4\sqrt{3}}\right)^{|V|} P(G^*; 4).$$

In this thesis we see how Alon and Tarsi's theorem and Matiyasevich's theorem are related and develop in Chapter 3 a theoretical framework from which many similar results can be derived.

Chapter 2

The Tutte polynomial modulo an integer

2.1 Definitions

Let $G = (V, E)$ denote a graph, with loops and parallel edges permitted, and $\mathcal{G}$ the collection of all such graphs. The *size* of a graph $G = (V, E)$ is $|E|$. If G has $c(G)$ connected components, then the *rank* of G, denoted $r(E)$, is $|V| - c(G)$. The rank $r(A)$ of a subset of edges $A \subseteq E$ is the rank of the subgraph (V, A).

Let X, Y be commuting indeterminates. The Tutte polynomial $T(G; X, Y)$ is a map $T : \mathcal{G} \to \mathbb{Z}[X, Y]$ defined for all graphs G by

$$T(G; X, Y) = \sum_{A \subseteq E} (X-1)^{r(E)-r(A)}(Y-1)^{|A|-r(A)}. \tag{2.1}$$

An *evaluation* of the Tutte polynomial in a commutative ring R with 1 is a map $T(x, y) : \mathcal{G} \to \mathbb{Z}[x, y] \subseteq R$ obtained from T by substituting $(x, y) \in R \times R$ for the indeterminate pair (X, Y) in (2.1).

2.2 On the parity of colourings and flows

Tarsi [62] proves that for a graph G the set of proper 3-colourings and the set of nowhere zero 3-flows are of the same parity. We show that this is a special case of a more general result.

As pointed out by Tarsi, the set $C_3(G)$ of proper 3-colourings has the property that $|C_3(G)|$ is always divisible by 6 (permutations of the 3 colours), while flows come in pairs (obtained by reversing the entire orientation). Thus, if $NZF_3(G)$ denotes the set of nowhere-zero 3-flows on G, what Tarsi actually shows (his Theorem 1.3) is that

$$|C_3(G)| \equiv |NZF_3(G)| \bmod 4. \tag{2.2}$$

We change notation to that of [68] and write $P(G;k)$ for the chromatic polynomial and $F(G;k)$ for the flow polynomial. Thus (2.2) can be rewritten as

$$P(G;3) \equiv F(G;3) \bmod 4. \tag{2.3}$$

To prove our results we use the spanning tree expansion of the Tutte polynomial (see for example [14])

$$T(G;x,y) = \sum t_{i,j} x^i y^j, \tag{2.4}$$

and the identities

$$P(G;k) = (-1)^{r(E)} k^{c(G)} T(G;1-k,0), \tag{2.5}$$

$$F(G;k) = (-1)^{|E|-r(E)} T(G;0,1-k). \tag{2.6}$$

Two well known properties of the Tutte polynomial are noted: for $|E| > 0$ we have $t_{0,0} = 0$ and for $|E| > 1$ we have $t_{1,0} = t_{0,1}$.

From (2.5) and (2.6)

$$P(G;k) = (-1)^{r(E)} k^{c(G)} \sum_{i \geq 1} t_{i,0} (1-k)^i \tag{2.7}$$

and

$$F(G;k) = (-1)^{|E|-r(E)} \sum_{j \geq 1} t_{0,j} (1-k)^j. \tag{2.8}$$

Theorem 2.2.1 *For a graph $G = (V,E)$, $|E| \geq 2$, and integer k, $|k| \geq 2$,*

$$P(G;1+k) \equiv (-1)^{|E|} F(G;1+k) \bmod k^2.$$

Proof. Using (2.7) and (2.8) we have

$$P(G;1+k) \equiv (-1)^{r(E)} (1+k)^{c(G)} (-k) t_{1,0} \bmod k^2,$$

and

$$F(G;1+k) \equiv (-1)^{|E|-r(E)} (-k) t_{0,1} \bmod k^2.$$

The result follows since $(1+k)^{c(G)}(-k) \equiv (-k) \bmod k^2$ and $t_{1,0} = t_{0,1}$.□

Putting $k = 2$ gives as a corollary (2.3) above. Since $P(G;3)$ and $F(G;3)$ are even and $2 \equiv -2 \bmod 4$ the sign $(-1)^{|E|}$ of the theorem is redundant in this instance.

2.3 The Tutte polynomial modulo a prime

In this section we consider the question of when it is possible to compute an evaluation of the Tutte polynomial modulo a fixed integer in polynomial time. Although in §2.2 we noted that if G has at least one edge then $T(G;-k,0) \equiv T(G;0,-k) \equiv (-k)t_{0,1} \bmod k^2$, it is still left open by [8] and by the theorems of this chapter whether it is possible to evaluate the Tutte polynomial at $(-k,0)$ and $(0,-k)$ modulo k^2 in polynomial time.

The principal concern of the remainder of this chapter is the evaluation of the Tutte polynomial in the field F_p for p a prime. In other words, our problem is to determine for which points $(x,y) \in \mathbb{Z} \times \mathbb{Z}$ it is possible to compute the Tutte polynomial modulo p.

We begin with the observation that for all commutative rings R with unity 1 and $(x,y) \in R \times R$, we have

$$T(G;x,y) = x^{|E|}(x-1)^{r(E)-|E|} \qquad \text{if} \quad (x-1)(y-1) = 1, \tag{2.9}$$

so that the Tutte polynomial is polynomial-time computable at each of the points in the set $\{(x,y) \in R \times R : (x-1)(y-1) = 1\}$.

A theorem of [37] determines *all* the evaluations of the Tutte polynomial in $\mathbb{C}$ which are polynomial-time computable in the size of the graph G. Apart from those covered by (2.9) they are at the points

$$(-1,0), (0,-1), (1,1), (-1,-1), \tag{2.10}$$

and

$$(i,-i), (-i,i), (j,j^2), (j^2,j), \tag{2.11}$$

where $i = \sqrt{-1}$ and $j = (-1+\sqrt{-3})/2$.

In [68, §7.5] it is shown that all evaluations of the Tutte polynomial in F_2 are polynomial-time computable. All four evaluations in F_2 reduce to finding the parity of evaluations in $\mathbb{Z}$ at points in (2.9) and (2.10).

Annan [8] proved the following result:

Theorem 2.3.1 [8, §3.6] *Provided random polynomial time $\mathcal{RP}$ is not equal to $\mathcal{NP}$, the only polynomial-time computable evaluations of the Tutte polynomial in F_3 are at the points* $(-1,0), (0,-1), (1,1), (-1,-1)$ *and* $(0,0)$.

He conjectured that similarly, for any prime $p > 3$, the only polynomial-time computable evaluations of the Tutte polynomial in F_p correspond to the points covered by (2.9) and the points $(-1,0),(0,-1),\ (1,1),(-1,-1)$ in $F_p \times F_p$ corresponding to the points (2.10) in $\mathbb{C} \times \mathbb{C}$. However, it will be shown that this conjecture needs to be modified to include further pairs of points corresponding to (2.11) when -1 or -3 is a square in F_p.

We call a point $(x, y) \in R \times R$ *easy* if the evaluation $T(G; x, y)$ in R is polynomial-time computable in the size of G. If π is a ring homomorphism then $\pi(T(G; x, y)) = T(G; \pi(x), \pi(y))$. The easy points (2.10) in $\mathbb{Z} \times \mathbb{Z}$ and the homomorphism $\pi : \mathbb{Z} \to R$, $z \mapsto z1$, ensure that $(-1,0),(0,-1),(1,1),(-1,-1)$ are easy in any commutative ring R with unity 1. This observation is made in [8, §3.6] for $R = F_p$.

The following shows that the points of (2.11) yield further easy points in F_p for $p \equiv 1 \bmod 4$ or $p \equiv 1 \bmod 3$. The *Legendre symbol* (a/p) is defined to be $+1$ when a is a non-zero square in F_p and -1 when a is not a square.

Theorem 2.3.2 *Let $p > 3$ be a prime. There are (at least) $p + 5 + (-1/p) + (-3/p)$ polynomial-time computable evaluations of the Tutte polynomial in F_p. These are at the following points in $F_p \times F_p$:*

$$\{(x, y) \in F_p \times F_p : (x - 1)(y - 1) = 1\}, \tag{2.12}$$

$$(-1, 0), (0, -1), (1, 1), (-1, -1), \tag{2.13}$$

$$(a, -a), (-a, a), \tag{2.14}$$

if $p \equiv 1 \bmod 4$ and $a^2 + 1 = 0$ in F_p; and,

$$(b, b^2), (b^2, b), \tag{2.15}$$

if $p \equiv 1 \bmod 3$ and $b^2 + b + 1 = 0$ in F_p.

Proof. For each odd prime $p \in \mathbb{N}$ there are ideals of norm p in $\mathbb{Z}[i]$ if and only if $(-1/p) = +1$, or $p \equiv 1 \bmod 4$, and in $\mathbb{Z}[j]$ if and only if $(-3/p) = +1$, or $p \equiv 1 \bmod 3$ (see e.g. [24] for proofs of these facts).

Hence, for $p \equiv 1 \bmod 4$, there is a prime $r + si \in \mathbb{Z}[i]$ with norm $r^2 + s^2 = p$ and the homomorphism $\pi : \mathbb{Z}[i] \to \mathbb{Z}[i]/(r+si)$ gives images of the easy points $(i, -i), (-i, i) \in \mathbb{Z}[i]$ in the quotient ring. The homomorphic images of the points $(i, -i), (-i, i)$ do not coincide with the points given by (2.12) since $(i-1)(-i-1) = 2$, nor with the points $(1, 1), (-1, -1)$ of (2.13), since $p > 2$, nor with $(-1, 0), (0, -1)$ since $\pm i$ are units and cannot be mapped to 0. The ideal $(r + si)$ is prime in the ring of ideals of $\mathbb{Z}[i]$, so the quotient $\mathbb{Z}[i]/(r + si)$ is a field, has p elements, and hence is isomorphic to F_p.

Similarly, for $p \equiv 1 \bmod 3$, there is prime $r + sj \in \mathbb{Z}[j]$ with norm $r^2 - rs + s^2 = p$ and the homomorphism $\pi : \mathbb{Z}[j] \to \mathbb{Z}[j]/(r + sj)$ onto a field isomorphic to F_p has in its domain the easy points $(j, j^2), (j^2, j) \in \mathbb{Z}[j]$. The homomorphic images of these two points cannot coincide with any points in (2.12), (2.13), (2.14) since $(j-1)(j^2-1)=3$ and $p>3$.□

When $p \in \mathbb{Z}$ does not split in the larger ring $\mathbb{Z}[i]$ or $\mathbb{Z}[j]$ it generates a prime ideal (p) of norm p^2. The quotient ring is then isomorphic with F_{p^2}. The proof of Theorem 2.3.2 gives the following:

Corollary 2.3.3 *For odd prime $p \equiv -1 \bmod 4$ or $p \equiv -1 \bmod 3$, the points listed in* (2.9), (2.10), (2.11) *provide $p^2+5-(-1/p)-(-3/p)$ easy points in $F_{p^2} \times F_{p^2}$ via the homomorphism(s) $\mathbb{Z}[i] \to \mathbb{Z}[i]/(p)$ and/or $\mathbb{Z}[j] \to \mathbb{Z}[j]/(p)$.*

For $p \equiv 1 \bmod 12$, p^2+7 easy points in $F_{p^2} \times F_{p^2}$ arise from (2.9) and evaluation in the subfield isomorphic to F_p: all 8 points in (2.13), (2.14), (2.15) exist for this p. It is a small step to deduce the following:

Corollary 2.3.4 *Let $p>3$ be prime and $n \geq 1$. If n is odd then there are $p^n+5+(-1/p)+(-3/p)$ polynomial-time computable evaluations of the Tutte polynomial in F_{p^n}. If n is even then there are p^n+7 polynomial-time computable evaluations of the Tutte polynomial in F_{p^n}.*

Proof. In $F_{p^n} \times F_{p^n}$ there are p^n-1 points satisfying (2.9). Further easy points correspond to the $6+(-1/p)+(-3/p)$ points (2.13), (2.14), (2.15) of Theorem 2.3.2, evaluation being in the subfield F_p of F_{p^n}. For n even, Corollary 2.3.3 provides polynomial-time evaluations in the subfield F_{p^2} for the $2-(-1/p)-(-3/p)$ remaining points.□

There are 2^n+2 easy evaluations in F_{2^n} for $n \geq 1$. There are no elements of multiplicative order 4, and for even n, when there are two elements b, b^2 of order 3, the points $(b, b^2), (b^2, b)$ are such that $(b-1)(b^2-1)=3=1$ in F_{2^n} and so are counted already under (2.9).

There are $3^n+3+(-1)^n$ easy evaluations in F_{3^n} for $n \geq 1$, with no elements order 3 and, for even n, two elements order 4. The point $(-1,-1)$ is already counted under (2.9) since $(-1-1)(-1-1)=4=1$ in F_{3^n}.

Interpreting evaluation in F_p as "counting modulo p" it is natural to extend Theorem 2.3.2 from evaluation in $\mathbb{Z}/p\mathbb{Z}$ to evaluation in $\mathbb{Z}/m\mathbb{Z}$ for composite m by use of the Chinese Remainder Theorem. For prime $p>3$ there are $1+(-1/p)$ elements of multiplicative order 4 in $\mathbb{Z}/p^n\mathbb{Z}$ and $1+(-3/p)$ elements of order 3. In $\mathbb{Z}/2^n\mathbb{Z}$, -1 is a square only if $n=1$ and -3 is a square only if $n=1,2$. In $\mathbb{Z}/3^n\mathbb{Z}$, -1 is not a square and -3 is only a square for $n=1$. We obtain the following corollary by counting the number of solutions to $a^2 \equiv -1 \bmod m$ and $(2b+1)^2 \equiv -3 \bmod m$.

Corollary 2.3.5 *Let $3<m \in \mathbb{N}$ have $s \geq 0$ distinct prime factors greater than 3 and let $\phi(m)$ denote Euler's totient function. Denote by $e(m)$ the number of polynomial-time*

computable evaluations of the Tutte polynomial in $\mathbb{Z}/m\mathbb{Z}$. *Then Theorem 2.3.2 yields the following lower bounds on* $e(m)$, *the number of easy points in* $\mathbb{Z}/m\mathbb{Z} \times \mathbb{Z}/m\mathbb{Z}$.

If $m \not\equiv 0 \bmod 4$ *and each of the odd prime factors* $p > 2$ *of* m *satisfy* $p \equiv 1 \bmod 12$, *then*

$$e(m) \geq \phi(m) + 4 + 2^{s+1}. \tag{2.16}$$

If $m \not\equiv 0 \bmod 4$ *and each of the odd prime factors* $p > 2$ *of* m *satisfy* $p \equiv 1 \bmod 4$, *at least one of which satisfies* $p \equiv 5 \bmod 12$,

or *if* $m \not\equiv 0 \bmod 8$, $m \not\equiv 0 \bmod 9$ *and there are* $s \geq 1$ *distinct prime factors* $p > 3$ *of* m *each satisfying* $p \equiv 1 \bmod 3$, *at least one of which satisfies* $p \equiv 7 \bmod 12$ *if* $m \not\equiv 0 \bmod 4$ *and* $m \not\equiv 0 \bmod 3$, *then*

$$e(m) \geq \phi(m) + 4 + 2^{s}. \tag{2.17}$$

Otherwise,

$$e(m) \geq \phi(m) + 4. \tag{2.18}$$

Whether these inequalities for $e(m)$ can be improved to equalities depends on whether Theorem 2.3.2 describes *all* the easy points in $\mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}$. We finish this chapter with a brief discussion of this question.

For prime $p > 3$ it remains an open problem to determine if all the polynomial-time computable evaluations of the Tutte polynomial in F_p have been found. A revised version of the conjecture made in [8, §3.6] is the following:

Conjecture 2.3.6 *Let* $p > 3$ *be prime. Provided* $\mathcal{RP} \neq \mathcal{NP}$, *any evaluation of the Tutte polynomial in* F_p *not listed in Theorem 2.3.2 is not computable by a randomised polynomial-time algorithm.*

In [8, §3.7] some partial confirmation for this conjecture is adduced. Annan shows that evaluating the Tutte polynomial at the following points cannot be random polynomial time unless $\mathcal{RP} = \mathcal{NP}$:

$$\{(1, y) \in F_p \times F_p : y \neq 1\}, \tag{2.19}$$

and,

$$\{(x, y) \in F_p \times F_p : (x-1)(y-1) \neq 0, 1, 2;\ \langle x \rangle = F_p^* \text{ or } \langle y \rangle = F_p^*\}, \tag{2.20}$$

where F_p^* is the multiplicative group of units and $\langle z \rangle$ denotes the set generated multiplicatively by $z \in F_p$.

Apart from evaluation at the points

$$\{(x, 1) \in F_p \times F_p : x \neq 1\}, \tag{2.21}$$

$$\{(x, y) \in F_p \times F_p : (x-1)(y-1) = 2\}, \tag{2.22}$$

the author has verified that the statement of Conjecture 2.3.6 is true for all evaluations of the Tutte polynomial in F_p for $3 < p \leq 37$.

For $p = 5$, the points listed in (2.19) and (2.20) account for all the points not shown to be easy by Theorem 2.3.2, except for the points in (2.21). All four points of (2.22) are easy for $p = 5$.

For the restricted problem of evaluating the Tutte polynomial of *planar* graphs in F_p, the points of (2.22) will be easy by a theorem of [65]: evaluating the Tutte polynomial of planar graphs at the points $\{(x, y) \in \mathbb{C} \times \mathbb{C} : (x-1)(y-1) = 2\}$ is polynomial time.

We note that $T(G; 2, 1)$ counts forests in G and $(-1)^{|E|-r(E)}T(G; 0, 1-p)$ counts nowhere-zero p-flows in G when evaluating the Tutte polynomial in $\mathbb{Z}$. Interpreting evaluation in F_p as counting modulo a prime, the following question in particular arises from (2.21).

Problem 2.3.7 [8, §3.8] *For prime $p > 3$, is there a randomised polynomial-time algorithm for counting the number of forests of a graph modulo p? Can the number of nowhere-zero p-flows modulo p be found in random polynomial time?*

Chapter 3

Half-edges: partitions, colourings and weights

3.1 Introduction

The framework constructed in this chapter provides the basis for unifying and developing the results of Alon, Tarsi [62, 7, 6] and Matiyasevich [43, 45], two of which were presented in the introductory Chapter 1. In Chapter 4 we explore some properties of the Matiyasevich polynomial by interpreting its coefficients in terms of the *weights* and *induced colourings* introduced in §3.7 and §3.9 of this chapter. These weights and induced colourings will be further used in Chapter 5 to extend the probabilistic results of Matiyasevich [45] on edge 3-colouring cubic graphs; analogous results for vertex 3-colouring arbitrary graphs are deduced, yielding probabilistic interpretations and extensions of the results of Alon and Tarsi [7, 62]. In Chapter 6 we will use the weighted half-edge colourings defined in this chapter to give some enumerative results analogous to Tarsi's [62] expression for the ℓ_2-norm of the graph polynomial with exponents reduced modulo k as a sum over weighted "$k, 1$-flows". The content of this chapter consists entirely of preparatory material for Chapters 4-6. In the final section §3.12 the three lemmas marked A,B,C are the basis for many of the proofs in these later chapters.

3.2 Half-edges and edge double covers

Suppose G is an arbitrary graph embedded on a surface, so that vertices are points and edges are smooth curves with endpoints their incident vertices. Edges only meet at points representing vertices and a loop is a closed curve with just one vertex as a single endpoint. The *half-edges* of a graph G may be regarded as the result of cutting each edge across the middle, so that every edge comprises two half-edges and every vertex is incident with as

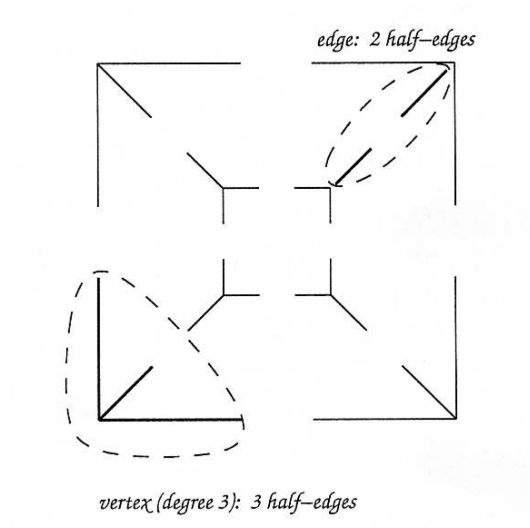


Figure 3.1: Half-edges by cutting edges across their middle. *For any graph, each edge comprises two half-edges. For a cubic graph, each vertex comprises three half-edges.*

many half-edges as its degree. (See Figure 3.1 above.)

We follow [17] and begin with half-edges as primitive undefined elements, gluing them together to make the edges of G, rather than beginning with G and a given embedding of G and then cutting edges in half to obtain half-edges. The advantage of this approach is that the interpretation of half-edges is not fixed in terms of a given embedding, nor need half-edges be edges cut across their middle (in §5.5, for example, we use a different interpretation of half-edges).

Definition 3.2.1 *Let $G = (V, E)$ be a graph. The* half-edges *of G are defined by a pair $(H, \mathcal{E})$, where H is a set of $2|E|$ elements and $\mathcal{E}$ is a set $\{H(e) : e \in E\}$ partitioning H into 2-sets indexed by the edge set E.*

Apart from the finest partition $\mathcal{H} = \{\{h\} : h \in H\}$ of H into singletons, the other two partitions of H which we shall consider will be defined by reference to a double cover of the edge set E of the graph G. For a given index set S, a multiset $\{E(s) : s \in S\}$ of subsets of E forms a *double cover* of E if each edge in E belongs to exactly two members of $\{E(s) : s \in S\}$. We then let $\mathcal{S} := \{H(s) : s \in S\}$ be the set of *blocks* in a partition of H with the following properties:

(1) for each $s \in S$, $|H(s)| = |E(s)|$,

(2) for each $s \in S$, $H(s) \cap H(e) \neq \emptyset$ if and only if $e \in E(s)$.

For a given double cover $\{E(s) : s \in S\}$ of E there are $2^{|E|}$ partitions $\{H(s) : s \in S\}$ of H satisfing (1) and (2), since for each edge $e \in E(s)$ there are two choices for which half-edge from $H(e)$ is to belong to $H(s)$. In order that one of these $2^{|E|}$ partitions is uniquely determined by the double cover $\{E(s) : s \in S\}$ one of these choices must be forced, determining a unique half-edge in $H(e) \cap H(s)$ whenever $e \in E(s)$. This can be achieved by interpreting the half-edges as objects in the embedding of the graph G. In §§3.3-3.4 below we use the definition of the set of half-edges H in a given embedding of G as the set of $2|E|$ simple curves obtained by cutting each edge in half (at a point not equal to a vertex in the embedding of G).

3.3 Partitioning half-edges of a graph by vertices

Let $G = (V, E)$ be any graph, with edges considered as subsets of vertices of size 2 (non-loops) or size 1 (loops). The set $E(v) = \{e \in E : v \in e\}$ of edges incident with a common vertex v is a cutset of G. The collection $\{E(v) : v \in V\}$ of such cutsets forms a double cover of the edges of G.

Given an arbitrary embedding of G on a surface with half-edge set H, we interpret half-edges as being obtained by cutting the edges of G across their middle. The blocks of the edge partition $\mathcal{E}$ of H comprise pairs of half-edges which are obtained from the same edge. We define the vertex partition $\mathcal{V} = \{H(v) : v \in V\}$ of H to be the unique partition of H which satisfies

(1) for each $v \in V$, $|H(v)|$ equals the degree of the vertex v in G,

(2) for each $v \in V$ and $e \in E$, $H(v) \cap H(e) \neq \emptyset$ if and only if $e \in E(v)$ (in other words, v is incident with e in G),

(3) in the embedding of G on a surface, the half-edges in $H(v)$ meet in a common point (representing v in the embedding).

For an edge $e = \{u, v\}$ not a loop, the intersection $H(v) \cap H(e)$ gives the half-edge which has endpoint v and is half of the edge e. When $e = \{v\}$ is a loop, $H(v) \cap H(e)$ comprises the two halves of the loop.

Two vertices $u, v \in V$ are adjacent in G if and only if there is some $e \in E$ such that $H(u) \cup H(v) \supseteq H(e)$ and r edges $e_1, e_2, \ldots e_r \in E$ contain the cutset $E(v) = \{e \in E : v \in e\}$ of G if and only if $H(e_1) \cup H(e_2) \cup \ldots \cup H(e_r) \supseteq H(v)$.

3.4 Partitioning half-edges of an edge-2-connected graph by faces

We require some preliminary topological definitions in order to define the face partition $\mathcal{F}$ of the half-edge set H of a given edge-2-connected graph G. The partition of the half-edge set of a graph by faces is used in §5.5, where we consider the problem of proper face 3- and 4-colouring bridgeless cubic graphs 2-cell embedded on a surface. See for example [10] for more on topological graph theory.

An embedding of G on a surface $\mathbb{S}$ (a compact Hausdorff topological space locally homeomorphic to $\mathbb{R}^2$) is a 2*-cell embedding* if the connected components of the surface complement $\mathbb{S}\backslash G$ of G are each homeomorphic to an open disk. Only connected graphs have 2-cell embeddings; in what follows we take G to be not only connected but edge-2-connected. Edge-2-connectivity ensures that each edge of G is incident with two distinct faces. An embedding of a connected bridgeless cubic planar graph on a sphere is an example of a 2-cell embedding.

The connected components of $\mathbb{S}\backslash G$ are the *faces* of the 2-cell embedded graph G. A face f of G will as usual be identified with the subset of edges which lie in its closure (edges on its boundary) and the set of faces denoted by F. If a 2-cell embedding of a planar graph G on the sphere $\mathbb{S}$ is punctured at a point in $\mathbb{S}\backslash G$ to obtain an embedding of G on the plane, the face which is punctured is called the *outer face* in this embedding.

The face boundaries $E(f) = \{e \in E : e \text{ lies on the boundary of } f\}$ in a 2-cell embedding of an edge-2-connected graph G together form a cycle double cover of the edges of G. MacLane's Theorem [47] says that an edge-2-connected graph is planar if and only if it has a cycle double cover which spans the cycle space of G. This implies that the faces in a 2-cell embedding of an edge-2-connected graph G span the cycle space of G only when G is planar. By contrast, the cutset double cover $\{E(v) : v \in V\}$ of the edges of any graph G spans the cutset space of G.

The *surface dual* (or *geometric dual*) G^* of an edge-2-connected graph G which has been 2-cell embedded on a surface has vertices the faces of G, and two faces are adjacent if they share a common boundary (the closures of the two faces meet in an edge). If G is embedded in the plane, then the surface dual of G is a plane embedding of the planar dual of G.

We present one way to define a unique face partition $\mathcal{F}$ of the half-edge set H of G by using the edge double cover $\{E(f) : f \in F\}$ by faces in a 2-cell embedding of G on an *orientable* surface, which has the advantage of using the interpretation of half-edges as edges cut across their middle into two halves, as used for the partitions $\mathcal{E}$ and $\mathcal{V}$ by edges and vertices in §3.3. (In Chapter 5, a different definition for $\mathcal{F}$ is given that also applies to 2-cell embedded graphs on non-orientable surfaces; this is achieved by giving a different interpretation to half-edges in the embedding of G.)

Given a 2-cell embedded edge-2-connected graph G on an orientable surface with half-edge set H and partition $\mathcal{E}$ by edges, we define the face partition $\mathcal{F} = \{H(f) : f \in F\}$ of H as the unique partition satisfying the following properties:

(1) for each $f \in F$, $|H(f)|$ equals the size of the face f in G,

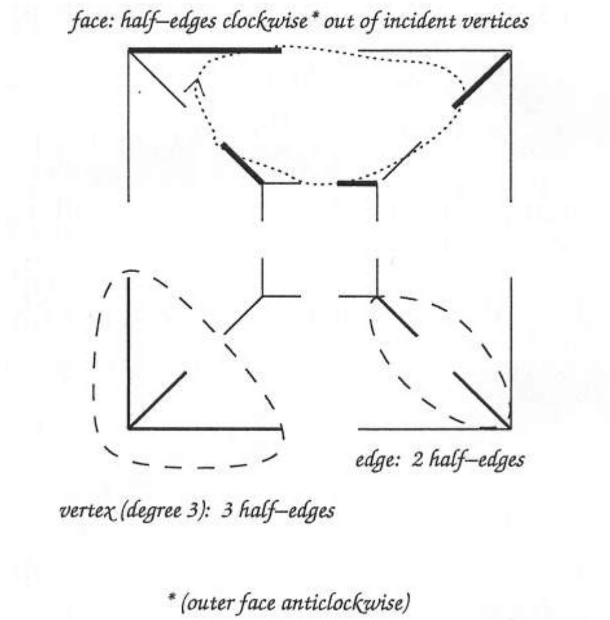


Figure 3.2: Faces as subsets of half-edges obtained by cutting edges across the middle. *The half-edges belonging to a face (1) belong to the edges on the boundary of the face, and (2) when the edges of the face are traversed clockwise (anticlockwise for the outer face) the half-edges belonging to the face are those which are first on each edge.*

(2) for each $f \in F$ and $e \in E$, $H(f) \cap H(e) \neq \emptyset$ if and only if e is on the boundary of f in the embedding of G,

(3) if e is on the boundary of f, a half-edge h belonging to $H(e)$ belongs to $H(f)$ only if it is the first half-edge encountered when traversing the face in a clockwise direction. If the embedding of G is on the plane, then the outer face is traversed in an anticlockwise direction rather than clockwise. (See Figure 3.2 above.)

3.5 Orientations and local vertex rotations

Let G be a graph with half-edge set H and $\mathcal{E} = \{H(e) : e \in E\}$ the partition of H by edges and $\mathcal{V} = \{H(v) : v \in V\}$ the partition of H by vertices.

An arbitrary linear order is put on each block $H(e)$ of $\mathcal{E}$, denoted by $<_{\mathcal{E}}$, and this linear order $<_{\mathcal{E}}$ on each block of $\mathcal{E}$ defines an *orientation* of G. If $e = \{u, v\}$ is an edge in G, and $H(e) = \{a, b\}$ with $a \in H(u)$ and $b \in H(v)$, then say u is directed towards v if $a <_{\mathcal{E}} b$. Thus, the partitions $\mathcal{V}, \mathcal{E}$ and the order $<_{\mathcal{E}}$ on each block of $\mathcal{E}$ determine an orientation of G.

Similarly, a linear order $<_{\mathcal{V}}$ on each block $H(v)$ of $\mathcal{V}$ determines a *local rotation* at each vertex v of G, which is a cyclic order on the set of edges incident with v. Taking the half-edges in $H(v)$ in $<_{\mathcal{V}}$-order and following the last half-edge in $H(v)$ by the first half-edge in $H(v)$ gives a cyclic order on $H(v)$. A cyclic order on the edges incident with v is then defined by taking the edges incident with v in the cyclic order of the corresponding set of half-edges at v: the position of a non-loop edge e incident with v in the cyclic order of edges around v corresponds to the position of the half-edge $H(e) \cap H(v)$ in the cyclic order of $H(v)$. For a loop e incident with v, both half-edges in $H(e)$ belong to $H(v)$ and for definiteness we take the position of e in the cyclic order of edges at v to be the position of the first half-edge of $H(e)$ in the cyclic order on $H(v)$. A set of local vertex rotations for G corresponds to an *orientable embedding* of G. The local vertex rotations describe the order of edges incident with a vertex when taken in a clockwise sense as viewed from a fixed side of the orientable surface on which G is embedded. (See for example [10, 51] for more on the combinatorial definition of embeddings by local vertex rotations and edge signatures.)

Thus, given a graph G with half-edge set H, partition $\mathcal{E}$ by edges and partition $\mathcal{V}$ by vertices, the $<_{\mathcal{V}}$-order on each block of $\mathcal{V}$ determines an orientable embedding of G.

In later chapters we shall only consider the order $<_{\mathcal{V}}$ for a cubic graph G, where all the blocks in the partition $\mathcal{V} = \{H(v) : v \in V\}$ of H by vertices have size 3. There are then three linear orders $<_{\mathcal{V}}$ on a block $H(v)$ which determine the same rotation at v in the embedding of G. The particular linear order which is chosen may be given a graphical interpretation as follows.

Petersen [53] showed that every bridgeless cubic graph G has a 1-*factor* (1-regular spanning subgraph). Given a set of local vertex rotations of a bridgeless cubic graph G and a fixed 1-factor of G, we define a unique linear order on each block of $\mathcal{V}$ as follows:

(1) use the clockwise local vertex rotation of the 3-set $H(v)$ of half-edges at each vertex v to determine a cyclic order on each block $H(v)$ belonging to $\mathcal{V}$,

(2) for each vertex v there is a unique edge e incident with v and which lies on the fixed 1-factor of G. The half-edge $b \in H(e) \cap H(v)$ is put between the two other half-edges in $H(v)$ by the linear order $<_{\mathcal{V}}$.

In this way, not only is the cyclic order of half-edges at each vertex in the given embedding of G encoded by $<_{\mathcal{V}}$, but also the position of the edge which lies on the given 1-factor of G (see Figure 3.3 below).

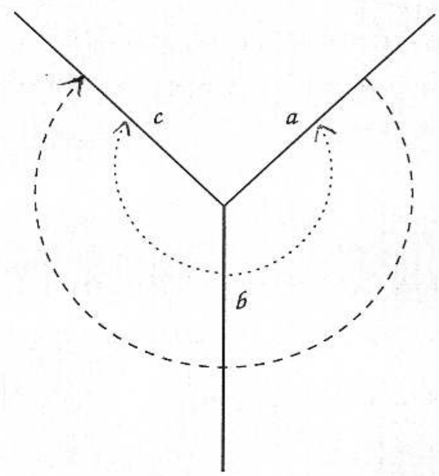


Figure 3.3: A linear order on the three half-edges $\{a, b, c\}$ at a vertex of a bridgeless cubic graph with a fixed 1-factor. *The first half-edge a in* $<_\nu$ *order is in an anticlockwise direction from the second half-edge b which belongs to the edge on the* 1*-factor, and the third half-edge c in a clockwise direction from b.*

3.6 Colouring half-edges

A proper colouring of the vertices or edges of a graph G is independent of any embedding or orientation of G, although the existence of a proper vertex- or edge-colouring does depend on the set of surfaces on which it is possible to embed G. For example, a bridgeless cubic graph has a proper edge 3-colouring if and only if it does not have a Petersen minor [55]; in particular, any bridgeless planar cubic graph has a proper edge 3-colouring (the Four Colour Theorem).

The set H of half-edges of G together with its partitions by edges and vertices have been defined in section §§3.3-3.4 in terms of a fixed embedding of G (and could also have been defined in terms of a fixed orientation of G). The theme of Chapters 4-6 is that colouring the half-edges of G tells us about proper colouring the vertices and edges of G *in terms of the particular surface on which G is embedded (or, in terms of a fixed orientation of G).*

In order to develop this connection, the structure of an additive Abelian group is put on the set of k colours. In the subsequent technical details of this chapter the case of an arbitrary Abelian group of order k follows a similar pattern to the simplest case of assuming the group is cyclic, although there are differences in graphical interpretation according to the choice of group structure (alluded to in the concluding Chapter 7). These differences in graphical interpretation are suggested by the case of nowhere-zero $Z_2 \times Z_2$-flows and nowhere-zero Z_4-flows of a cubic graph. Nowhere-zero $Z_2 \times Z_2$-flows of a cubic graph G are proper edge 3-colourings of G with colours the non-zero elements of $Z_2 \times Z_2$, while nowhere-zero Z_4-flows of G are in bijective correspondence with totally cyclic orientations of those 2-factors of G whose components are all of even size.

Henceforth, we take our set of colours be the additive cyclic group Z_k of order k, for some fixed $k \in \mathbb{N}$. The elements of the set Z_k will be taken to be the integers $0, 1, \ldots, k-1$ modulo k and these integers are put in the linear order $0 < 1 < \cdots < k-1$. Fixing k, the set Z_k^H of *half-edge colourings* of a graph with half-edge set H is the set of all maps $\mu : H \to Z_k$ from the set of half-edges into Z_k. We will refer to a map into Z_k as a "colouring", and, for example, a map $\mu : E \to Z_k$ as a "colouring of E" or an "edge colouring".

In §3.5 we introduced a linear order $<_{\mathcal{E}}$ on each block of the partition $\mathcal{E}$ and a linear order $<_{\mathcal{V}}$ on each block of the partition $\mathcal{V}$. A colouring of the half-edges with elements of Z_k assigns an element of Z_k^2 to each linearly ordered block of $\mathcal{E}$ and, for a cubic graph, an element of Z_k^3 to a linearly ordered block of $\mathcal{V}$. In other words, $Z_k^H \cong (Z_k^2)^{\mathcal{E}}$ when the blocks of $\mathcal{E}$ have been $<_{\mathcal{E}}$-ordered and for a cubic graph we have the isomorphism $Z_k^H \cong (Z_k^3)^{\mathcal{V}}$ when the blocks of $\mathcal{V}$ have been $<_{\mathcal{V}}$-ordered.

Ordered sets (tuples) of elements from $\{0, 1, \ldots, k-1\}$ will be concatenated to form a string, wherever it is possible to make this abbreviation without confusion. Thus the pair $(1, 0)$ is abbreviated to 10 and the triple $(0, 2, 1)$ is abbreviated to 021, while for example $(0, k-1)$ is left as it is. For $r \in \mathbb{N}$, the set of r-tuples on Z_k is denoted by Z_k^r.

For any given $(l_0, l_1, \ldots, l_{r-1}) \in Z_k^r$ we denote by $\underline{(l_0, l_1, \ldots, l_{r-1})}$ the k-set $\{(l_0 + m, l_1 + m, \ldots, l_{r-1} + m) : m \in Z_k\}$ of r-tuples obtained from $(l_0, l_1, \ldots, l_{r-1})$ by cyclically permuting the colour set Z_k. We shall also use the notation $\overline{(l_0, l_1, \ldots, l_{r-1})}$ for the r-set $\{(l_0, l_1, \ldots, l_{r-1}), (l_{r-1}, l_0, l_1, \ldots, l_{r-2}), \ldots, (l_1, l_2, \ldots, l_{r-1}, l_0)\}$ of r-tuples obtained from $(l_0, l_1, \ldots, l_{r-1})$ by cyclically permuting its components.

This notation will be most frequently used for $k = 3$ in Chapters 4-5, where, for example,

$$\underline{00} = \{00, 11, 22\}, \quad \underline{01} = \{01, 12, 20\}, \quad \underline{10} = \{10, 21, 02\},$$

$$\underline{000} = \{000, 111, 222\}, \quad \underline{012} = \{012, 120, 201\} = \overline{012}, \quad \underline{021} = \{021, 102, 210\} = \overline{021},$$

and $\overline{01} = \{01, 10\}$, $\overline{12} = \{12, 21\}$, $\overline{20} = \{20, 02\}$.

Given a partition $\mathcal{S} = \{H(s) : s \in S\}$ of the half-edge set H of a graph G with a fixed linear order on each block of $\mathcal{S}$, the restriction of a half-edge colouring $\mu : H \to Z_k$ to the set $H(s)$ may be regarded interchangeably as a map $H(s) \to Z_k$ or as the ordered set of values taken by μ on $H(s)$. In other words, for a linearly ordered set $H(s)$ we use the isomorphism $Z_k^{H(s)} \cong Z_k^{|H(s)|}$. Only the linear order of the elements in $H(s)$ will be needed, the identity of the elements themselves may be "forgotten", for we are ultimately interested in the properties of G as an unlabelled graph. We write μ_s both for the restriction of μ to $H(s)$ and for the ordered set of colours taken by μ on $H(s)$.

The space of half-edge colourings, and more generally for a set U (such as a block of a partition $\mathcal{S}$ of H, or one of E, V, F) the space of U-colourings Z_k^U, has a naturally defined inner product, for which we introduce the following notation.

Definition 3.6.1 *Let U be a set. The* inner product $\langle\, , \,\rangle$ *of two colourings $\lambda, \mu : U \to Z_k$ is defined by*

$$\langle \lambda, \mu \rangle = \sum_{u \in U} \lambda(u)\mu(u),$$

where the arithmetic is carried out in Z_k.

By taking U in Definition 3.6.1 to be the set of blocks in a partition $\mathcal{S}$ of H, we have in particular that, for any half-edge colourings $\lambda, \mu : H \to Z_k$,

$$\langle \lambda, \mu \rangle = \sum_{s \in S} \sum_{h \in H(s)} \lambda(h)\mu(h) = \sum_{s \in S} \langle \lambda_s, \mu_s \rangle,$$

expressing the inner product of two half-edge colourings in terms of their "local" inner products on the blocks of $\mathcal{S}$.

3.7 Refined and induced colourings

In our notation, Matiyasevich in [45] considers half-edge colourings $\mu : H \to Z_3$ of a cubic graph which have the property that for each block $H(v)$ in the partition $\mathcal{V}$ of H by vertices, μ_v assigns distinct colours to $H(v)$. He then defines that a half-edge colouring μ *induces* a given edge colouring $\lambda : E \to Z_3$ by adding together the two values taken by μ on each block of the partition $\mathcal{E}$ by edges.

This definition of induced colourings generalises to other partitions of H.

Definition 3.7.1 *Let $\mathcal{T} = \{H(t) : t \in T\}$ be any partition of H.*

Given a half-edge colouring $\mu : H \to Z_k$, the colouring $\mu_{\mathcal{T}} : T \to Z_k$ induced *on T by μ is defined for each $t \in T$ by*

$$\mu_{\mathcal{T}}(t) = \sum_{h \in H(t)} \mu(h).$$

A colouring $\mu : T \to Z_k$ is refined *to the colouring $\mu_{\mathcal{H}} : H \to Z_k$, defined for each $h \in H$ by*

$$\mu_{\mathcal{H}}(h) = \mu(t) \qquad \text{if } h \in H(t).$$

We note that, for $\lambda, \mu : T \to Z_k$,

$$\langle \lambda_{\mathcal{H}}, \mu_{\mathcal{H}} \rangle = \sum_{t \in T} |H(t)| \langle \lambda_t, \mu_t \rangle.$$

In particular, for two edge colourings $\lambda, \mu : E \to Z_k$,

$$\langle \lambda_{\mathcal{H}}, \mu_{\mathcal{H}} \rangle = \sum_{e \in E} 2 \langle \lambda_e, \mu_e \rangle = 2 \langle \lambda, \mu \rangle.$$

For $k = 3$ we then have $\langle \lambda_{\mathcal{H}}, \mu_{\mathcal{H}} \rangle = -\langle \lambda, \mu \rangle$, a fact which will be used in Chapter 4.

The definition of induced colourings may be further extended to any pair of partitions $\mathcal{S}, \mathcal{T}$ of H as follows. If $\mu : S \to Z_k$ is an S-colouring, then the T-colouring *induced* by μ is obtained by first refining μ to $\mu_{\mathcal{H}} : H \to Z_k$. This half-edge colouring then induces $\mu_{\mathcal{H}\mathcal{T}} : T \to Z_k$ according to Definition 3.7.1 by setting

$$\mu_{\mathcal{H}\mathcal{T}}(t) = \sum_{h \in H(t)} \mu_{\mathcal{H}}(h) = \sum_{s \in S} \sum_{h \in H(s) \cap H(t)} \mu(s).$$

For example, given a vertex colouring $\mu : V \to Z_k$, the edge colouring induced on E by μ is defined by

$$\begin{aligned} \mu_{\mathcal{H}\mathcal{E}}(e) &= \sum_{v \in V} \sum_{h \in H(v) \cap H(e)} \mu(v) \\ &= \mu(u) + \mu(v), \qquad \text{where } u, v \text{ are the endpoints of } e \text{ in } G. \end{aligned}$$

In Chapters 4-5 we consider half-edge colourings and their induced T-colourings, where T is one of E, V or F. We will be interested in which half-edge colourings induce the same T-colouring, which motivates the following definition:

Definition 3.7.2 *Let $\mathcal{T} = \{H(t) : t \in T\}$ be a partition of the half-edge set H. For a fixed T-colouring $\lambda : T \to Z_k$, the class of half-edge colourings* equivalent to λ *is defined by*

$$[\lambda]_{\mathcal{T}} = \{\mu \in Z_k^H : \mu_{\mathcal{T}} = \lambda\}.$$

The equivalence classes $\{[\lambda]_{\mathcal{T}} : \lambda \in Z_k^T\}$ partition the set Z_k^H of all half-edge colourings, where two half-edge colourings lie in the same class if they induce the same T-colouring.

3.8 Colouring the blocks of a half-edge partition

In this section vertex, edge and face colourings of a graph G are represented as particular types of half-edge colourings, defined in terms of the appropriate partitions of the half-edge set of G. First we begin by defining three special subsets of Z_k^r.

Definition 3.8.1 *A* monochrome *r-tuple is an element of Z_k^r whose components are all the same. A* proper *r-tuple is an element of Z_k^r whose components are distinct. A* null *r-tuple is an element of Z_k^r whose components sum to zero in Z_k.*

The set of monochrome r-tuples in Z_k^r form a subspace of Z_k^r isomorphic to Z_k. The set of k^{r-1} null r-tuples form a subspace of Z_k^r isomorphic to Z_k^{r-1}. For example, for $r = 2$, the null pairs in Z_k^2 are elements in the set $\{(0,0),(1,k-1),(2,k-2),\dots,(k-1,1)\}$.

Permuting the order of the components in an r-tuple preserves the property of being monochrome, proper or null, so that we extend the terms monochrome, proper and null to unordered multisets of r colours. We use the following definition to represent vertex, edge and face colourings of a graph as special types of half-edge colourings.

Definition 3.8.2 *Let $G = (V, E)$ be a graph with half-edge set H and $\mathcal{S} = \{H(s) : s \in S\}$ any partition of H.*

*An $\mathcal{S}$-*monochrome *half-edge colouring is a half-edge colouring $\mu : H \to Z_k$ which is monochrome on each block of $\mathcal{S}$. Similarly, an $\mathcal{S}$-*proper *half-edge colouring is proper on each block of $\mathcal{S}$ and an $\mathcal{S}$-*null *half-edge colouring is null on each block of $\mathcal{S}$.*

Taking $G = (V, E, F)$ to be a graph with half-edge set H, vertex partition $\mathcal{V}$, edge partition $\mathcal{E}$ and, for a given 2-cell embedding of G, face partition $\mathcal{F}$, in the terminology of Definition 3.8.2 we have bijections between the following:

(1) *(proper) vertex k-colourings* and ($\mathcal{E}$-proper) $\mathcal{V}$-monochrome half-edge Z_k-colourings,

(2) *(proper) edge k-colourings* and ($\mathcal{V}$-proper) $\mathcal{E}$-monochrome half-edge Z_k-colourings,

(3) *(proper) face k-colourings* and ($\mathcal{E}$-proper) $\mathcal{F}$-monochrome half-edge Z_k-colourings; alternatively, ($\mathcal{V}$-proper) $\mathcal{F}$-monochrome half-edge Z_k-colourings.

For example, when G is a cubic graph and $k = 3$, the set of proper edge 3-colourings of G is identifiable with the set of half-edge colourings $\mu : H \to Z_3$ with the property that $\mu : \mathcal{E} \to \underline{00}$ and $\mu : \mathcal{V} \to \underline{012} \cup \underline{021}$.

We also have the following correspondences (see for example [18, 13, 35] for definitions and more on flows and tensions):

(4) *Z_k-flows* and $\mathcal{E}$-null $\mathcal{V}$-null half-edge Z_k-colourings,

(5) for plane graphs, *Z_k-tensions* and $\mathcal{E}$-null $\mathcal{F}$-null half-edge Z_k-colourings.

If we call a half-edge colouring $\mu : H \to Z_k$ *nowhere-zero* if $\mu(h) \neq 0$ for all $h \in H$, then we have the usual definition of *nowhere-zero flows* and *nowhere-zero tensions*. Planarity of G is required for our bijection between tensions and $\mathcal{E}$-null $\mathcal{F}$-null half-edge colourings since the faces in an embedding of G span the cycle space of G only for plane graphs.

We finish this section by recording the following proposition (see for example [34, 13]), which relates flows and tensions of a plane graph to its face and vertex colourings.

Proposition 3.8.3 *For a plane graph G with $c(G)$ components,*

(i) every Z_k-tension of G uniquely determines $k^{c(G)}$ vertex Z_k-colourings of G and every nowhere-zero Z_k-tension of G uniquely determines $k^{c(G)}$ proper vertex Z_k-colourings of G,

(ii) every Z_k-flow on G uniquely determines k face Z_k-colourings of G and every nowhere-zero Z_k-flow uniquely determines k proper face Z_k-colourings of G.

3.9 Weights

For a given partition $\mathcal{S}$ of the half-edge set H of a graph G, we introduce some terminology for the set of functions from the space of half-edge colourings Z_k^H into the field of complex numbers $\mathbb{C}$ which look at each half-edge colouring "locally" on each block $H(s)$ of $\mathcal{S}$, giving a value which depends only on the ordered set of colours assigned to $H(s)$.

We have seen in §3.8 that, for example, the colours assigned by a half-edge colouring on the blocks of the partitions $\mathcal{E}$ and $\mathcal{V}$ of H by edges and vertices are determining factors in the representation of proper vertex colourings of G as a subset of the half-edge colourings of G. In the terminology of the following definition, the indicator function of the set of proper vertex k-colourings of G is then the product of a $\mathcal{V}$-weight (indicator function of $\mathcal{V}$-monochrome half-edge colourings) and an $\mathcal{E}$-weight (indicator function of $\mathcal{E}$-proper half-edge colourings).

Definition 3.9.1 *Let G be a graph with half-edge set H and $\mathcal{S} = \{H(s) : s \in S\}$ a partition of H, where each of the sets $H(s)$ have been put in some fixed linear order.*

*An $\mathcal{S}$-*weight *is a function $\gamma^{\mathcal{S}} : Z_k^H \to \mathbb{C}$ which satisfies the following:*

(1) for each $s \in S$, the restriction $\gamma^{(s)}$ of the function $\gamma^{\mathcal{S}}$ to $Z_k^{H(s)}$ depends only on $|H(s)|$,

(2) the function $\gamma^{\mathcal{S}}$ is multiplicative over the blocks of $\mathcal{S}$; in other words,

$$\gamma^{\mathcal{S}}(\mu) = \prod_{s \in S} \gamma^{(s)}(\mu_s).$$

Property (1) says that the "local" value of an $\mathcal{S}$-weight on a block of $\mathcal{S}$ only depends on the size of the block, not on its particular elements. In (2) the map $\mu_s \in Z_k^{H(s)}$ may be identified with its image in $Z_k^{|H(s)|}$, and by property (1) an $\mathcal{S}$-weight is then simply a product of complex-valued functions on Z_k^r for each $r \in \{|H(s)| : s \in S\}$. Letting $\mathcal{S}$ be the coarsest partition $\{H\}$ of H, an $\mathcal{S}$-weight is just an arbitrary function from Z_k^H into $\mathbb{C}$.

An $\mathcal{S}$-weight extends additively to subsets of Z_k^H. In particular, if $\mathcal{T}$ is a partition of H and $[\lambda]_{\mathcal{T}}$ is the equivalence class of half-edge colourings inducing the colouring $\lambda : T \to Z_k$, then

$$\gamma^{\mathcal{S}}([\lambda]_{\mathcal{T}}) = \sum_{\mu \in [\lambda]_{\mathcal{T}}} \gamma^{\mathcal{S}}(\mu).$$

We record two special cases of Definition 3.9.1 which are used in Chapters 4-5.

For an arbitrary graph G with edge partition $\mathcal{E}$ of its half-edge set and each block of $\mathcal{E}$ in a fixed linear order, an $\mathcal{E}$-weight is a function $\gamma : Z_k^2 \to \mathbb{C}$ extending multiplicatively to a function $\gamma^{\mathcal{E}} : (Z_k^2)^{\mathcal{E}} \to \mathbb{C}$ by setting, for $\mu \in Z_k^H$,

$$\gamma^{\mathcal{E}}(\mu) = \prod_{e \in E} \gamma(\mu_e).$$

For a cubic graph G with vertex partition $\mathcal{V}$ of its half-edge set H and each block of $\mathcal{V}$ in a fixed linear order, a $\mathcal{V}$-weight is a function $\gamma : Z_k^3 \to \mathbb{C}$ extending multiplicatively to a function $\gamma^{\mathcal{V}} : (Z_k^3)^{\mathcal{V}} \to \mathbb{C}$ by setting, for $\mu \in Z_k^H$,

$$\gamma^{\mathcal{V}}(\mu) = \prod_{v \in V} \gamma(\mu_v).$$

Example 3.9.2 *The edge weight $\sigma^{\mathcal{E}}$ (used in a different notation by Alon and Tarsi* [6]*) is defined on the set of orientations of a graph given a fixed orientation (determined by the order $<_{\mathcal{E}}$ on each block of $\mathcal{E}$). Orientations of a graph G can be represented as half-edge Z_k-colourings μ with the property that $\mu : \mathcal{E} \to \overline{01}$. (The half-edge coloured* 0 *is the tail and the half-edge coloured* 1 *is the head of the edge in the orientation μ.) Then, in our notation, the edge weight σ of Alon and Tarsi is given for each $e \in E$ by*

$$\sigma(\mu_e) = \begin{cases} +1 & \mu_e = 01, \\ -1 & \mu_e = 10, \\ 0 & \textit{otherwise}. \end{cases}$$

Example 3.9.3 *The vertex weight $\rho^{\mathcal{V}}$ (implicit in Matiyasevich* [43, 45]*) is defined on half-edge Z_3-colourings μ of a cubic graph G with local vertex rotations (determined by* *the order $<_{\mathcal{V}}$ on the blocks of $\mathcal{V}$). The vertex weight ρ is given for each $v \in V$ by*

$$\rho(\mu_v) = \begin{cases} +1 & \mu_v \in \underline{012}, \\ -1 & \mu_v \in \underline{021}, \\ 0 & \textit{otherwise}. \end{cases}$$

3.10 Discrete Fourier transforms

The material of this section is adapted from such accounts of Fourier analysis on the roots of unity as that given in [41], which may be consulted for proofs of those standard results which are only stated here.

Let U be a set (which in practice will be either H or a block in a partition of H) and denote by $\mathbb{C}[Z_k^U]$ the vector space over $\mathbb{C}$ of functions from Z_k^U into $\mathbb{C}$. The vector space $\mathbb{C}[Z_k^U]$ is an inner product space with the inner product (ϕ, ψ) of two functions $\phi, \psi : Z_k^U \to \mathbb{C}$ defined by

$$(\phi, \psi) = \sum_{\mu : U \to Z_k} \phi(\mu)\overline{\psi(\mu)},$$

the bar denoting complex conjugation. Defining, for a function $\phi : Z_k^U \to \mathbb{C}$,

$$\|\phi\|_2^2 = (\phi, \phi) = \sum_{\mu : U \to Z_k} |\phi(\mu)|^2,$$

the ℓ_2-*norm* $\|\phi\|_2$ of ϕ is the nonnegative real number $(\phi, \phi)^{\frac{1}{2}}$. The ℓ_2-norm has the property that $\|\phi\|_2 = 0$ if and only if $\phi = \mathit{0}$, the function taking all zero values.

In §3.7 we defined, for a given partition $\mathcal{T} = \{H(t) : t \in T\}$ of the half-edge set H and a T-colouring $\lambda : T \to Z_k$, the equivalence class of half-edge colourings inducing λ by

$$[\lambda]_{\mathcal{T}} = \{\mu \in Z_k^H : \mu_{\mathcal{T}} = \lambda\},$$

where

$$\mu_{\mathcal{T}}(t) = \sum_{h \in H(t)} \mu(h).$$

Analogously, for given $\phi \in \mathbb{C}[Z_k^H]$, the function $\phi_{\mathcal{T}} \in \mathbb{C}[Z_k^T]$ is defined for any T-colouring $\lambda : T \to Z_k$ by

$$\phi_{\mathcal{T}}(\lambda) = \sum_{\mu \in [\lambda]_{\mathcal{T}}} \phi(\mu) = \phi([\lambda]_{\mathcal{T}}),$$

and for functions $\phi, \psi \in \mathbb{C}[Z_k^H]$ we have the inner product of $\phi_{\mathcal{T}}$ and $\psi_{\mathcal{T}}$ given by

$$(\phi_{\mathcal{T}}, \psi_{\mathcal{T}}) = \sum_{\lambda:T\to Z_k} \phi([\lambda]_{\mathcal{T}})\overline{\psi([\lambda]_{\mathcal{T}})}.$$

In this section we derive expressions for $\phi([\lambda]_{\mathcal{T}}) = \phi_{\mathcal{T}}(\lambda)$ and $(\phi_{\mathcal{T}}, \psi_{\mathcal{T}})$ which will find wide application in Chapters 4-6 when taking $\phi, \psi \in \mathbb{C}[Z_k^H]$ to be $\mathcal{S}$-weights, for $\mathcal{S}$ one of the partitions $\mathcal{E}, \mathcal{V}$ or $\mathcal{F}$ of the half-edge set H of a graph by edges, vertices or faces respectively. In Chapter 4 we show that the coefficients of the graph polynomial modulo the ideal $(x_v^k - 1)$ of $\mathbb{C}[(x_v)]$ are given by $\sigma^{\mathcal{E}}([\lambda]_{\mathcal{V}})$ for vertex colourings $\lambda : V \to Z_k$, where σ is defined as in Example 3.9.2. Similarly the coefficients of the Matiyasevich polynomial of a cubic graph are given by $\rho^{\mathcal{V}}([\lambda]_{\mathcal{E}})$ for edge colourings $\lambda : E \to Z_3$, where ρ is defined as in Example 3.9.3. In Chapter 5 we intepret inner products of the form $(\phi_{\mathcal{T}}, \phi_{\mathcal{T}})$ and $(\phi_{\mathcal{T}}, |\phi|_{\mathcal{T}})$ in terms of a probability distribution on the space of half-edge colourings of a graph, which lead to results analogous to Matiyasevich's [45, 46] probabilistic restatements of the Four Colour Theorem.

The $\mathcal{S}$-weights introduced in §3.9 form a subspace of $\mathbb{C}[Z_k^H]$, equal to the direct sum of the vector spaces $\{\mathbb{C}[Z_k^{H(s)}] : s \in S\}$. This decomposition of the space of $\mathcal{S}$-weights as a direct sum of smaller vector spaces facilitates the calculation of their Fourier transforms (see Lemma 3.10.6 below). In section §3.11 we introduce notation for two bases for the vector space $\mathbb{C}[Z_k^H]$ which helps us to calculate the discrete Fourier transform of $\mathcal{S}$-weights as and when required in Chapters 4-6.

Definition 3.10.1 *Let U be a set, $\phi : Z_k^U \to \mathbb{C}$ a function and $j = e^{2\pi i/k}$. The (discrete Fourier)* transform *operator* $\mathtt{t}$ *is defined for all $\lambda : U \to Z_k$ by*

$$\mathtt{t}\phi(\lambda) = \sum_{\mu:U\to Z_k} \phi(\mu) j^{\langle\lambda,\mu\rangle}.$$

The notation $\mathtt{t}$ for the transform contains its dependence on k and U implicitly in that $\mathtt{t}$ operates on a function $\phi : Z_k^H \to \mathbb{C}$.

Lemma 3.10.2 *For all functions $\phi : Z_k^U \to \mathbb{C}$ and colourings $\lambda : U \to Z_k$,*

$$\mathtt{t}^{-1}\phi(\lambda) = k^{-|U|} \sum_{\mu:U\to Z_k} \phi(\mu) j^{-\langle\lambda,\mu\rangle}.$$

In other words,

$$\mathtt{t}^{-1}\phi(\lambda) = k^{-|U|}\mathtt{t}\phi(-\lambda).$$

Proof. We shall verify that the transform of the right-hand side of the given identity is equal to $\phi(\lambda)$. Since $\mathtt{t}$ is linear, this transform is

$$k^{-|U|} \sum_{\mu:U\to Z_k} \mathtt{t}\phi(\mu) j^{-\langle\lambda,\mu\rangle} = k^{-|U|} \sum_{\mu:U\to Z_k} \sum_{\mu':U\to Z_k} \phi(\mu') j^{\langle\mu,\mu'\rangle-\langle\lambda,\mu\rangle}$$

$$= k^{-|U|} \sum_{\mu':U\to Z_k} (\, \phi(\mu') \sum_{\mu:U\to Z_k} j^{\langle\mu,\mu'-\lambda\rangle}\,)$$

$$= k^{-|U|} k^{|U|} \phi(\lambda) = \phi(\lambda),$$

where we have used the fact that

$$\sum_{\mu:U\to Z_k} j^{\langle\mu,\mu'-\lambda\rangle} = \begin{cases} k^{|U|} & \mu' = \lambda, \\ 0 & \mu' \neq \lambda. \end{cases}$$

(This latter fact is readily verified and is a particular case of Lemma 3.11.4 proved below.) □

For a function $\phi : Z_k^U \to \mathbb{C}$, if we define the operator $\mathtt{s}$ by

$$\mathtt{s}\phi(\lambda) = \phi(-\lambda),$$

for each $\lambda \in Z_k^U$, Lemma 3.10.2 then says that

$$\mathtt{t}^{-1}\phi = k^{-|U|}\mathtt{ts}\phi.$$

We have the following useful *inner product formula*:

Lemma 3.10.3 *For all functions $\phi, \psi : Z_k^U \to \mathbb{C}$,*

$$(\phi, \psi) = k^{-|U|}(\mathtt{t}\phi, \mathtt{t}\psi).$$

Proof. We use the inversion formula of Lemma 3.10.2 to rewrite $\phi = \mathtt{t}^{-1}(\mathtt{t}\phi)$:

$$(\phi, \psi) = \sum_{\mu:U\to Z_k} \phi(\mu)\overline{\psi(\mu)} = k^{-|U|} \sum_{\mu:U\to Z_k} \overline{\psi(\mu)} \sum_{\lambda:U\to Z_k} \mathtt{t}\phi(\lambda) j^{-\langle\lambda,\mu\rangle}$$

$$= k^{-|U|} \sum_{\lambda:U\to Z_k} \mathtt{t}\phi(\lambda) \sum_{\mu:U\to Z_k} \overline{\psi(\mu)} j^{-\langle\lambda,\mu\rangle} \qquad \text{(reversing the order of summation)}$$

$$= k^{-|U|} \sum_{\lambda:U\to Z_k} \mathtt{t}\phi(\lambda)\overline{\mathtt{t}\psi(\lambda)} \qquad \text{(using the linearity of complex conjugation and } \overline{j} = j^{-1}\text{)},$$

$$= k^{-|U|}(\mathtt{t}\phi, \mathtt{t}\psi).$$

□

On taking $\phi = \psi$ in Lemma 3.10.3, the following result often known as *Parseval's Formula* is obtained:

Corollary 3.10.4 *For all functions $\phi : Z_k^U \to \mathbb{C}$,*

$$(\phi, \phi) = k^{-|U|}(\mathtt{t}\phi, \mathtt{t}\phi).$$

(Alternatively, $\|\phi\|_2^2 = k^{-|U|}\|\mathtt{t}\phi\|_2^2$.)

The *dot product* $\phi \cdot \psi$ of two functions $\phi, \psi : Z_k^U \to \mathbb{C}$ is defined by

$$\phi \cdot \psi = \sum_{\mu : U \to Z_k} \phi(\mu)\psi(\mu) = (\phi, \overline{\psi}),$$

and will appear extensively in proving the results of Chapter 6. Another useful corollary of Lemma 3.10.3 which will be much used in this later chapter is the following:

Corollary 3.10.5 *For all functions $\phi, \psi : Z_k^U \to \mathbb{C}$,*

$$\phi \cdot \psi = \mathtt{t}\phi \cdot \mathtt{t}^{-1}\psi.$$

(Alternatively, $\phi \cdot \mathtt{t}\phi = \mathtt{t}\phi \cdot \psi$.)

Proof. By Lemma 3.10.3,

$$\phi \cdot \psi = (\phi, \overline{\psi}) = k^{-|U|}(\mathtt{t}\phi, \mathtt{t}\overline{\psi}).$$

With

$$\begin{aligned} \mathtt{t}\overline{\psi}(\lambda) = \sum_{\mu : U \to Z_k} \overline{\psi(\mu)} j^{\langle \lambda, \mu \rangle} &= \overline{\sum_{\mu : U \to Z_k} \psi(-\mu) j^{\langle \lambda, \mu \rangle}} \\ &= \overline{\mathtt{ts}\psi}(\lambda), \end{aligned}$$

and, by Lemma 3.10.2,

$$\mathtt{t}^{-1}\psi = k^{-|U|}\mathtt{ts}\psi,$$

we then have

$$\phi \cdot \psi = (\mathtt{t}\phi, \overline{\mathtt{t}^{-1}\psi}) = \mathtt{t}\phi \cdot \mathtt{t}^{-1}\psi.$$

□

The final lemma of this section is specific to $\mathcal{S}$-weights. Just as an $\mathcal{S}$-weight is multiplicative over the blocks of $\mathcal{S}$, so too the transform of an $\mathcal{S}$-weight can be found by multiplying the transforms of its restrictions to the blocks of $\mathcal{S}$.

Lemma 3.10.6 *Let $\mathcal{S} = \{H(s) : s \in S\}$ be a partition of H and $\gamma^{\mathcal{S}} : Z_k^H \to \mathbb{C}$ an $\mathcal{S}$-weight. Then, for all $\lambda \in Z_k^H$,*

$$\mathtt{t}\gamma^{\mathcal{S}}(\lambda) = \prod_{s\in S} \mathtt{t}\gamma^{(s)}(\lambda_s).$$

Proof.

$$\mathtt{t}\gamma^{\mathcal{S}}(\lambda) = \sum_{\mu:H\to Z_k} \gamma^{\mathcal{S}}(\mu) j^{\langle\lambda,\mu\rangle}$$

$$= \prod_{s\in S} \sum_{\mu_s:H(s)\to Z_k} \gamma^{(s)}(\mu_s) j^{\langle\lambda_s,\mu_s\rangle} = \prod_{s\in S} \mathtt{t}\gamma^{(s)}(\lambda_s).$$

□

When $\mathcal{S}$ is a partition into blocks of the same size r, an $\mathcal{S}$-weight takes the form

$$\gamma^{\mathcal{S}}(\mu) = \prod_{s\in S} \gamma(\mu_s),$$

where $\gamma : Z_k^r \to \mathbb{C}$ is a fixed function. By Lemma 3.10.6, the transform of γ then determines the transform of the $\mathcal{S}$-weight $\gamma^{\mathcal{S}} : (Z_k^r)^{\mathcal{S}} \to \mathbb{C}$ as follows:

$$\mathtt{t}\gamma^{\mathcal{S}} = (\mathtt{t}\gamma)^{|S|}.$$

This observation enables us to easily calculate the transforms of edge weights (the partition $\mathcal{E}$ into blocks of size 2) on half-edge colourings of arbitrary graphs and, for half-edge colourings of cubic graphs, the transform of vertex weights (the partition $\mathcal{V}$ into blocks of size 3).

3.11 Two bases for weights

In order to calculate the discrete Fourier transform of $\mathcal{S}$-weights, we introduce some notation for two orthogonal bases for the vector space of $\mathcal{S}$-weights. The first is the basis of indicator functions, the second the image of this basis under the Fourier transform.

Definition 3.11.1 *Let $\mathcal{S} = \{H(s) : s \in S\}$ be a partition of H and $\lambda : H \to Z_k$ a half-edge colouring with the property that*

$$\forall s, s' \in S \qquad |H(s)| = |H(s')| \quad \Rightarrow \quad \lambda_s = \lambda_{s'},$$

where λ_s denotes the restriction of λ to the block $H(s)$. The $\mathcal{S}$-weights α_ℓ for $\ell \in Z_k^{|H(s)|}$

are defined on colourings $\mu_s : H(s) \to Z_k$ by

$$\alpha_\ell(\mu_s) = \begin{cases} 1 & \mu_s = \ell, \\ 0 & \mu_s \neq \ell, \end{cases}$$

and extend multiplicatively as $\mathcal{S}$-weights $\alpha_\lambda^{\mathcal{S}}$ on colourings $\mu : H \to Z_k$ to

$$\alpha_\lambda^{\mathcal{S}}(\mu) = \prod_{s \in S} \alpha_{\lambda_s}(\mu_s).$$

Clearly, the set of weights $\{\alpha_\lambda^{\mathcal{S}} : \lambda \in Z_k^H, |H(s)| = |H(s')| \Rightarrow \lambda_s = \lambda_{s'}\}$ forms a basis for the vector space of $\mathcal{S}$-weights. In particular, when $\mathcal{S}$ is a partition of H into blocks of equal size r, the set of $\mathcal{S}$-weights $\{\alpha_\ell^{\mathcal{S}} : \ell \in Z_k^r\}$ form a basis, where

$$\alpha_\ell^{\mathcal{S}}(\mu) = \prod_{s \in S} \alpha_\ell(\mu_s).$$

In the notation of Definition 3.11.1, $\alpha_\ell^{\mathcal{S}} = \alpha_\lambda^{\mathcal{S}}$ where $\lambda_s = \ell$ for each $s \in S$.

For a subset $L \subseteq Z_k^{|H(s)|}$ we write α_L for the weight defined by

$$\alpha_L(\mu_s) = \sum_{\ell \in L} \alpha_\ell(\mu_s) = \begin{cases} 1 & \mu_s \in L, \\ 0 & \mu_s \notin L. \end{cases}$$

In other words, α_L is the indicator function of the subset L. For example, in the partition $\mathcal{E}$ of H into 2-sets, $\alpha_{\underline{00}}$ extends multiplicatively over the blocks of $\mathcal{E}$ to the edge weight $\alpha_{\underline{00}}^{\mathcal{E}}$ which counts just those half-edge colourings $\mu : H \to Z_k$ which are monochrome on each block of $\mathcal{E}$. For a cubic graph and $k = 3$ the vertex weight $\alpha_{\underline{012},\underline{021}}^{\mathcal{V}}$ counts just those half-edge colourings $\mu : H \to Z_3$ which are proper on each block of $\mathcal{V}$.

Alon and Tarsi's edge weight $\sigma^{\mathcal{E}}$ (Example 3.9.2) is given by $\sigma = \alpha_{01} - \alpha_{10}$ on Z_k^2 and extends multiplicatively to a function on $(Z_k^2)^{\mathcal{E}}$. Matiyasevich's vertex weight $\rho^{\mathcal{V}}$ (Example 3.9.3) is given by $\rho = \alpha_{\underline{012}} - \alpha_{\underline{021}}$ on Z_3^3, extended multiplicatively to a function on $(Z_3^3)^{\mathcal{V}}$.

The next definition introduces a second basis for $\mathcal{S}$-weights, which is the image of the first basis of indicator weights under the discrete Fourier transform.

Definition 3.11.2 *For a partition $\mathcal{S}$ of H, the $\mathcal{S}$-weights β_ℓ for $\ell \in Z_k^{|H(s)|}$ are defined on colourings $\mu_s : H(s) \to Z_k$ by*

$$\beta_\ell(\mu_s) = j^{\langle \ell, \mu_s \rangle},$$

where the inner product on the right-hand side is on $Z_k^{|H(s)|}$. These maps extend multi-

plicatively to $\mathcal{S}$-weights on Z_k^H by setting, for $\mu : H \to Z_k$,

$$\beta_\lambda^{\mathcal{S}}(\mu) = \prod_{s \in S} \beta_{\lambda_s}(\mu_s) = j^{\sum_{s\in S}\langle \lambda_s, \mu_s\rangle} = j^{\langle \lambda, \mu\rangle},$$

where $\lambda : H \to Z_k$ has the property that if $H(s), H(s') \in \mathcal{S}$ are of the same size then $\lambda_s = \lambda_{s'}$.

In particular, when $\mathcal{S}$ is a partition of H into blocks of equal size r, the $\mathcal{S}$-weights $\{\beta_\ell^{\mathcal{S}} : \ell \in Z_k^r\}$ form a basis, where

$$\beta_\ell^{\mathcal{S}}(\mu) = \prod_{s \in S} \beta_\ell(\mu_s).$$

In the notation of Definition 3.11.2, $\beta_\ell^{\mathcal{S}} = \beta_\lambda^{\mathcal{S}}$ where $\lambda_s = \ell$ for each $s \in S$.

For $L \subseteq Z_k^r$ we let β_L denote the weight given by

$$\beta_L(\mu_s) = \sum_{\ell \in L} \beta_\ell(\mu_s).$$

Having set up notation, the following proposition records how the transform $\mathtt{t}$ and its inverse $\mathtt{t}^{-1}$ act on the bases $(\alpha_\lambda^{\mathcal{S}})$ and $(\beta_\lambda^{\mathcal{S}})$ for $\mathcal{S}$-weights, and is simply a consequence of the definition of these bases and of the transform $\mathtt{t}$.

Proposition 3.11.3 *Let $\lambda : H \to Z_k$ be such that $\lambda_s = \lambda_{s'}$ whenever $|H(s)| = |H(s')|$, so that $\alpha_{\pm\lambda}^{\mathcal{S}}, \beta_{\pm\lambda}^{\mathcal{S}}$ are $\mathcal{S}$-weights. Then,*

$$\mathtt{t}\alpha_\lambda^{\mathcal{S}} = \beta_\lambda^{\mathcal{S}}, \qquad \mathtt{t}\beta_\lambda^{\mathcal{S}} = k^{|H|}\alpha_{-\lambda}^{\mathcal{S}},$$

and

$$\mathtt{t}^{-1}\alpha_\lambda^{\mathcal{S}} = k^{-|H|}\beta_{-\lambda}^{\mathcal{S}}, \qquad \mathtt{t}^{-1}\beta_\lambda^{\mathcal{S}} = \alpha_\lambda^{\mathcal{S}}.$$

Given a partition $\mathcal{S}$ of the half-edge set H of a graph G with a linear order on each block of $\mathcal{S}$, the set of half-edge colourings Z_k^H is isomorphic to the direct product of rings of the form Z_k^r, where r is the size of a block in $\mathcal{S}$.

The subsets of Z_k^r which are of most interest in Chapter 6 all have the property that their description is independent of k and r. For example, the "monochrome" elements of Z_k^r can be described as those elements whose components are all the same. We use generic names for the following subsets of Z_k^r, some of which have already been introduced in Definition 3.8.1:

$$\text{“zero”} = \{(0,0,\ldots,0)\}, \qquad \text{“all”} = Z_k^r,$$

$$\text{“monochrome”} = \{(l_0, l_1, \ldots, l_{r-1}) \in Z_k^r : l_0 = l_1 = \ldots l_{r-1}\},$$

$$\text{“null”} = \{(l_0, l_1, \ldots, l_{r-1}) \in Z_k^r : l_0 + l_1 + \cdots l_{r-1} = 0\},$$

$$\text{“null no zeroes”} = \{(l_0, l_1, \ldots, l_{r-1}) \in Z_k^r : l_0 + l_1 + \cdots l_{r-1} = 0,\ \ l_0 \neq 0, \ldots, l_{r-1} \neq 0\},$$

and

$$\text{“proper”} = \{(l_0, l_1, \ldots, 1_{r-1}) \in Z_k^r : l_0, l_1, \ldots, l_{r-1} \text{ distinct}\}.$$

Recall that the indicator function $\alpha_M : Z_k^r \to \mathbb{C}$ of a subset M of Z_k^r is defined by $\alpha_M(\ell) = 1$ if $\ell \in M$ and $\alpha_M(\ell) = 0$ otherwise. We define the subgroup $M^\perp$ *orthogonal* to M by

$$M^\perp := \{\ell \in Z_k^r : \forall_{m \in M}\ \langle \ell, m\rangle = 0\}.$$

For example, taking M as the set “monochrome”, $M^\perp$ is the set “null”.

The discrete Fourier transform relates the indicator function of a subgroup M of Z_k^r to the indicator function of the subgroup $M^\perp$ orthogonal to M.

Lemma 3.11.4 *For given $k, r \geq 1$ and any additive subgroup M of Z_k^r,*

$$\mathtt{t}\alpha_M = |M| \alpha_{M^\perp}.$$

Proof. For $j = e^{2\pi i/k}$ and any $\ell \in M^\perp$,

$$\mathtt{t}\alpha_M(\ell) = \sum_{m \in M} j^{\langle \ell, m\rangle} = \sum_{m \in M} 1 = |M|,$$

while if $\ell \notin M^\perp$ there is $m_1 \in M$ such that $\langle \ell, m_1\rangle = d \neq 0$. Then

$$\sum_{m \in M} j^{\langle \ell, m\rangle} = k^{-1} \sum_{m \in M} \sum_{z \in Z_k} j^{\langle \ell, m + z m_1\rangle}$$

$$= k^{-1} \sum_{z \in Z_k} j^{z\langle \ell, m_1\rangle} \sum_{m \in M} j^{\langle \ell, m\rangle},$$

and since $j^{\langle \ell, m_1\rangle} \neq 1$ is a kth root of unity

$$\sum_{z \in Z_k} j^{z\langle \ell, m_1\rangle} = \frac{(j^{\langle \ell, m_1\rangle})^k - 1}{j^{\langle \ell, m_1\rangle} - 1} = 0$$

and the result follows. □

In particular, we have the following relation between the two trivial subgroups of Z_k^r:

$$\mathtt{t}\alpha_{\text{zero}} = \alpha_{\text{all}}, \quad \text{and} \quad \mathtt{t}\alpha_{\text{all}} = k^r \alpha_{\text{zero}},$$

and, of importance to the later results of this thesis,

$$\mathtt{t}\alpha_{\text{monochrome}} = k\alpha_{\text{null}}, \quad \text{and} \quad \mathtt{t}\alpha_{\text{null}} = k^{r-1}\alpha_{\text{monochrome}}.$$

We now extend the scope of Lemma 3.11.4 from subgroups of Z_k^r to subgroups of the set of half-edge colourings Z_k^H of a graph G with half-edge set H. Let $\mathcal{T} = \{H(t) : t \in T\}$ be any partition of H with each block of $\mathcal{T}$ linearly ordered so that we can identify $Z_k^{H(t)}$ with $Z_k^{|H(t)|}$. For each $t \in T$ we let M_t be a subgroup of $Z_k^{|H(t)|}$, subject to the condition that if $|H(t)| = |H(t')|$ then $M_t = M_{t'}$. Defining the subgroup M of Z_k^H to be the direct product of the groups $\{M_t : t \in T\}$, the function $\alpha_M : Z_k^H \to \mathbb{C}$ defined by

$$\alpha_M = \prod_{t \in T} \alpha_{M_t},$$

is a $\mathcal{T}$-weight and is the indicator function of the subgroup M.

Since $-M := \{-m : m \in M\} = M$ for any group M, we have $\alpha_{-M} = \alpha_M$ and the following result by Corollary 3.10.5:

Lemma 3.11.5 *Let $\mathcal{T} = \{H(t) : t \in T\}$ be a partition of H and $\phi : Z_k^H \to \mathbb{C}$. Then for any subgroup M of Z_k^H whose restriction M_t to $Z_k^{H(t)}$ depends only on $|H(t)|$,*

$$\phi \cdot \alpha_M = k^{-|H|}|M|\, \mathtt{t}\phi \cdot \alpha_{M^\perp}.$$

In particular, if each block of $\mathcal{T}$ has size r and N is a subgroup of Z_k^r, then

$$\phi \cdot \alpha_N^{\mathcal{T}} = k^{-|H|}|N|^{|T|}\, \mathtt{t}\phi \cdot \alpha_{N^\perp}^{\mathcal{T}}.$$

Taking N to be the subgroup "monochrome" or "null" in Lemma 3.11.5 we have, for a partition $\mathcal{T}$ of H,

$$\mathtt{t}\alpha^{\mathcal{T}}_{\text{monochrome}} = k^{|T|}\alpha^{\mathcal{T}}_{\text{null}}, \quad \text{and} \quad \mathtt{t}\alpha^{\mathcal{T}}_{\text{null}} = k^{|H|-|T|}\alpha^{\mathcal{T}}_{\text{monochrome}}.$$

3.12 Main lemmas

We finish this chapter with our three main lemmas for use in Chapters 4-6, collected together here for ease of reference.

Lemma A *Let $\mathcal{T} = \{H(t) : t \in T\}$ be a partition of H and $\phi \in \mathbb{C}[Z_k^H]$. For any colouring $\lambda : T \to Z_k$,*

$$\phi([\lambda]_{\mathcal{T}}) = k^{-|T|} \sum_{\mu : T \to Z_k} \mathtt{t}\phi(\mu_{\mathcal{H}}) j^{-\langle \lambda, \mu \rangle}.$$

In particular,

$$\phi([\mathit{0}]_{\mathcal{T}}) = k^{-|T|} \sum_{\mu:T\to Z_k} \mathtt{t}\phi(\mu_{\mathcal{H}}),$$

where 0 denotes the all-zero colouring of T.

Proof. Let a T-colouring $\lambda : T \to Z_k$ be given. Choose any fixed half-edge colouring $\nu \in [\lambda]_{\mathcal{T}}$. Then

$$[\lambda]_{\mathcal{T}} = \nu + [\mathit{0}]_{\mathcal{T}},$$

for if $\mu \in [\lambda]_{\mathcal{T}}$ then, for all $t \in T$,

$$\sum_{h\in H(t)} (\mu - \nu)(h) = 0,$$

which is to say $\mu - \nu \in [\mathit{0}]_{\mathcal{T}}$.

Thus

$$\alpha_{[\lambda]_{\mathcal{T}}} = \alpha_{[\mathit{0}]_{\mathcal{T}}+\nu}$$

and

$$\mathtt{t}^{-1}\alpha_{[\mathit{0}]_{\mathcal{T}}+\nu} = k^{-|H|}\beta_{[\mathit{0}]_{\mathcal{T}}-\nu} = \beta_{[\mathit{0}]_{\mathcal{T}}}\beta_{-\nu}.$$

From Lemma 3.11.5,

$$\beta_{[\mathit{0}]_{\mathcal{T}}} = \beta^{\mathcal{T}}_{\text{null}} = k^{|H|-|T|}\alpha^{\mathcal{T}}_{\text{monochrome}},$$

and we then have by Corollary 3.10.5

$$\phi([\lambda]_{\mathcal{T}}) = \phi \cdot \alpha_{[\lambda]_{\mathcal{T}}} = \mathtt{t}\phi \cdot \mathtt{t}^{-1}\alpha_{[\lambda]_{\mathcal{T}}} = \phi \cdot k^{-|H|+|H|-|T|}\alpha^{\mathcal{T}}_{\text{monochrome}}\beta_{-\nu}.$$

This may be written as

$$\phi([\lambda]_{\mathcal{T}}) = k^{-|T|} \sum_{\mu:T\to Z_k} \beta_{-\nu}(\mu_{\mathcal{H}}) = k^{-|T|} \sum_{\mu:T\to Z_k} j^{-\langle \nu,\mu_{\mathcal{H}}\rangle}$$

and since, for $\mu : T \to Z_k$,

$$\langle \nu, \mu_{\mathcal{H}}\rangle = \sum_{t\in T}\sum_{h\in H(t)} \nu(h)\mu_{\mathcal{H}}(h)$$

$$= \sum_{t\in T} \mu(t) \sum_{h\in H(t)} \nu(h) = \sum_{t\in T} \mu(t)\lambda(t) = \langle \lambda, \mu\rangle,$$

the result now follows.

□

The next lemma is a restatement of Corollary 3.10.5 in a form which will be used in Chapter 6.

Lemma B *For all functions $\phi, \psi : Z_k^H \to \mathbb{C}$,*

$$\phi \cdot \psi = k^{-|H|} \mathtt{t}\phi \cdot \mathtt{ts}\psi,$$

where $\mathtt{s}\psi$ is defined for each $\lambda : H \to Z_k$ by $\mathtt{s}\psi(\lambda) = \psi(-\lambda)$.

Proof. This is Corollary 3.10.5 with $\mathtt{t}^{-1}\psi$ replaced by $k^{-|H|}\mathtt{ts}\psi$ by Lemma 3.10.2.□

Note that if $M = -M$ then $\mathtt{s}\alpha_M = \alpha_M$; in applications of Lemma B the function ψ will often satisfy $\mathtt{s}\psi = \psi$.

Recall that for $\phi \in \mathbb{C}[Z_k^H]$ we defined the function $\phi_{\mathcal{T}} \in \mathbb{C}[Z_k^T]$ for $\lambda : T \to Z_k$ by $\phi_{\mathcal{T}}(\lambda) = \phi([\lambda]_{\mathcal{T}})$. By the inversion formula of Lemma 3.10.2, we have

$$\phi_{\mathcal{T}}(\lambda) = k^{-|T|} \sum_{\mu : T \to Z_k} \mathtt{t}\phi_{\mathcal{T}}(\mu) j^{-\langle \lambda, \mu \rangle}$$

so that Lemma A implies that, for a T-colouring $\mu : T \to Z_k$,

$$\mathtt{t}\phi_{\mathcal{T}}(\mu) = \mathtt{t}\phi(\mu_{\mathcal{H}}).$$

This observation is useful for the proof of the following lemma, which is used in order to obtain many of the probabilistic results of Chapter 5:

Lemma C *Let $\mathcal{T} = \{H(t) : t \in T\}$ be a partition of H and $\phi, \psi \in \mathbb{C}[Z_k^H]$. Then*

$$\sum_{\lambda : T \to Z_k} \phi([\lambda]_{\mathcal{T}}) \overline{\psi([\lambda]_{\mathcal{T}})} = k^{-|T|} \sum_{\mu : T \to Z_k} \mathtt{t}\phi(\mu_{\mathcal{H}}) \overline{\mathtt{t}\phi(\mu_{\mathcal{H}})},$$

where the bar denotes complex conjugation.

In particular,

$$\sum_{\lambda : T \to Z_k} |\phi([\lambda]_{\mathcal{T}})|^2 = k^{-|T|} \sum_{\mu : T \to Z_k} |\mathtt{t}\phi(\mu_{\mathcal{H}})|^2.$$

Proof. We have, by Lemma 3.10.3,

$$(\phi_{\mathcal{T}}, \psi_{\mathcal{T}}) = k^{-|T|} (\mathtt{t}\phi_{\mathcal{T}}, \mathtt{t}\psi_{\mathcal{T}})$$

Using the previously observed identity $\mathtt{t}\phi_{\mathcal{T}}(\mu) = \mathtt{t}\phi(\mu_{\mathcal{H}})$ for functions $\phi \in \mathbb{C}[Z_k^H]$ and $\mu : T \to Z_k$,

$$k^{-|T|} (\mathtt{t}\phi_{\mathcal{T}}, \mathtt{t}\psi_{\mathcal{T}}) = k^{-|T|} \sum_{\mu : T \to Z_k} \mathtt{t}\phi(\mu_{\mathcal{H}}) \overline{\mathtt{t}\psi(\mu_{\mathcal{H}})}.$$

□

Chapter 4

The Matiyasevich polynomial

4.1 Introduction

The primary motivation for this chapter is [43], where Matiyasevich lists, without including his proofs, many interesting properties of his "polynomial related to a triangulation of the sphere". Matiyasevich in [44] gives dual results for the class of 3-connected planar cubic graphs embedded in the sphere. In this chapter we consider cubic graphs rather than surface triangulations, allowing our graphs to be non-planar and with an arbitrary fixed set of local vertex rotations. Included in this chapter are proofs of all the theorems Matiyasevich gives in [43] which are not proved in his related papers [42, 44, 45, 46]. Theorems which are already known are given a reference to the paper(s) in which they are to be found.

The main new results of this chapter concern the coefficients of the Matiyasevich polynomial, in particular Theorems 4.4.4, 4.4.6, 4.4.7, 4.4.9 and 4.5.3. The remaining new material comprises extensions of Matiyasevich's results to non-plane cubic graphs and alternative proofs of some of his results by use of the discrete Fourier transform.

4.2 Defining the Matiyasevich polynomial

Throughout this chapter $G = (V, E)$ will be a cubic graph with a set of local vertex rotations, half-edge set H, edge partition $\mathcal{E}$ of H into blocks of size 2 and vertex partition $\mathcal{V}$ of H into blocks of size 3. Each block of $\mathcal{V}$ is put in one of the three linear orders $<_{\mathcal{V}}$ which are consistent with the fixed set of local vertex rotations of G.

The *line graph* $L(G) = (E, L)$ of G has vertices the edges of G and two vertices of $L(G)$ are adjacent if the corresponding edges of G are incident. Since G is cubic, the line graph $L(G)$ is 4-regular.

For a given set of local vertex rotations of G, we define an orientation of $L(G)$ as follows. For a vertex v of G incident with edges e_0, e_1, e_2 taken in a cyclic order according to the local vertex rotation at v, a totally cyclic orientation is given to the triangle of edges $\{e_0, e_1\}, \{e_1, e_2\}, \{e_2, e_0\}$ in $L(G)$ so that e_0 is directed towards e_1, e_1 towards e_2 and e_2 towards e_0. In other words, following cyclically consecutive edges e_0, e_1, e_2 at a vertex v according to the local vertex rotation scheme of G corresponds to traversing the triangle $\{e_0, e_1\}, \{e_1, e_2\}, \{e_2, e_0\}$ by following the direction around the triangle according to the orientation of $L(G)$. We note that this is an Eulerian orientation of $L(G)$.

The *Matiyasevich polynomial* of the cubic graph G is given by

$$f_3(L(G); (x_e)) = \prod_{e_0 < e_1} (x_{e_1} - x_{e_0}) \mod (x_e^3 - 1),$$

where $(x_e^3 - 1)$ is the ideal generated by the polynomials $\{x_e^3 - 1 : e \in E\}$ and the product is over all directed edges $e_0 < e_1$ (e_0 directed towards e_1) in the orientation of $L(G)$ according to the local vertex rotations of G.

By grouping together in this product three factors which correspond to the directed edges of a triangle $\{e_0, e_1\}, \{e_1, e_2\}, \{e_2, e_0\}$ for which e_0, e_1, e_2 are mutually incident edges in G, we have Matiyasevich's polynomial as presented (for plane embeddings) in [43, 44, 45] and Chapter 1:

$$f_3(L(G); (x_e)) = \prod_{e_0 < e_1 < e_2} (x_{e_0} - x_{e_1})(x_{e_1} - x_{e_2})(x_{e_2} - x_{e_0}) \mod (x_e^3 - 1),$$

where the product is over all 3-sets of mutually incident edges $\{e_0, e_1, e_2\}$ of G where the order $e_0 < e_1 < e_2$ follows the local vertex rotation scheme of G. (The signs of the factors $x_{e_1} - x_{e_0}$ and $x_{e_2} - x_{e_1}$ have been reversed.)

Matiyasevich then interprets the coefficients of $f_3(L(G); (x_e))$ not in terms of the edge orientations $e_0 < e_1$ taken singly, but taken three at a time $e_0 < e_1, e_1 < e_2, e_2 < e_0$ from the triangles $\{e_0, e_1\}, \{e_1, e_2\}, \{e_2, e_0\}$ of $L(G)$. In this way he is able to interpret the coefficients of $f_3(L(G))$ in terms of half-edge colourings of G and the local vertex rotations of G, rather than in terms of orientations of $L(G)$ or Eulerian subgraphs of $L(G)$ (Alon and Tarsi [6, 62]).

We begin this chapter by translating the interpretations of Matiyasevich and Alon and Tarsi of the coefficients of $f(L(G); (x_e))$ into the language of Chapter 3.

Lemma 4.2.1 *Let $G = (V, E)$ be a cubic graph with a set of local vertex rotations. Let G have half-edge set H, partition $\mathcal{E}$ of H by edges and partition $\mathcal{V}$ of H by vertices into blocks of size 3, with blocks of $\mathcal{V}$ linearly ordered consistently with the local vertex rotations of G. Then the Matiyasevich polynomial of G is given by*

$$f_3(L(G); (x_e)) = \sum_{\lambda : E \to Z_3} \rho^{\mathcal{V}}([\lambda]_{\mathcal{E}}) \prod_{e \in E} x_e^{\lambda(e)},$$

where the vertex weight $\rho^{\mathcal{V}} : Z_3^H \to \{-1, 0, +1\}$ on half-edge colourings of G is defined on the restriction of half-edge colourings to blocks of $\mathcal{V}$ by $\rho = \alpha_{\underline{012}} - \alpha_{\underline{021}}$.

Proof. Let (y_h) be commuting indeterminates indexed by the half-edge set H and define, for each $v \in V$,

$$g_v((y_h)) = \sum_{\mu_v : H(v) \to Z_3} \rho(\mu_v) \prod_{h \in H} y_h^{\mu_v(h)}.$$

If $H(v) = \{h_0 <_{\mathcal{V}} h_1 <_{\mathcal{V}} h_2\}$ then

$$g_v((y_h)) = y_{h_0} y_{h_1}^2 + y_{h_1} y_{h_2}^2 + y_{h_2} y_{h_0}^2 - y_{h_0}^2 y_{h_1} - y_{h_1}^2 y_{h_2} - y_{h_2}^2 y_{h_0} = (y_{h_0} - y_{h_1})(y_{h_1} - y_{h_2})(y_{h_2} - y_{h_0}).$$

Since $\mathcal{V} = \{H(v) : v \in V\}$ is a partition of H, we have

$$\prod_{h_0 <_{\mathcal{V}} h_1 <_{\mathcal{V}} h_2} (y_{h_0} - y_{h_1})(y_{h_1} - y_{h_2})(y_{h_2} - y_{h_0}) = \prod_{v \in V} g_v((y_h)) = \sum_{\mu : H \to Z_3} \rho^{\mathcal{V}}(\mu) \prod_{h \in H} y_h^{\mu(h)},$$

where the product is over all triples of half-edges in blocks of $\mathcal{V}$. Given the partition $\mathcal{E} = \{H(e) : e \in E\}$ of H by edges, setting $y_h = x_e$ for each $e \in E$ and $h \in H(e)$ and working modulo the ideal $(x_e^3 - 1)$, the term

$$\rho^{\mathcal{V}}(\mu) \prod_{h \in H} y_h^{\mu(h)}$$

becomes

$$\rho^{\mathcal{V}}(\mu) \prod_{e \in E} x_e^{\mu_{\mathcal{E}}(e)}$$

where $\mu_{\mathcal{E}} : E \to Z_3$ is the edge colouring induced by $\mu : H \to Z_3$. By definition of $[\lambda]_{\mathcal{E}}$ as the set of half-edge colourings inducing the edge colouring λ, this yields

$$\sum_{\lambda : E \to Z_3} \rho^{\mathcal{V}}([\lambda]_{\mathcal{E}}) \prod_{e \in E} x_e^{\lambda(e)} = \prod_{e_0 < e_1 < e_2} (x_{e_0} - x_{e_1})(x_{e_1} - x_{e_2})(x_{e_2} - x_{e_0}) \mod (x_e^3 - 1),$$

where the product is over all 3-sets of edges e_0, e_1, e_2 incident with a common vertex $v \in V$ (a loop appears twice in such a 3-set and contributes a zero factor) and the order $e_0 < e_1 < e_2$ follows the vertex rotation of G. (The order $e_0 < e_1 < e_2$ is inherited from the $<_{\mathcal{V}}$-order of the corresponding half-edges in the subset $H(v)$ of H: $H(e_0) \cap H(v) <_{\mathcal{V}} H(e_1) \cap H(v) <_{\mathcal{V}} H(e_2) \cap H(v)$.) $\square$

Lemma 4.2.1 gives an interpretation of the coefficients of $f(L(G); (x_e))$ in terms of half-edge colourings of G inducing edge colourings of G. We also have an interpretation in terms of half-edge colourings of $L(G)$ inducing vertex colourings of $L(G)$, derived from a more general result of Alon and Tarsi [6] which interprets the coefficients of the graph polynomial of an arbitrary directed graph. For $k \in \mathbb{N}$, the graph polynomial of a directed graph G modulo the ideal generated by the polynomials $\{x_v^k - 1 : v \in V\}$ is defined by

$$f_k(G;(x_v)) = \prod_{v_0<v_1} (x_{v_1} - x_{v_0}) \mod (x_v^k - 1),$$

where the product is over all edges $\{v_0, v_1\}$ with v_0 directed towards v_1 ($v_0 < v_1$).

Lemma 4.2.2 *Let G be a graph with a fixed orientation and $k \in \mathbb{N}$. Let G have half-edge set H, half-edge partition $\mathcal{V}$ by vertices and half-edge partition $\mathcal{E}$ by edges, with each block of $\mathcal{E}$ linearly ordered consistently with the orientation of G. Then the graph polynomial of G modulo the ideal generated by the polynomials $\{x_v^k - 1 : v \in V\}$ is given by*

$$f_k(G;(x_v)) = \sum_{\lambda:V\to Z_k} \sigma^{\mathcal{E}}([\lambda]_{\mathcal{V}}) \prod_{v\in V} x_v^{\lambda(v)},$$

where the edge weight $\sigma^{\mathcal{E}} : Z_k^H \to \{-1, 0, +1\}$ on half-edge colourings of G is defined on half-edge colourings restricted to blocks of $\mathcal{E}$ by $\sigma = \alpha_{01} - \alpha_{10}$.

Proof. Let $\mathcal{V} = \{H(v) : v \in V\}$ be the partition of H by vertices and (y_h) be commuting indeterminates indexed by the half-edge set H and define, for each $e \in E$,

$$g_e((y_h)) = \sum_{\mu_e:H(e)\to Z_k} \sigma(\mu_e) \prod_{h\in H} y_h^{\mu_e(h)}.$$

If $H(e) = \{h_0 <_{\mathcal{E}} h_1\}$ then

$$g_e((y_h)) = y_{h_1} - y_{h_0}.$$

Since $\mathcal{E} = \{H(e) : e \in E\}$ is a partition of H, we have

$$\prod_{h_0<_{\mathcal{E}}h_1} (y_{h_1} - y_{h_0}) = \prod_{e\in E} g_e((y_h)) = \sum_{\mu:H\to Z_k} \sigma^{\mathcal{E}}(\mu) \prod_{h\in H} y_h^{\mu(h)},$$

where the product is over all pairs of half-edges in blocks of $\mathcal{E}$. Given the partition $\mathcal{V} = \{H(v) : v \in V\}$ of H by vertices, setting $y_h = x_v$ for each $v \in V$ and $h \in H(v)$ and working modulo the ideal $(x_v^k - 1)$, the term

$$\sigma^{\mathcal{E}}(\mu) \prod_{h\in H} y_h^{\mu(h)}$$

becomes

$$\sigma^{\mathcal{E}}(\mu) \prod_{v\in V} x_v^{\mu_{\mathcal{V}}(v)}$$

where $\mu_{\mathcal{V}} : V \to Z_k$ is the vertex colouring induced by $\mu : H \to Z_k$. By definition of $[\lambda]_{\mathcal{V}}$

as the set of half-edge colourings inducing the vertex colouring λ, this yields

$$\sum_{\lambda:V\to Z_k} \sigma^{\mathcal{E}}([\lambda]_{\mathcal{V}}) \prod_{v\in V} x_v^{\lambda(v)} = \prod_{v_0<v_1} (x_{v_1} - x_{v_0}) \mod (x_v^k - 1),$$

where the product is over all 2-sets of vertices v_0, v_1 incident with a common edge $e \in E$ (a loop appears twice in such a 2-set and contributes a zero factor) and the order $v_0 < v_1$ follows the orientation of G (v_0 is directed towards v_1). The order $v_0 < v_1$ is inherited from the $<_{\mathcal{E}}$-order of the corresponding half-edges in the subset $H(e)$ of H: $H(v_0) \cap H(e) \ <_{\mathcal{E}}\ H(v_1) \cap H(e)$. □

Lemma 4.2.2 with $k = 3$ applied to the line graph $L(G)$ of a cubic graph G with vertex rotations yields the following alternative description of the Matiyasevich polynomial:

Corollary 4.2.3 *Let G be a cubic graph with a set of local vertex rotations and $L(G) = (E, L)$ its line graph with orientation determined by the set of local vertex rotations of G. Let $L(G)$ have half-edge set $\widetilde{H}$, half-edge partition $\widetilde{\mathcal{E}}$ by the vertices of $L(G)$ into blocks of size 4 and half-edge partition $\mathcal{L}$ by the edges of $L(G)$, with each block of $\mathcal{L}$ linearly ordered consistently with the orientation of $L(G)$. Then the Matiyasevich polynomial of G is given by*

$$f_3(L(G); (x_e)) = \sum_{\lambda:E\to Z_3} \sigma^{\mathcal{L}}([\lambda]_{\widetilde{\mathcal{E}}}) \prod_{e\in E} x_e^{\lambda(e)},$$

where the edge weight $\sigma^{\mathcal{L}} : Z_3^{\widetilde{H}} \to \{-1, 0, +1\}$ on half-edge colourings of $L(G)$ is defined on half-edge colourings restricted to blocks of $\mathcal{L}$ by $\sigma = \alpha_{01} - \alpha_{10}$.

In this chapter we will work with $f_3(L(G))$ as given by Lemma 4.2.1, although in Theorem 4.3.6 the form given by Corollary 4.2.3 is used. We note that changing the set of local vertex rotations of G or changing the orientation of $L(G)$ can only change the sign of $f_3(L(G))$, so that any assertion about $f_3(L(G))$ which also holds for $-f_3(L(G))$ (such as the existence of non-zero coefficients) holds for any choice of local vertex rotations for G and any orientation of its line graph $L(G)$. However, in order that $f_3(L(G))$ is the Matiyasevich polynomial of G, for which we have the interpretation of the coefficients of $f_3(L(G); (x_e))$ as given in Lemma 4.2.1, we shall always stipulate that the orientation of $L(G)$ accords with the set of local vertex rotations of G.

4.3 Evaluations and ℓ_2-norm

A proper edge 3-colouring $\lambda : E \to Z_3$ of G refines to a half-edge colouring $\lambda_{\mathcal{H}} : H \to Z_3$ monochrome on each block of $\mathcal{E}$ and proper on each block of $\mathcal{V}$. We call a vertex v of a cubic graph G with local vertex rotations *anticlockwise* in the proper colouring λ if the colours $0, 1, 2$ assigned by λ to the edges incident with v appear in an opposite rotational sense to the given rotation at v. (Thinking of G with local vertex rotations as an embedding of G on an orientable surface, following the three edges incident with a vertex v in the order of the rotation at v corresponds to a clockwise sense on the surface, and following the edges incident v in the opposite sense corresponds to an anticlockwise sense on the surface.) A proper edge 3-colouring is *even* if it has an even number of anticlockwise vertices and *odd* if this number is odd. An edge colouring λ of G is even if and only if $\rho(\lambda_{\mathcal{H}}) = +1$ and odd if and only if $\rho(\lambda_{\mathcal{H}}) = -1$.

Letting $j = e^{2\pi i/3}$, note that $|j - j^2| = |j^2 - 1| = |1 - j| = \sqrt{3}$. It is clear that the evaluation $f_3(L(G); (j^{\lambda(e)}))$ of the Matiyasevich polynomial at a point $(j^{\lambda(e)})$ is non-zero if and only if $\lambda : E \to Z_3$ is a proper vertex 3-colouring of $L(G)$, and that in this case $|f_3(L(G); (j^{\lambda(e)}))| = \sqrt{3}^{|L|} = 3^{|E|}$. We begin with a proposition which will be useful later when we need to know the sign of a non-zero evaluation of $f_3(L(G); (j^{\lambda(e)}))$, which is given here in terms of the parity of the proper edge 3-colouring λ of G.

In the proof of Proposition 4.3.1 and many other later theorems in this chapter we use the following easily verified transforms of the indicator functions of clockwise $(\underline{012})$ and anticlockwise $(\underline{021})$ half-edge colourings at a vertex of a cubic graph:

$$\mathtt{t}\alpha_{\underline{012}} = \beta_{\underline{012}} = 3(\alpha_{\underline{000}} + j^2\alpha_{\underline{012}} + j\alpha_{\underline{021}}),$$

$$\mathtt{t}\alpha_{\underline{021}} = \beta_{\underline{021}} = 3(\alpha_{\underline{000}} + j\alpha_{\underline{012}} + j^2\alpha_{\underline{021}}).$$

Proposition 4.3.1 *Let $G = (V, E)$ be a cubic graph with local vertex rotations and $L(G)$ its line graph with orientation determined by the local vertex rotations of G.*

For any given $\lambda : E \to Z_3$, the Matiyasevich polynomial has the following evaluation at the point $(j^{\lambda(e)})$:

$$f_3(L(G); (j^{\lambda(e)})) = (-3)^{|E|}\rho^{\mathcal{V}}(\lambda_{\mathcal{H}}),$$

where the half-edge colouring $\lambda_{\mathcal{H}} : H \to Z_3$ refines the edge colouring $\lambda : E \to Z_3$ to a half-edge colouring monochrome on each block of the partition $\mathcal{E}$ of the half-edge set by edges.

Proof. The vertex weight $\rho = \alpha_{\underline{012}} - \alpha_{\underline{021}} : Z_3^3 \to \mathbb{C}$ has transform given by

$$\mathtt{t}\rho = \beta_{\underline{012}} - \beta_{\underline{021}} = 3(j^2 - j)(\alpha_{\underline{012}} - \alpha_{\underline{021}}) = -(-3)^{\frac{3}{2}}\rho.$$

Thus, by Lemma 3.10.6 and with $|V|$ even and $\frac{3}{2}|V| = |E|$, the transform of $\rho^{\mathcal{V}}$ is given by

$$\mathtt{t}\rho^{\mathcal{V}} = (-3)^{|E|}\rho^{\mathcal{V}}.$$

Setting $\phi = \rho^{\mathcal{V}}$, recall from §3.12 that by Lemma A the function $\phi_{\mathcal{E}} : Z_3^E \to \mathbb{C}$ defined by $\phi_{\mathcal{E}}(\lambda) = \phi([\lambda]_{\mathcal{E}})$ has transform given for $\lambda : E \to Z_3$ by $\mathtt{t}\phi_{\mathcal{E}}(\lambda) = \mathtt{t}\phi(\lambda_{\mathcal{H}})$.

Since, by Lemma 4.2.1,

$$f_3(L(G);(x_e)) = \sum_{\mu:E\to Z_3} \phi_{\mathcal{E}}(\mu) \prod_{e\in E} x_e^{\mu(e)},$$

we have

$$\begin{aligned} f_3(L(G);(j^{\lambda(e)})) &= \sum_{\mu:E\to Z_3} \phi_{\mathcal{E}}(\mu) j^{\langle\lambda,\mu\rangle} \\ &= \mathtt{t}\phi_{\mathcal{E}}(\lambda) = \mathtt{t}\phi(\lambda_{\mathcal{H}}). \end{aligned}$$

This gives us the result that

$$f_3(L(G);(j^{\lambda(e)})) = \mathtt{t}\rho^{\mathcal{V}}(\lambda_{\mathcal{H}}) = (-3)^{|E|}\rho^{\mathcal{V}}(\lambda_{\mathcal{H}}).$$

□

For an arbitrary graph $G = (V, E)$ given any orientation, Alon and Tarsi [7, Theorem 1.4] show that $\|f_3(G)\|_2^2$ equals $3^{|E|-|V|}P(G;3)$. Taking the special case of the 4-regular graph $L(G) = (E, L)$ (with orientation determined by the local vertex rotations of G so that $f_3(L(G))$ is the Matiyasevich polynomial of G), Alon and Tarsi's theorem yields the following corollary, which shows that Theorem 1 in [43] does not require the cubic graph G to be planar. Matiyasevich in his recent paper [46] observes this fact independently of Alon and Tarsi's result, drawing on the results of his earlier paper [42].

Theorem 4.3.2 [46] *Let G be a cubic graph with a set of local vertex rotations and $L(G)$ its line graph with orientation determined by the local vertex rotations of G. Then*

$$\|f_3(L(G))\|_2^2 = 3^{|E|}P(L(G);3).$$

The expression for the ℓ_2-norm of the graph polynomial $f_3(G)$ of an arbitrary graph G deduced by Tarsi [62] from his Theorem 1.2, gives, alternatively,

$$\|f_3(L(G))\|_2^2 = \sum_{Z_3\text{-flows of } L(G)} (-2)^{\#\text{zero edges}},$$

where a *zero edge* in a Z_3-flow of $L(G)$ is an edge of the directed graph $L(G)$ to which the value zero is given.[1]

Our second combinatorial interpretation of the ℓ_2-norm of $f_3(L(G))$ is in terms of Z_3-flows of the cubic graph G rather than Z_3-flows of its line graph $L(G)$. We use the definition of Chapter 3 of a Z_3-flow of G as a half-edge Z_3-colouring which is null on each block of $\mathcal{E}$ and null on each block of $\mathcal{V}$. In other words, for a cubic graph G, a

[1] We will always qualify "flow" by "nowhere-zero" when zero values are not permitted, just as we will always qualify "colouring" by "proper" when adjacent/incident colours are to be distinct.

Z_3-flow is a half-edge colouring $\mu : H \to Z_3$ with the property that $\mu : \mathcal{E} \to \{00, \overline{12}\}$ and $\mu : \mathcal{V} \to \{\underline{000}, \underline{012}, \underline{021}\}$. We call a vertex $v \in V$ *monochromatic* in a Z_3-flow μ of G if μ is monochrome on the block $H(v)$ of $\mathcal{V}$. Following Tarsi [62], we interpret a Z_3-flow of G as a partial orientation of G with the property that the indegree is congruent to the outdegree modulo 3. A monochromatic vertex in a given Z_3-flow of G is then a vertex with either all or none of its incident edges directed. Our alternative expression for the ℓ_2-norm of the Matiyasevich polynomial is given by the following theorem.

Theorem 4.3.3 *Let G be a cubic graph with a set of vertex rotations and $L(G)$ its line graph with orientation determined by the local vertex rotations of G. Then*

$$\|f_3(L(G))\|_2^2 = 3^{|V|} \sum_{Z_3\text{-flows of } G} (-2)^{\#\text{ monochromatic vertices}}.$$

Proof. Using the result of Theorem 4.3.2,

$$\|f_3(L(G))\|_2^2 = 3^{|E|} \sum_{\mu : E \to Z_3} \alpha^{\mathcal{V}}_{\underline{012},\underline{021}}(\mu_{\mathcal{H}}).$$

The weight $|\rho|^{\mathcal{V}} = \alpha^{\mathcal{V}}_{\underline{012},\underline{021}}$ has inverse transform given by

$$\begin{aligned} \mathtt{t}^{-1}|\rho|^{\mathcal{V}} = 3^{-|H|}\beta^{\mathcal{V}}_{\underline{012},\underline{021}} &= 3^{-|H|}3^{|V|}(2\alpha_{\underline{000}} - \alpha_{\underline{012},\underline{021}})^{\mathcal{V}} \\ &= 3^{-2|V|}(\alpha_{\underline{012},\underline{021}} - 2\alpha_{\underline{000}})^{\mathcal{V}}, \end{aligned}$$

using $|H| = 3|V|$ and the fact that $\gamma^{\mathcal{V}} = (-\gamma)^{\mathcal{V}}$ for any vertex weight γ since $|V|$ is even.

By Lemma 3.12.C,

$$\begin{aligned} \|f_3(L(G))\|_2^2 &= 3^{2|E|} \cdot 3^{-2|V|}(\alpha_{\underline{012},\underline{210}} - 2\alpha_{\underline{000}})^{\mathcal{V}}([\mathit{0}]_{\mathcal{E}}) \\ &= 3^{|V|} \sum_{\mu : \mathcal{E} \to \{00,\overline{12}\}, \mathcal{V} \to \{\underline{000}, \pm\underline{012}\}} (-2)^{\#\text{monochrome blocks of } \mathcal{V}}, \end{aligned}$$

where we have used the fact that $2|E| = 3|V|$. The result follows. □

Matiyasevich [45] proves that for a cubic graph G embedded in the plane and line graph $L(G)$ with orientation according to the local vertex rotations determined by this plane embedding of G, the constant term of $f_3(L(G))$ equals $P(L(G); 3)$ by using Theorem 4.3.4 below, which he proves by induction on the number of edges of G. [2]

[2] The result of Theorem 4.3.4 was known to Penrose [52], although he states that his proof is too unwieldy to include. Kaufmann [38] gives an elegant proof using the Jordan Curve Theorem for intersecting simple smooth closed curves in the plane. See also Jaeger [35] for a short proof.

Theorem 4.3.4 [52],[38],[45] *Let $G = (V, E)$ be a plane cubic graph and let $\mu : E \to Z_3$ be any proper edge 3-colouring of G. If $\mu_{\mathcal{H}} : H \to Z_3$ is the half-edge colouring refining μ to H, then*

$$\rho^{\mathcal{V}}(\mu_{\mathcal{H}}) = (-1)^{|E|}.$$

In other words, the number of anticlockwise vertices in any proper edge 3-colouring of G has the same parity as $|E|$.

Using Theorem 4.3.4, we obtain the following known theorem, which includes Matiyasevich's evaluation of the constant term of $f_3(L(G); (x_e))$ for a plane cubic graph G as a special case.

Theorem 4.3.5 [43, 11, 21] *Let G be a cubic graph with a set of local vertex rotations and $L(G)$ its line graph with orientation determined by the local vertex rotations of G. Then the constant term of $f_3(L(G))$ is given by*

$$(-1)^{|E|}\,(\ \#\{\text{Even proper edge 3-colourings of } G\} - \#\{\text{Odd proper edge 3-colourings of } G\}\ ).$$

In particular, if G is a plane cubic graph and $L(G)$ its line graph with orientation determined by the local vertex rotations in the plane embedding of G, then the constant term of $f_3(L(G); (x_e))$ equals $P(L(G); 3)$.

Proof. We use the identity of Lemma 4.2.1 for $f_3(L(G))$ which tells us that the constant term is given by $\rho^{\mathcal{V}}([0]_{\mathcal{E}})$. By Lemma 3.12.A,

$$\rho^{\mathcal{V}}([0]_{\mathcal{E}}) = 3^{-|E|} \sum_{\mu:E\to Z_k} \mathtt{t}\rho^{\mathcal{V}}(\mu_{\mathcal{H}}).$$

With $\mathtt{t}\rho^{\mathcal{V}} = (-3)^{|E|}\rho^{\mathcal{V}}$, it follows that

$$\rho^{\mathcal{V}}([0]_{\mathcal{E}}) = (-1)^{|E|} \sum_{\mu:E\to Z_k} \rho^{\mathcal{V}}(\mu_{\mathcal{H}}),$$

and this gives the theorem. If G is embedded in the plane then all proper edge 3-colourings have the same parity, each contributing $(-1)^{|E|}$ to the sum by Theorem 4.3.4. □

It is interesting to compare the result of Theorem 4.3.5 with the following theorem, which is not difficult to deduce from a more general theorem of Alon and Tarsi [6, 62] (proved by using the interpretation of the coefficients of $f_3(L(G))$ in terms of edge orientations of $L(G)$ in Corollary 4.2.3).

Theorem 4.3.6 [6], [62, proof of Theorem 1.2] *Let G be a cubic graph with a set of local vertex rotations and $L(G)$ its line graph with orientation determined by the local vertex* *rotations of* G*. Then the constant term of* $f_3(L(G))$ *is given by*

$$\#\{\text{Even Eulerian subgraphs of } L(G)\} - \#\{\text{Odd Eulerian subgraphs of } L(G)\},$$

where an Eulerian subgraph of the directed graph $L(G)$ *is even or odd according to the parity of its size (number of edges).*

For a cubic graph G which has no proper edge 3-colouring and $L(G)$ its line graph with an arbitrary orientation, Theorem 4.3.5 and Theorem 4.3.6 imply that there are as many even Eulerian subgraphs as odd Eulerian subgraphs of $L(G)$. On the other hand, if G is planar then the difference between the number of even and odd Eulerian subgraphs is equal to $P(L(G); 3)$.

It is easily checked that half of the 12 proper edge colourings of $K_{3,3}$ are even and the other half are odd, so that for any set of local vertex rotations of $K_{3,3}$ and any orientation of $L(K_{3,3})$ the polynomial $f_3(L(K_{3,3}))$ has constant term equal to zero by Theorem 4.3.5. Which graphs share this property with $K_{3,3}$ is still open:

Problem 4.3.7 *For which cubic graphs* G *does the constant term of* $f_3(L(G))$ *equal zero? Equivalently, which cubic graphs have as many even proper edge* 3*-colourings as odd proper edge* 3*-colourings?*

Other than the following, little seems to be known about connected cubic graphs for which the constant term vanishes (but see [22] for another sufficient criterion in terms of "Kempe equivalent" proper edge 3-colourings). All the coefficients of $f_3(L(G))$ are equal to zero when G has no proper edge 3-colouring; by [55] these graphs are precisely those with a bridge or a Petersen minor. The Four Colour Theorem and Theorem 4.3.5 imply that for bridgeless planar graphs the constant term is non-zero. For any cubic graph, if it is given that $P(L(G); 3) \equiv 6 \bmod 12$ then the constant term of $f_3(L(G))$ cannot be zero, for the difference between the number of even and odd proper edge 3-colourings must be an odd multiple of 6 since the same is true of of their sum $P(L(G); 3)$. Finally, for any cubic graph G such as $K_{3,3}$ which has a pair of vertices u, v which are both incident with the same three vertices, the constant term is zero. Given a proper edge 3-colouring of such a graph, a proper edge 3-colouring with number of anticlockwise vertices of opposite parity is obtained by swapping the colours on a pair of edges $\{u, w\}, \{v, w\}$ for each of the three vertices w incident with both u and v. This operation is self-inverse and no proper edge 3-colouring is fixed, so that we may pair off proper edge 3-colourings into even and odd.

4.4 The multiset of coefficients

For any edge colouring $\lambda : E \to Z_3$, the coefficient $[(x_e^{\lambda(e)})]f_3(L(G))$ of the Matiyasevich polynomial of G, which by Lemma 4.2.1 is equal to $\rho^{\mathcal{V}}([\lambda]_{\mathcal{E}})$, is zero if

$$\sum_{e\in E} \lambda(e) = \langle \lambda, \mathit{1} \rangle \neq 0,$$

where $\mathit{1}$ denotes the colouring which is always equal to 1. This is due to the fact that $\mathcal{E}, \mathcal{V}$ are partitions of H and any half-edge colouring $\mu \in [\lambda]_{\mathcal{E}}$ which is proper on each block of $\mathcal{V}$ is also null on each block of $\mathcal{V}$ (the proper triples $\underline{012}, \underline{021}$ have colours which sum to zero modulo 3). Supposing that $\mu \in [\lambda]_{\mathcal{E}}$, we have

$$\sum_{h\in H} \mu(h) = \sum_{e\in E} \sum_{h\in H(e)} \mu(h) = \sum_{e\in E} \lambda(e),$$

while also

$$\sum_{h\in H} \mu(h) = \sum_{v\in V} \sum_{h\in H(v)} \mu(h) = 0.$$

Thus we see that at least $2 \cdot 3^{|E|-1}$ of the $3^{|E|}$ coefficients of $f_3(L(G))$ are zero.

The following lemma is key for proving a succession of theorems which describe in further detail the multiset of coefficients of $f_3(L(G))$.

In the statement of Lemma 4.4.1 we use the expression for $f_3(L(G))$ of Lemma 4.2.1 which tells us that the coefficient $[(x_e^{\lambda(e)})]f_3(L(G);(x_e))$ is equal to $\rho^{\mathcal{V}}([\lambda]_{\mathcal{E}})$, and its notation we use the fact that any weight on the set of half-edge colourings extends additively to a function defined on subsets of half-edge colourings.

Lemma 4.4.1 *Let G be a cubic graph with a set of local vertex rotations and $L(G)$ its line graph with orientation determined by the local vertex rotations of G. Then for any edge colouring $\lambda : E \to Z_3$, the coefficient $[(x_e^{\lambda(e)})]f_3(L(G);(x_e))$ of the Matiyasevich polynomial of G is given by*

$$(-1)^{|E|}\rho^{\mathcal{V}}([\lambda]_{\mathcal{E}}) =$$
$$\rho^{\mathcal{V}}\left(\{\mu_{\mathcal{H}} : \mu \in Z_3^E,\, \langle \lambda, \mu \rangle = 0\}\right) - \frac{1}{2}\rho^{\mathcal{V}}\left(\{\mu_{\mathcal{H}} : \mu \in Z_3^E,\, \langle \lambda, \mu \rangle \neq 0\}\right).$$

Proof.

By Lemma 3.12.A,

$$\rho^{\mathcal{V}}([\lambda]_{\mathcal{E}}) = 3^{-|E|} \sum_{\mu:E\to Z_3} \mathtt{t}\rho^{\mathcal{V}}(\mu_{\mathcal{H}}) j^{-\langle\lambda,\mu\rangle}$$

$$= (-1)^{|E|} \sum_{\mu:E\to Z_3} \rho^{\mathcal{V}}(\mu_{\mathcal{H}}) j^{-\langle\lambda,\mu\rangle}.$$

With

$$\langle \lambda, -\mu \rangle = -\langle \lambda, \mu \rangle, \quad \text{and} \quad \rho^{\mathcal{V}}(-\mu_{\mathcal{H}}) = \rho^{\mathcal{V}}(\mu_{\mathcal{H}}),$$

a pair $\{\mu, -\mu\}$ of edge colourings together contribute 0 as their weight to the sum if μ is not proper, together contribute $2\rho^{\mathcal{V}}(\mu_{\mathcal{H}})$ if $\langle \lambda, \mu \rangle = 0$ and together contribute $(j+j^2)\rho^{\mathcal{V}}(\mu_{\mathcal{H}}) = -\rho^{\mathcal{V}}(\mu_{\mathcal{H}})$ if $\langle \lambda, \mu \rangle \neq 0$.□

Our first consequence of Lemma 4.4.1 is a bound on the size of the coefficients of $f_3(L(G))$. We recall that the parity of a proper edge 3-colouring of a cubic graph embedded in an orientable surface is the parity of the number of anticlockwise vertices.

Theorem 4.4.2 *Let $G = (V, E)$ be a cubic graph with a set of local vertex rotations and $L(G)$ its line graph with orientation determined by the local vertex rotations of G.*

Then the coefficients of $f_3(L(G))$ lie in the interval $[-P(L(G); 3), P(L(G); 3)]$ of $\mathbb{Z}$.

Furthermore, some coefficient equals $P(L(G); 3)$ if and only if all proper edge 3-colourings of G are of the same parity as $|E|$, in which case the constant term is $P(L(G); 3)$ and the minimum coefficient is $-\frac{1}{2}P(L(G); 3)$.

Similarly, some coefficient equals $-P(L(G); 3)$ if and only if all proper edge 3-colourings of G are of opposite parity to $|E|$, in which case the constant term is $-P(L(G); 3)$ and the maximum coefficient is $\frac{1}{2}P(L(G); 3)$.

Proof. We wish to show that

$$\max_{\lambda : E \to Z_3} \rho^{\mathcal{V}}([\lambda]_{\mathcal{E}}) \leq P(L(G); 3),$$

with equality if and only if for each proper edge 3-colouring $\mu : E \to Z_3$ of G the weight $\rho^{\mathcal{V}}(\mu_{\mathcal{H}})$ equals $(-1)^{|E|}$, where μ is refined to a half-edge colouring $\mu_{\mathcal{H}} : \mathcal{E} \to \underline{00}, \mathcal{V} \to \{\underline{012}, \underline{021}\}$. Also, if equality holds, then $\max \rho^{\mathcal{V}}([\lambda]_{\mathcal{E}}) = -2 \min \rho^{\mathcal{V}}([\lambda]_{\mathcal{E}}) = \rho^{\mathcal{V}}([0]_{\mathcal{E}})$.

By Lemma 4.4.1, if $\rho^{\mathcal{V}}(\mu_{\mathcal{H}}) = (-1)^{|E|}$ for each proper edge colouring μ then

$$\rho^{\mathcal{V}}([\lambda]_{\mathcal{E}}) = \left|\{\mu \in Z_3^E : \rho^{\mathcal{V}}(\mu_{\mathcal{H}}) \neq 0, \langle \lambda, \mu \rangle = 0\}\right| - \frac{1}{2}\left|\{\mu \in Z_3^E : \rho^{\mathcal{V}}(\mu_{\mathcal{H}}) \neq 0, \langle \lambda, \mu \rangle \neq 0\}\right|.$$

Certainly $\langle 0, \mu \rangle = 0$ for all $\mu : E \to Z_3$, so that

$$\rho^{\mathcal{V}}([0]_{\mathcal{E}}) = \left|\{\mu \in Z_3^E : \rho^{\mathcal{V}}(\mu_{\mathcal{H}}) \neq 0\}\right| = P(L(G); 3).$$

Remaining under the hypothesis that all proper edge 3-colourings have weight $(-1)^{|E|}$, the minimum value a coefficient can take is

$$-\frac{1}{2}\left|\{\mu \in Z_3^E : \rho^{\mathcal{V}}(\mu_{\mathcal{H}}) \neq 0\}\right| \geq -\frac{1}{2}P(L(G); 3),$$

with equality if and only if there is some colouring $\lambda : E \to Z_3$ for which $\langle \lambda, \mu \rangle \neq 0$ for each proper edge 3-colouring μ. An example of such an edge colouring λ is that which assigns the colour zero to all edges except for a pair of edges e_1, e_2 incident in G, to which it gives the values 1 and 2. For all proper edge colourings $\mu : E \to Z_3$ we have

$$\langle \lambda, \mu \rangle = \lambda(e_1)\mu(e_1) + \lambda(e_2)\mu(e_2) = 1\mu(e_1) + 2\mu(e_2) = \mu(e_1) - \mu(e_2) \neq 0,$$

with $2 = -1$ in Z_3 and $\mu(e_1) \neq \mu(e_2)$ for incident edges e_1, e_2.

It is similarly shown that

$$\min_{\lambda:E\to Z_3} \rho^{\mathcal{V}}([\lambda]_{\mathcal{E}}) \qquad \geq -P(L(G); 3),$$

with equality if and only if all proper edge 3-colourings of G have weight $-(-1)^{|E|}$, in which case $\min \rho^{\mathcal{V}}([\lambda]_{\mathcal{E}}) = -2 \max \rho^{\mathcal{V}}([\lambda]_{\mathcal{E}}) = \rho^{\mathcal{V}}([\mathit{0}]_{\mathcal{E}})$.□

Theorem 4.3.4 says that it is sufficient for a cubic graph G to be planar in order that all proper edge 3-colourings have the same parity for any fixed set of local vertex rotations of G. Thus, for planar graphs one of the bounds of Theorem 4.4.2 is attained and as a corollary we have a theorem of Matiyasevich:

Theorem 4.4.3 [43] *Let G be a plane cubic graph and $L(G)$ its line graph with orientation determined by the local vertex rotations of G. Then $f_3(L(G))$ has largest coefficient equal to its constant term $P(L(G); 3)$ and smallest coefficient equal to $-\frac{1}{2}P(L(G); 3)$.*

Trivially, a non-planar graph with just 6 proper edge 3-colourings also has the property that all proper edge 3-colourings have the same parity, an example of which is given by one of the "generalised Petersen graphs" defined by Watkins [67] (see Figure 4.1 below, and [23, §18]).

A further consequence of Lemma 4.4.1 is the following extension of a result of Matiyasevich [43] for plane cubic graphs:

Theorem 4.4.4 *Let G be a cubic graph with a set of local vertex rotations and $L(G)$ its line graph with orientation determined by the local vertex rotations of G.*

Then every coefficient of $f_3(L(G))$ is divisible by 3 and all its non-zero coefficients are congruent to each other modulo 9.

Proof. Firstly we show that $\rho^{\mathcal{V}}([\lambda]_{\mathcal{E}}) \equiv 0 \mod 3$ for all $\lambda : E \to Z_3$, and secondly that the elements in the set of non-zero coefficients (which by the first observation of this section is contained in $\{\rho^{\mathcal{V}}([\lambda]_{\mathcal{E}}) : \lambda \in Z_3^E,\ \langle \lambda, \mathit{1} \rangle = 0\}$) are congruent to each other modulo 9.

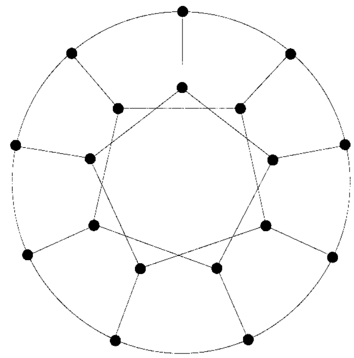


Figure 4.1: A generalised Petersen graph G for which $P(L(G);3) = 6$ is a coefficient of $f_3(L(G))$ in some orientable embedding of G.

Using Lemma 4.4.1, with $-2^{-1} \equiv 1 \bmod 3$ we have

$$\rho^{\mathcal{V}}([\lambda]_{\mathcal{E}}) \equiv \rho^{\mathcal{V}}(\{\mu_{\mathcal{H}} : \mu \in Z_3^E\}) \equiv \rho^{\mathcal{V}}(\underline{00}^{\mathcal{E}}) \mod 3.$$

This is congruent to zero modulo 3, since for $\mu \in \underline{00}^{\mathcal{E}}$,

$$\rho^{\mathcal{V}}(\mu) = \rho^{\mathcal{V}}(\mu + \mathit{1}) = \rho^{\mathcal{V}}(\mu - \mathit{1}).$$

This gives a partition of $\underline{00}^{\mathcal{E}}$ into triples $\{\mu, \mu + \mathit{1}, \mu - \mathit{1}\}$ of half-edge colourings which between them contribute $-3, 0$ or 3 to $\rho^{\mathcal{V}}([\lambda]_{\mathcal{E}})$.

We have, for any proper edge colouring $\mu \in \underline{00}^{\mathcal{E}} \cap \{\underline{012}, \underline{021}\}^{\mathcal{V}}$ and any edge colouring $\lambda \in \underline{00}^{\mathcal{E}}$ satisfying $\langle \lambda, \mathit{1} \rangle = 0$,

$$\langle \lambda, \mu \rangle = \langle \lambda, \mu + \mathit{1} \rangle = \langle \lambda, \mu - \mathit{1} \rangle = -\langle \lambda, -\mu \rangle = -\langle \lambda, -\mu + \mathit{1} \rangle = -\langle \lambda, -\mu - \mathit{1} \rangle,$$

so that the six proper colourings $\{\mu, \mu + \mathit{1}, \mu - \mathit{1}, -\mu, -\mu + \mathit{1}, -\mu - \mathit{1}\}$ either all belong to $\{\mu \in \underline{00}^{\mathcal{E}} : \langle \lambda, \mu \rangle = 0\}$ or all belong to $\{\mu \in \underline{00}^{\mathcal{E}} : \langle \lambda, \mu \rangle \neq 0\}$. Furthermore, $\rho^{\mathcal{V}}(\mu) = \rho^{\mathcal{V}}(\pm\mu \pm \mathit{1})$. This implies that both $\rho^{\mathcal{V}}\left(\{\mu \in \underline{00}^{\mathcal{E}} : \langle \lambda, \mu \rangle = 0\}\right)$ and $\rho^{\mathcal{V}}\left(\{\mu \in \underline{00}^{\mathcal{E}} : \langle \lambda, \mu \rangle \neq 0\}\right)$ are divisible by 6 and thus the difference between any two non-zero coefficients is a multiple of 9 (using the fact that if $a, b \in \mathbb{Z}$ are divisible by 6, then $(a \pm 6) - \frac{1}{2}(b \mp 6) = (a - \frac{1}{2}b) \pm 9$). In other words, all non-zero coefficients must lie in one of the arithmetic progressions $9\mathbb{Z}, 9\mathbb{Z} + 3$ or $9\mathbb{Z} + 6$. $\square$

We have seen that at least $2 \cdot 3^{|E|-1}$ coefficients of $f_3(L(G))$ must be zero. The following is a partial converse:

Theorem 4.4.5 *Let G be a cubic graph with a set of local vertex rotations and $L(G)$ its* *line graph with orientation determined by the local vertex rotations of G. If $f_3(L(G))$ has a coefficient not divisible by 9, then*

$$[(x_e^{\lambda(e)})]f_3(L(G)) \neq 0 \qquad \Leftrightarrow \qquad \langle \lambda, \mathit{1} \rangle = 0.$$

In particular, if G is planar and $P(L(G);3) \not\equiv 0 \bmod 18$, then $f_3(L(G))$ has $3^{|E|-1}$ non-zero coefficients.

Proof. It has already been remarked that it is necessary that $\langle \lambda, \mathit{1} \rangle = 0$ in order for the coefficient $\rho^V([\lambda]_\mathcal{E})$ of $(x_e^{\lambda(e)})$ in $f_3(L(G))$ to be non-zero (with $L(G)$ given the orientation determined by the local vertex rotations of G); the largest number of non-zero coefficients possible is $3^{|E|-1}$.

By Lemma 4.4.1 all non-zero coefficients are congruent modulo 9. Thus, if there is a coefficient not congruent to 0 modulo 9 then all coefficients $\{\rho^V([\lambda]_\mathcal{E}) : \langle \lambda, \mathit{1} \rangle = 0\}$ are not congruent to 0 modulo 9. In particular, they are non-zero as integers.

For the second statement of the theorem, if G is planar then the constant term of $f_3(L(G))$ is $\pm P(L(G);3)$ which is not a multiple of 9 iff $P(L(G);3) \not\equiv 0 \bmod 18$.□

Theorems 4.4.2 and 4.4.4 together say that the coefficients of $f_3(L(G))$ for a cubic graph G with a set of local vertex rotations lie in the interval

$$[-P(L(G);3), P(L(G);3)] = \{m \in \mathbb{Z} : -P(L(G);3) \leq m \leq P(L(G);3)\},$$

and the non-zero coefficient values m all lie in one of the arithmetic progressions $9\mathbb{Z}, 9\mathbb{Z}+3$ or $9\mathbb{Z}+6$ according to the difference between the number of even and odd proper edge 3-colourings modulo 9.

Furthermore, when G is plane and $L(G)$ has the orientation according to the local vertex rotations of G the coefficients lie in

$$\{\, 9\mathbb{Z} + P(L(G);3) \,\} \cap [-\frac{1}{2}P(L(G);3), P(L(G);3)].$$

In the remainder of this section we will explore the distribution of the coefficients of $f_3(L(G))$ amongst the integers in more detail.

We begin by observing that for an arbitrary directed graph G and $k \in \mathbb{N}, j = e^{2\pi i/k}$ the polynomial $f_k(G)$ has coefficient sum $f_k(G;(j^{\mathit{0}(v)}))$ equal to zero since the all-zero colouring is not a proper vertex k-colouring. In particular, for a cubic graph G with any set of local vertex rotations and line graph $L(G)$ with orientation according to the local vertex rotations of G, the sum of the coefficients of $f_3(L(G))$ is zero. Consequently, the distribution of the coefficients of $f_3(L(G))$ has mean zero, although it is not necessarily symmetric about zero. Indeed, if G is planar and has no bridges it has just been seen that the distribution is not symmetric about zero (with $P(L(G);3) \neq 0$ by the Four Colour Theorem).

In order to aid further discussion of the multiset of the coefficients of the Matiyasevich polynomial we introduce the following notation. For a cubic graph G with a set of local vertex rotations and line graph $L(G)$ with orientation according to the local vertex rotations of G, we denote by $N(G;m)$ the number of coefficients in $f_3(L(G))$ which are equal to $m \in \mathbb{Z}$. In other words,

$$N(G;m) = \#\{\lambda \in Z_3^E : \rho^{\mathcal{V}}([\lambda]_{\mathcal{E}}) = m\}.$$

Since there are $3^{|E|}$ edge colourings $\lambda : E \to Z_3$, we have

$$\sum_{m\in\mathbb{Z}} N(G;m) = 3^{|E|},$$

and since the sum of the coefficients of $f_3(L(G))$ is zero we also have

$$\sum_{m\in\mathbb{Z}} mN(G;m) = 0.$$

Theorem 4.3.2 says that

$$\sum_{m\in\mathbb{Z}} m^2 N(G;m) = 3^{|E|}P(L(G);3).$$

These relations on $\{N(G;m) : m \in \mathbb{Z}\}$ imply the following result:

Theorem 4.4.6 *Let G be a cubic graph with a set of local vertex rotations and $L(G)$ its line graph with orientation determined by the local vertex rotations of G.*

If $P(L(G);3) = 6$ then the multiset of coefficients of $f_3(L(G))$ is only a function of the size $|E|$ of G and the set of local vertex rotations (the latter determining the signs of the coefficients).

Proof. Without loss of generality we may take a set of local vertex rotations of G which makes the greatest positive coefficient of $f_3(L(G))$ the maximum coefficient in absolute value. The three independent relations above determine $N(G;6), N(G;0), N(G;-3)$ uniquely. Specifically, $N(G;6) + N(G;0) + N(G;-3) = 3^{|E|}$, $6N(G;6) - 3N(G;-3) = 0$ and $6^2N(G;6) + 3^2N(G;3) = 6 \cdot 3^{|E|}$. This implies that $N(G;6) = 3^{|E|-2}, N(G;-3) = 2 \cdot 3^{|E|-2}$ and $N(G;0) = 2 \cdot 3^{|E|-1}$.□

The further information provided by Theorem 4.4.5 that $N(G;0) = 2\cdot 3^{|E|-1}$ whenever G is planar and $P(L(G);3) \not\equiv 0 \bmod 18$ gives the following:

Theorem 4.4.7 *Let G be a plane cubic graph and $L(G)$ the line graph of G with orientation determined by the local vertex rotations of G.*

If $P(L(G);3) = 12$ then each coefficient of $f_3(L(G))$ belongs to $\{-6,0,3,12\}$ and the multiplicity of any one of these integers in the multiset of coefficients depends only on $|E|$.

Proof. Under the hypotheses of the proposition,

$$N(G;12) + N(G;3) + 2\cdot 3^{|E|-1} + N(G;-6) = 3^{|E|},$$

$$12N(G;12) + 3N(G;3) - 6N(G;-6) = 0,$$

$$12^2 N(G;12) + 3^2 N(G;3) + 6^2 N(G;6) = 12\cdot 3^{|E|}.$$

Solving these equations yields $N(G;12) = 3^{|E|-2}$, $N(G;3) = 4\cdot 3^{|E|-3}$ and $N(G;-6) = 4\cdot 3^{|E|-3}$.$\square$

When $G = K_{3,3}$, where $P(L(G);3) = 12$, it can be checked that $N(G;9) = N(G;-9) = 2\cdot 3^6, N(G;0) = 23\cdot 3^6$, so that there are at least two possibilities for the multiset of coefficients of $f_3(L(G))$ amongst graphs with exactly 12 proper edge 3-colourings.

This prompts the following:

Problem 4.4.8 *Given $P(L(G);3)$, what are the possible coefficient multisets of the Matiyasevich polynomial $f_3(L(G))$ of a cubic graph G? How far does knowing that G is planar limit the possibilities?*

The following proposition puts some limitation on what is possible.

Theorem 4.4.9 *Let G be a connected non-bipartite cubic graph with a set of local vertex rotations and $L(G)$ its line graph with orientation determined by the local vertex rotations of G. Then the number of coefficients of $f_3(L(G))$ which are equal to a given non-zero integer is divisible by $3^{|V|}$.*

Proof. The statement of the proposition is vacuous for graphs with loops or bridges for in these cases there are no proper edge 3-colourings and all coefficients are zero. So assume G has no loops or bridges.

We use the expression for $f_3(L(G))$ of Lemma 4.2.1 which tells us that the coefficients of $f_3(L(G))$ are equal to $\rho([\lambda]_\varepsilon)$ for $\lambda : E \to Z_3$. We wish to show that for any given $\lambda : E \to Z_3$ there are $3^{|V|}$ edge colourings $\kappa : E \to Z_3$ such that $\rho^{\mathcal{V}}([\lambda]_\varepsilon) = \rho^{\mathcal{V}}([\kappa]_\varepsilon)$.

For any $\lambda : E \to Z_3$, Lemma 4.4.1 says that

$$(-1)^{|E|}\rho^{\mathcal{V}}([\lambda]_\varepsilon) =$$

$$\rho^{\mathcal{V}}\left(\{\mu_{\mathcal{H}} : \mu \in Z_3^E,\, \langle\lambda,\mu\rangle = 0\}\right) - \frac{1}{2}\rho^{\mathcal{V}}\left(\{\mu_{\mathcal{H}} : \mu \in Z_3^E,\, \langle\lambda,\mu\rangle \neq 0\}\right),$$

so that it suffices to find $3^{|V|}$ edge colourings κ with the property that $\langle\kappa,\mu\rangle = \pm\langle\lambda,\mu\rangle$ for all proper edge colourings μ. We recall that the refinement of a proper edge colouring μ to a half-edge colouring $\mu_{\mathcal{H}}$ has the property that $\mu_{\mathcal{H}}$ is monochrome on each block of $\mathcal{E}$ and proper on each block of $\mathcal{V}$.

We define another partition $\mathcal{V}' = \{H(v)' : v \in V\}$ of H by setting, for each $v \in V$,

$$H(v)' = \{h \in H(e)\backslash(\,H(e) \cap H(v)\,) : e \in E,\, H(e) \cap H(v) \neq \emptyset\,\}.$$

In other words, for each vertex $v \in V$ the block $H(v)'$ consists of the halves of the edges incident with v which are not the half-edges in $H(v)$. (Note that no vertex of G is incident with a loop by our previous assumption.) Each block $H(v)' = \{h_0', h_1', h_2'\}$ of $\mathcal{V}'$ is put in the order $h_0' < h_1' < h_2'$ so that if $h_0 < h_1 < h_2$ are the half-edges in the corresponding block $H(v)$ of $\mathcal{V}$ then $\{h_0, h_0'\}, \{h_1, h_1'\}, \{h_2, h_2'\}$ are blocks of $\mathcal{E}$. In this way a half-edge colouring whose restrictions to $H(v)$ and $H(v)'$ are the same colouring for each $v \in V$ is monochrome on the blocks of $\mathcal{E}$. For each proper edge colouring $\mu : E \to Z_3$ not only is $\mu_{\mathcal{H}}$ proper on each block of $\mathcal{V}$ but also $\mu_{\mathcal{H}}$ is proper on each block of $\mathcal{V}'$. Conversely, a half-edge colouring monochrome on the blocks of $\mathcal{E}$ and proper on blocks of $\mathcal{V}'$ is also proper on blocks of $\mathcal{V}$.

For each $v \in V$ let $\ell_v \in \underline{000}$ be a monochrome triple and denote by $\ell_v^{\mathcal{V}}$ the half-edge colouring whose restriction to the block $H(v)$ is ℓ_v and denote by $\ell_v^{\mathcal{V}'}$ the half-edge colouring whose restriction to the block $H(v)'$ is also ℓ_v. Then an edge colouring $\kappa : E \to Z_3$ obtained from $\lambda : E \to Z_3$ by adding for each $v \in V$ the three equal colours of ℓ_v to the three *edges* incident with v is given (in terms of refined half-edge colourings) by

$$\kappa_{\mathcal{H}} = \lambda_{\mathcal{H}} + \ell_v^{\mathcal{V}} + \ell_v^{\mathcal{V}'}.$$

For each proper edge colouring μ of G we then have,

$$-\langle\kappa,\mu\rangle = \langle\lambda_{\mathcal{H}} + \ell_v^{\mathcal{V}} + \ell_v^{\mathcal{V}'}, \mu_{\mathcal{H}}\rangle = \langle\lambda_{\mathcal{H}}, \mu_{\mathcal{H}}\rangle + \langle\ell_v^{\mathcal{V}}, \mu_{\mathcal{H}}\rangle + \langle\ell_v^{\mathcal{V}'}, \mu_{\mathcal{H}}\rangle = \langle\lambda_{\mathcal{H}}, \mu_{\mathcal{H}}\rangle = -\langle\lambda,\mu\rangle,$$

with $\mu_{\mathcal{H}}$ null on each block of $\mathcal{V}$ and each block of $\mathcal{V}'$ (since the proper triples $\pm\underline{012}$ are null) and the inner products of null triples with monochrome triples are all zero in Z_3. Thus the coefficient $\rho^{\mathcal{V}}([\lambda]_{\mathcal{E}})$ of $f_3(L(G))$ is equal to the coefficient $\rho^{\mathcal{V}}([\kappa]_{\mathcal{E}})$.

In order to show that the $3^{|V|}$ possible choices for $(\ell_v : v \in V)$ give distinct edge colourings κ as defined above we need to show that if $(m_v : v \in V)$ is another set of $|V|$ monochrome triples then

$$\lambda_{\mathcal{H}} + \ell_v^{\mathcal{V}} + \ell_v^{\mathcal{V}'} \neq \lambda_{\mathcal{H}} + m_v^{\mathcal{V}} + m_v^{\mathcal{V}'}.$$

Supposing $\lambda_{\mathcal{H}} + \ell_v^{\mathcal{V}} + \ell_v^{\mathcal{V}'} = \lambda_{\mathcal{H}} + m_v^{\mathcal{V}} + m_v^{\mathcal{V}'}$, then

$$(\ell_v - m_v)^{\mathcal{V}} = (m_v - \ell_v)^{\mathcal{V}'}.$$

If the partitions $\mathcal{V}$ and $\mathcal{V}'$ share two blocks, so that $H(u) = H(v)', H(u)' = H(v)$ for some vertices $u \neq v$, then this equation is soluble by setting $\ell_u = m_v, \ell_v = m_u$ and colouring all other blocks 000. However, the partitions $\mathcal{V}, \mathcal{V}'$ only have two blocks in common if two vertices $u \neq v$ of G are incident with the same three edges. The only connected cubic graph for which this holds is the bipartite graph with two vertices and three parallel edges.

We claim then that the above equation cannot hold unless the monochrome colouring $\ell_v - m_v$ of the block $H(v)$ and the monochrome colouring $m_v - \ell_v$ of the block $H(v)'$ is the zero colouring for each $v \in V$.

If $u \neq v$ are adjacent vertices incident with a common edge e then

$$H(e) = (\, H(u) \cap H(v)'\,) \cup (\, H(u)' \cap H(v)\,).$$

The blocks $H(u), H(v)'$ receive the same monochrome colouring through the common half-edge in their overlap $H(v) \cap H(u)'$ and similarly the blocks $H(u)', H(v)$ receive the same monochrome colouring. Now $H(v)$ is coloured $\ell_v - m_v$ while $H(u)$ is coloured with its negative $m_v - \ell_v$. Since it is not the case that $H(u)' = H(v)$ and $H(u) = H(v)'$, we may repeat the argument for vertices u, w and onward around a cycle of edges to return to v. Since G is not bipartite, for any vertex v we can find an odd cycle containing v (by assumption G is connected) and this leads to $\ell_v - m_v = 000$. Hence, $\ell_v - m_v$ is the zero colouring on all blocks of $\mathcal{V}$. $\square$

By Proposition 4.4.6 the complete graph $G = K_4$ on four vertices has $N(K_4; 6) = 3^4, N(K_4; 0) = 2 \cdot 3^5$ and $N(K_4; -3) = 2 \cdot 3^4$ so that Theorem 4.4.9 is the best possible result in this direction. In Corollary 4.5.3 below we deduce a similar result for bipartite cubic graphs.

4.5 Contracting triangles

For a fixed cubic graph G with a set of local vertex rotations (which determines an embedding of G on an orientable surface) we have considered the multiplicity of a given integer value amongst the coefficients of $f_3(L(G))$. We now consider the effect on the multiset of coefficients when the graph G is changed.

The operation we shall consider is expanding a vertex of a G into a triangle, thereby obtaining another cubic graph G' embedded in the same surface as G. [3] The operation of

[3] See Duzhin et al. [21, 17] for a consideration of other operations and the interpretation of the Matiyasevich polynomial as an antisymmetric knot invariant satisfying the "IHX" relation.

contracting a triangle and expanding a vertex to a triangle are inverse operations which preserve the number of proper edge 3-colourings of a cubic graph (see for example [23]). It is not difficult to show the following:

Proposition 4.5.1 *Let G be a cubic graph with a set of local vertex rotations and let G' be obtained from G by contracting a triangle. Then*

$$P(L(G');3) = P(L(G);3).$$

Further, if λ' is a proper edge 3-colouring of G' which coincides with the proper edge 3-colouring λ of G restricted to the edges of G', then λ and λ' are of opposite parities.

The operation of contracting a triangle features in the conjecture [23] that every "uniquely edge 3-colourable" cubic planar graph G (i.e. $P(L(G);3) = 6$) can by contracting triangles be reduced to the graph with two vertices and three parallel edges. We have seen in Theorem 4.4.6 that the polynomial $f_3(L(G))$ is up to sign uniquely determined when $P(L(G);3) = 6$, but the example already cited of a non-planar cubic graph with just 6 proper edge 3-colourings shows that we cannot move from the uniqueness of $f_3(L(G))$ to uniqueness of G modulo contracting triangles.

Proposition 4.5.2 below says that the effect on $f_3(L(G))$ of expanding a vertex to a triangle is to enlarge the multiset of coefficients of $f_3(L(G))$ by a factor of 27. This implies that symmetric functions of the coefficients of $f_3(L(G))$ are preserved under the operation of contracting a triangle/expanding a vertex when multiplied by a suitable power of $3^{-|E|}$. For example, we have

$$\|f_3(L(G))\|_2^2 = \sum_{m\in\mathbb{Z}} N(G,m)m^2,$$

and then

$$\|f_3(L(G'))\|_2^2 = \sum_{m\in\mathbb{Z}} N(G',m)m^2 = 27\sum_{m\in\mathbb{Z}} N(G,m)m^2.$$

Thus

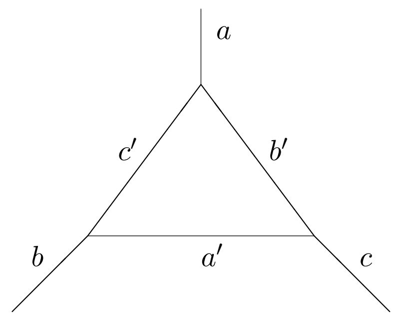
$$3^{-|E|}\sum_{m\in\mathbb{Z}} N(G,m)m^2$$

is preserved under the operation of expanding a vertex to a triangle, since there are 3 extra edges in the graph G' obtained from G. By Theorem 4.3.2 this agrees with the already quoted fact that $P(L(G);3)$ is preserved under contracting triangles.

Another pair of examples of a function symmetric in the coefficients of $f_3(L(G))$ are the maximum and minimum coefficients, and these are invariant under contraction of triangles. For planar G this reduces by Theorem 4.4.3 to the invariance of $P(L(G);3)$, but for non-planar graphs it gives a different invariant under contracting triangles.

Theorem 4.5.2 *Let G be a cubic graph with a set of local vertex rotations and suppose that G' is a cubic graph obtained from G by expanding a vertex v of G into a triangle whose three vertices have the same rotational sense as v. Then $N(G';m) = 27N(G,m)$.*

Proof. Let the vertex of $G = (V,E)$ be adjacent to the three edges $a,b,c \in E$ and let the triangle which expands this vertex have sides $a',b',c' \in E'$ in $G' = (V',E')$ opposite to a,b,c respectively, as illustrated in the diagram below.



Let G' have half-edge set H', partition $\mathcal{E}'$ of H' by edges and partition $\mathcal{V}'$ of H' by vertices. By Proposition 4.5.1, for each proper edge colouring $\mu : E \to Z_3$ of G there is a unique proper edge colouring $\mu' : E' \to Z_3$ of G' which coincides with μ on E. We have $\mu'(a') = \mu(a), \mu'(b') = \mu(b), \mu(c') = \mu(c)$ and μ, μ' are of opposite parity:

$$\rho^{\mathcal{V}}(\mu'_{\mathcal{H}}) = -\rho^{\mathcal{V}'}(\mu'_{\mathcal{H}}).$$

We will show that for each edge colouring $\lambda : E \to Z_3$ there are 27 edge colourings $\lambda' : E' \to Z_3$ uniquely determined by λ for which

$$\langle \lambda', \mu' \rangle = 0 \quad \Leftrightarrow \quad \langle \lambda, \mu \rangle = 0.$$

The result will then follow, since Lemma 4.4.1 says that the coefficients of $f_3(L(G))$ are given by

$$(-1)^{|E|}\rho^{\mathcal{V}}([\lambda]_{\mathcal{E}}) = \rho^{\mathcal{V}}(\{\mu_{\mathcal{H}} : \mu \in Z_3^E,\ \langle \lambda, \mu \rangle = 0\}) - \frac{1}{2}\ \rho^{\mathcal{V}}(\{\mu_{\mathcal{H}} : \mu \in Z_3^E,\ \langle \lambda, \mu \rangle \neq 0\}),$$

and the coefficients of $f_3(L(G'))$ by

$$-(-1)^{|E|}\rho^{\mathcal{V}'}([\lambda']_{\mathcal{E}'}) = \rho^{\mathcal{V}'}(\{\mu'_{\mathcal{H}} : \mu' \in Z_3^{E'},\ \langle \lambda', \mu' \rangle = 0\}) - \frac{1}{2}\ \rho^{\mathcal{V}'}(\{\mu'_{\mathcal{H}} : \mu \in Z_3^{E'},\ \langle \lambda', \mu' \rangle \neq 0\}).$$

In other words, we partition the set $Z_3^{E'}$ of edge colourings of G' into 27-sets of edge colourings $\{C(\lambda) : \lambda \in Z_3^E\}$ with the property that the 27 coefficients $\{\rho^{\mathcal{V}}([\lambda']_{E'}) : \lambda' \in C(\lambda)\}$ of $f_3(L(G'))$ are all equal to the coefficient $\rho^{\mathcal{V}}([\lambda]_{\mathcal{E}})$ of $f_3(L(G))$.

For given $\lambda : E \to Z_3$, set $\lambda' = \lambda$ on $E \backslash \{a, b, c\}$. Define nine possible values for λ' on the three edges a, b, c by setting

$$\lambda'(abc) = \lambda(abc) + \ell'$$

where ℓ' lies in a transversal of the 9 cosets of $\underline{000}$ in the additive group Z_3^3.

Fixing ℓ', define three possible values for λ' on the triangle $a'b'c'$ by putting

$$\lambda'(a'b'c') \in -\ell' + \underline{000}.$$

We claim that this gives a set $C(\lambda) \subseteq Z_3^{E'}$ of 27 possible edge colourings λ' of G' which cannot be obtained from any edge colouring of G other than λ. In other words, if ν is another edge colouring of G, then $C(\lambda) \cap C(\nu) = \emptyset$.

For suppose that there is $\lambda' \in C(\lambda)$ which equals $\nu' \in C(\nu)$. Then $\lambda = \nu$ on $E \backslash \{a, b, c\}$ while $\lambda(abc) \neq \nu(abc)$. Suppose $\nu(abc) = \lambda(abc) + \ell$ for some $000 \neq \ell \in Z_3^3$. For it to be possible that $\lambda' = \nu'$ we must have

$$\lambda'(abc) = \lambda(abc) + \ell', \qquad \nu'(abc) = \nu(abc) + \ell' - \ell,$$

where $\ell', \ell' - \ell$ both lie in the transversal of cosets by $\underline{000}$, and $\ell' \neq \ell' - \ell$ since $\ell \neq 000$. But this implies $\ell' - \ell \notin \ell' + \underline{000}$, which means that $\lambda'(a'b'c') \in -\ell' + \underline{000}$ cannot equal $\nu'(a'b'c') \in \ell - \ell' + \underline{000}$. Thus when $\lambda \neq \nu$ we have $\lambda' \neq \nu'$, even if $\lambda' = \nu'$ on E. This establishes that $C(\lambda) \cap C(\nu) = \emptyset$ if $\lambda \neq \nu$.

For any edge colouring λ' of G' we have

$$\langle \lambda', \mu' \rangle = \langle \lambda, \mu \rangle + \langle (\lambda' - \lambda)(abc), \mu(abc) \rangle + \langle \lambda'(a'b'c'), \mu'(a'b'c') \rangle.$$

If μ' is the unique proper edge 3-colouring extending to G' the proper edge 3-colouring μ of G, then $\mu'(a'b'c') = \mu(abc)$. Also, for $\lambda' \in C(\lambda)$, we have $\lambda'(a'b'c') \in -(\lambda' - \lambda)(abc) + \underline{000}$, so that

$$\langle (\lambda' - \lambda)(abc), \mu(abc) \rangle + \langle \lambda'(a'b'c'), \mu'(a'b'c') \rangle = 0 \qquad (\text{ in } Z_3. \text{ })$$

This shows that we have $\langle \lambda', \mu' \rangle = \langle \lambda, \mu \rangle$ for any edge colouring λ of G and $\lambda' \in C(\lambda)$, all proper edge colourings μ of G and μ' the unique extension of μ to a proper edge colouring of G'.□

We finish this section with a corollary of Proposition 4.5.2 which covers the cases left out by Proposition 4.4.9.

Corollary 4.5.3 *Let G be a connected bipartite cubic graph with a set of local vertex rotations and $L(G)$ its line graph with orientation determined by the local vertex rotations* *of G. Then the the number of coefficients of $f_3(L(G))$ which are equal to a given non-zero integer is divisible by $3^{|V|-1}$.*

Proof. We may assume G has no loops or bridges. Expanding a vertex of G into a triangle to obtain a non-bipartite graph G', the multiplicity $N(G';m)$ of a non-zero integer m is by Theorem 4.5.2 equal to $3^3 N(G;m)$, and this is divisible by $3^{|V|+2}$ by applying Theorem 4.4.9 to G'. This yields the result.□

The plane bipartite graph G with two vertices and three parallel edges has $N(G;6) = 3, N(G;0) = 2{\cdot}3^2, N(G;-3) = 3^2$ so that the result of Corollary 4.5.3 cannot be improved.

It is clear from its definition that $f_3(L(G))$ is multiplicative over the connected components of $L(G)$, which correspond to the connected components of G. Thus Theorem 4.4.9 and Corollary 4.5.3 between them provide conditions on the coefficient multisets for the Matiyasevich polynomial of any cubic graph.

4.6 Coefficients modulo a prime

Theorem 4.4.4 says that the coefficients of $f_3(L(G))$ for a cubic graph G with directed line graph $L(G)$ are all zero modulo 3 and that the non-zero coefficients are either all zero modulo 9 or all non-zero modulo 9. Clearly, if at least one coefficient of $f_3(L(G))$ is non-zero modulo some prime p then at least one coefficient is non-zero in $\mathbb{Z}$. Theorem 4.6.1 below says that for any fixed prime p other than 3 the converse holds: if there is at least one coefficient non-zero in $\mathbb{Z}$ then there is at least one coefficient non-zero modulo p. This extends a result of Matiyasevich [43] given for planar cubic graphs.

Theorem 4.6.1 *Let G be a cubic graph with a set of local vertex rotations, $L(G)$ its line graph with orientation determined by the local vertex rotations of G and let $p \neq 3$ be a prime in $\mathbb{Z}$. Then*

$$f_3(L(G)) \neq 0 \qquad \Leftrightarrow \qquad f_3(L(G)) \not\equiv 0 \bmod\ p.$$

In other words, $f_3(L(G))$ is non-zero if and only if it has a coefficient not divisible by p.

Proof. With $f_3(L(G)) \neq 0$ if and only if $\|f_3(L(G))\|_2^2 \neq 0$, it follows by Theorem 4.3.2 that $f_3(L(G)) \neq 0$ if and only if there is an edge colouring $\lambda : E \to Z_3$ such that $\rho^{\mathcal{V}}(\lambda_{\mathcal{H}}) \neq 0$. By Proposition 4.3.1, $f_3(L(G);(j^{\lambda(e)})) = (-3)^{|E|}\rho^{\mathcal{V}}(\lambda_{\mathcal{H}})$, which is non-zero modulo p for any prime $p \neq 3$ provided $\rho^{\mathcal{V}}(\lambda_{\mathcal{H}}) \neq 0$, since $\rho^{\mathcal{V}}(\lambda_{\mathcal{H}}) \in \{-1, 0, +1\}$. This implies $f_3(L(G)) \not\equiv 0 \mod p$.□

Theorem 4.6.1 with $p = 2$ yields the following corollary, in which by a *proper half-edge colouring* of G is meant a $\mathcal{V}$-proper half-edge colouring (proper on each block of the partition of half-edges by vertices).

Corollary 4.6.2 *Let G be a cubic graph with a set of local vertex rotations and $L(G)$ its line graph with orientation determined by the local vertex rotations of G. Then $f_3(L(G))$ is non-identically zero if and only if there exists an edge colouring of G which is induced by an odd number of proper half-edge colourings of G.*

We remark that if $P(L(G);3) > 0$ then the edge colourings which are induced by an odd number of half-edge colourings may not include any of the proper edge 3-colourings themselves. For example, the graph $K_2 \times K_3$ obtained from K_4 by expanding one vertex into a triangle has the property that all proper edge colourings are induced by exactly 6 proper half-edge colourings (all of the same parity).

Theorem 4.6.3 below sharpens Corollary 4.6.2 when G is planar and is the final theorem of the chapter, for which we give a proof since Matiyasevich has only published its statement.

Theorem 4.6.3 [43] *Let $G = (V, E)$ be a plane cubic graph. Then there are an even number of proper half-edge 3-colourings of G which induce a given edge colouring of G and whose number of anticlockwise vertices is of opposite parity to $|E|$.*

Proof. The statement of the theorem is equivalent to asserting that, for all edge colourings $\lambda : E \to Z_3$,

$$(|\rho|^{\mathcal{V}} - (-1)^{|E|}\rho^{\mathcal{V}})([\lambda]_{\mathcal{E}}) \equiv 0 \bmod 4,$$

where $|\rho| = \alpha_{\underline{012},\underline{021}}$.

By Lemma 3.12.A, for any function $\phi : Z_3^H \to \mathbb{C}$,

$$3^{|E|}\phi([\lambda]_{\mathcal{E}}) = \sum_{\mu:E\to Z_3} \mathtt{t}\phi(\mu_{\mathcal{H}})j^{-\langle\lambda,\mu\rangle}.$$

If $\phi(-\mu_{\mathcal{H}}) = \phi(\mu_{\mathcal{H}})$ for all edge colourings $\mu : E \to Z_3$ then this gives

$$3^{|E|}\phi([\lambda]_{\mathcal{E}}) = \mathtt{t}\phi(\{\mu_{\mathcal{H}} : \mu \in Z_3^E,\, \langle\lambda,\mu\rangle = 0\}) - \frac{1}{2}\mathtt{t}\phi(\{\mu_{\mathcal{H}} : \mu \in Z_3^E,\, \langle\lambda,\mu\rangle \neq 0\}).$$

The condition $\phi(-\mu_{\mathcal{H}}) = \phi(\mu_{\mathcal{H}})$ is satisfied by $\phi = (|\rho|^{\mathcal{V}} - (-1)^{|E|}\rho^{\mathcal{V}})$, so we will use the above identity for $\phi([\lambda]_{\mathcal{E}})$ in terms of the transform $\mathtt{t}\phi$.

We have

$$\mathtt{t}|\rho|^{\mathcal{V}}(\mu_{\mathcal{H}}) = (6\alpha_{\underline{000}} - 3\alpha_{\underline{012},\underline{021}})^{\mathcal{V}}(\mu_{\mathcal{H}}),$$

and by Theorem 4.3.4, since $\mu_{\mathcal{H}}$ is monochrome on blocks of $\mathcal{E}$,

$$\mathtt{t}\rho^{\mathcal{V}}(\mu_{\mathcal{H}}) = (-3)^{|E|}\rho^{\mathcal{V}}(\mu_{\mathcal{H}}) = 3^{|E|}|\rho|^{\mathcal{V}}(\mu_{\mathcal{H}}).$$

Then, for all edge colourings $\mu : E \to Z_3$,

$$\mathtt{t}(|\rho|^{\mathcal{V}} - (-1)^{|E|}\rho^{\mathcal{V}})(\mu_{\mathcal{H}}) = \mathtt{t}|\rho|^{\mathcal{V}}(\mu_{\mathcal{H}}) - (-1)^{|E|}\mathtt{t}\rho^{\mathcal{V}}(\mu_{\mathcal{H}})$$

$$= 3^{|V|}(2\alpha_{\underline{000}} - |\rho|)^{\mathcal{V}}(\mu_{\mathcal{H}}) - (-1)^{|E|}3^{|E|}|\rho|^{\mathcal{V}}(\mu_{\mathcal{H}})$$

$$\equiv (2\alpha_{\underline{000}} - |\rho|)^{\mathcal{V}}(\mu_{\mathcal{H}}) - |\rho|^{\mathcal{V}}(\mu_{\mathcal{H}}) \mod 4.$$

For a given edge colouring $\mu \in Z_3^E$, we write $\#\underline{000}$ for the number of vertices whose incident edges are all of the same colour (monochrome vertices) and $\# \pm \underline{012}$ for the number of vertices whose incident edges are of distinct colours (proper vertices). By the previous and Lemma 4.4.1 we have

$$(-1)^{|E|}(\rho^{\mathcal{V}} - (-1)^{|E|}|\rho|^{\mathcal{V}})([\lambda]_{\mathcal{E}}) \equiv$$

$$-2\#\{\mu \in Z_3^E : \#\underline{000} = 1, \# \pm \underline{012} = |V| - 1, \langle\lambda, \mu\rangle = 0\}$$

$$-\#\{\mu \in Z_3^E : \#\underline{000} = 1, \# \pm \underline{012} = |V| - 1, \langle\lambda, \mu\rangle \neq 0\}$$

$$+2\#\{\mu \in Z_3^E : \#\underline{000} = 2, \# \pm \underline{012} = |V| - 2, \langle\lambda, \mu\rangle \neq 0\} \mod 4.$$

We will first show that for any cubic graph G,

$$\#\{\mu \in Z_3^E : \#\underline{000} = 1, \# \pm \underline{012} = |V| - 1\} = 0.$$

For suppose an edge colouring $\mu : E \to Z_3$ has one monochrome vertex, and the rest are proper. Without loss of generality, suppose the edges incident with the monochrome vertex are coloured 0. Deleting edges coloured 1 then leaves a subgraph of G, where vertices of G which received distinct colours now all have degree 2 and the one monochrome vertex remains of degree 3. This is impossible since the number of vertices of odd degree in any graph must be even. This establishes that there cannot be just one monochrome vertex when the rest are proper.

We now have

$$(-1)^{|E|}(\rho^{\mathcal{V}} - (-1)^{|E|}|\rho|^{\mathcal{V}})([\lambda]_{\mathcal{E}}) \equiv$$

$$2\#\{\{\mu \in Z_3^E : \#\underline{000} = 2, \# \pm \underline{012} = |V| - 2, \langle\lambda, \mu\rangle \neq 0\} \mod 4.$$

This is congruent to zero modulo 4, for if an edge colouring $\mu \in Z_3^E$ is such that there are two monochrome vertices and the rest are proper, then this property is shared by $-\mu$.

Further, $\langle\lambda,\mu\rangle \neq 0 \;\Leftrightarrow\; \langle\lambda,-\mu\rangle \neq 0$. Hence the set $\{\mu \in Z_3^E : \#\underline{000} = 2, \# \pm \underline{012} = |V| - 2, \langle\lambda,\mu\rangle \neq 0\}$ partitions into pairs $\{\mu,-\mu\}$ and is of even size. (If $\mu = -\mu$ then $\langle\lambda,\mu\rangle = 0$.) $\square$

Chapter 5

Probabilistic interpretations

5.1 Introduction

In this chapter we will consider a wide variety of probability distributions on the set of half-edge colourings of a graph, developing the approach taken by Matiyasevich [45] to give his probabilistic restatements of the Four Colour Theorem. We find all the probability distributions which lead to criteria for the existence of proper edge 3-colourings of an arbitrary cubic graph with a set of local vertex rotations in terms of the correlation between two naturally defined events. In §5.2 a generally applicable definition of the two key events of *same parity* and *same induced colouring* is given for any graph and any two partitions of its half-edge set. This allows us in §5.3 to derive criteria for proper edge 3-colouring a cubic graph in terms of a given set of local vertex rotations and allows us in §5.4 to deduce analogous criteria for proper vertex 3-colouring an arbitrary graph in terms of a given orientation of its edges.

In §5.5 we deduce from §5.3 criteria for proper face 3-colouring 2-cell embedded bridgeless cubic graphs (Theorem 5.5.1) and in Theorem 5.5.2 describe a criterion for the existence of proper face 4-colourings.

The main new results on edge 3-colouring cubic graphs in §5.3 are Theorems 5.3.1, 5.3.2, and 5.3.3 and Theorems 5.3.11 and 5.3.13. In §5.4 the main new results on vertex 3-colouring arbitrary graphs are Theorems 5.4.2, 5.4.4, 5.4.5 and 5.4.6.

5.2 Parity

Let G be an arbitrary graph with half-edge set H and $\mathcal{S} = \{H(s) : s \in S\}$, $\mathcal{T} = \{H(t) : t \in T\}$ two fixed partitions of H. The half-edges in each block of $\mathcal{S}$ are put in an arbitrary fixed linear order. The half-edges in a block of $\mathcal{T}$ do not need to be linearly ordered. In §§5.3-5.5, $\mathcal{S}$ will be either the partition $\mathcal{V}$ of H by vertices or the partition $\mathcal{E}$ of H by edges and $\mathcal{T}$ will be either $\mathcal{E}, \mathcal{V}$ or, when G is a bridgeless cubic graph 2-cell embedded on a surface, the partition $\mathcal{F}$ of H by faces.

An $\mathcal{S}$-weight $\gamma^{\mathcal{S}} : Z_k^H \to \mathbb{C}$ extends additively to subsets of Z_k^H. In particular, the $\mathcal{S}$-weight $|\gamma^{\mathcal{S}}|$ maps the set of half-edge colourings into the nonnegative reals $\mathbb{R}_+ \cup \{0\}$ and is a finite measure on Z_k^H. By scaling we can make the absolute value of any $\mathcal{S}$-weight into a probability measure, for which we introduce the terminology of the following definition.

Definition 5.2.1 *A* probability $\mathcal{S}$-weight *is an $\mathcal{S}$-weight $\gamma^{\mathcal{S}}$ for which*

$$|\gamma^{\mathcal{S}}|(Z_k^H) = \sum_{\mu:H\to Z_k} |\gamma^{\mathcal{S}}(\mu)| = 1.$$

The probability distribution on the set of half-edge colourings Z_k^H determined by $\gamma^{\mathcal{S}}$ is defined for each $\mu \in Z_k^H$ by

$$\Pr(\mu) = |\gamma^{\mathcal{S}}(\mu)|.$$

We emphasise that a probability weight $\gamma^{\mathcal{S}}$ need not take positive real values: it is its absolute value $|\gamma^{\mathcal{S}}|$ which defines a probability distribution on the set of half-edge colourings. We will be particularly interested in probability $\mathcal{S}$-weights of the following special type:

Definition 5.2.2 *A probability weight $\gamma^{\mathcal{S}} : Z_k^H \to \mathbb{C}$ is a* parity weight *if its restriction to each block of $\mathcal{S}$ is real-valued and furthermore*

$$\gamma^{\mathcal{S}}(Z_k^H) = \sum_{\mu:H\to Z_k} \gamma^{\mathcal{S}}(\mu) = 0.$$

The second condition for a parity weight implies that half-edge colourings are as likely to have a negative weight as a positive weight, which will be formally proved in Proposition 5.2.4 below.

Example 5.2.3 *Let G be a cubic graph with a set of local vertex rotations, partition $\mathcal{V}$ of its half-edge set by vertices and each block of $\mathcal{V}$ put in a linear order consistent with the local vertex rotations of G. Matiyasevich's vertex weight $\rho^{\mathcal{V}}$ signs a $\mathcal{V}$-proper half-edge colouring positively or negatively according as the parity of the number of anticlockwise vertices is even or odd.*

By dividing by the total number of $\mathcal{V}$-proper half-edge colourings we obtain the parity weight $(\frac{1}{6}\rho)^{\mathcal{V}} = 6^{-|V|} \cdot (\alpha_{\underline{012}} - \alpha_{\underline{021}})^{\mathcal{V}}$.

The set of half-edge colourings Z_k^H is partitioned into $k^{|T|}$ equivalence classes defined for each $\lambda \in Z_k^T$ by $[\lambda]_{\mathcal{T}} = \{\mu \in Z_k^H : \mu_{\mathcal{T}} = \lambda\}$.[1] By definition, for a probability $\mathcal{S}$-weight

[1] See §3.7 for the definition of induced colourings and equivalence classes.

$\gamma^{\mathcal{S}}$,

$$|\gamma^{\mathcal{S}}|([\lambda]_{\mathcal{T}}) = \Pr(\mu \in [\lambda]_{\mathcal{T}}),$$

when choosing a half-edge colouring μ at random with probability $|\gamma^{\mathcal{S}}|(\mu)$.

The set Z_k^H of half-edge colourings also has a partition into the following *parity classes*, defined in terms of a fixed parity weight $\gamma^{\mathcal{S}}$:

$$\text{Even } = \{\mu \in Z_k^H : \gamma^{\mathcal{S}}(\mu) \in \mathbb{R}_+\}, \quad \text{Odd} = \{\mu \in Z_k^H : \gamma^{\mathcal{S}}(\mu) \in -\mathbb{R}_+\},$$

$$\text{Neither} = \{\mu \in Z_k^H : \gamma^{\mathcal{S}}(\mu) = 0\}.$$

Thus a half-edge colouring is *even* if its weight is strictly positive and *odd* if its weight is strictly negative. (The fixed parity weight $\gamma^{\mathcal{S}}$ on which the definition of "Even" and "Odd" depend will be clear from the context in which these terms are used.)

The events "Even" and "Odd" can be defined naturally in terms of "local events" on the blocks of $\mathcal{S}$. For a given half-edge colouring $\mu : H \to Z_k$, the restriction $\gamma^{(s)}$ of the weight $\gamma^{\mathcal{S}}$ to the block $H(s)$ of $\mathcal{S}$ determines that $H(s)$ is a *positive* block of $\mathcal{S}$ if $\gamma^{(s)}(\mu_s) > 0$ and a *negative* block if $\gamma^{(s)}(\mu_s) < 0$. For half-edge colourings chosen at random according to the probability weight $|\gamma^{\mathcal{S}}|$, "Even" is the event that there are an even number of negative blocks of $\mathcal{S}$, and "Odd" is the event that there are an odd number of negative blocks of $\mathcal{S}$.

It is not possible to choose the half-edge colourings in the set "Neither" under the probability distribution determined by $\gamma^{\mathcal{S}}$. In other words, $\Pr(\text{Neither}) = 0$. We now prove our earlier remark that because a parity weight $\gamma^{\mathcal{S}}$ must satisfy $\gamma^{\mathcal{S}}(Z_k^H) = 0$ the events "Even" and "Odd" have equal probability.

Proposition 5.2.4 *Let $\gamma^{\mathcal{S}} : Z_k^H \to \mathbb{R}$ be a parity weight. Then*

$$\Pr(\text{Even}) = \frac{1}{2} = \Pr(\text{Odd}).$$

Proof. We have,

$$\Pr(\text{Even}) = \Pr(\{\mu \in Z_k^H : \gamma^{\mathcal{S}}(\mu) \in \mathbb{R}_+\}) = \sum_{\gamma^{\mathcal{S}}(\mu) \in \mathbb{R}_+} \gamma^{\mathcal{S}}(\mu),$$

$$\Pr(\text{Odd}) = \Pr(\{\mu \in Z_k^H : \gamma^{\mathcal{S}}(\mu) \in -\mathbb{R}_+\}) = \sum_{\gamma^{\mathcal{S}}(\mu) \in -\mathbb{R}_+} -\gamma^{\mathcal{S}}(\mu).$$

On subtracting we obtain

$$\Pr(\text{Even}) - \Pr(\text{Odd}) = \sum_{\mu \in Z_k^H} \gamma^{\mathcal{S}}(\mu) = 0.$$

Clearly $\Pr(\text{Even}) + \Pr(\text{Odd}) = 1$ and the statement of the proposition now follows. $\square$

Our next proposition says that there is non-zero correlation between the events "Even" and "Odd" and the event of inducing a fixed T-colouring if and only if there is some equivalence class $[\lambda]_{\mathcal{T}}$ for which $\gamma^{\mathcal{S}}([\lambda]_{\mathcal{T}}) \neq 0$.

Proposition 5.2.5 *Let $\gamma^{\mathcal{S}}$ be a parity weight. Then*

$$2\gamma^{\mathcal{S}}([\lambda]_{\mathcal{T}}) = \Pr(\,[\lambda]_{\mathcal{T}} \mid \text{Even}\,) - \Pr(\,[\lambda]_{\mathcal{T}} \mid \text{Odd}\,).$$

Proof. By Proposition 5.2.4,

$$\Pr([\lambda]_{\mathcal{T}} \mid \text{Even}) - \Pr([\lambda]_{\mathcal{T}} \mid \text{Odd}\,) = 2\Pr([\lambda]_{\mathcal{T}} \cap \text{Even}) - 2\Pr([\lambda]_{\mathcal{T}} \cap \text{Odd}),$$

which by definition is equal to

$$= 2 \sum_{\mu\in[\lambda]_{\mathcal{T}},\gamma^{\mathcal{S}}(\mu)>0} \gamma^{\mathcal{S}}(\mu) \;-2 \sum_{\mu\in[\lambda]_{\mathcal{T}},\gamma^{\mathcal{S}}(\mu)<0} -\gamma^{\mathcal{S}}(\mu)$$

$$= 2 \sum_{\mu\in[\lambda]_{\mathcal{T}}} \gamma^{\mathcal{S}}(\mu) = 2\gamma^{\mathcal{S}}([\lambda]_{\mathcal{T}}).$$

$\square$

The correlation between the equivalence class events and the parity events will be explored in terms of the "diagonal copies" of these events in the product space $Z_k^H \times Z_k^H$. The advantage of considering this product space lies in the fact that the existence of *some* equivalence class $[\lambda]_{\mathcal{T}}$ such that $\gamma^{\mathcal{S}}([\lambda]_{\mathcal{T}}) \neq 0$ may be established by considering *all* the equivalence classes together as a non-exhaustive event in this product space and conditioning this event on a parity event.

Firstly, the event that two half-edge colourings induce the same T-colouring is defined by

$$\text{Equivalent} := \bigcup_{\lambda:T\to Z_k} [\lambda]_{\mathcal{T}} \times [\lambda]_{\mathcal{T}} \quad = \{(\mu,\mu') \in Z_k^H \times Z_k^H : \mu_{\mathcal{T}} = \mu'_{\mathcal{T}}\}.$$

Secondly, the *parity events* are defined by

$$\text{Both Even} := \text{Even} \times \text{Even} \quad = \{(\mu,\mu') \in Z_k^H \times Z_k^H : \gamma^{\mathcal{S}}(\mu) \in \mathbb{R}_+, \gamma^{\mathcal{S}}(\mu') \in \mathbb{R}_+\},$$

$$\text{Both Odd} := \text{Odd} \times \text{Odd} \quad = \{(\mu,\mu') \in Z_k^H \times Z_k^H : \gamma^{\mathcal{S}}(\mu) \in -\mathbb{R}_+, \gamma^{\mathcal{S}}(\mu') \in -\mathbb{R}_+\},$$

$$\text{Same Parity} := \text{Both Even} \cup \text{Both Odd} \quad = \{(\mu,\mu') \in Z_k^H \times Z_k^H : \gamma^{\mathcal{S}}(\mu)\gamma^{\mathcal{S}}(\mu') \in \mathbb{R}_+\}.$$

Implicit in the definition of "Equivalent" is a given partition $\mathcal{T} = \{H(t) : t \in T\}$ of H, just as the parity events implicitly depend on the parity weight $\gamma^{\mathcal{S}}$.

If we choose two half-edge colourings μ, μ' independently at random, selecting μ, μ' with probabilities $|\gamma^{\mathcal{S}}|(\mu), |\gamma^{\mathcal{S}}|(\mu')$ respectively, then, by definition,

$$|\gamma^{\mathcal{S}}|([\lambda]_{\mathcal{T}})^2 = \Pr((\mu, \mu') \in [\lambda]_{\mathcal{T}} \times [\lambda]_{\mathcal{T}}).$$

This, together with Lemma 3.12.C, yields the following expression for the probability that two half-edge colourings induce the same T-colouring when they are randomly chosen under the probability distribution defined by $|\gamma^{\mathcal{S}}|$.

Theorem 5.2.6 *Let $\gamma^{\mathcal{S}} : Z_k^H \to \mathbb{C}$ be a probability weight and $\mathcal{T} = \{H(t) : t \in T\}$ a partition of H. Then,*

$$\Pr(\text{Equivalent}) = \sum_{\lambda: T \to Z_k} |\gamma^{\mathcal{S}}|([\lambda]_{\mathcal{T}})^2$$

$$= k^{-|T|} \sum_{\mu: T \to Z_k} \left| \mathtt{t}|\gamma^{\mathcal{S}}|(\mu_{\mathcal{H}}) \right|^2,$$

where $\mu_{\mathcal{H}} : H \to Z_k$ is the refinement of $\mu : T \to Z_k$ to a half-edge colouring monochrome on each block of $\mathcal{T}$.

The first of our correlations between "Equivalent" and the parity events is given by the following, which determines the extent to which knowing that two half-edge colourings are of the same parity affects the chances of them inducing the same colouring:

Theorem 5.2.7 *Let $\gamma^{\mathcal{S}}$ be a parity weight and $\mathcal{T} = \{H(t) : t \in T\}$ a partition of H. Then*

$$\Pr(\text{Equivalent } \mid \text{Same Parity}) - \Pr(\text{Equivalent }) = \sum_{\lambda: T \to Z_k} \gamma^{\mathcal{S}}([\lambda]_{\mathcal{T}})^2$$

$$= k^{-|T|} \sum_{\mu: T \to Z_k} |\,\mathtt{t}\gamma^{\mathcal{S}}(\mu_{\mathcal{H}})\,|^2.$$

Proof. The event "Same Parity" is a disjoint union of the events "Both Even" and "Both Odd", and

$$\Pr(\text{Same Parity}) = \Pr(\text{Both Even}) + \Pr(\text{Both Odd}) = \Pr(\text{Even})^2 + \Pr(\text{Odd})^2 = \frac{1}{2}.$$

Thus, for any $\lambda : T \to Z_k$,

$$\Pr([\lambda]_{\mathcal{T}} \times [\lambda]_{\mathcal{T}} \mid \text{Same Parity}) = \frac{\Pr([\lambda]_{\mathcal{T}} \times [\lambda]_{\mathcal{T}} \cap \text{Both Even}) + \Pr([\lambda]_{\mathcal{T}} \times [\lambda]_{\mathcal{T}} \cap \text{ Both Odd})}{\frac{1}{2}}$$

$$= \frac{\Pr([\lambda]_{\mathcal{T}} \cap \text{Even})^2 + \Pr([\lambda]_{\mathcal{T}} \cap \text{Odd})^2}{\frac{1}{2}}.$$

If we write $p_0 = \Pr([\lambda]_{\mathcal{T}} \cap \text{Even})$ and $p_1 = \Pr([\lambda]_{\mathcal{T}} \cap \text{Odd})$, then

$$|\gamma^{\mathcal{S}}|([\lambda]_{\mathcal{T}}) = \Pr([\lambda]_{\mathcal{T}}) = \Pr([\lambda]_{\mathcal{T}} \cap \text{Even}) + \Pr([\lambda]_{\mathcal{T}} \cap \text{Odd}) = p_0 + p_1,$$

and by Proposition 5.2.5, $\gamma^{\mathcal{S}}([\lambda]_{\mathcal{T}}) = p_0 - p_1$. It follows that

$$\Pr([\lambda]_{\mathcal{T}} \times [\lambda]_{\mathcal{T}} \mid \text{Same Parity}) - \Pr([\lambda]_{\mathcal{T}} \times [\lambda]_{\mathcal{T}}) = 2(p_0^2 + p_1^2) - (p_0 + p_1)^2$$

$$= (p_0 - p_1)^2 = \gamma^{\mathcal{S}}([\lambda]_{\mathcal{T}})^2.$$

The first equation of the theorem results on summing over all $\lambda : T \to Z_k$, and the second equation follows by Lemma 3.12.C. $\square$

Theorem 5.2.7 yields some related correlations between "Equivalent" and other parity events. The event "Different Parity" is the complement of "Same Parity" in the product space of half-edge colourings with non-zero probability,

$$\text{Different Parity} = \{(\mu, \mu') \in Z_k^H \times Z_k^H : \gamma^{\mathcal{S}}(\mu)\gamma^{\mathcal{S}}(\mu') \in -\mathbb{R}_+\}.$$

Since $\Pr(\text{Same Parity}) = \frac{1}{2} = \Pr(\text{Different Parity})$ it is easy to derive the following:

$$\Pr(\text{Equiv.} \mid \text{Different Parity}) - \Pr(\text{Equiv.}) = -[\Pr(\text{Equiv.} \mid \text{Same Parity}) - \Pr(\text{Equiv.})],$$

$$\Pr(\text{Equiv.} \mid \text{Same Parity}) - \Pr(\text{Equiv.} \mid \text{Different Parity}) = 2[\Pr(\text{Equiv.} \mid \text{Same Parity}) - \Pr(\text{Equiv.})].$$

Theorem 5.2.7 enabled us to answer the question of whether knowing that two half-edge colourings are of the same parity affects the chances of them being equivalent. The next theorem allows us to answer questions such as whether knowing that two half-edge colourings are both even makes the chances of them being equivalent better than they would be were they both odd.

Theorem 5.2.8 *Let $\gamma^{\mathcal{S}}$ be a parity weight and $\mathcal{T} = \{H(t) : t \in T\}$ a partition of H. Then,*

$$\Pr(\text{Equivalent} \mid \text{Both Even}) - \Pr(\text{Equivalent} \mid \text{Both Odd}) = 4 \sum_{\lambda: T \to Z_k} \gamma^{\mathcal{S}}([\lambda]_{\mathcal{T}}) |\gamma^{\mathcal{S}}|([\lambda]_{\mathcal{T}})$$

$$= 4k^{-|T|} \sum_{\mu: T \to Z_k} \mathtt{t}|\gamma^{\mathcal{S}}| \overline{\mathtt{t}\gamma^{\mathcal{S}}}(\mu_{\mathcal{H}}),$$

where the bar denotes complex conjugation.

Proof. In the notation of the proof of Theorem 5.2.7,

$$\Pr([\lambda]_{\mathcal{T}} \times [\lambda]_{\mathcal{T}} \mid \text{Both Even}) - \Pr([\lambda]_{\mathcal{T}} \times [\lambda]_{\mathcal{T}} \mid \text{Both Odd})$$

$$= \frac{p_0^2 - p_1^2}{\frac{1}{4}} = 4(p_0 - p_1)(p_0 + p_1) = 4\gamma^{\mathcal{S}}([\lambda]_{\mathcal{T}})|\gamma^{\mathcal{S}}|([\lambda]_{\mathcal{T}}).$$

Summing over all $\lambda : T \to Z_k$ gives the first equation, and Lemma 3.12.C gives the second equation. $\square$

Between them Theorems 5.2.7 and 5.2.8 enable us to answer questions of the form *Does the event A increase the chances of "Equivalent" (to a greater extent than the event B)?* where A and B are parity events. Questions of the form *Does the event "Equivalent" increase the chances of event A?* may then be answered by using the identity

$$\Pr(A \mid \text{Equivalent}) - \Pr(A) = \Pr(A)\,\frac{\Pr(\text{Equivalent} \mid A) - \Pr(\text{Equivalent})}{\Pr(\text{Equivalent})}.$$

We finish this section by displaying a number of identities which illustrate how Theorems 5.2.7 and 5.2.8 yield correlations between "Equivalent" and other parity events. All these identities are used by Matiyasevich [45] to generate a number of his probabilistic restatements of the Four Colour Theorem. Again, they are straightforward consequences of the fact that the various parity events have probability either $\frac{1}{2}$ or $\frac{1}{4}$.

$$\Pr(\text{Equiv.} \mid \text{Both Even}) - \Pr(\text{Equiv.})$$

$$= [\Pr(\text{Equiv.} \mid \text{Same Parity}) - \Pr(\text{Equiv.})] + \frac{1}{2}[\Pr(\text{Equiv.} \mid \text{Both Even}) - \Pr(\text{Equiv.} \mid \text{Both Odd})],$$

$$\Pr(\text{Equiv.} \mid \text{Both Odd}) - \Pr(\text{Equiv.})$$

$$= [\Pr(\text{Equiv.} \mid \text{Same Parity}) - \Pr(\text{Equiv.})] - \frac{1}{2}[\Pr(\text{Equiv.} \mid \text{Both Even}) - \Pr(\text{Equiv.} \mid \text{Both Odd})],$$

$$\Pr(\text{Equiv.} \mid \text{Both Even}) - \Pr(\text{Equiv.} \mid \text{Different Parity})$$

$$= 2[\Pr(\text{Equiv.} \mid \text{Same Parity}) - \Pr(\text{Equiv.})] + \frac{1}{2}[\Pr(\text{Equiv.} \mid \text{Both Even}) - \Pr(\text{Equiv.} \mid \text{Both Odd})],$$

$$\Pr(\text{Equiv.} \mid \text{Both Odd}) - \Pr(\text{Equiv.} \mid \text{Different Parity})$$

$$= 2[\Pr(\text{Equiv.} \mid \text{Same Parity}) - \Pr(\text{Equiv.})] - \frac{1}{2}[\Pr(\text{Equiv.} \mid \text{Both Even}) - \Pr(\text{Equiv.} \mid \text{Both Odd})].$$

5.3 Cubic graphs

In this section we apply the results of §5.2 to extend the results of Matiyasevich [45, 46] on proper edge 3-colouring planar cubic graphs. We take G to be a cubic graph with a set of local vertex rotations, half-edge set H, partition $\mathcal{E} = \{H(e) : e \in E\}$ of H by edges into blocks of size 2 and partition $\mathcal{V} = \{H(v) : v \in V\}$ of H by vertices into blocks of size 3. A linear order is put on each block of $\mathcal{V}$ consistent with the local vertex rotations of G.

We denote the line graph of G by $L(G)$ and the number of proper edge 3-colourings of G by $P(L(G);3)$, the evaluation of the chromatic polynomial of $L(G)$ at 3.

A half-edge colouring $\mu : H \to Z_3$ is the same as a Z_3^3-colouring of $\mathcal{V}$ and a probability weight $\gamma^{\mathcal{V}} : Z_3^H \to \mathbb{R}$, defined by a function $\gamma : Z_3^3 \to \mathbb{R}$ with the property that $|\gamma|(Z_3^3) = 1$, extended multiplicatively over the blocks of $\mathcal{V}$ to $\gamma^{\mathcal{V}} : (Z_3^3)^{\mathcal{V}} \to \mathbb{R}$, is a parity weight if and only if $\gamma(Z_3^3) = 0$.

In the terminology of §5.2, a triple $\ell \in Z_3^3$ is *positive* or *negative* according as $\gamma(\ell) > 0$ or $\gamma(\ell) < 0$ respectively. Given a half-edge colouring $\mu : \mathcal{V} \to Z_3^3$, we call a vertex v *positive* in the colouring μ if and only if $\gamma(\mu_v) > 0$ and *negative* if $\gamma(\mu_v) < 0$. The *parity* of a half-edge colouring $\mu \in Z_3^H$ for which each vertex is either positive or negative is then *even* if the number of negative vertices is even and *odd* if the number of negative vertices is odd. A half-edge colouring $\mu : H \to Z_3$ *induces* a (not necessarily proper) edge 3-colouring $\mu_{\mathcal{E}} : E \to Z_3$ of G defined by adding together the two colours on the two halves of an edge and two half-edge colourings are *equivalent* if they induce the same edge colouring.

For a triple $\ell \in Z_3^3$ we use the notation introduced in §3.6 and write $\underline{\ell}$ for the set of three triples $\{\ell, \ell+111, \ell+222\}$. For example, $\underline{001} = \{011, 122, 200\}$. We will use names for the following special subsets of Z_3^3:

$$\text{“proper”} = \{\underline{012}, \underline{021}\}, \qquad \text{“clockwise”} = \underline{012}, \qquad \text{“anticlockwise”} = \underline{021},$$

$$\text{“monochrome”} = \underline{000}, \qquad \text{“null”} = \{\underline{000}, \underline{012}, \underline{021}\} = \text{“monochrome” or “proper”}.$$

Our first example of a parity weight is the function $\frac{1}{6}\rho : Z_3^3 \to \mathbb{R}$, where ρ is defined by

$$\rho(\ell) = (\alpha_{\text{clockwise}} - \alpha_{\text{anticlockwise}})(\ell) = \begin{cases} +1 & \ell \in \underline{012} \ \text{(clockwise)}, \\ -1 & \ell \in \underline{021} \ \text{(anticlockwise)}, \\ 0 & \ell \notin \underline{012}, \underline{021} \ \text{(not proper)}. \end{cases}$$

The probability distribution on Z_3^H determined by $|\rho| = \frac{1}{6}\alpha_{\text{proper}}$ corresponds to randomly selecting a half-edge colouring by assigning to each block of $\mathcal{V}$ any of the six proper colourings uniformly at random. In §4.3 we found [2] that $\mathtt{t}\rho = -3\sqrt{-3}\rho$ and that $\mathtt{t}|\rho| = 6\alpha_{\text{monochrome}} - 3\alpha_{\text{proper}}$. The parity of a proper edge 3-colouring $\mu : E \to Z_3$ (relative to the parity weight ρ) is the parity of the number of vertices $v \in V$ for which the colours of the edges incident with v appear in an anticlockwise sense relative to the fixed set of local vertex rotations of G. Theorem 4.3.4 says that for a plane cubic graph G with set

[2] These and other transforms can be calculated by using the matrix for the discrete Fourier transform taking the basis $\{\alpha_\ell : \ell \in Z_3^3\}$ to the basis $\{\beta_\ell : \ell \in Z_3^3\}$ of §3.11 or by methods similar to those used in Chapter 6.

of local vertex rotations determined by rotational sense of edges in the plane embedding of G, if μ is any proper edge 3-colouring of G then $\rho^{\mathcal{V}}(\mu_{\mathcal{H}}) = (-1)^{|E|}$.

Theorems 5.2.6, 5.2.7 and 5.2.8 immediately yield the following extension of a theorem of Matiyasevich [45] to non-planar graphs:

Theorem 5.3.1 *Let G be a cubic graph with a set of local vertex rotations, half-edge set H, partition $\mathcal{E}$ of H by edges and partition $\mathcal{V}$ of H by vertices, each block of $\mathcal{V}$ put in a linear order consistent with the local vertex rotations of G. Suppose that a uniform distribution is put on the set of half-edge Z_3-colourings which give distinct colours to three half-edges if they lie in the same block of $\mathcal{V}$. If we call a colouring of a block of $\mathcal{V}$ positive when the three colours appear in a clockwise sense and negative when the colours appear in an anticlockwise sense, then*

$$\Pr(\text{Equivalent} \mid \text{Same Parity}) - \Pr(\text{Equivalent}) = (\frac{1}{4\sqrt{3}})^{|V|} P(L(G);3),$$

$$\Pr(\text{Equivalent} \mid \text{Both Even}) - \Pr(\text{Equivalent} \mid \text{Both Odd})$$

$$= (-1)^{|E|} 4 (\frac{1}{12})^{|V|} (\,\#\{\text{Even proper edge 3-colourings}\} - \#\{\text{ Odd proper edge 3-colourings}\}\,),$$

and

$$\Pr(\text{Equivalent}) = (\frac{1}{12\sqrt{3}})^{|V|} \sum_{\text{edge 3-colourings, null vertices}} 4^{\#\text{monochrome vertices}},$$

where the last sum is over all edge 3-colourings of G which have the property that each vertex is either incident with edges of distinct colours (proper) or with edges of all the same colour (monochrome).

By considering the parity weight $\frac{1}{12}\gamma$ where γ is defined by

$$\gamma(\ell) = (\alpha_{\text{proper}} - 2\alpha_{\text{monochrome}})(\ell) = \begin{cases} +1 & \ell \in \underline{012}, \underline{021} \;\; \text{(proper)}, \\ -2 & \ell \in \underline{000} \;\; \text{(monochrome)}, \\ 0 & \text{otherwise}, \end{cases}$$

which has transform $\mathtt{t}\gamma = -9\alpha_{\text{proper}}$, and for which we have $\mathtt{t}|\gamma| = 12\alpha_{\text{monochrome}} + 3\alpha_{\text{proper}}$, we obtain our second example of a parity colouring scheme where the probability that two half-edge colourings induce the same edge colouring is increased given the knowledge that they have the same parity, by an amount proportional to the number of proper edge 3-colourings:

Theorem 5.3.2 *Let G be a cubic graph with half-edge set H, partition $\mathcal{E}$ of H by edges and partition $\mathcal{V}$ of H by vertices. Call a half-edge colouring of a block in $\mathcal{V}$ positive if there are three distinct colours and negative if all three colours are the same. Suppose half-edge colourings are chosen randomly by first deciding u.a.r. whether to colour a vertex* *positively or negatively and then, having chosen the sign, choosing u.a.r. which particular colouring of this sign to take. Then*

$$\Pr(\text{Equivalent} \mid \text{Same Parity}) - \Pr(\text{Equivalent}) = (\frac{\sqrt{3}}{16})^{|V|} P(L(G);3),$$

$$\Pr(\text{Equivalent} \mid \text{Both Even}) - \Pr(\text{Equivalent} \mid \text{Both Odd}) = 4(\frac{1}{16\sqrt{3}})^{|V|} P(L(G);3),$$

and

$$\Pr(\text{Equivalent}) = (\frac{1}{48\sqrt{3}})^{|V|} \sum_{\text{edge 3-colourings,null vertices}} 16^{\#\text{monochrome vertices}},$$

where the last sum is over all edge 3*-colourings of* G *which have the property that each vertex is either incident with edges of distinct colours (proper) or with edges of all the same colour (monochrome).*

Note that in Theorem 5.3.2 there is no linear order on the blocks of $\mathcal{V}$ since the monochrome triples and proper triples remain respectively monochrome or proper under permutation.

Our final example of a parity colouring scheme depends not only on the cyclic order determined by the linear order put on each block of $\mathcal{V}$, which is the case for Theorem 5.3.3, but also on the particular linear order chosen.

A multiset permutation on the ordered set $Z_3 = \{0 < 1 < 2\}$ is a not necessarily bijective function $\pi : Z_3 \to Z_3$. If π has image $(z_0, z_1, z_2) \in Z_3^3$, an *inversion* occurs if either $z_0 > z_1$ or $z_0 > z_2$ or $z_1 > z_2$. Then π is an *even* permutation if the number of inversions is even, and *odd* otherwise. We accordingly give names to the following subsets of Z_3^3:

$$\text{“even”} = \{\underline{000}, \underline{012}, \underline{100}, \underline{001}, \underline{200}, \underline{002}\}, \qquad \text{“odd”} = \{\underline{021}, \underline{010}, \underline{020}\}.$$

By considering the parity weight $\frac{1}{36}\gamma$, where γ is defined by

$$\gamma(\ell) = (\alpha_{\text{even}} - 2\alpha_{\text{odd}})(\ell) = \begin{cases} +1 & \ell \in \{\underline{000}, \underline{012}, \underline{100}, \underline{001}, \underline{200}, \underline{002}\} \ \ \text{(even)}, \\ -2 & \ell \in \{\underline{021}, \underline{010}, \underline{020}\} \ \ \text{(odd)}, \\ 0 & \text{otherwise}, \end{cases}$$

for which

$$\mathtt{t}\gamma = 9\sqrt{-3}\left(j(\alpha_{201} - \alpha_{021,210}) + j^2(\alpha_{012,120} - \alpha_{102}) \right),$$

and

$$\mathtt{t}|\gamma| = 36\alpha_{000} + 3\sqrt{-3}\left(j(\alpha_{021,210} - \alpha_{201}) + j^2(\alpha_{102} - \alpha_{012,120} \right) = 36\alpha_{000} + \frac{1}{3}\mathtt{t}\gamma,$$

we obtain the following:

Theorem 5.3.3 *Let G be a cubic graph with half-edge set H, partition $\mathcal{E}$ of H by edges and partition $\mathcal{V}$ by vertices, and with an arbitrary linear order on each block of $\mathcal{V}$. Call a colouring of a block in $\mathcal{V}$ positive if it is an "even" colour triple and negative if it is an "odd" colour triple. Suppose half-edge colourings are chosen randomly by first deciding u.a.r. whether to colour a vertex positively or negatively and then, having chosen the sign, choosing u.a.r. which particular colouring of this sign to take.*

Then

$$\Pr(\text{Equivalent} \mid \text{Same Parity}) - \Pr(\text{Equivalent}) = (\frac{1}{16\sqrt{3}})^{|V|} P(L(G);3),$$

$$\Pr(\text{Equivalent} \mid \text{Both Even}) - \Pr(\text{Equivalent} \mid \text{Both Odd}) = 4(\frac{1}{48\sqrt{3}})^{|V|} P(L(G);3),$$

$$\Pr(\text{Equivalent}) = (\frac{1}{144\sqrt{3}})^{|V|} \sum_{\text{edge 3-colourings proper/zero vertices}} 12^{\#\text{zero vertices}},$$

where the last sum is over all edge Z_3-colourings with the property that each vertex is either incident with edges of distinct colours (proper) or with edges all coloured 0 (zero).

Having exhibited three parity weights for which "Equivalence" and "Parity" are correlated positively by an amount proportional to $P(L(G);3)$ it is natural to consider the following problem.

Problem 5.3.4 *Which parity weights determine probability distributions for which*

$$\Pr(\text{Equivalent} \mid \text{Same Parity}) - \Pr(\text{Equivalent}) = C(|V|)P(L(G);3)$$

for some non-zero function $C(|V|)$ depending only on $|V|$?

Referring to the result of Theorem 5.2.7, a parity weight $\gamma^{\mathcal{V}} : (Z_3^3)^{\mathcal{V}} \to [-\frac{1}{2}, \frac{1}{2}]$ will lead to a solution to Problem 5.3.4 if and only if

$$\mathtt{t}\gamma = \sum_{p \text{ proper}} c_p \alpha_p,$$

where the sum is over proper triples in Z_3^3 and $c_p \in \mathbb{C}$ are of constant absolute value $c \in \mathbb{R}_+$. In this case, Theorem 5.2.7 says that

$$\Pr(\text{Equivalent} \mid \text{Same Parity}) - \Pr(\text{Equivalent}) = k^{-|E|} c^{|V|} P(L(G);3).$$

Let $\gamma : Z_3^3 \to [-\frac{1}{2}, \frac{1}{2}]$ define a parity weight $\gamma^{\mathcal{V}}$ with the property that $|\mathtt{t}\gamma|$ is the constant $c \in \mathbb{R}_+$ on $\{p \in Z_3^3 : p \text{ proper}\}$ and zero otherwise. Then γ satisfies the hypotheses of the following proposition, which provides a necessary condition on a parity weight to be a solution to Problem 5.3.4. The *support* $\mathrm{supp}(\gamma)$ of a function $\gamma : Z_3^3 \to \mathbb{C}$ is the set $\{\ell \in Z_3^3 : \gamma(\ell) \neq 0\}$.

Proposition 5.3.5 *Let $\gamma : Z_3^3 \to \mathbb{C}$ be a vertex weight with the property that* $\mathrm{supp}(\mathtt{t}\gamma) \subseteq \{p \in Z_3^3 : p \text{ monochrome or proper}\}$. *Then $\gamma(\ell) = \gamma(\ell')$ whenever $\ell - \ell'$ is monochrome.*

Proof. The monochrome and proper triples in Z_3^3 are precisely the null triples, i.e. their components sum to zero modulo 3. Thus for any monochrome m and null p in Z_3^3 we have $\langle m, p\rangle = 0$. Thus, if $\ell - \ell' \in \underline{000}$ is a monochrome triple, then for any null triple p we have $\langle \ell', p\rangle = \langle \ell - (\ell - \ell'), p\rangle = \langle \ell, p\rangle - \langle \ell - \ell', p\rangle = \langle \ell, p\rangle$. Hence

$$\gamma(\ell') = 3^{-3} \sum_{p \text{ null}} \mathtt{t}\gamma(p) j^{\langle \ell', p\rangle} = 3^{-3} \sum_{p \text{ null}} \mathtt{t}\gamma(p) j^{\langle \ell, p\rangle} = \gamma(\ell).$$

□

Consequently, any solution γ to Problem 5.3.4 takes the form

$$\gamma = \sum_{\ell} b_\ell \alpha_{\underline{\ell}}, \qquad b_\ell \in [-\frac{1}{2}, \frac{1}{2}], \ \sum_{\ell} |b_\ell| = 1, \ \sum_{\ell} b_\ell = 0,$$

where the summations range over the transversal $\{000, \pm 012, \pm 100, \pm 010, \pm 001\}$ of Z_3^3 modulo its additive subgroup $\underline{000}$ of monochrome triples. In other words, in the random colouring scheme determined by γ, triples which differ by a monochrome triple in Z_3^3 must have the same sign (positive, negative, neither) and be chosen with the same probability.

In the notation introduced above for the parity weight γ, the probability of choosing a particular colour triple in $\underline{\ell}$ for a block of $\mathcal{V}$ is $|b_\ell|/3$. If $b_\ell > 0$ then the three triples in $\underline{\ell}$ are positive colourings, if $b_\ell < 0$ then the triples in $\underline{\ell}$ are negative colourings, and finally if $b_\ell = 0$ then the triples in $\underline{\ell}$ are neither positive nor negative.

In order to describe the set of parity weights which are solutions to Problem 5.3.4 it will be convenient to represent a parity weight $\gamma = \sum_\ell b_\ell \alpha_{\underline{\ell}}$ by its coefficients placed in a matrix as follows:

$$\begin{pmatrix} b_{000} & b_{012} & b_{021} \\ b_{100} & b_{001} & b_{010} \\ b_{200} & b_{020} & b_{002} \end{pmatrix}.$$

For example, the coefficient matrices of the parity weights of Theorems 5.3.1,5.3.2 and

5.3.3 are respectively given by

$$\begin{pmatrix} 0 & \frac{1}{2} & -\frac{1}{2} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} -\frac{1}{2} & \frac{1}{4} & \frac{1}{4} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \frac{1}{12}\begin{pmatrix} 1 & 1 & -2 \\ 1 & 1 & -2 \\ 1 & -2 & 1 \end{pmatrix}.$$

There are three types of operation on the coefficient matrices of parity weights which preserve the correlation between "Equivalent" and "Same Parity". In particular, these operations preserve the property of a parity weight being a solution to Problem 5.3.4.

The first is quite straightforward to verify: since $|V|$ is even we have $\mathtt{t}(-\gamma)^{\mathcal{V}} = \mathtt{t}\gamma^{\mathcal{V}}$ so that if γ is a solution to Problem 5.3.4 then so is $-\gamma$. Thus negating each entry in the coefficient matrix preserves the property of being a solution to Problem 5.3.4.

The second operation arises as a consequence of the following proposition:

Proposition 5.3.6 *Let $\gamma : Z_3^3 \to \mathbb{C}$ be defined by*

$$\gamma = \sum_{\ell \in Z_3^3} a_\ell \alpha_\ell, \qquad \textit{where} \quad a_\ell \in \mathbb{C},$$

and for a fixed $\ell' \in Z_3^3$ let $\gamma' : Z_3^3 \to \mathbb{C}$ be defined by $\gamma'(\ell) = \gamma(\ell - \ell')$ for each $\ell \in Z_3^3$. In other words,

$$\gamma' = \sum_{\ell \in Z_3^3} a_\ell \alpha_{\ell + \ell'}.$$

Then

$$|\mathtt{t}\gamma| = |\mathtt{t}\gamma'|.$$

Proof. Since $\mathtt{t}\alpha_{\ell+\ell'} = \beta_{\ell+\ell'} = \beta_{\ell'}\beta_\ell$ we have $\mathtt{t}\gamma'(\ell) = \beta_{\ell'}\mathtt{t}\gamma$, and $|\beta_{\ell'}| = 1$ gives the result. □

For any fixed $\ell' \in Z_3^3$, the cosets $\underline{\ell}$ of the monochrome triples $\underline{000}$ are wholly contained in the orbits of the permutation $\ell \mapsto \ell + \ell'$ on Z_3^3. Hence, this permutation on Z_3^3 induces a well-defined permutation on the set $\{\underline{\ell} : \ell \in Z_3^3\}$ and the map $\ell \mapsto \ell + \ell'$ permutes the coefficients of the parity weight

$$\gamma = \sum_{\ell} b_\ell \alpha_{\underline{\ell}}.$$

In terms of the coefficient matrix of γ, the action $\underline{\ell} \mapsto \underline{\ell} + \ell'$ transforming γ into

$$\gamma' = \sum_{\ell} b_\ell \alpha_{\underline{\ell} + \ell'}$$

permutes rows cyclically ($\ell' = 000, 100, 200$), columns cyclically ($\ell' = 000, 012, 021$), or is a combination of these two permutations.

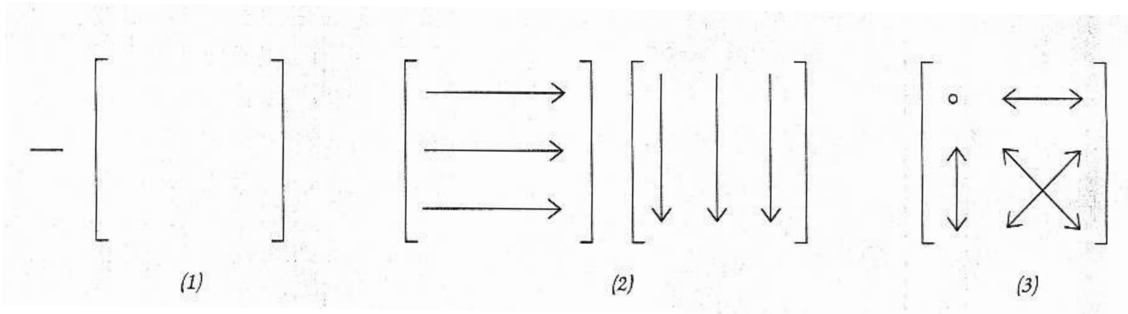


Figure 5.1: Operations on the coefficient matrix $[\,b_\ell\,]$ of a parity weight which preserve Pr(Equivalent | Same Parity) – Pr(Equivalent): *(1) Negation of each entry* $b_\ell \mapsto -b_\ell$, *(2) cyclic permutation of rows and columns* $b_\ell \mapsto b_{\ell+\ell'}$, *(3)* $b_\ell \mapsto b_{-\ell}$.

The third operation on the coefficient matrices of parity weights which preserves the property of being a solution to Problem 5.3.4 is a consequence of the following proposition:

Proposition 5.3.7 *Let* $\gamma : Z_3^3 \to \mathbb{R}$ *be a parity weight. Then the weight* $\mathtt{s}\gamma$ *defined for each* $\ell \in Z_3^3$ *by* $\mathtt{s}\gamma(\ell) = \gamma(-\ell)$ *satisfies* $\mathtt{ts}\gamma = \overline{\mathtt{t}\gamma}$, *where the bar denotes complex conjugation.*

Proof. We have

$$\mathtt{ts}\gamma(\ell) = \mathtt{t}\gamma(-\ell) = \sum_{m \in Z_3^3} \gamma(-m) j^{\langle \ell, m\rangle} = \sum_{m \in Z_3^3} \gamma(m) j^{-\langle \ell, m\rangle} = \overline{\mathtt{t}\gamma}(\ell),$$

the last step using the fact that γ is real-valued. $\square$

The permutation $\underline{\ell} \mapsto -\underline{\ell}$ on the cosets of $\underline{000}$ in Z_3^3, which maps the parity weight γ to the parity weight $\mathtt{s}\gamma$ of Proposition 5.3.7, switches the second and third entries of the first column, the second and third entries of the first row and cross-changes the four entries in the lower right corner of the coefficient matrix (see Figure 5.1 above).

It is readily calculated that the discrete Fourier transform $\mathtt{t}\gamma$ of any given parity weight of the form

$$\gamma = \sum_\ell b_\ell \alpha_{\underline{\ell}}$$

and with the property that $\mathrm{supp}(\mathtt{t}\gamma) \subseteq \{p \in Z_3^3 : p \text{ proper}\}$ is given in terms of the coefficients b_ℓ of γ as follows:

$$
\mathtt{t}\gamma \;=\; 3 \begin{pmatrix} b_{000} \\ b_{012} \\ b_{021} \\ \hline b_{100} \\ b_{010} \\ b_{001} \\ \hline b_{200} \\ b_{020} \\ b_{002} \end{pmatrix}^{\top} \left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 1 & 1 \\ j^2 & j^2 & j^2 & j & j & j \\ j & j & j & j^2 & j^2 & j^2 \\ \hline 1 & j & j^2 & 1 & j^2 & j \\ j & j^2 & 1 & j^2 & j & 1 \\ j^2 & 1 & j & j & 1 & j^2 \\ \hline 1 & j^2 & j & 1 & j & j^2 \\ j^2 & j & 1 & j & j^2 & 1 \\ j & 1 & j^2 & j^2 & 1 & j \end{array}\right) \begin{pmatrix} \alpha_{012} \\ \alpha_{120} \\ \alpha_{201} \\ \hline \alpha_{021} \\ \alpha_{210} \\ \alpha_{102} \end{pmatrix}.
$$

The column sums of the matrix obtained by multiplying the transposed vector $(b_\ell)^\top$ with the matrix representing the transform $\mathtt{t}$ are the coefficients $c_{012}, c_{120}, c_{201}, c_{021}, c_{210}, c_{102}$ of $\mathtt{t}\gamma$ as defined by

$$
\mathtt{t}\gamma = \sum_{p \text{ proper}} c_p \alpha_p.
$$

Lemma 5.3.8 *A necessary condition for a parity weight γ to be a solution to Problem 5.3.4, when given by its matrix of coefficients*

$$
\begin{pmatrix} b_{000} & b_{012} & b_{021} \\ b_{100} & b_{001} & b_{010} \\ b_{200} & b_{020} & b_{002} \end{pmatrix},
$$

is that the entries in each row sum to zero.

Proof. Since γ is real-valued, $\gamma(-\ell) = \overline{\gamma(\ell)}$. The inverse of the matrix above representing the discrete Fourier transform $\mathtt{t}$ is then by the relation $\mathtt{t}^{-1}\gamma(\ell) = 3^{-3}\mathtt{t}\gamma(-\ell)$ its conjugate transpose scaled by 3^{-3}:

$$
\gamma \;=\; 3^{-2} \begin{pmatrix} c_{012} \\ c_{120} \\ c_{201} \\ \hline c_{102} \\ c_{210} \\ c_{021} \end{pmatrix}^{\top} \left(\begin{array}{ccc|ccc|ccc} 1 & j & j^2 & 1 & j^2 & j & 1 & j & j^2 \\ 1 & j & j^2 & j^2 & j & 1 & j & j^2 & 1 \\ 1 & j & j^2 & j & 1 & j^2 & j^2 & 1 & j \\ \hline 1 & j^2 & j & 1 & j & j^2 & 1 & j^2 & j \\ 1 & j^2 & j & j & j^2 & 1 & j^2 & j & 1 \\ 1 & j^2 & j & j^2 & 1 & j & j & 1 & j^2 \end{array}\right) \begin{pmatrix} \alpha_{\underline{000}} \\ \alpha_{\underline{012}} \\ \alpha_{\underline{210}} \\ \hline \alpha_{\underline{100}} \\ \alpha_{\underline{010}} \\ \alpha_{\underline{001}} \\ \hline \alpha_{\underline{200}} \\ \alpha_{\underline{020}} \\ \alpha_{\underline{002}} \end{pmatrix}
$$

The column sums of the transposed vector $(c_p)^\top$ multiplied with the matrix representing the inverse transform $\mathtt{t}^{-1}$ are the coefficients b_ℓ of the weight γ. The sum of the first three columns in the matrix above representing $\mathtt{t}^{-1}$ is zero. Similarly, the sum of the middle three columns is zero and the sum of the final three is also zero. This implies that $b_{000} + b_{012} + b_{021} = 0$, $b_{100} + b_{010} + b_{001} = 0$ and $b_{200} + b_{020} + b_{002} = 0$.□

If we define the triple of points $(A, B, C) \in \mathbb{C}^3$ by

$$\begin{pmatrix} A \\ B \\ C \end{pmatrix} = \begin{pmatrix} b_{000} & b_{012} & b_{021} \\ b_{100} & b_{001} & b_{010} \\ b_{200} & b_{020} & b_{002} \end{pmatrix} \begin{pmatrix} 1 \\ j^2 \\ j \end{pmatrix},$$

and relate to the points (A, B, C) another triple $(a, b, c) \in \mathbb{C}^3$ through the bijective linear transformation

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & j^2 & j \\ 1 & j & j^2 \end{pmatrix} \begin{pmatrix} A \\ B \\ C \end{pmatrix},$$

then the discrete Fourier transform

$$\mathtt{t} : \gamma = \sum_\ell b_\ell \alpha_{\underline{\ell}} \quad \mapsto \quad \sum_{p \text{ proper}} c_p \alpha_p,$$

is given in terms of the points (a, b, c) by

$$(\, c_{012} \;\; c_{120} \;\; c_{201} \;\; c_{021} \;\; c_{210} \;\; c_{102} \,) = 3(\, a \;\; c \;\; b \;\; \overline{a} \;\; \overline{c} \;\; \overline{b} \,).$$

The following lemma summarises our progress thus far towards resolving Problem 5.3.4.

Lemma 5.3.9 *Let G be a cubic graph embedded in an arbitrary surface with half-edge set H, partition $\mathcal{E}$ by edges and partition $\mathcal{V}$ by vertices into blocks of size 3, each block of $\mathcal{V}$ given an arbitrary linear order.*

Let the parity weight $\gamma^{\mathcal{V}}$ be defined on any fixed block of $\mathcal{V}$ by

$$\gamma = \sum_\ell b_\ell \alpha_{\underline{\ell}}, \qquad \sum_\ell b_\ell = 0, \quad \sum_\ell |b_\ell| = 1,$$

where, for each block of $\mathcal{V}$, $|b_\ell|$ gives the probability of choosing one of the three colourings in $\underline{\ell}$ and, when non-zero, the sign of b_ℓ determines whether the three colour triples in $\underline{\ell}$ are positive or negative half-edge colourings.

Let $a, b, c \in \mathbb{C}$ be defined by

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & j^2 & j \\ 1 & j & j^2 \end{pmatrix} \begin{pmatrix} b_{000} & b_{012} & b_{021} \\ b_{100} & b_{001} & b_{010} \\ b_{200} & b_{020} & b_{002} \end{pmatrix} \begin{pmatrix} 1 \\ j^2 \\ j \end{pmatrix}.$$

Then we have

$$\Pr(\text{Equiv.} \mid \text{Same Parity}) - \Pr(\text{Equiv.}) \begin{array}{ll} > 0 & \text{if} \quad P(L(G);3) > 0, \\ = 0 & \text{if} \quad P(L(G);3) = 0, \end{array}$$

only if

$$\begin{pmatrix} b_{000} & b_{012} & b_{021} \\ b_{100} & b_{001} & b_{010} \\ b_{200} & b_{020} & b_{002} \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

If this condition holds then,

$$\Pr(\text{Equivalent} \mid \text{Same Parity}) - \Pr(\text{Equivalent})$$
$$= 3^{-|E|} \sum_{\text{proper edge } Z_3\text{-colourings}} (|a|^2)^{\#\pm 012} (|b|^2)^{\#\pm 201} (|c|^2)^{\#\pm 120},$$

where for example $\# \pm 012$ means the number of vertices in the proper edge Z_3-colouring of G whose incident edges in linear order are either coloured 012 *or* 021*.*

In particular

$$\Pr(\text{Equiv.} \mid \text{Same Parity}) - \Pr(\text{Equiv.}) = C(|V|)P(L(G);3),$$

for some non-zero $C(|V|)$ depending only on $|V|$ if and only if

$$0 \neq |a| = |b| = |c|.$$

Cyclic permutation of the components of a colour triple in Z_3^3 is transitive on the set $\{\underline{100}, \underline{001}, \underline{010}\}$, transitive on the set $\{\underline{200}, \underline{020}, \underline{002}\}$ and fixes each of $\underline{000}$, $\underline{012}$ and $\underline{021}$. If a parity weight gives equal probability to colour triples which are cyclic permutations of each other, then we have $|b_{100}| = |b_{001}| = |b_{010}|$ and, by Lemma 5.3.8, $b_{100}+b_{001}+b_{010} = 0$. Since $b_{100}, b_{001}, b_{010}$ are real, this forces $b_{100} = b_{001} = b_{010} = 0$. Similarly $b_{200} = b_{020} = b_{002} = 0$. Hence by Lemma 5.3.9 we deduce the following:

Corollary 5.3.10 *Let G be a cubic graph with half-edge set H, partition $\mathcal{E}$ of H by edges and partition $\mathcal{V}$ of H by vertices, each of whose blocks have been linearly ordered.*

Then the most general parity weight giving equal probability to colour triples which are cyclic permutations of each other and for which the correlation between "Equivalence" *and* "Same Parity" *is equal to* $P(L(G);3)$ *up to a factor dependent only on* $|V|$*, has coefficient matrix (up to permutation of columns and negation) equal to*

$$\begin{pmatrix} q & \frac{1}{2}-q & -\frac{1}{2} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad \frac{1}{4} \leq q \leq \frac{1}{2}.$$

For such a parity weight,

$$\Pr(\text{Equivalent} \mid \text{Same Parity}) - \Pr(\text{Equivalent}) = (\frac{4q^2-2q+1}{4\sqrt{3}})^{|V|} P(L(G);3).$$

The quantity $4q^2-2q+1$ is maximised on $[\frac{1}{4},\frac{1}{2}]$ at $q=\frac{1}{2}$ and minimised for $q=\frac{1}{4}$. The former gives the coefficient matrix of the parity weight of Theorem 5.3.1, the latter the coefficient matrix of the parity weight of Theorem 5.3.2 (after suitably shuffling the columns of the matrix given by Corollary 5.3.10).

In the remainder of this section we find all parity weights for which the correlation between the events "Equivalent" and "Same Parity" is up to a factor dependent only on $|V|$ equal to $P(L(G);3)$. There are two cases to consider, the first of which is more straightforward and this we describe in the following:

Theorem 5.3.11 *Let* $G=(V,E)$ *be a cubic graph with half-edge set* H*, partition* $\mathcal{E}$ *by edges and partition* $\mathcal{V}$ *by vertices into blocks of size* 3*, each block of* $\mathcal{V}$ *given an arbitrary linear order.*

Let there be given the second and third rows of a 3×3 *matrix over* $\mathbb{R}$

$$\begin{pmatrix} \cdot & \cdot & \cdot \\ x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \end{pmatrix}$$

with the properties that $x_1+y_1+z_1=0=x_2+y_2+z_2$ *and the row* $(x_1\ \ y_1\ \ z_1)$ *is not a cyclic permutation of the row* $(x_2\ \ y_2\ \ z_2)$.

Then there exists a unique $(x_0,y_0,z_0)\in\mathbb{R}^3$ *such that the matrix*

$$\pm\begin{pmatrix} qx_0 & qy_0 & qz_0 \\ qx_1 & qy_1 & qz_1 \\ qx_2 & qy_2 & qz_2 \end{pmatrix},$$

is, for a uniquely determined positive real number q*, the coefficient matrix of a parity* *weight for which the correlation between the events "Equivalent" and "Same Parity" is equal to $P(L(G);3)$ up to a factor dependent only on $|V|$.*

Proof. First we find $(x_0, y_0, z_0) \in \mathbb{R}^3$ (which must satisfy $x_0 + y_0 + z_0 = 0$ by Lemma 5.3.8) so that the matrix

$$\begin{pmatrix} x_0 & y_0 & z_0 \\ x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \end{pmatrix}$$

has the property that

$$|A + B + C| = |A + j^2 B + jC| = |A + jB + j^2 C| \neq 0,$$

where

$$\begin{pmatrix} A \\ B \\ C \end{pmatrix} = \begin{pmatrix} x_0 & y_0 & z_0 \\ x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \end{pmatrix} \begin{pmatrix} 1 \\ j^2 \\ j \end{pmatrix}.$$

We then choose the unique positive value of $q \in \mathbb{R}$ such that

$$q(\,|x_0| + |y_0| + |z_0| + |x_1| + |y_1| + |z_1| + |x_2| + |y_2| + |z_2|\,) = 1,$$

so that the matrix

$$\begin{pmatrix} qx_0 & qy_0 & qz_0 \\ qx_1 & qy_1 & qz_1 \\ qx_2 & qy_2 & qz_2 \end{pmatrix}$$

defines the requisite parity weight of Theorem 5.3.11.

We are given $B = x_1 + j^2 y_1 + jz_1, C = x_2 + j^2 y_2 + jz_2$ as fixed points in the complex plane and we wish to construct a point $A = x_0 + j^2 y_0 + jz_0$ satisfying $|A + B + C| = |A + j^2 B + jC| = |A + jB + j^2 C| \neq 0$. Using $-z_0 = x_0 + y_0$ to write $A = (2x_0 + y_0) + j^2(x_0 + 2y_0)$, we use the basis $\{2 + j^2, 1 + 2j^2\}$ for $\mathbb{C}$ to give the coordinates of a point in the complex plane. The notation (x, y) is used to denote a point in the complex plane relative to this basis $(2 + j^2, 1 + 2j^2)$, so that for example $B = (x_1, y_1), C = (x_2, y_2)$.

A row in the coefficient matrix of a parity weight is zero if and only if one of A, B, C equals zero and two rows are cyclic permutations of each other if and only if multiplying one of A, B, C by j gives either one of the other two. Corollary 5.3.10 covers the case where the given rows $(x_1\ y_1\ z_1)$, $(x_2\ y_2\ z_2)$ are both zero. The following lemma says that if we exclude this case then we may assume that neither of the given rows $(x_1\ y_1\ z_1)$, $(x_2\ y_2\ z_2)$ are zero, i.e. $B, C \neq 0$.

Lemma 5.3.12 *Let $A, B, C \in \mathbb{C}$ satisfy*

$$|A+B+C| = |A+j^2B+jC| = |A+jB+j^2C|.$$

Then if one of A, B, C equals zero, exactly two of A, B, C are zero.

In particular, a coefficient matrix representing a parity weight for which the correlation between "Equivalent" and "Same Parity" is equal to $P(L(G);3)$ up to a factor dependent only on $|V|$ has either no zero rows or exactly two zero rows.

Proof. Take for example $C = 0$. Then

$$|A+B| = |A+j^2B| = |A+jB|.$$

If $B \neq 0$ then the points $-B, -j^2B, -jB$ are the vertices of an equilateral triangle with circumcentre the origin. The circumcentre of a triangle is uniquely determined as the point equidistant from its vertices. Hence A coincides with the origin, i.e. $A = 0$.□

Since the rows $(x_1\, y_1\, z_1)$, $(x_2\, y_2\, z_2)$ of the matrix given in Theorem 5.3.11 are not cyclic permutations of each other, the points B, jB, j^2B, C, jC, j^2C are distinct. It follows that the points $-(B+C), -(j^2B+jC), -(jB+j^2C)$ are distinct, forming the vertices of a triangle Δ in the complex plane. The relations $|A+B+C| = |A+j^2B+jC| = |A+jB+j^2C|$ then determine A uniquely as the circumcentre of the triangle Δ, since A is equidistant from its vertices.

If A has coordinates (x_0, y_0) relative to our basis $(2+j^2, 1+2j^2)$ for $\mathbb{C}$, then the matrix with given second and third rows $(x_1\, y_1\, z_1)$, $(x_2\, y_2\, z_2)$ and first row equal to $(x_0 \;\; y_0 \;\; -x_0-y_0)$ has rows with zero sums and satisfies $|A+B+C| = |A+j^2B+jC| = |A+jB+j^2C| \neq 0$. It just remains to multiply the matrix by the unique $q \in \mathbb{R}_+$ which makes the sum of the absolute values of all the entries equal to 1. □

We now present the final result of this section, completing our solution to Problem 5.3.4.

Theorem 5.3.13 *Let $G = (V, E)$ be a cubic graph with half-edge set H, partition $\mathcal{E}$ by edges and partition $\mathcal{V}$ by vertices into blocks of size 3, each block of $\mathcal{V}$ given an arbitrary linear order.*

Let there be given the second and third rows of a 3×3 matrix over $\mathbb{R}$

$$\begin{pmatrix} \cdot & \cdot & \cdot \\ x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \end{pmatrix}$$

with the properties that $x_1 + y_1 + z_1 = 0 = x_2 + y_2 + z_2$, $(x_1, y_1, z_1) \neq (0,0,0)$ *and the row* $(x_1\ \ y_1\ \ z_1)$ *is a cyclic permutation of the row* $(x_2\ \ y_2\ \ z_2)$.

Then for an infinite set of values of $(x_0, y_0, z_0) \in \mathbb{R}^3$ *and* $q \in \mathbb{R}_+$ *uniquely determined by* (x_0, y_0, z_0),

$$\pm \begin{pmatrix} qx_0 & qy_0 & qz_0 \\ qx_1 & qy_1 & qz_1 \\ qx_2 & qy_2 & qz_2 \end{pmatrix}$$

is the coefficient matrix of a parity weight for which the correlation between the events "Equivalent" and "Same Parity" is equal to $P(L(G); 3)$ *up to a factor dependent only on* $|E|$. *The values of* (x_0, y_0, z_0) *and* q *satisfy the following:*

(1) if $(x_1\ \ y_1\ \ z_1)$ *is not a scalar multiple of* $(1\ \ 1\ \ -2)$ *or one of its cyclic permutations, then there exists a line* ℓ *and an octagon in the complex plane such that* $x_0 + j^2 y_0 + j z_0$ *lies on* ℓ *and* $\pm q(x_0 + j^2 y_0 + j z_0)$ *lie on the boundary of the octagon,*

(2) if $(x_1\ \ y_1\ \ z_1)$ *is a scalar multiple of* $(1\ \ 1\ \ -2)$ *or one of its cyclic permutations, then there exists a line* ℓ *and a hexagon in the complex plane such that* $x_0 + j^2 y_0 + j z_0$ *lies on* ℓ *and* $\pm q(x_0 + j^2 y_0 + j z_0)$ *lie on the boundary of the hexagon.*

In particular, the first row of the coefficent matrix of a parity weight of the above form has each of its three entries given by one of three or four linear functions of q, *the values of* q *ranging over some subinterval of* $\mathbb{R}_+$.

Before embarking on the proof of Theorem 5.3.13 we remark that the result of Corollary 5.3.10 is the excluded case $(x_1\ \ y_1\ \ z_1) = (\,0\ \ 0\ \ 0\,)$ in this theorem. The locus of all points in the complex plane which correspond to the first row of the coefficient matrices for the parity weights of Corollary 5.3.10 traces the boundaries of a pair of hexagons, which compares with the single hexagon described in case *(2)* of Theorem 5.3.13. The points $\{q + j^2(\frac{1}{2} - q) - \frac{1}{2}j : 0 < q \leq \frac{1}{2}\}$ trace one side of a hexagon in the complex plane, the points $\{-\frac{1}{2} + j^2 q - j(\frac{1}{2} - q) : 0 < q \leq \frac{1}{2}\}$ and $\{(\frac{1}{2} - q) - j^2\frac{1}{2} - jq : 0 < q \leq \frac{1}{2}\}$ two further sides, and the negatives the opposite sides to these three. The conjugates of all these points form a distinct congruent hexagon.

Proof. We divide the proof of Theorem 5.3.13 into two steps. First, we find all points (x_0, y_0) in the complex plane (coordinates relative to the basis $(2 + j^2, 1 + 2j^2)$) which lead to the condition $|A + B + C| = |A + j^2 B + jC| = |A + jB + j^2 C| \neq 0$ being satisfied, where A, B, C are defined as in the proof of Theorem 5.3.11. Second, we describe the points (x_0, y_0) in terms of a real parameter r and then use the equation $q^{-1} = |x_0| + |y_0| + |z_0| + |x_1| + |y_1| + |z_1| + |x_2| + |y_2| + |z_2|$ defining $q \in \mathbb{R}_+$ to express r in terms of q and thereby obtaining a "piecewise linear" parametrisation of (x_0, y_0) in q.

Since the second and third rows $(x_1\ y_1\ z_1)$, $(x_2\ y_2\ z_2)$ given in the hypothesis of Theorem 5.3.11 are cyclic permutations of each other, we have either $B = C$ or $B = jC$ or $B = j^2C$. By Lemma 5.3.12 we may assume $B, C \neq 0$. Then two of the points $-(B+C), -(j^2B+jC), -(jB+j^2C)$ coincide and the third point is different. If $B = C$ then the two points are $-2B, B$, if $B = jC$ the two points are $-2jB, jB$ and if $B = j^2C$ then the two points are $-2j^2B, j^2B$. We will denote by B', C' a fixed pair of distinct points amongst these three possible pairs. The locus of points A equidistant from B' and C' is the perpendicular bisector of B' and C', and we will denote this line by ℓ. The line ℓ does not pass through the origin O since the point on ℓ closest to O is the midpoint of the line segment $B'C'$ passing through O, and this midpoint is one of $\frac{3}{2}B, \frac{3}{2}jB$, or $\frac{3}{2}j^2B$ according as $B = C, B = jC$ or $B = j^2C$.

We choose a point $A = (x_0, y_0)$ on ℓ arbitrarily, and then take the unique value of q satisfying

$$q^{-1} = |x_0| + |y_0| + |-x_0 - y_0| + 2|x_1| + 2|y_1| + 2|z_1|.$$

The signs of x_1, y_1, z_1 are fixed since the point $B = (x_1, y_1)$ is fixed. We define the *sign function* by $\mathrm{sgn}(x) = +$ if $x \geq 0$ and $\mathrm{sgn}(x) = -$ if $x < 0$ and note that there are one of six possible *sign triples* associated with the three reals $x_0, y_0, -x_0 - y_0$:

$$\mathrm{sgn}(x_0), \mathrm{sgn}(y_0), \mathrm{sgn}(-x_0 - y_0) \in \{++-, +--, +-+, --+, -++, -+-\}.$$

When (x_0, y_0) ranges over the points on the line ℓ the number of possible sign triples is reduced, this number depending on whether or not ℓ is parallel to one of the lines $x = 0, y = 0, x + y = 0$ in the (x, y)-plane with basis $(2 + j^2, 1 + 2j^2) = (ij^2\sqrt{3}, -i\sqrt{3})$.

Lemma 5.3.14 *The perpendicular bisector ℓ of B' and C' is parallel with one of the two axes $x = 0, y = 0$ or the line $x + y = 0$ if and only if B is a real scalar multiple of either $1, j$ or j^2. The latter occurs if and only if the row $(x_1\ \ y_1\ z_1)$ is a scalar multiple of $(1\ \ 1\ \ -2)$ or one of its cyclic permutations.*

When ℓ is parallel to one of the lines $x = 0, y = 0$ or $x + y = 0$ there are exactly 3 sign triples in the set

$$\{\mathrm{sgn}(x_0),\ \mathrm{sgn}(y_0),\ \mathrm{sgn}(-x_0 - y_0) : \text{ points } (x_0, y_0) \text{ lying on } \ell\}.$$

Otherwise, when ℓ is not parallel to any of the three lines $x = 0, y = 0, x + y = 0$, there are exactly 4 sign triples in this set.

Proof. We will use complex numbers to describe the direction of a line in the (x, y)-plane: two lines ℓ', ℓ'' in the complex plane lie in the same direction if the difference between any two points on ℓ' as a complex number is a scalar multiple of the difference between any two points on ℓ'' as a complex number.

According as $B = C, jC, j^2C$ we have the three cases $(B', C') = (B, -2B), (jB, -2jB)$ or $(j^2B, -2j^2B)$, for which the direction of the perpendicular bisector ℓ is given respectively by iB, ijB or ij^2B. The direction of the axis $y = 0$ is given by $-i\sqrt{3}$, the direction of the axis $x = 0$ is given by $ij^2\sqrt{3}$, and the direction of the line $x + y = 0$ is given by $1+2j^2-2-j^2 = -ij\sqrt{3}$. This yields the stated condition on B. By equating coefficients of 1 and j^2 in the equations $2x_1 + y_1 + j^2(x_1 + 2y_1) = 1, j(= -1 - j^2)$ or j^2 respectively we find that B is a scalar multiple of $1, j$ or j^2 if and only if $(x_1 \;\; y_1 \; z_1)$ is a scalar multiple of $(1 \;\; -2 \;\; 1), (1 \;\; 1 \;\; -2)$ or $(-2 \;\; 1 \;\; 1)$ respectively.

Two non-parallel lines in the complex plane meet in exactly one point. Since ℓ does not pass through the origin it does not coincide with any of the lines $x = 0, y = 0, x+y = 0$. If ℓ is parallel to one of $x = 0, y = 0, x+y = 0$ then it meets each of the other two, otherwise it meets all three lines, each in one point. This implies that ℓ passes through respectively 3 or 4 of the 6 regions of the plane bounded by the lines $x = 0, y = 0, x + y = 0$ removed. Corresponding to these 3 or 4 regions are 3 respectively 4 sign triples for $x_0, y_0, -x_0-y_0$.$\square$

We now move to the second stage of the proof of Theorem 5.3.13, using a parametrisation of the line ℓ in order to parametrise the point (qx_0, qy_0) in terms of q and x_1, y_1.

A point (x_0, y_0) lies on the perpendicular bisector ℓ of B' and C' if and only if $2x_0+y_0+j^2(x_0+2y_0)-\frac{B'+C'}{2}$ is a scalar multiple of $i(B'-C')$. The line ℓ thus has a parametrisation of the form

$$x_0 = a_0r + b_0, \quad y_0 = c_0r + d_0, \qquad r \in \mathbb{R},$$

where $a_0, b_0, c_0, d_0 \in \mathbb{R}$ depend on x_1, y_1. Then q is defined by

$$q^{-1} = (\pm a_0 \pm c_0 \pm (-a_0 - c_0))r + (\pm b_0 \pm d_0 \pm (-b_0 - d_0)) + 2|x_1| + 2|y_1| + 2|x_1 + y_1|,$$

where both choices of the three signs $\pm\pm\pm$ correspond to the sign triple of $x_0, y_0, -x_0-y_0$.

First we consider the case where ℓ is not parallel to any of the lines $x = 0, y = 0, x+y = 0$, when we have $a_0 \neq 0, c_0 \neq 0$ and $a_0 + c_0 \neq 0$. Then $\pm a_0 \pm c_0 \pm (-a_0 - c_0) \neq 0$ since the sign triples $+++, ---$ are impossible for $x_0, y_0, -x_0 - y_0$.

The line ℓ passes through 4 of the 6 regions of the plane bounded by the lines $x = 0, y = 0, x+y = 0$. A set of points on ℓ for which the triple of signs $\mathrm{sgn}(x_0), \mathrm{sgn}(y_0), \mathrm{sgn}(z_0)$ is constant is a line segment. In terms of the parametrisation of ℓ in r, the set of $r \in \mathbb{R}$ for which the sign triple $\mathrm{sgn}(a_0r + b_0), \mathrm{sgn}(c_0r + d_0), \mathrm{sgn}(-(a_0 + c_0)r - b_0 - d_0)$ is constant is one amongst two bounded half-open intervals, one open unbounded interval and one closed unbounded interval, which together partition $\mathbb{R}$. Restricting r to any one of these 4 intervals, we can solve the equation for q to obtain an equation for r linear in q^{-1}. Since x_0, y_0 are linear in r, we then have $qx_0 = ar + b, qy_0 = cq + d$ for some a, b, c, d dependent on x_1, y_1 and for q lying in a bounded subinterval of $(0, \infty)$. In fact, since $q > 2|x_1| + 2|y_1| + 2|x_1 + y_1|$, q must lie in a subinterval of $(0, q_0)$ where

$q_0^{-1} = 2|x_1| + 2|y_1| + 2|x_1 + y_1|$

Taking each of the 4 intervals on which $x_0, y_0, -x_0 - y_0$ have constant sign triple, we obtain 4 parametrisations for the row $(qx_0 \;\; qy_0 \;\; qz_0)$ with entries in terms of x_1, y_1 and linear in q, where q ranges over one of the 4 subintervals of $(0, q_0)$. By taking the points (x_0, y_0) on ℓ continuously the values of q are taken continuously. Hence the 4 different parametrisations for the point (qx_0, qy_0) in the complex plane describe a connected curve consisting of 4 straight line segments and the locus of the points $\pm(qx_0, qy_0)$ is the boundary of an octagon.

The case where ℓ is parallel to one of $x = 0, y = 0, x + y = 0$ is proved in an entirely similar way. The parameter q is similarly defined and the locus of the points $\pm(qx_0, qy_0)$ as (x_0, y_0) ranges over ℓ is the boundary of a hexagon since there are now just 3 different sign triples. We note that $a_0 = 0$ if ℓ is parallel with $x = 0$, $c_0 = 0$ if ℓ is parallel with $y = 0$ and $a_0 + c_0 = 0$ if ℓ is parallel to $x + y = 0$. This implies that for one interval of values of r for which the sign triple $\mathrm{sgn}(a_0r + b_0)$, $\mathrm{sgn}(c_0r + d_0)$, $\mathrm{sgn}(-(a_0 + c_0)r - b_0 - d_0)$ is constant, the parameter q is not a function of r^{-1} but equal to a constant (dependent on x_1, y_1). This constant coincides with the appropriate constant value of $x_0 = a_0r + b_0$, $y_0 = c_0r + d_0$ or $z_0 = -(a_0 + c_0)r - b_0 - d_0$ according as ℓ is parallel to $x = 0, y = 0$ or $x + y = 0$.□

We finish this section with an illustration of Theorem 5.3.13.

Example 5.3.15 *We find all parity weights with the property that the correlation between the events "Equivalent" and "Same Parity" is equal to $P(L(G); 3)$ up to a factor dependent only on $|V|$ and whose coefficient matrix (up to negation and permutation of rows and columns) have two rows given up to scalar multiples by*

$$\begin{pmatrix} \cdot & \cdot & \cdot \\ 1 & 1 & -2 \\ 1 & -2 & 1 \end{pmatrix}.$$

Here we have $B = -3j = j^2C \neq 0$, and the perpendicular bisector ℓ of the points $j^2B = -3, -2j^2B = 6$ in the complex plane is parallel to the axis $x = 0$ (relative to the basis $(2 + j^2, 1 + 2j^2)$). By Theorem 5.3.13 we will find 3 parametrisations in terms of a positive real q. The line ℓ has parametrisation $\{(1, r) : r \in \mathbb{R}\}$ in the complex plane (relative to the basis $(2 + j^2, 1 + 2j^2)$) and the sign triples of $1, r, -1 - r$ switch to different sign triples at $r = 0$ and $r = -1$. We then solve for r each of the 3 equations

$$q^{-1} = 2r + 10, \qquad 0 \leq r,$$

$$q^{-1} = -2r + 10, \qquad -1 \leq r < 0,$$

$$q^{-1} = -2r + 6, \qquad r < -1,$$

finding that the first row $(qx_0 \ qy_0 \ qz_0) = (q, qr, -q - qr)$ *has one of the following* 3 *parametrisations:*

$$(q \quad \frac{1}{2} - 5q \quad 4q - \frac{1}{2}), \qquad 0 < q \leq \frac{1}{10},$$

$$(q \quad 5q - \frac{1}{2} \quad \frac{1}{2} - 6q), \qquad \frac{1}{12} < q \leq \frac{1}{10},$$

$$(q \quad 3q - \frac{1}{2} \quad \frac{1}{2} - 4q), \qquad 0 < q \leq \frac{1}{8}.$$

We thus find that the parity weight of Theorem 5.3.3 is the special case $q = \frac{1}{12}$ *of the set of parity weights with coefficient matrix given by*

$$\begin{pmatrix} q & \frac{1}{2} - 5q & 4q - \frac{1}{2} \\ q & q & -2q \\ q & -2q & q \end{pmatrix}, \qquad 0 < q \leq \frac{1}{10}.$$

5.4 Vertex 3-colouring arbitrary graphs

In this section we reverse the rôles of edges and vertices to those played in §5.3 by considering parity edge weights and induced vertex colourings, rather than parity vertex weights and induced edge colourings, and by doing so derive probabilistic criteria for a graph to have a proper vertex 3-colouring.

Let $G = (V, E)$ be an arbitrary graph with half-edge set H, $\mathcal{E} = \{H(e) : e \in E\}$ the partition of H by edges into blocks of size 2, and $\mathcal{V} = \{H(v) : v \in V\}$ the partition of H by the vertices of G. For each $v \in V$, the block $H(v)$ has size equal to the degree of v. A linear order on each block of $\mathcal{E}$ is determined by any fixed orientation of G, much as a linear order on each block of $\mathcal{V}$ in §5.3 was made consistently with a fixed set of local vertex rotations of G.

A half-edge colouring $\mu : H \to Z_3$ is the same as a Z_3^2-colouring of $\mathcal{E}$. We recall from Definition 5.2.1 that any given probability weight $\delta^{\mathcal{E}} : Z_3^H \to \mathbb{R}$ is defined on each block of $\mathcal{E}$ by a function $\delta : Z_3^2 \to \mathbb{R}$ with the property that $|\delta|(Z_3^2) = 1$ and that $\delta^{\mathcal{E}}$ is a parity weight if and only if $\delta(Z_3^2) = 0$ (Definition 5.2.2).

A pair of colours $\ell \in Z_3^2$ is *positive* or *negative* with respect to δ according as $\delta(\ell) > 0$ or $\delta(\ell) < 0$. Given a half-edge colouring $\mu : \mathcal{E} \to Z_3^2$, we call an edge e *positive* in the colouring μ if $\delta(\mu_e) > 0$ and *negative* if $\delta(\mu_e) < 0$. The *parity* of a half-edge colouring $\mu \in Z_3^H$ for which each edge is either positive or negative is then *even* if the number of negative edges is even and *odd* otherwise. A half-edge colouring $\mu : H \to Z_3$ *induces* a (not necessarily proper) vertex 3-colouring $\mu_{\mathcal{V}} : V \to Z_3$ of G which colours a vertex $v \in V$ with the sum in Z_3 of the colours of the half-edges in $H(v)$. Two half-edge colourings are *equivalent* if they induce the same vertex colouring.

We ask a similar question to that posed in §5.3: *Which parity edge weights on the half-edge colourings of a graph G with a fixed orientation have the property that the correlation between* "Equivalent" *(same induced vertex colouring) and* "Same Parity" *(same number of negative edges modulo 2) is up to a factor, dependent only on the size of G, equal to the number of proper vertex 3-colourings of G?* We use Lemma 5.4.1 below to show how the parity vertex weights of Theorems 5.3.10, 5.3.11 and 5.3.13 allow us to immediately deduce the answer.

Lemma 5.4.1 *For $\ell = (l_0, l_1, l_2) \in Z_3^3$ and a null triple $m = (m_0, m_1, -m_0 - m_1) \in Z_3^3$ we have*

$$\mathtt{t}\alpha_{\underline{\ell}}(m) = 3\mathtt{t}\alpha_{\ell'}(m'),$$

where $\ell' = (l_0 - l_2, l_1 - l_2) \in Z_3^2$ is uniquely determined by $\underline{\ell}$ and $m' = (m_0, m_1) \in Z_3^2$.

Moreover, the function

$$\gamma = \sum_{\underline{\ell} \in Z_3^3/\underline{000}} b_\ell \alpha_{\underline{\ell}}$$

satisfies $\gamma(Z_3^3) = 0$ and $|\mathtt{t}\gamma| = 3d\alpha_{\pm\underline{012}}$ for some $d \neq 0$ if and only if the function

$$\delta = \sum_{\ell' \in Z_3^2} b_\ell \alpha_{\ell'}$$

satisfies $\delta(Z_3^2) = 0$ and $|\mathtt{t}\delta| = d\alpha_{\pm\underline{01}}$.

Proof. When m is not a null triple we have $\mathtt{t}\alpha_{\underline{\ell}}(m) = j^{\langle \ell, m\rangle}(1 + j^{\langle 111, m\rangle} + j^{-\langle 111, m\rangle}) = 0$, and if m is null this equation gives $\mathtt{t}\alpha_{\underline{\ell}}(m) = 3\mathtt{t}\alpha_\ell(m)$.

The map $m \mapsto m'$ removing the last component of m is an isomorphism of the subspace of null triples in Z_3^3 with Z_3^2 and maps the set $\underline{012} \cup \underline{021}$ to the set $\underline{01} \cup \underline{10}$. Explicitly, $m \mapsto m'$ is given by

$$(000, 111, 222,\ 012, 120, 201,\ 021, 210, 102) \leftrightarrow (00, 11, 22,\ 01, 12, 20,\ 02, 21, 10),$$

where the elements in Z_3^2 have been grouped by cosets of the subspace of monochrome pairs $\{00, 11, 22\}$, and we see that proper triples are mapped to proper pairs and monochrome triples to monochrome pairs.

The map $(l_0, l_1, l_2) \mapsto (l_0 - l_2, l_1 - l_2)$ is well-defined as a map $\underline{(l_0, l_1, l_2)} \mapsto (l_0 - l_2, l_1 - l_2)$ on the cosets $\underline{\ell}$ of monochrome triples and the latter is an isomorphism of the vector space $Z_3^3/\underline{000}$ with the vector space Z_3^2. The bijection $\underline{\ell} \mapsto \ell'$ is given by

$$(\underline{000}, \underline{012}, \underline{021},\ \underline{100}, \underline{001}, \underline{010},\ \underline{200}, \underline{020}, \underline{002}) \leftrightarrow (00, 12, 21,\ 10, 22, 01,\ 20, 02, 11),$$

where we have grouped the elements of Z_3^2 by cosets of the set of null pairs $\{00, 12, 21\}$.

For null $m = (m_0, m_1, -m_0 - m_1) \in Z_3^3$,

$$\mathtt{t}\alpha_{\underline{(l_0,l_1,l_2)}}((m_0, m_1, -m_0 - m_1)) = 3\mathtt{t}\alpha_{(l_0,l_1,l_2)}((m_0, m_1, -m_0 - m_1)),$$

and since

$$\langle\, (l_0, l_1, l_2), (m_0, m_1, -m_0 - m_1)\,\rangle = l_0m_0 + l_1m_1 - l_2m_0 - l_2m_1$$

$$= (l_0 - l_2)m_0 + (l_1 - l_2)m_1 = \langle\, (l_0 - l_2, l_1 - l_2), (m_0, m_1)\rangle,$$

it follows that

$$\mathtt{t}\alpha_{\underline{(l_0,l_1,l_2)}}((m_0, m_1, -m_0 - m_1)) = 3\mathtt{t}\alpha_{(l_0-l_2,l_1-l_2)}((m_0, m_1)).$$

This gives our result that $\mathtt{t}\alpha_{\underline{\ell}}(m) = \mathtt{t}\alpha_{\ell'}(m')$ where ℓ' is obtained from $\underline{\ell}$ by the isomorphism $\underline{(l_0, l_1, l_2)} \mapsto (l_0 - l_2, l_1 - l_2)$ and m' from m by the isomorphism $(m_0, m_1, -m_0 - m_1) \mapsto (m_0, m_1)$.

The remainder of the lemma follows since $m \mapsto m'$ maps proper triples to proper pairs and so given the hypothesis that $|\mathtt{t}\gamma(m)|$ is constant for $m \in \underline{012} \cup \underline{021}$ the same is true of $|\mathtt{t}\delta(m')|$ for $m' \in \underline{01} \cup \underline{10}$. Similarly, $|\mathtt{t}\gamma(m)| = 0$ when $m \in Z_3^3$ is monochrome if and only if $|\mathtt{t}\delta(m')| = 0$ for monochrome $m' \in Z_3^2$. □

As a consequence of Lemma 5.4.1, we see that if the vertex weight $\gamma^{\mathcal{V}}$ on the set of half-edge colourings of a cubic graph is given on blocks of $\mathcal{V}$ by

$$\gamma = \sum_{\underline{\ell} \in Z_3^3/\underline{000}} b_\ell \alpha_{\underline{\ell}},$$

and the edge weight $\delta^{\mathcal{E}}$ on the set of half-edge colourings of an arbitrary graph is given on blocks of $\mathcal{E}$ by

$$\delta = \sum_{\ell' \in Z_3^2} b_\ell \alpha_{\ell'},$$

where ℓ' is given by the isomorphism $\underline{\ell} \mapsto \ell'$ in the proof of Lemma 5.4.1, then there is $d \neq 0$ such that $|\mathtt{t}\gamma^{\mathcal{V}}| = (3d)^{|V|}\alpha^{\mathcal{V}}_{\text{proper}}$ if and only if $|\mathtt{t}\delta^{\mathcal{E}}| = d^{|E|}\alpha^{\mathcal{E}}_{\text{proper}}$ for the same value of d. Thus, if γ is a parity vertex weight on the half-edge colourings of an *cubic graph* $G = (V, E)$ (with a linear order on each block of its half-edge partition $\mathcal{V}$ by vertices), so that we have, by Theorem 5.2.7,

$$\Pr(\text{Equivalent} \mid \text{Same Parity}) - \Pr(\text{Equivalent}) = 3^{-|E|}d^{2|V|}P(L(G); 3),$$

where "Equivalent" means the same induced edge colouring and "Same Parity" means the same number of negative vertices modulo 2, then δ as defined above is a parity weight on the half-edge colourings of an *arbitrary graph* $G = (V, E)$ (with a linear order on each block of its half-edge partition $\mathcal{E}$ by edges, *i.e.* an orientation of its edges) and, by Theorem 5.2.7,

$$\Pr(\text{Equivalent} \mid \text{Same Parity}) - \Pr(\text{Equivalent}) = 3^{-|V|} d^{2|E|} P(G; 3),$$

where "Equivalent" in the last statement means the same induced vertex colouring and "Same Parity" means the same number of negative edges modulo 2.

We summarise these conclusions in Theorem 5.4.2 below. For an arbitrary parity edge weight $\delta^{\mathcal{E}}$ given on each block of $\mathcal{E}$ by the function

$$\delta = \sum_{\ell \in Z_3^2} a_\ell \alpha_\ell,$$

we write the coefficients $\{a_\ell : \ell \in Z_3^2\}$ in a 3×3 matrix

$$\begin{pmatrix} a_{00} & a_{12} & a_{21} \\ a_{10} & a_{22} & a_{01} \\ a_{20} & a_{02} & a_{11} \end{pmatrix},$$

so that the entry $a_{\ell'}$ is in the same place as the entry $b_{\underline{\ell}}$ in the coefficient matrix of a parity vertex weight, where $\underline{\ell} \mapsto \ell'$ is the isomorphism of Lemma 5.4.1.

Theorem 5.4.2 *Let $\delta^{\mathcal{E}}$ be a parity $\mathcal{E}$-weight on the set of half-edge colourings of a graph $G = (V, E)$ with a fixed orientation which determines a probability distribution on the set of half-edge colourings of G for which*

$$\Pr(\text{Equivalent} \mid \text{Same Parity}) - \Pr(\text{Equivalent}) = 3^{-|V|} d^{2|E|} P(G; 3),$$

where $d \neq 0$ is constant.

Then δ has coefficient matrix given by

$$\begin{pmatrix} a_{00} & a_{12} & a_{21} \\ a_{10} & a_{22} & a_{01} \\ a_{20} & a_{02} & a_{11} \end{pmatrix} = \begin{pmatrix} b_{000} & b_{012} & b_{021} \\ b_{100} & b_{001} & b_{010} \\ b_{200} & b_{020} & b_{002} \end{pmatrix}$$

where the matrix on the right-hand side is the coefficient matrix of a parity $\mathcal{V}$-weight determining a probability distribution on the set of half-edge colourings of a cubic graph $G = (V, E)$ for which $\Pr(\text{Equivalent} \mid \text{Same Parity}) - \Pr(\text{Equivalent}) = 3^{-|E|} d^{2|V|} P(L(G); 3)$ *for the same constant d.*

We now consider an example of a parity edge weight satisfying the conditions of Theorem 5.4.2 which is related to Matiyasevich's parity vertex weight $\frac{1}{6}\rho = \frac{1}{6}(\alpha_{\underline{012}} - \alpha_{\underline{021}})$ on the set of half-edge colourings of a cubic graph. Through the isomorphism $\underline{\ell} \mapsto \ell'$ of Lemma 5.4.1 we obtain from the parity vertex weight $\frac{1}{6}\rho$ the parity edge weight $\delta = \frac{1}{2}(\alpha_{12} - \alpha_{21})$, which has transform $\mathtt{t}\delta = -\frac{i\sqrt{3}}{2}(\alpha_{\underline{01}} - \alpha_{\underline{10}})$. Before proceeding to deduce the correlations between "Equivalent" and the various parity events from Theorem 5.3.1 by appealing to Theorem 5.4.2, we introduce some preliminary terminology in order to be able to interpret our results.

We consider the elements $0, 1, 2$ of Z_3 in the cyclic order $(\,0\ \ 1\ \ 2\,)$. If the elements of Z_3 are the beads of a necklace with $0, 1, 2$ appearing in clockwise order then cyclically consecutive elements of Z_3 are obtained by moving clockwise around the necklace.[3] Each block $H(e)$ of the partition $\mathcal{E}$ of H by edges is put in a linear order consistent with the fixed orientation ω of G. A half-edge colouring μ *weakly preserves* the orientation ω of the edge e if μ_e is monochrome and *preserves* the orientation ω if μ_e colours $H(e)$ properly with two colours in cyclic order (i.e. $\mu_e \in \underline{01}$). Otherwise, when μ_e properly colours $H(e)$ against the cyclic order of Z_3 (i.e. $\mu_e \in \underline{10}$), we say that μ *reverses* the orientation ω of the edge e .

A $\mathcal{V}$-monochrome $\mathcal{E}$-proper half-edge colouring μ is a proper vertex 3-colouring of G which on each edge of G either preserves or reverses the orientation ω. Under the parity weight $\frac{1}{6}(\alpha_{\underline{01}} - \alpha_{\underline{10}})$, a proper vertex colouring is *even* or *odd* according as it reverses an even or odd number of edges. This is comparable to a proper edge 3-colouring of a cubic graph being even or odd according as it preserves or reverses the clockwise rotation of an even or odd number of vertices.

Interpreting half-edge colourings of G which colour each block of $\mathcal{E}$ either with 12 (preserving ω) or with 21 (reversing ω) as orientations of G, we have the following consequence of Theorem 5.4.2 and Theorem 5.3.1:

Theorem 5.4.3 *Let G be any graph and let a given orientation of G induce a colouring of the vertices of G with elements of Z_3 by colouring a vertex with its indegree minus its outdegree modulo 3. If we choose two orientations of G u.a.r., then*

$$\Pr(\text{Equivalent} \mid \text{Same Parity}) - \Pr(\text{Equivalent}) = 3^{|E|-|V|}4^{-|E|}P(G;3),$$

where two orientations are equivalent if they induce the same vertex colouring and have the same parity if they differ on an even number of edges.

By permuting rows and columns of the coefficient matrix to obtain the parity edge

[3]In Chapter 6 we shall use the identification of elements of Z_k with beads of a necklace in order to distinguish different types of k-colourings, of which the clockwise/anticlockwise order of two or three distinct colours is an instance.

weight $\frac{1}{2}(\alpha_{01}-\alpha_{10})$, the same theorem holds with "indegree minus outdegree modulo 3" replaced by "indegree modulo 3". This provides a probabilistic interpretation of Alon and Tarsi's result [7] that $\|f_3(G)\|_2^2 = 3^{|E|-|V|}P(G;3)$. By Lemma 4.2.2 of Chapter 4, the coefficients of $f_3(G)$ are given by $\sigma^{\mathcal{E}}([\lambda]_{\mathcal{V}})$ for vertex colourings $\lambda : V \to Z_3$, where $\sigma = \alpha_{01}-\alpha_{10}$. Note also that this extends a result of Matiyasevich [46], who has Theorem 5.4.3 with "outdegree modulo 3", but only considers planar cubic graphs (due to the fact that he is seeking equivalents to the Four Colour Theorem).

An immediate corollary of Theorem 5.4.3 is that the correlation between "Equivalent" and "Same Parity" is always positive for a graph which has a proper vertex 3-colouring. Welsh [69] has an alternative proof of this fact, with independent proofs of both Theorem 5.4.3 and Theorem 5.4.5 below.

Given a graph G with a fixed orientation ω, for two orientations chosen u.a.r. the event "Both Even" occurs when both orientations switch the direction of an even number of edges in the orientation ω. The event "Both Odd" is similarly defined. The weight $|\delta| = \frac{1}{2}\alpha_{\pm 12}$ has transform given by $\mathtt{t}|\delta| = \frac{1}{2}(2\alpha_{\underline{00}} - \alpha_{\pm\underline{01}})$. By Theorem 5.2.8, if $|E|$ is even then

$$\Pr(\text{Equivalent} \mid \text{Both Even}) - \Pr(\text{Equivalent} \mid \text{Both Odd})$$

$$= (-1)^{|E|/2}3^{|E|/2-|V|}4^{1-|E|}(\#\{\text{Even proper vertex 3-colourings}\}-\#\{\text{Odd proper vertex 3-colourings}\}),$$

If $|E|$ is odd this correlation equals 0 for the following reason. If the pair of half-edge colourings $\mu, \mu' : \mathcal{E} \to \{\pm 12\}$ are equivalent and both even, inducing the vertex colouring $\lambda : V \to Z_3$, then the pair of half edge colourings $-\mu, -\mu'$ are still equivalent (inducing the vertex colouring $-\lambda$) but now both of them are odd since the sign of the parity weight $\delta^{\mathcal{E}}$ has changed on each of the odd number of blocks of $\mathcal{E}$. In other words, for each $\lambda : V \to Z_3$ we have a bijection between the set $[\lambda]_{\mathcal{V}} \times [\lambda]_{\mathcal{V}} \cap$ Both Even and the set $[-\lambda]_{\mathcal{V}} \times [-\lambda]_{\mathcal{V}} \cap$ Both Odd. Since Pr(Both Even) = Pr(Both Odd), this implies

$$\sum_{\lambda:V\to Z_3} [\Pr([\lambda]_{\mathcal{V}} \times [\lambda]_{\mathcal{V}} \mid \text{Both Even}) - \Pr([-\lambda]_{\mathcal{V}} \times [-\lambda]_{\mathcal{V}} \mid \text{Both Odd})] = 0.$$

With

$$\text{Equivalent} = \bigcup_{\lambda:V\to Z_3} [\lambda]_{\mathcal{V}} \times [\lambda]_{\mathcal{V}} = \bigcup_{\lambda:V\to Z_3} [-\lambda]_{\mathcal{V}} \times [-\lambda]_{\mathcal{V}},$$

we conclude that when $|E|$ is odd

$$\Pr(\text{Equivalent} \mid \text{Both Even}) - \Pr(\text{Equivalent} \mid \text{Both Odd}) = 0.$$

We will interpret the correlation $\Pr(\text{Equivalent} \mid \text{Both Even})-\Pr(\text{Equivalent} \mid \text{Both Odd})$ for the special case of a line graph of a cubic graph, thereby exhibiting the connection with the similar correlation of Theorem 5.3.1. Let $L(G) = (E, L)$ be the line graph of a cubic graph $G = (V, E)$ with a set of local vertex rotations. If we give $L(G)$ the Eulerian orientation ω determined by the local vertex rotations of G as described in Chapter 4, §4.2, then a proper vertex 3-colouring of $L(G)$ either preserves ω on all three edges of a triangle $\{e_0, e_1\}, \{e_1, e_2\}, \{e_2, e_0\}$ of $L(G)$ for which e_0, e_1, e_2 are three mutually incident edges in G, or reverses ω on all three edges of such a triangle of $L(G)$. Three preserved edges on a triangle $\{e_0, e_1\}, \{e_1, e_2\}, \{e_2, e_0\}$ of $L(G)$ corresponds to the three colours $0, 1, 2$ appearing in a clockwise order on the vertices e_0, e_1, e_2 around the triangle, and three reversed edges to an anticlockwise order. A proper vertex 3-colouring of $L(G)$ is a proper edge 3-colouring of G, and clockwise (anticlockwise) triangles $\{e_0, e_1\}, \{e_1, e_2\}, \{e_2, e_0\}$ of $L(G)$ in the proper vertex 3-colouring of $L(G)$ correspond to clockwise (anticlockwise) vertices in the proper edge 3-colouring of G.

Since $|L| = 2|E|$ is even, in the probability distribution determined by the edge parity weight $\frac{1}{2}(\alpha_{12} - \alpha_{21})$ on half-edge colourings of $L(G)$ we have

$$\Pr(\text{Equivalent} \mid \text{Both Even}) - \Pr(\text{Equivalent} \mid \text{Both Odd})$$

$$= (-1)^{|E|} 4^{1-2|E|} (\#\{\text{Even proper vertex 3-colourings of } L(G)\} - \#\{\text{Odd proper vertex 3-colourings of } L(G)\}).$$

This yields the following theorem, which provides a probabilistic interpretation of the constant term of $f_3(L(G))$ (see Theorems 4.3.5 and 4.3.6):

Theorem 5.4.4 *Let $G = (V, E)$ be a cubic graph with a set of local vertex rotations, $L(G)$ its line graph and ω the orientation of $L(G)$ determined by the local vertex rotations of G. An orientation of $L(G)$ is even or odd according as it differs from ω on an even or odd number of edges and induces a vertex 3-colouring of $L(G)$ by colouring a vertex with its indegree minus its outdegree modulo 3.*

If we choose any two orientations of $L(G)$ u.a.r. then

$$\Pr(\text{Equivalent} \mid \text{Both Even}) - \Pr(\text{Equivalent} \mid \text{Both Odd})$$

$$= (-1)^{|E|} 4^{1-2|E|} (\#\{\text{Even proper edge 3-colourings of } G\} - \#\{\text{Odd proper edge 3-colourings of } G\}),$$

where a proper edge 3-colouring of G is even or odd according as the number of anticlockwise vertices is even or odd. In particular, if G has set of local vertex rotations corresponding to an embedding of G on the plane then this correlation equals $4^{1-2|E|} P(L(G); 3)$.

Finally, we determine the value of Pr(Equivalent) under the probability distribution defined by the probability edge weight $|\delta| = \frac{1}{2}\alpha_{\pm 12}$ in terms of an evaluation of the Tutte polynomial of G, thus completing our derivation of expressions analogous to those of Theorem 5.3.1. In order to do this we introduce the *monochrome polynomial* of a graph.

For an arbitrary graph G, positive integer k and indeterminate s, the monochrome polynomial $B(G;k,s)$ of G is defined by

$$B(G;k,s) = \sum_{\text{vertex } k\text{-colourings of } G} s^{\#\text{monochrome edges}},$$

where an edge is monochrome if its endpoints receive the same colour. Evaluating $B(G;k,s)$ at $s=0$ counts vertex k-colourings with no monochrome edges so that $B(G;k,0) = P(G;k)$. Evaluating the monochrome polynomial at $s=1$ counts all $k^{|V|}$ vertex k-colourings of G. In [68] it is shown that for any connected graph G the monochrome polynomial has the evaluation

$$B(G;k,s) = k(s-1)^{|V|-1}T(G;\frac{s+k-1}{s-1},s),$$

so that $B(G;k,s)$ is a specialisation of the Tutte polynomial. The definition of the monochrome polynomial closely relates it to the Potts partition function, where the k colours are "states" (see for example [68, §4.4]).

Theorem 5.4.5 *Let G be any graph and let a given orientation of G induce a 3-colouring of the vertices of G by colouring a vertex with its indegree minus its outdegree modulo 3. If we choose two orientations of G u.a.r., then the probability that they induce the same colouring is given by*

$$\Pr(\text{Equivalent}) = 3^{-|V|}4^{-|E|}B(G;3,4) = 4^{-|E|}T(G;2,4).$$

Proof. For the parity edge weight $\delta = \frac{1}{2}(\alpha_{12} - \alpha_{21})$ we have $\mathtt{t}|\delta| = \frac{1}{2}(2\alpha_{\underline{00}} - \alpha_{\underline{01},\underline{10}})$, so that $|\,\mathtt{t}|\delta|\,| = \frac{1}{2}(2\alpha_{\text{monochrome}} + \alpha_{\text{proper}})$. By Theorem 5.2.6,

$$\Pr(\text{Equivalent}) = 3^{-|V|}2^{-2|E|} \sum_{\mu:V\to Z_3} \left|\,(2\alpha_{\text{monochrome}} + \alpha_{\text{proper}})^{\mathcal{E}}(\mu_{\mathcal{H}})\,\right|^2$$

$$= 3^{-|V|}4^{-|E|} \sum_{\text{vertex 3-colourings}} 4^{\#\text{ monochrome edges}} = 3^{-|V|}4^{-|E|}B(G;3,4).$$

□

We remark that "indegree modulo 3" or "outdegree modulo 3" may be substituted for "indegree minus outdegree modulo 3" in Theorem 5.4.5 since $\left|\,\mathtt{t}|\alpha_{01} - \alpha_{10}|^{\mathcal{E}}\,\right|^2 = \left|\,\mathtt{t}|\alpha_{12} - \alpha_{21}|^{\mathcal{E}}\,\right|^2$.

We finish this section with an analogue of Corollary 4.6.2 which provides a criterion for the existence of a proper vertex 3-colouring of an arbitrary graph. Since $\left|\,\mathtt{t}(\alpha_{12} - \alpha_{21})^{\mathcal{E}}\,\right|^2 = 3^{|E|} = \left|\,\mathtt{t}(\alpha_{01} - \alpha_{10})^{\mathcal{E}}\,\right|^2$ we obtain the following in an entirely similar way to the proof of Theorem 4.6.1:

Theorem 5.4.6 *Let G be any graph and let a given orientation of G induce a 3-colouring of the vertices of G by colouring a vertex with its indegree modulo 3. Then G has a proper vertex 3-colouring if and only if there is a 3-colouring of the vertices of G which is induced by an odd number of orientations.*

We remark that even when $P(G;3) > 0$ the vertex 3-colourings induced by an odd number of orientations may not include any of the proper vertex 3-colourings themselves. For example, it is easily checked that $K_2 \times K_3$ has the property that all its proper vertex 3-colourings are induced by exactly 2 orientations.[4]

5.5 Face 3- and 4-colouring 2-cell embedded cubic graphs

For a cubic 2-cell embedded graph G,[5] we can deduce probabilistic criteria for the existence of proper face 3-colourings of G directly from §5.3 by using the same parity vertex weights but inducing face colourings of G rather than edge colourings of G. The existence of a proper face 3-colouring of a plane cubic graph G is by planar duality equivalent to the existence of nowhere-zero 3-flows of G, and cubic graphs only have nowhere-zero 3-flows if they are bipartite (see e.g. [34]). Since the faces of a 2-cell embedded plane graph G span the cycle space of G, a 2-cell embedded cubic plane graph has a proper face 3-colouring if and only if all the faces are of even size.

The problem of characterising graphs which have a proper face 4-colouring is for planar graphs settled by the Four Colour Theorem. At the end of this section we obtain a probabilistic criterion different to those deducible from §§5.3-5.4 for the existence of a proper face 4-colouring of a cubic graph 2-cell embedded on an orientable surface. For non-planar graphs we do not have the correspondence between proper edge 3-colourings (or nowhere-zero 4-flows) and proper face 4-colourings, since the surface dual G^* of G is not the matroidal dual of G for non-planar graphs. However, if a graph has a proper face k-colourable 2-cell embedding in some orientable surface then it has a nowhere-zero k-flow [34, Theorem 3.1], so that by [55] we know that cubic graphs with a Petersen minor cannot have a proper face 4-colouring.

For a cubic graph G with a 2-cell embedding on a surface $\mathbb{S}$ and with half-edge set H, we define the partition $\mathcal{F}$ of H by faces dually to the way we defined the partition by vertices in §3.3, in the sense that the vertices of the surface dual $G^* = (F, E, V)$ are the faces of $G = (V, E, F)$. This has the advantage that the blocks of $\mathcal{F}$ correspond more naturally to the faces as components of $\mathbb{S}\backslash G$ than in their definition of §3.4. Half-edges are obtained not by cutting edges across their middle but by splitting them lengthways.

[4]This graph also has the property that each proper edge 3-colouring is induced by exactly 6 proper half-edge colourings. See the observation following Corollary 4.6.2.

[5]See §3.4 for definitions of 2-cell embeddings, faces and surface duals.

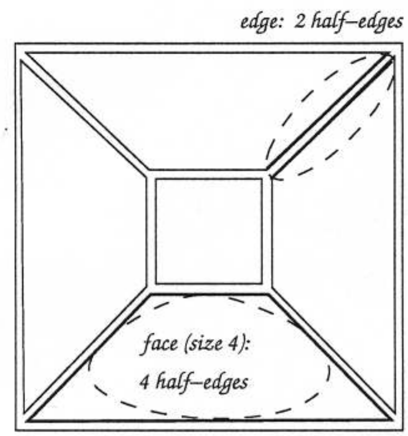


Figure 5.2: Edges and faces as subsets of half-edges obtained by splitting edges lengthways. *For any graph, each edge comprises two half-edges and each face as many half-edges as its size.*

For purposes of illustration we take a plane cubic graph G whose edges are straight line segments. The faces of G become a set of disjoint, non-overlapping polygons (see Figure 5.2 above). Each face comprises half-edges forming a polygon with as many sides as edges bounding the face in the embedding of G. Each edge of G comprises two half-edges belonging to the polygons corresponding to its two incident faces.

In order to have a natural definition of the partition $\mathcal{V}$ by vertices we adapt this definition of $\mathcal{F}$, making it into a partition of the set of *face corners* (see Figure 5.3 below). A face corner at a vertex v is the intersection of a neighbourhood of v (containing no vertices other than v) in the embedding of G and a face whose closure contains v. A block $H(f)$ in the partition $\mathcal{F}$ of H by faces is identified with the set of face corners contained in the closure of f. A block $H(v)$ in the partition $\mathcal{V}$ by vertices is identified with the set of face corners containing v.

We define a *proper face-corner colouring* of a 2-cell embedded graph G to be a colouring of the face corners of G with the property that the face corners incident with a common vertex receive distinct colours. A face colouring of G which gives distinct colours to faces incident with a common vertex also has the property that each pair of faces incident with a common edge receive distinct colours, and vice versa.

Given a proper face-corner Z_3-colouring of a 2-cell embedded cubic graph G on an orientable surface, we define as *negative* those vertices where the colours $0, 1, 2$ appear in an anticlockwise rotational sense in the surface and as *positive* those vertices where the colours are clockwise. The *parity* of a proper face-corner colouring is the number of

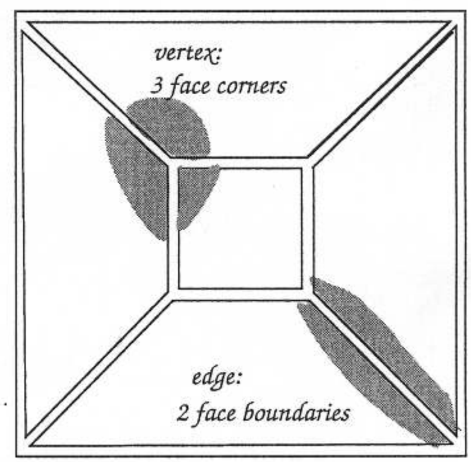


Figure 5.3: Edges and vertices as face boundaries and face corners. *For any graph, each edge comprises two face boundaries and each vertex (face) as many corners as its degree (size).*

negative vertices modulo 2. If the surface on which G is embedded is not orientable, we can alternatively define two proper face-corner colourings to be of the *same parity* if there are an even number of vertices at which the cyclic order of colours in one face-corner colouring is different to the cyclic order in the other colouring. A face 3-colouring of G is *induced* by a face-corner colouring by colouring a face with the sum of the colours of its face corners modulo 3. Corresponding to Theorem 5.3.1 we have the following by Theorem 5.2.7:

Theorem 5.5.1 *Let $G = (V, E, F)$ be a cubic graph 2-cell embedded on a surface $\mathbb{S}$, set of faces F and surface dual G^*. If two proper face-corner Z_3-colourings of G are chosen u.a.r. then*

$$\Pr(\text{Equivalent} \mid \text{Same Parity}) - \Pr(\text{Equivalent}) = 3^{|V|-|F|}4^{-|V|}P(G^*; 3),$$

where two face-corner colourings are equivalent if they induce the same face colouring and have the same parity if they differ in the rotational sense of their colours at an even number of vertices.

More generally, suppose $\gamma^{\mathcal{V}}$ is a parity weight on the half-edge colourings of a cubic graph with the property that $\gamma^{\mathcal{V}}$ determines a probability distribution in such a way that

$$\Pr(\text{Equivalent} \mid \text{Same Parity}) - \Pr(\text{Equivalent}) = 3^{-|E|}d^{2|V|}P(L(G); 3),$$

where half-edge colourings are equivalent if they induce the same edge colouring of G.

By the identification of half-edges with face corners, we can then deduce that $\gamma^{\mathcal{V}}$ is a parity weight on the face-corner colourings of a 2-cell embedded cubic graph with the property that

$$\Pr(\text{Equivalent} \mid \text{Same Parity}) - \Pr(\text{Equivalent}) = 3^{-|F|}d^{2|V|}P(G^*;3).$$

where two face-corner colourings are equivalent if they induce the same face colouring of G.

When we have a 2-cell embedding of G on an orientable surface it is possible to avoid mention of half-edges and face-corners altogether by using a bijection from half-edges of $G=(V,E)$ to the edges of the medial graph $\widetilde{G}=(E,L)$ of G. The *medial graph* $\widetilde{G}$ is an embedding of the line graph $L(G)$ on the same surface $\mathbb{S}$ on which G is 2-cell embedded: a vertex of $\widetilde{G}$ is placed at the middle of each edge of G, and edges of $\widetilde{G}$ join two vertices of $\widetilde{G}$ if they lie on cyclically consecutive edges of G in its local vertex rotation scheme (i.e mutually incident edges for a cubic graph G). The medial graph $\widetilde{G}$ is a 2-cell embedding of $L(G)$: the faces of the medial graph $\widetilde{G}$ are coloured black if they contain a vertex of G and white if they are contained in a face of G. The orientation is given to $\widetilde{G}$ which has the property that the edges on black faces are cyclically oriented in a clockwise sense. (This is the same as the orientation of the edges of $L(G)$ determined by a set of local vertex rotations of G which was used to define the Matiyasevich polynomial in Chapter 4, the set of local vertex rotations here defined by taking the three edges of G incident with a vertex of G in the clockwise sense on $\mathbb{S}$.)

The blocks of the partitions $\mathcal{V}$ and $\mathcal{F}$ of the half-edges or corners of G by vertices and faces become the black faces and white faces of $\widetilde{G}$ respectively, and the edges on a black face are cyclically ordered by taking edges in the order which follows the cyclic orientation of the black face in the orientation of $\widetilde{G}$. A face-corner colouring of G by this correspondence becomes a colouring of the edges of $\widetilde{G}$ and a vertex weight $\gamma^{\mathcal{V}}$ on the half-edge colourings of G becomes a function on the set of edge colourings of $\widetilde{G}$ defined locally on each black face of $\widetilde{G}$.

This medial graph interpretation is illustrated in our final theorem of this section, which considers face 4-colouring a 2-cell embedded cubic graph G on an orientable surface $\mathbb{S}$. We take our set of colours to be $Z_4=\{0,1,2,-1\}$ with cyclic order $(0\ \ 1\ \ 2\ \ -1)$ but we will in fact only use three of the four colours to colour the edges of $\widetilde{G}$.

Call a colouring of the three edges of a black face of $\widetilde{G}$ with the colours $0,1,-1$ *negative* if the colours of the edges in cyclic order $(0\ \ 1\ \ -1)$ appear in an anticlockwise rotational sense on $\mathbb{S}$ and *positive* if the colours appear in a clockwise sense. An edge colouring of $\widetilde{G}$ using the colours $0,1,-1$ and assigning distinct colours to each black triangle is *even* or *odd* according as it has an even or odd number of negative (anticlockwise) black triangles. An edge colouring *induces* a colouring of the white faces of $\widetilde{G}$ by colouring a face with the sum of the colours on its bounding edges modulo 4. A vertex in a proper face 4-colouring of G is anticlockwise if the colours of its incident faces appear in an anticlockwise rotational sense on $\mathbb{S}$.

Theorem 5.5.2 *Let $G = (V, E, F)$ be a cubic graph 2-cell embedded on an orientable surface $\mathbb{S}$ and $\widetilde{G} = (E, L)$ its medial graph. Suppose two edge colourings of $\widetilde{G}$ with the colours $0, 1, -1$ appearing on each black triangle of $\widetilde{G}$ are chosen u.a.r. Then,*

$$\Pr(\text{Equivalent} \mid \text{Same Parity}) - \Pr(\text{Equivalent}) = 4^{|V|-|F|}9^{-|V|}P(G^*; 4),$$

where two edge colourings of $\widetilde{G}$ are equivalent if on each white face of $\widetilde{G}$ the sum of their colours is the same modulo 4, and are of the same parity if they differ in the rotational sense of their colours at an even number of black triangles.

Futhermore, if $\mathbb{S}$ has Euler genus $2g$ then

$$\Pr(\text{Equivalent} \mid \text{Both Even}) - \Pr(\text{Equivalent} \mid \text{Both Odd})$$

$$= (-1)^{|E|}16^{g-1}9^{-|V|}(\,\#\{\text{Even proper face 4-colourings of } G\} - \#\{\text{Odd proper face 4-colourings of } G\}\,)$$

where an edge colouring is even or odd according to the number of anticlockwise black triangles modulo 2 and a proper face 4-colouring of G is even or odd according as it has an even or odd number of anticlockwise vertices.

Proof. The weight $\alpha_{\overline{(0,1,-1)}} - \alpha_{-\overline{(0,1,-1)}}$ has discrete Fourier transform $-4i(\alpha_{\text{clockwise}} - \alpha_{\text{anticlockwise}})$ where the clockwise vertex colourings are $\{\underline{012}.\underline{013}, \underline{023}\} = \overline{012}$ and the anticlockwise $\{\underline{210}, \underline{310}, \underline{320}\} = \overline{210}$ (in Chapter 6 we describe the calculations for this and other transforms) and the weight $\alpha_{\pm\overline{(0,1,-1)}}$ has transform taking the value -2 on all each proper colouring. The theorem results on applying Theorems 5.2.7 and 5.2.8. In order to simplify the exponent of 16 in the second correlation we use Euler's formula, which says that for an orientable surface $\mathbb{S}$ of Euler genus $2g$ (twice the number of holes in $\mathbb{S}$) and for a graph $G = (V, E, F)$ embedded on $\mathbb{S}$, $|V| - |E| + |F| = 2 - 2g$. $\square$

Chapter 6

Enumerations

6.1 Introduction

In this chapter we derive in §6.2 a number of identities for the number of proper edge k-colourings and nowhere-zero k-flows of a cubic graph G and deduce as a result three expressions for the number of proper edge 3-colourings of G. The set of proper edge k-colourings of G can be represented as a set of half-edge Z_k-colourings of G and the same is true for the set of nowhere-zero k-flows of G (see §3.8). The formulae in §6.2 are derived by using the discrete Fourier transform of the characteristic function of the appropriate set of half-edge Z_k-colourings of G. We give just those expressions which do not appear to be known already or which do not seem to have an obvious combinatorial interpretation, for example Theorems 6.2.2 and 6.2.5 and Corollaries 6.2.4 and 6.2.6.

In §6.3 we use the well-known equivalence between the existence of a proper edge 3-colouring of a cubic graph G and the existence of a 2-factor of G whose components are all of even size. We define two polynomials for a given cubic graph G as a sum over weighted 2-factors of G, both of which have a non-zero evaluation only if G has a proper edge 3-colouring. Expressions for the evaluation of these polynomials at any positive integer are derived and we finish with some particular evaluations. In particular Theorem 6.3.4 gives a correspondence between proper edge 3-colourings of a cubic graph and proper edge 4-colourings, and in the final theorems of the chapter we explore the relation between the existence of proper edge 3-colourings of a cubic graph to the non-vanishing of sums over weighted proper edge 5-colourings.

6.2 Nowhere-zero flows and proper edge colourings of cubic graphs

We begin by tailoring Lemma 3.11.5 and Lemma 3.12.B for our purposes in this chapter.

Lemma 6.2.1 *Let $k \geq 1$ be an integer and G any graph with half-edge set H, partition $\mathcal{E}$ of H by edges and partition $\mathcal{V}$ of H by vertices. Then for any vertex weight $\gamma^{\mathcal{V}} : Z_k^H \to \mathbb{C}$ we have the identities*

$$\gamma^{\mathcal{V}} \cdot \alpha^{\mathcal{E}}_{\text{null}} = k^{-|E|}\, \mathtt{t}\gamma^{\mathcal{V}} \cdot \alpha^{\mathcal{E}}_{\text{monochrome}}, \qquad \textit{and}$$

$$\gamma^{\mathcal{V}} \cdot \alpha^{\mathcal{E}}_{\text{monochrome}} = k^{-|E|}\, \mathtt{t}\gamma^{\mathcal{V}} \cdot \alpha^{\mathcal{E}}_{\text{null}}.$$

Also, for any edge weight $\gamma^{\mathcal{E}} : Z_k^H \to \mathbb{C}$,

$$\gamma^{\mathcal{E}} \cdot \alpha^{\mathcal{V}}_{\text{monochrome}} = k^{|V|-|H|}\, \mathtt{t}\gamma^{\mathcal{E}} \cdot \alpha^{\mathcal{V}}_{\text{null}}, \qquad \textit{and}$$

$$\gamma^{\mathcal{E}} \cdot \alpha^{\mathcal{V}}_{\text{null}} = k^{-|V|}\, \mathtt{t}\gamma^{\mathcal{E}} \cdot \alpha^{\mathcal{V}}_{\text{monochrome}}.$$

Proof. The edge weights $\alpha^{\mathcal{E}}_{\text{monochrome}}$ and $\alpha^{\mathcal{E}}_{\text{null}}$ on Z_k^H have by Lemma 3.11.5 transforms given by

$$\mathtt{t}\alpha^{\mathcal{E}}_{\text{monochrome}} = k^{|E|}\alpha^{\mathcal{E}}_{\text{null}}, \qquad \mathtt{t}\alpha^{\mathcal{E}}_{\text{null}} = k^{|H|-|E|}\alpha^{\mathcal{E}}_{\text{monochrome}} = k^{|E|}\alpha^{\mathcal{E}}_{\text{monochrome}},$$

and the vertex weights $\alpha^{\mathcal{V}}_{\text{monochrome}}$ and $\alpha^{\mathcal{V}}_{\text{null}}$ on Z_k^H have transforms

$$\mathtt{t}\alpha^{\mathcal{V}}_{\text{monochrome}} = k^{|V|}\alpha^{\mathcal{V}}_{\text{null}}, \qquad \mathtt{t}\alpha^{\mathcal{E}}_{\text{null}} = k^{|H|-|V|}\alpha^{\mathcal{E}}_{\text{monochrome}}.$$

We now just need to apply Lemma 3.12.B.□

Let G be a graph with half-edge set H, partition $\mathcal{E}$ of H by edges and partition $\mathcal{V}$ of H by vertices. In §3.8 we saw how the set of nowhere-zero k-flows of G can be represented as the set of half-edge colourings $\mu : H \to Z_k \backslash \{0\}$ which are both null on blocks of $\mathcal{E}$ and null on blocks of $\mathcal{V}$.

Thus,

$$F(G;k) = \alpha^{\mathcal{V}}_{\text{null}} \cdot \alpha^{\mathcal{E}}_{\text{null no zeroes}},$$

where the weights are on Z_k^H.

For $r = 2$ we have $\alpha_{\text{null no zeroes}} = \alpha_{\text{null}} - \alpha_{\text{zero}}$ so that

$$\mathtt{t}\alpha_{\text{null no zeroes}} = \mathtt{t}(\alpha_{\text{null}} - \alpha_{\text{zero}}) = (k-1)\alpha_{\text{monochrome}} - \alpha_{\text{not monochrome}}.$$

By Lemma 6.2.1 this yields the following:

$$F(G;k) = k^{-|V|}\alpha^{\mathcal{V}}_{\text{monochrome}} \cdot ((k-1)\alpha_{\text{monochrome}} - \alpha_{\text{not monochrome}})^{\mathcal{E}}$$

$$= (-1)^{|E|}k^{-|V|} \sum_{\text{vertex } k\text{-colourings of } G} (1-k)^{\#\text{ monochrome edges}}.$$

We recognise this as the evaluation

$$B(G;k,1-k)=k(-k)^{|V|-c(G)}T(G;0,1-k)$$

$$=(-1)^{|V|-c(G)}k^{|V|}\cdot(-1)^{|E|-|V|+c(G)}F(G;k)=(-1)^{|E|}k^{|V|}F(G;k)$$

of the monochrome polynomial $B(G;k,s)$ at $s=1-k$, where $c(G)$ is the number of components of G. (See §5.4 and [68] for the relation between the monochrome polynomial and the Tutte polynomial.)

Alternatively, we also have the identity

$$F(G;k)=\alpha^{\mathcal{V}}_{\text{null no zeroes}}\cdot\alpha^{\mathcal{E}}_{\text{null}},$$

where the weights are on Z_k^H. For this expression to be useful in enumerating nowhere-zero Z_k-flows of an arbitrary graph we need to calculate $\mathtt{t}\alpha_{\text{null no zeroes}}$ for any value of r (the size of a block of $\mathcal{V}$, equal to the degree of a vertex in G). Here we will consider the special case $r=3$, where the graph G is cubic ($\mathcal{V}$ is a partition of H into blocks of size 3).

Theorem 6.2.2 *Let $G=(V,E)$ be a cubic graph and $k\in\mathbb{N}$. Then*

$$F(G;k)=k^{-|E|}\sum_{\text{edge }k\text{-colourings of }G}(1-k)^{\#\text{ monochrome vertices}}(2-k)^{|V|-\#\text{ proper vertices}}2^{\#\text{ proper vertices}},$$

where the sum is over all edge k-colourings of G and vertices are "monochrome" or "proper" in a given edge colouring according as their incident edges have the same colour or have distinct colours.

Proof. For $\alpha_{\text{null no zeroes}}:Z_k^3\to\mathbb{C}$ we have

$$\mathtt{t}\alpha_{\text{null no zeroes}}=\mathtt{t}(\alpha_{\text{null}}-\alpha_{\text{zero}}-\alpha_{\text{null, one zero}})$$

The null elements of Z_k^3 with just one zero are given by $\{(m,-m,0),(0,m,-m),(-m,0,m):0\neq m\in Z_k\}$, so that for any $(l_0,l_1,l_2)\in Z_k^3$,

$$\mathtt{t}\alpha_{\text{null, one zero}}((l_0,l_1,l_2))=\sum_{0\neq m\in Z_k}j^{m(l_0-l_1)}+j^{m(l_1-l_2)}+j^{m(l_2-l_0)}.$$

Since j is a kth root of unity,

$$\mathtt{t}\alpha_{\text{null, one zero}}((l_0,l_1,l_2))=\begin{cases}3(k-1) & l_0-l_1,l_1-l_2,l_2-l_0\quad\text{all equal to zero,}\\(k-1)-2 & \text{exactly one zero amongst}\quad l_0-l_1,l_1-l_2,l_2-l_0,\\-3 & \text{no zeroes amongst}\quad l_0-l_1,l_1-l_2,l_2-l_0,\end{cases}$$

so that

$$\mathtt{t}\alpha_{\text{null, one zero}} = 3(k-1)\alpha_{\text{monochrome}} + (k-3)\alpha_{\text{two colours}} - 3\alpha_{\text{proper}}.$$

Hence

$$\mathtt{t}\alpha_{\text{null no zeroes}} = k^2\alpha_{\text{monochrome}} - \alpha_{\text{all}} + 3\alpha_{\text{proper}} + (3-k)\alpha_{\text{two}} - 3(k-1)\alpha_{\text{monochrome}}$$

$$= (k-1)(k-2)\alpha_{\text{monochrome}} + 2\alpha_{\text{proper}} + (2-k)\alpha_{\text{two colours}}.$$

We then have by Lemma 6.2.1,

$$F(G;k) = \alpha^{\mathcal{V}}_{\text{null no zeroes}} \cdot \alpha^{\mathcal{E}}_{\text{null}} = k^{-|E|}\mathtt{t}\alpha^{\mathcal{V}}_{\text{null no zeroes}} \cdot \alpha^{\mathcal{E}}_{\text{monochrome}},$$

and this yields the result (using the fact that for a cubic graph $G = (V, E)$ we have $(2-k)^{|V|} = (k-2)^{|V|}$ since $|V|$ is even). We remark that for even integer $2k \geq 2$ we may rewrite the identity as

$$F(G;2k) = 2^{|V|-|E|}k^{-|E|} \sum_{\text{edge } 2k\text{-colourings of } G} (1-2k)^{\#\text{ monochrome vertices}}(1-k)^{|V|-\#\text{ proper vertices}}.$$

In particular, putting $2k = 2$ in the latter, the expression obtained is the trivial $F(G;2) = 2^{|V|-|E|}P(\widetilde{G};2) = 0$. $\square$

For an arbitrary graph G with line graph $L(G)$, the number of proper edge k-colourings of G is by definition given by

$$P(L(G);k) = \alpha^{\mathcal{V}}_{\text{proper}} \cdot \alpha^{\mathcal{E}}_{\text{monochrome}},$$

where the weights are on Z_k^H. For a cubic graph, where $\mathcal{V}$ has blocks of size 3, we will deduce the value of $\mathtt{t}\alpha_{\text{proper}}$ via the transform $\mathtt{t}\alpha_{\text{null no zeroes}}$ already calculated in the proof of Theorem 6.2.2.

The map $\pi : (l_0, l_1, l_2) \mapsto (l_0 - l_1, l_1 - l_2, l_2 - l_0)$ is a homomorphism from Z_k^3 onto its subring of null triples with kernel equal to the subring of monochrome triples. The function π has the property that the set "proper" is mapped onto the set "null no zeroes". We shall temporarily denote the set of proper triples by P and the set of null triples with no zeroes by Q.

For each $\ell = (l_0, l_1, l_2) \in Z_k^3$ and $p = (p_0, p_1, p_2) \in P$, the homomorphism π satisfies

$$\langle \pi(\ell), p \rangle = \langle (l_0 - l_1, l_1 - l_2, l_2 - l_0), (p_0, p_1, p_2) \rangle$$

$$= l_0(p_0 - p_2) + l_1(p_1 - p_0) + l_2(p_2 - p_1)$$

$$= \langle (l_0, l_1, l_2), -(p_2 - p_0, p_0 - p_1, p_1 - p_2) \rangle = \langle \ell, \pi(p^*) \rangle$$

where $p^* = -(p_2, p_0, p_1)$ lies in P since P is closed under negation and cyclic permutation of components. The set of proper triples P is a union of cosets of the monochrome subring of Z_k^3 and we write $P = \{\underline{p'} : p' \in P'\}$, where P' is any transversal of these cosets. We note that π is constant on each set $\underline{p'}$ for $p' \in P'$ and that $\pi : P' \to Q$ is a bijection. Then for each $\ell \in Z_k^3$,

$$\mathtt{t}\alpha_P(\ell) = \sum_{p \in P} j^{\langle \ell, \pi(p) \rangle}$$

$$= k \sum_{p' \in P'} j^{\langle \ell, \pi(p') \rangle} = k \sum_{q \in Q} j^{\langle \ell, q \rangle} = k\mathtt{t}\alpha_Q(\ell).$$

In other words,

$$\mathtt{t}\alpha_{\text{proper}} = k\mathtt{t}\alpha_{\text{null no zeroes}}$$

$$= k\left((k-1)(k-2)\alpha_{\text{zero}} + 2\alpha_{\text{null no zeroes}} + (2-k)\alpha_{\text{null, one zero}} \right).$$

We deduce by Lemma 6.2.1 the following counterpart to Theorem 6.2.2:

Theorem 6.2.3 *Let G be a cubic graph, $L(G)$ its line graph and $k \geq 3$ any integer. Then*

$$P(L(G); k) = k^{|V|-|E|} \sum_{k\text{-flows of } G} (1-k)^{\#\text{ all zero vertices}} (2-k)^{|V|-\#\text{ no zero vertices}} 2^{\#\text{no zero vertices}},$$

where the sum is over all k-flows of G and a vertex is "all zero" in a k-flow if its three incident edges all have zero flow value and a "no zero" vertex is incident with no edges with flow value equal to zero.

Taking $k = 3$ in Theorem 6.2.3, the result of Theorem 5.3.3 is obtained as a special case:

$$P(L(G); 3) = 3^{|V|-|E|} \sum_{3\text{-flows of } G} (-2)^{\#\text{ vertices with three or no zeroes}}.$$

For cubic graphs $F(G;4)$ counts proper edge 3-colourings as well as nowhere-zero Z_4-flows. In other words, for $r = 3$ we have

$$F(G;4) = \alpha^{\mathcal{V}}_{\text{null}} \cdot \alpha^{\mathcal{E}}_{\text{null no zeroes}}$$

where the weights are on Z_4^H, and also

$$F(G;4) = P(L(G);3) = \alpha^{\mathcal{V}}_{\text{proper}} \cdot \alpha^{\mathcal{E}}_{\text{monochrome}},$$

where the weights are now on Z_3^H. Theorem 6.2.2 yields a different expression for the number of proper edge 3-colourings of G to that of Theorem 5.3.3:

Corollary 6.2.4 *Let G be a cubic graph and $L(G)$ its line graph. Then*

$$P(L(G);3) = 4^{-|V|} \sum_{\text{edge 4-colourings of } G} (-3)^{\#\text{ monochrome vertices}} (-1)^{\#\text{ proper vertices}},$$

where the summation is over all edge 4-colourings of G and a vertex is "proper" in a given edge 4-colouring if it is incident with edges of distinct colours and "monochrome" if it is incident with edges all of the same colour.

It is interesting to compare the expression of Corollary 6.2.4 for $F(G;4)$ for a cubic graph G obtained by taking $k = 4$ in Theorem 6.2.2 with the expression for $F(G;3)$ obtained by taking $k = 3$ in the same theorem:

$$F(G;3) = 3^{-|E|} \sum_{\text{edge 3-colourings of } G} (-2)^{\#\text{ monochrome vertices}} (-2)^{\#\text{ proper vertices}}.$$

Subsequent to Theorem 6.2.5 below, we deduce for a cubic graph G another expression for $P(L(G);3)$, over weighted vertex 4-colourings of G rather than the weighted edge 4-colourings of Corollary 6.2.4.

Theorem 6.2.5 *Let G be a cubic graph. Then the number of nowhere-zero 4-flows of G satisfies the following identity for all non-zero $z \in \mathbb{C}$:*

$$F(G;4) = (-1)^{|E|}(2z)^{-|V|} \sum_{\text{vertex } Z_4\text{-colourings of } G} (-1-z)^{\#\text{ monochrome edges}} (-1+z)^{\#\text{ nonconsecutive edges}},$$

where the summation is over all vertex Z_4-colourings of G and where an edge is "monochrome" in a given vertex colouring if its endpoints receive the same colour and an edge is "non-consecutive" if its endpoints receive colours whose difference is equal to 2 in Z_4.

Proof. Let $G = (V, E)$ be any cubic graph with half-edge set H, partition $\mathcal{E}$ of H by edges and partition $\mathcal{V}$ by vertices. An orientation of G corresponds to a linear order on each block of $\mathcal{E}$. A nowhere-zero Z_4-flow of G represented as a half-edge colouring $\mu : H \to Z_4\backslash\{0\}$ is null on blocks of $\mathcal{E}$ and null on blocks of $\mathcal{V}$. In other words $\mu : \mathcal{V} \to \pm\overline{(1,1,2)}$, $\mathcal{E} \to \{\pm(1,3),(2,2)\}$. Then μ has the property that the blocks of $\mathcal{E}$ which are coloured with $(2,2) \in Z_4^2$ correspond to the edges on a 1-factor of G, so that there are always $|E|/3$ blocks of $\mathcal{E}$ with colour pair $(2,2)$. Hence, for any $s \in \mathbb{C}$,

$$s^{|E|/3} F(G;4) = \alpha_{\text{null}}^{\mathcal{V}} \cdot (\alpha_{\pm(1,3)} + s\alpha_{(2,2)})^{\mathcal{E}},$$

where the weights are on Z_4^H. (This result holds vacuously when G has a bridge: since G does not have any nowhere-zero 4-flows, both left and right-hand side are always zero.)

For any $(l_0, l_1) \in Z_4^2$,

$$\mathtt{t}\alpha_{\pm(1,3)}((l_0,l_1)) = i^{l_0-l_1} + i^{l_1-l_0} = \begin{cases} 2 & l_0-l_1=0, \\ -2 & l_0-l_1=2, \\ 0 & l_0-l_1=\pm 1, \end{cases}$$

and

$$\mathtt{t}\alpha_{(2,2)}((l_0,l_1)) = i^{2(l_0+l_1)} = \begin{cases} 1 & l_0+l_1=0,2, \\ -1 & l_0+l_1=1,3. \end{cases}$$

In other words, $\mathtt{t}\alpha_{\pm(1,3)} = \alpha_{\underline{(0,0)}} - 2\alpha_{\underline{(0,2)}}$ and $\mathtt{t}\alpha_{(2,2)} = \alpha_{\underline{(0,0)},\underline{(0,2)}} - \alpha_{\underline{(0,1)},\underline{(0,3)}}$, so that

$$\mathtt{t}(\alpha_{\pm(1,3)} + s\alpha_{(2,2)}) = (s+2)\alpha_{\underline{(0,0)}} + (s-2)\alpha_{\underline{(0,2)}} - s\alpha_{\pm\underline{(0,1)}}.$$

Hence

$$s^{|E|/3}F(G;4) = 4^{-|V|} \sum_{\text{vertex } Z_4\text{-colourings}} (s+2)^{\#\text{ monochrome}}(s-2)^{\#\text{ nonconsecutive}}(-s)^{\#\text{ consecutive}},$$

and on dividing through by $(-s)^{|E|}$ for $s \neq 0$ and writing $z = 2s^{-1}$ we obtain the result.$\Box$

Taking $z = 2$, the identity of Theorem 6.2.5 coincides with the expression

$$F(G;4) = (-1)^{|E|}4^{-|V|} \sum_{\text{vertex 4-colourings of } G} (-3)^{\#\text{ monochrome edges}}$$

obtained for $k = 4$ from the previous expression for the number of nowhere-zero k-flows $F(G;k)$ of an arbitrary graph G as an evaluation of the monochrome polynomial. Taking $z = -1$ yields our third expression for $P(L(G);3)$:

Corollary 6.2.6 *Let G be a cubic graph and $L(G)$ its line graph. Then*

$$P(L(G);3) = (-1)^{|E|}2^{-|V|} \sum_{\text{proper vertex } Z_4\text{-colourings of } G} (-2)^{\#\text{ nonconsecutive edges}},$$

where the summation is over all proper vertex Z_4-colourings of G and an edge in a proper vertex Z_4-colouring is nonconsecutive if the colours of its endpoints lie 2 apart in Z_4.

6.3 Even 2-factors of cubic graphs

Throughout this section G is a cubic graph and $L(G)$ its line graph. A set of local vertex rotations of G will be interpreted as an embedding of G on an orientable surface $\mathbb{S}$, where three edges incident with a vertex v taken in the order determined by the local vertex rotation are in a clockwise sense on $\mathbb{S}$.

A 1-*factor* (or *perfect matching*) of G is a 1-regular spanning subgraph of G and a 2-*factor* is a 2-regular spanning subgraph of G. (A loop is counted as having degree 2.) Since G is cubic, the edge complement of a 1-factor of G is a 2-factor of G. A 1-*factorisation* of G is a set of 1-factors such that each edge lies in exactly one of the 1-factors. A 1-factorisation of G corresponds to a set of six proper edge 3-colourings of G equivalent to each other under permutation of the three colours. An *even* 2-*factor* of G is a 2-factor of G whose components all have even size and a *directed* 2-*factor* of G is a 2-factor of G with a totally cyclic orientation. Each component of a directed 2-factor has either a clockwise or anticlockwise rotational sense in $\mathbb{S}$ by following the direction of the edges given by the cyclic orientation.

We begin by enumerating the number of 2-factors of G as a sum over integer-weighted edge 2-colourings. The method used in the proof of Theorem 6.3.1 is a simple illustration of the technique which will be used to prove subsequent theorems. We use the bijective correspondence of the set of 2-factors of G with the set of edge Z_2-colourings of G having the property that each vertex is incident with two edges coloured 1 and one edge coloured 0.

Theorem 6.3.1 *For any cubic graph $G = (V, E)$,*

$$\#\{\text{2-factors of } G\} = 2^{-|E|} \sum_{\text{edge 2-colourings of } G} (-3)^{\#\text{monochrome vertices}},$$

where the sum ranges over all edge 2-colourings and a vertex is "monochrome" in a given edge 2-colouring of G if it is incident with three edges of the same colour.

Proof. Let G have half-edge set H, partition $\mathcal{E}$ by edges and partition $\mathcal{V}$ by vertices. Each 2-factor of G corresponds to a half-edge Z_2-colouring which is monochrome on each block of $\mathcal{E}$ and colours each block of $\mathcal{V}$ with a colour triple belonging to $\overline{(0,1,1)} = \{(0,1,1),(1,0,1),(1,1,0)\}$.

Hence

$$\#\,\{\text{2-factors of } G\} = (\alpha^{\mathcal{V}}_{\overline{(0,1,1)}} \cdot \alpha^{\mathcal{E}}_{\text{monochrome}})\,(Z_2^H).$$

For $(l_0, l_1, l_2) \in Z_2^3$,

$$\mathtt{t}\alpha_{\overline{(0,1,1)}}((l_0,l_1,l_2)) = (-1)^{l_0+l_1} + (-1)^{l_1+l_2} + (-1)^{l_2+l_0} = \begin{cases} 3 & \text{if } (l_0,l_1,l_2) \text{ is monochrome,} \\ -1 & \text{if } (l_0,l_1,l_2) \text{ is not monochrome.} \end{cases}$$

By Lemma 6.2.1,

$$\alpha^{\mathcal{V}}_{\overline{(0,1,1)}} \cdot \alpha^{\mathcal{E}}_{\text{monochrome}} = 2^{-|E|}\mathtt{t}\alpha^{\mathcal{V}}_{\overline{(0,1,1)}} \cdot \alpha^{\mathcal{E}}_{\text{null}}),$$

and in Z_2^2 the null pairs $(0,0),(1,1)$ are precisely the monochrome pairs so that $\alpha_{\text{null}} = \alpha_{\text{monochrome}}$ and the result now follows.□

In the remainder of this section we will be concerned with finding expressions involving the *even* 2-factors of a cubic graph, which are of interest due to the following well-known identity (see e.g. [23])

$$P(L(G);3) = \sum_{\text{directed even 2-factors of } G} 1 = \sum_{\text{even 2-factors of } G} 2^{\#\text{ components}}.$$

Given a cubic graph G we define the polynomial $E(G)$ in $\mathbb{Z}[x]$ by

$$E(G;x) = \sum_{\text{directed even 2-factors of } G} x^{\#\text{ components}} = \sum_{\text{even 2-factors of } G} (2x)^{\#\text{ components}},$$

which we have just seen has the evaluation $E(G;1) = P(L(G);3)$.

As well as the polynomial $E(G)$ we will be interested in a related polynomial, which we shall denote by $E^{\circlearrowleft}(G)$, which encodes some information about the orientable surface $\mathbb{S}$ on which G is embedded. In order to define the polynomial $E^{\circlearrowleft}(G)$ we use the rotational sense of the components of a given directed 2-factor of G on the surface $\mathbb{S}$.

A vertex v is anticlockwise in a directed 2-factor of G if, when traversing the component C of the 2-factor which contains v according to the cyclic orientation of C, the edge directed out of v is anticlockwise in $\mathbb{S}$ from the edge directed into v. The vertex v is *clockwise* in the directed 2-factor otherwise. Reversing the cyclic orientation of C switches the rotational sense of v. Consequently, if C has an odd number of vertices then reversing its cyclic orientation changes the parity of the number of anticlockwise vertices in C.

For a cubic graph G and a given embedding of G in an orientable surface, we define the polynomial $E^{\circlearrowleft}(G)$ in $\mathbb{Z}[x]$ by

$$E^{\circlearrowleft}(G;x) = \sum_{\text{directed 2-factors of } G} (-1)^{\#\circlearrowleft} x^{\#\text{ components}},$$

where the summation is over all directed 2-factors of G and, for a given directed 2-factor, the exponent $\#\circlearrowleft$ of -1 counts the number of anticlockwise vertices of the 2-factor. Embedding G on a different orientable surface $\mathbb{S}'$ can only change the sign of $E^{\circlearrowleft}(G)$, since the embedding of G in $\mathbb{S}'$ either switches the sign of $(-1)^{\#\circlearrowleft}$ for all the directed 2-factors of G from its sign when G is embedded in $\mathbb{S}$ or preserves the sign of $(-1)^{\#\circlearrowleft}$ for all directed 2-factors of G. If a 2-factor of G has c components and contains a component C of odd size then its 2^c totally cyclic orientations contribute zero to the polynomial $E^{\circlearrowleft}(G)$, for the reason already given that reversing the cyclic orientation of C switches the parity of the number of anticlockwise vertices.

Thus, in the definition of $E^{\circlearrowleft}(G)$ the summation can be restricted to directed even

2-factors of G:

$$E^{\circlearrowleft}(G;x) = \sum_{\text{directed even 2-factors of } G} (-1)^{\#\circlearrowleft} x^{\#\text{ components}}.$$

We note that each of the totally cyclic orientations of a fixed even 2-factor has the same sign $(-1)^{\#\circlearrowleft}$. Since even 2-factors of G exist if and only if $P(L(G);3) \neq 0$, the polynomial $E^{\circlearrowleft}(G)$ has the property that

$$E^{\circlearrowleft}(G;x) \neq 0 \qquad \Rightarrow \qquad P(L(G);3) \neq 0.$$

However, the converse does not hold. If for each $c \in \mathbb{N}$ there as many directed even 2-factors of G which have c components and an even number of anticlockwise vertices as there are directed even 2-factors of G with c components and an odd number of anticlockwise vertices then $E^{\circlearrowleft}(G;x) = 0$. An example of such a case is $K_{3,3}$, whose six 2-factors all have one component, but in any orientable embedding of $K_{3,3}$ three of the 2-factors have an even number of anticlockwise vertices and the other three an odd number of anticlockwise vertices.

For given $k \geq 3$ and $0 \neq l \in Z_k$ we define the vertex weight $\rho_l : Z_k^3 \to \mathbb{C}$ by

$$\rho_l = \alpha_{\overline{(0,l,-l)}} - \alpha_{\overline{(0,-l,l)}}.$$

In particular, for $k = 3$, the weight ρ_1 is equal to Matiyasevich's vertex weight $\rho = \alpha_{\text{clockwise}} - \alpha_{\text{anticlockwise}}$. For $L \subseteq Z_k$ with the property that $L \cap (-L) = \emptyset$ we define the vertex weight ρ_L by

$$\rho_L = \sum_{l \in L} \rho_l = \sum_{l \in L} \alpha_{\overline{(0,l,-l)}} - \alpha_{\overline{(0,-l,l)}}.$$

The condition on L is needed since $\rho_{-l} = -\rho_l$, and it implies that $|L| \leq \lfloor \frac{k-1}{2} \rfloor$.

For $0 \neq l \in Z_k$, we define the vertex weight $\tau_l : Z_k^3 \to \mathbb{C}$ by

$$\tau_l = \alpha_{\pm\overline{(0,l,l)}}$$

and, for $L \subseteq Z_k$ with the property that $L \cap (-L) = \emptyset$,

$$\tau_L = \sum_{l \in L} \tau_l = \sum_{l \in L} \alpha_{\pm\overline{(0,l,l)}}.$$

The vertex weights just defined allow us to express evaluations of $E(G)$ and $E^{\circlearrowleft}(G)$ at a positive integer m in terms of a sum of weighted edge k-colourings of G for any $k \geq 2m+1$.

Lemma 6.3.2 *Let $G = (V,E)$ be a cubic graph embedded on an orientable surface $\mathbb{S}$ with* *half-edge set H, partition $\mathcal{E}$ of H by edges and partition $\mathcal{V}$ by vertices, the half-edges in each block of $\mathcal{V}$ put in a linear order consistent with the clockwise rotation on $\mathbb{S}$.*

Let $k \geq 3$ be an integer and L a subset of Z_k with the property that $L \cap (-L) = \emptyset$. Then,

$$E^{\circlearrowleft}(G;|L|) = \rho_L^{\mathcal{V}} \cdot \alpha_{\text{null}}^{\mathcal{E}} = k^{-|E|}\mathtt{t}\rho_L^{\mathcal{V}} \cdot \alpha_{\text{monochrome}}^{\mathcal{E}},$$

and

$$E(G;|L|) = \tau_L^{\mathcal{V}} \cdot \alpha_{\text{null}}^{\mathcal{E}} = k^{-|E|}\mathtt{t}\tau_L^{\mathcal{V}} \cdot \alpha_{\text{monochrome}}^{\mathcal{E}},$$

where the weights are on Z_k^H.

Proof. We begin by proving the first identity. Suppose $\mu : H \to Z_k$ is any half-edge colouring which colours each block of $\mathcal{V}$ with a triple from $\{\pm\overline{(0,l,-l)} : l \in L\}$ and each block of $\mathcal{E}$ with a null pair (which is either $(0,0)$ or $\pm(l,-l)$ for some $l \in L$). Then μ has the property that for fixed $l \in L$ the edges in the set $\{e \in E : \mu_e \in \pm(l,-l)\}$ form a 2-regular subgraph of G. Since each vertex is coloured with a triple from $\pm\overline{(0,l,-l)}$ for some $l \in L$, the edges $\{e \in E : \mu_e \neq (0,0)\}$ form a 2-factor F of G. If μ' is obtained from μ by negating the colours of the half-edges on an odd component of F then $\rho_L(\mu') = -\rho_L(\mu)$. Hence, any 2-factor of G with a component of odd size contributes 0 to $\rho_L^{\mathcal{V}} \cdot \alpha_{\text{null}}^{\mathcal{E}}$. However, if F is the edge set of an even 2-factor with c components, then there are $2^c|L|^c$ half-edge colourings in the support of the product $\rho_L^{\mathcal{V}}\alpha_{\text{null}}^{\mathcal{E}}$ which colour the half-edges in $\{H(f) : f \in F\}$ with non-zero colours, and the sign of ρ_L is constant on this set of $(2|L|)^c$ half-edge colourings. These $(2|L|)^c$ half-edge colourings with support $\{H(f) : f \in F\}$ together contribute $(-1)^{\#\circlearrowleft}(2|L|)^c$ to $\rho_L^{\mathcal{V}} \cdot \alpha_{\text{null}}^{\mathcal{E}}$, where $\#\circlearrowleft$ is the number of blocks of $\mathcal{V}$ receiving a colour triple of the form $\overline{(0,-l,l)}$. Suppose $H(e) = \{h_{v_0}, h_{v_1}\}$ is a block of $\mathcal{E}$, where $h_{v_0} \in H(v_0), h_{v_1} \in H(v_1)$. For each $l \in L$, we interpret colouring $H(e)$ with the colour pair $(l,-l)$ as directing the edge $e = \{v_0, v_1\}$ from v_0 to v_1, and colouring $H(e)$ with $(-l,l)$ as the reverse direction. Then if the half-edge colouring μ colours the block $H(v)$ of $\mathcal{V}$ with a triple in $-\overline{(0,l,-l)}$, the vertex v is an anticlockwise vertex in the directed 2-factor F whose half-edges are coloured with non-zero colours by μ. This establishes that

$$\rho_L^{\mathcal{V}} \cdot \alpha_{\text{null}}^{\mathcal{E}} = \sum_{\text{directed even 2-factors of } G} (-1)^{\circlearrowleft}|L|^{\#\text{ components}} = E^{\circlearrowleft}(G;|L|).$$

We now prove the second identity. Any half-edge colouring $\mu : H \to Z_k$ which colours each block of $\mathcal{V}$ with a triple from $\{\pm\overline{(0,l,l)} : l \in L\}$ and each block of $\mathcal{E}$ with a null pair has the property that the edges in the set $\{e \in E : \mu_e \in \pm(l,-l)\}$ form a 2-regular subgraph of G which must be of even size, since vertices adjacent in the subgraph receive colour triples from $\pm\overline{(0,l,l)}$ of opposite signs (in order that the blocks of $\mathcal{E}$ receive null colour pairs). Since each vertex is coloured with a triple from $\pm\overline{(0,l,l)}$ for some $l \in L$, the edges $\{e \in E : \mu_e \neq (0,0)\}$ form a 2-factor of G whose components are all of even size. Each component of a given even 2-factor can be coloured in $2|L|$ different ways with non-zero colours on its half-edges, and this establishes that

$$\tau_L^{\mathcal{V}} \cdot \alpha_{\text{null}}^{\mathcal{E}} = \sum_{\text{even 2-factors of } G} (2|L|)^{\#\text{ components}} = E(G; |L|).$$

□

We finish the chapter with some illustrations of Lemma 6.3.2, beginning with its second identity and concluding with some special cases of the first identity.

If we define for a given triple $(l_0, l_1, l_2) \in Z_k^3$ the *pairwise sum multiset* $\{c_0, c_1, c_2\} = \{l_0 + l_1, l_1 + l_2, l_2 + l_0\}$, then for any given $\ell \in \overline{(l_0, l_1, l_2)}$

$$\mathtt{t}\tau_L(\ell) = \sum_{l \in L} j^{lc_0} + j^{-lc_0} + j^{lc_1} + j^{-lc_1} + j^{lc_2} + j^{-lc_2}$$

$$= 2\sum_{l \in L} \cos(\frac{2\pi l c_0}{k}) + \cos(\frac{2\pi l c_1}{k}) + \cos(\frac{2\pi l c_2}{k}).$$

In particular, $\mathtt{t}\tau_L(-\ell) = \mathtt{t}\tau_L(\ell)$.

Suppose L has the property that $L \cup (-L) \cup \{0\}$ is the subgroup Z_m of Z_k, where $1 \neq m \mid k$. If k is even then $k/2$ equals its negative $-k/2$ in Z_k, so that Z_m cannot contain $k/2$ when k is even. In other words, m must always be an odd divisor of k. Since j is a kth root of unity, for each $c \in Z_k$,

$$\sum_{l \in L} j^{lc} + j^{-lc} = \begin{cases} -1 & c \neq 0, \\ m - 1 & c = 0. \end{cases}$$

A given $(l_0, l_1, l_2) \in Z_k^3$ will be classified according to the number of zeroes in its pairwise sum multiset:

$$S_n = \{\ell \in Z_k^3 : \text{ exactly } n \text{ zeroes in the pairwise sum multiset of } \ell\}, \qquad n = 0, 1, 2, 3.$$

We note that for any $k \geq 3$, $S_3 = \{(0,0,0)\}$. For any $\ell \in Z_k^3$, the value of $\mathtt{t}\tau_L(\ell)$ only depends on how many zeroes there are in the pairwise sum multiset of ℓ:

$$\mathtt{t}\tau_L = 3(m-1)\alpha_{S_3} + (2m-3)\alpha_{S_2} + (m-3)\alpha_{S_1} - 3\alpha_{S_0}.$$

By Lemma 6.3.2 we then have the following:

Theorem 6.3.3 *Let G be a cubic graph, $k \geq 3$ an integer and $m \neq 1$ an odd divisor of*

k. Then

$$\sum_{\text{even 2-factors of } G} (m-1)^{\#\,\text{components}} = k^{-|E|} \sum_{\text{edge } Z_k\text{-colourings of } G} (3m-3)^{\#\,S_3}(2m-3)^{\#\,S_2}(m-3)^{\#\,S_1}(-3)^{\#\,S_0},$$

where $\#S_n$ counts for $n = 0, 1, 2, 3$ the number of vertices incident with edges of colours l_0, l_1, l_2 such that there are n zeroes amongst the elements $l_0 + l_1, l_1 + l_2, l_2 + l_0$ of Z_k.

For example, for $k = 3$ Theorem 6.3.3 says that

$$P(L(G); 3) = \sum_{\text{even 2-factors of } G} 2^{\#\,\text{components}} = 3^{|V|-|E|} \sum_{\text{edge } Z_3\text{-colourings of } G} 2^{\#\,S_3} 1^{\#\,S_2} 0^{\#\,S_1} (-1)^{\#\,S_0},$$

where

$$S_0 = \{\pm 111\} \cup \pm\overline{011}, \quad S_1 = \pm\overline{012} \cup \pm\overline{100}, \quad S_2 = \pm\overline{112}, \quad S_3 = \{000\}.$$

We finish this section by considering some specific cases of the vertex weight ρ_L for $L \subset Z_k, L \cap (-L) = \emptyset$. For $(l_0, l_1, l_2) \in Z_k^3$ we have

$$\mathtt{t}\rho_L((l_0, l_1, l_2)) = \sum_{l \in L} j^{l(l_0-l_1)} - j^{-l(l_0-l_1)} + j^{l(l_1-l_2)} - j^{-l(l_1-l_2)} + j^{l(l_2-l_0)} - j^{-l(l_2-l_0)}$$

We define the *difference multiset* of a given triple $\ell = (l_0, l_1, l_2) \in Z_k^3$ to be the multiset $\{d_0, d_1, d_2\} := \{l_0 - l_1, l_1 - l_2, l_2 - l_0\}$. Then

$$\mathtt{t}\rho_L(\ell) = 2i \sum_{l \in L} \sin(\frac{2\pi l d_0}{k}) + \sin(\frac{2\pi l d_1}{k}) + \sin(\frac{2\pi l d_2}{k}).$$

We use the fact that $d_0 + d_1 + d_2 \equiv 0 \bmod k$ and the identity $\sin 2x + \sin 2y + \sin 2z = -4 \sin x \sin y \sin z$ when $x + y + z$ is a multiple of 2π to rewrite the equation for $\mathtt{t}\rho_L$ as follows:

$$\mathtt{t}\rho_L(\ell) = -8i \sum_{l \in L} \sin(\frac{\pi l d_0}{k}) \sin(\frac{\pi l d_1}{k}) \sin(\frac{\pi l d_2}{k}).$$

In particular, $\mathtt{t}\rho_L(\ell) = 0$ if ℓ is not proper, $\mathtt{t}\rho_L(-\ell) = -\mathtt{t}\rho_L(\ell)$, and if ℓ' is a cyclic permutation of ℓ or if $\ell' \in \underline{\ell}$ then $\mathtt{t}\rho_L(\ell') = \mathtt{t}\rho_L(\ell)$. In other words, for two proper triples $\ell, \ell' \in Z_k^3$ which have the same difference multiset $\{d_0, d_1, d_2\}$ we have $\mathtt{t}\rho_L(\ell) = \mathtt{t}\rho_L(\ell')$.

It will be useful to have names for the following subsets of the proper triples in Z_k^3, where Z_k has the linear order $0 < 1 < \cdots < k - 1$:

$$\text{“clockwise”} = \{(l_0, l_1, l_2) \in Z_k^3 \text{ proper}: \text{ even \# inversions in } (0, 1, 2) \mapsto (l_0, l_1, l_2)\},$$

$$\text{“anticlockwise”} = \{(l_0, l_1, l_2) \in Z_k^3 \text{ proper}: \text{ odd \# inversions in } (0, 1, 2) \mapsto (l_0, l_1, l_2)\}.$$

An alternative interpretation for these two sets is obtained by taking the elements of Z_k as beads of a necklace with the order $0 < 1 < 2 \cdots < k-1$ clockwise around the necklace. The clockwise triples (l_0, l_1, l_2) have the property that l_0, l_1, l_2 are in a clockwise sense around the necklace when taken in the order (l_0, l_1, l_2) and the anticlockwise triples are in the reverse sense.

If we do not distinguish the clockwise/anticlockwise pair of difference multisets $\{d_0, d_1, d_2\}$, $\{-d_0, -d_1, -d_2\}$, the number of distinct difference multisets for proper triples in Z_k^3 is the number of free necklaces (bracelets) on k beads with three black beads and $k-3$ white beads. The black beads represent the colours chosen to form a colour triple in Z_k^3, and the difference multisets describe the spacing of these black beads around the necklace. By Polya counting (see e.g. [1]), the number of free necklaces using 3 black beads and $k-3$ white beads is given by $\lfloor k^2/12 \rfloor$ if $3 \not| \, k$ and $\lceil k^2/12 \rceil$ if $3 \mid k$. Thus the set of $k(k-1)(k-2)$ proper triples of Z_k^3 is partitioned into $\lfloor k^2/12 \rfloor$ (or $\lceil k^2/12 \rceil$) classes according to its difference multisets, and these classes are further divided into two subsets of equal size by the distinction between clockwise and anticlockwise rotational sense.For $k = 3, 4$ and 5 there are respectively $1, 1$ and 2 difference multisets modulo clockwise/anticlockwise. In particular, for $k = 3$ and 4 we have only the distinction between clockwise and anticlockwise proper triples.

Taking $k = 3$ and Matiyasevich's vertex weight $\rho = \rho_1$, whose transform is given by $\mathtt{t}\rho^{\mathcal{V}} = (-3)^{|E|}\rho^{\mathcal{V}}$, the first equation of Lemma 6.3.2 yields Theorem 5.3.5:

$$(-1)^{|E|}E^{\circlearrowleft}(G;1) = \#\{\text{Even proper edge 3-colourings of } G\} - \#\{\text{Odd proper edge 3-colourings of } G\},$$

and when G is a plane cubic graph we have $E^{\circlearrowleft}(G;1) = P(L(G);3) = E(G;1)$. This implies that $E^{\circlearrowleft}(G;x) = E(G;x)$ when G is a plane cubic graph.

For $k = 4$, the proper triples $(l_0, l_1, l_2) \in Z_4^3$ fall into two types, according as the difference multiset $\{l_0 - l_1, l_1 - l_2, l_2 - l_0\}$ equals $\{-1, -1, 2\}$ (the clockwise proper triples) or $\{1, 1, -2\}$ (the anticlockwise proper triples). We have

$$\sin(\frac{-\pi}{4})\sin(\frac{-\pi}{4})\sin(\frac{\pi}{2}) = \frac{1}{2}$$

so that

$$\mathtt{t}\rho_1 = -4i(\alpha_{\text{clockwise}} - \alpha_{\text{anticlockwise}}).$$

By Lemma 6.3.2 we deduce the following:

Theorem 6.3.4 *Let G be a cubic graph embedded on an orientable surface $\mathbb{S}$. Then*

$$E^{\circlearrowleft}(G;1) = (-4)^{|V|-|E|}(\#\{\text{Even proper edge } Z_4\text{-colourings of } G\} - \#\{\text{Odd proper edge } Z_4\text{-colourings of } G\}),$$

where a proper edge Z_4-colouring of G is even or odd according as the number of vertices whose incident colours appear in a clockwise order relative to the cyclic order $(0\ 1\ 2\ 3)$ *of Z_4 is even or odd.*

In particular, for a plane cubic graph G with line graph $L(G)$,

$$P(L(G);3) = (-4)^{|V|-|E|}(\#\{\text{Even proper edge } Z_4\text{-colourings of } G\} - \#\{\text{Odd proper edge } Z_4\text{-colourings of } G\}\,).$$

For $k = 5$ there are two types of proper colourings of a vertex to distinguish in addition to the clockwise or anticlockwise sense of the three colours. The four possible difference multisets of a proper triple in Z_5^3 are[1]

$$\{-1,-1,2\},\{1,1,-2\}, \qquad \{-1,-2,-2\},\{1,2,2\},$$

where a difference multiset and its negative (corresponding respectively to 15 clockwise and 15 anticlockwise proper triples in Z_5^3) have been paired together. We will call proper triples in Z_5^3 whose difference multiset is one of $\{-1,-2,-2\},\{1,2,2\}$ "nonconsecutive". Proper triples whose difference multiset is one of $\{-1,-1,2\},\{1,1,-2\}$ are called "consecutive" since the three colours in such a triple are consecutive elements of Z_5.

Then

$$\mathtt{t}\rho_1(\ell) = \begin{cases} -2i(\,2\sin(\frac{2\pi}{5}) - \sin(\frac{4\pi}{5})\,) & \ell \text{ consecutive clockwise,} \\ 2i(\,2\sin(\frac{2\pi}{5}) - \sin(\frac{4\pi}{5})\,) & \ell \text{ consecutive anticlockwise,} \\ -2i(\,\sin(\frac{2\pi}{5}) + 2\sin(\frac{4\pi}{5})\,) & \ell \text{ nonconsecutive clockwise,} \\ 2i(\,\sin(\frac{2\pi}{5}) + 2\sin(\frac{4\pi}{5})\,) & \ell \text{ nonconsecutive anticlockwise.} \end{cases}$$

We have

$$\sin(\frac{4\pi}{5}) = \frac{\sqrt{10-2\sqrt{5}}}{4}, \qquad \sin(\frac{2\pi}{5}) = \frac{\sqrt{10+2\sqrt{5}}}{4} = \phi\sin(\frac{4\pi}{5}),$$

where $\phi = \frac{1+\sqrt{5}}{2}$. By Lemma 6.3.2,

$$E^{\circlearrowleft}(G;1) = (\rho_1^{\mathcal{V}} \cdot \alpha_{\text{null}}^{\mathcal{E}})(Z_5^H) =$$

$$= 5^{-|E|}(\frac{-2i\sqrt{10-2\sqrt{5}}}{4})^{|V|} \sum_{\text{proper edge } Z_5\text{-colourings}} (2\phi-1)^{\#\text{consecutive}}(\phi+2)^{\#\text{nonconsecutive}}(-1)^{\#\text{anticlockwise}}$$

$$= (-1)^{|E|}(\frac{\sqrt{10-2\sqrt{5}}}{10})^{|V|} \sum_{\text{proper edge } Z_5\text{-colourings}} (-1)^{\#\text{anticlockwise}}\phi^{\#\text{nonconsecutive}},$$

[1]The differences are taken modulo 5 and we take $Z_5 = \{0,\pm1,\pm2\}$ rather than $Z_5 = \{0,1,2,3,4\}$.

since $2\phi - 1 = \sqrt{5}$ and $\phi + 2 = \sqrt{5}\phi$. With $|V|$ even,

$$(\frac{\sqrt{10-2\sqrt{5}}}{10})^{|V|} = (\frac{1}{5\sqrt{5}\phi})^{|V|/2},$$

and using $|E| = \frac{3}{2}|V|$ we finally obtain the first statement of Theorem 6.3.5 below.

The transform of $\rho_2 = \alpha_{\overline{(0,2,-2)}} - \alpha_{\overline{(0,-2,2)}}$ is computed as for ρ_1 except that the difference multisets of proper triples

$$\{-1,-1,2\},\{1,1,-2\}, \qquad \{-1,-2,-2\},\{1,2,2\},$$

are now multiplied by 2 to obtain respectively

$$\{-2,-2,-1\},\{2,2,1\}, \qquad \{-2,1,1\},\{2,-1,-1\}.$$

After calculations for $\mathtt{t}\rho_2$ similar to those carried out for $\mathtt{t}\rho_1$, Lemma 6.2.1 yields the second statement of Theorem 6.3.5 below.

We use the following terminology in Theorems 6.3.5 and 6.3.7 and their corollaries. A vertex of G is "anticlockwise" in a given proper edge Z_5-colouring of G if its three incident edges have colours which appear in a reverse order relative to the clockwise order $0 < 1 < 2 < 3 < 4$ of Z_5 and a vertex is "nonconsecutive" in a given proper edge Z_5-colouring if its incident colours are nonconsecutive elements of Z_5.

Theorem 6.3.5 *For a cubic graph G embedded on an orientable surface,*

$$E^{\circlearrowleft}(G;1) = (\sqrt{5})^{-|E|}(1-\phi)^{|E|/3} \sum_{\text{proper edge } Z_5\text{-colourings of } G} (-1)^{\#\text{anticlockwise}}\phi^{\#\text{nonconsecutive}},$$

where $\phi = \frac{1+\sqrt{5}}{2}$. Also,

$$E^{\circlearrowleft}(G;1) = (\sqrt{5})^{-|E|}(-\phi)^{|E|/3} \sum_{\text{proper edge } Z_5\text{-colourings of } G} (-1)^{\#\text{anticlockwise}}(1-\phi)^{\#\text{nonconsecutive}}.$$

The two equations of Theorem 6.3.5 can be combined so that summations are over integer-weighted proper edge Z_5-colourings. For integer n, we denote by $L(n)$ the nth Lucas number $L(n) = \phi^n + (-\phi)^{-n}$, the integer sequence $2, 1, 3, 4, 7, 11, 18, 29, \ldots$, whose terms are defined by $L(0) = 2, L(1) = 1$ and the recurrence $L(n+1) = L(n) + L(n-1)$. Likewise, we define $F(n)$ to be the nth Fibonacci number defined by $F(0) = 0, F(1) = 1, F(n+1) = F(n) + F(n-1)$, or $\sqrt{5}F(n) = \phi^n - (-\phi)^{-n}$.

By Theorem 6.3.5, and using $|V|/2 = |E|/3$, $1 - \phi = -\phi^{-1}$,

$$(-1)^{|E|}(\sqrt{5})^{|E|}(\phi^{|V|} \pm 1)E^{\circlearrowleft}(G;1)$$

$$= \phi^{|V|/2} \sum_{\text{proper edge } Z_5\text{-colourings}} (-1)^{\#\text{anticlockwise}} (\phi^{\#\text{ nonconsecutive}} \pm (1-\phi)^{\#\text{ nonconsecutive}}).$$

Then

$$(-1)^{|E|}(\sqrt{5})^{|E|}(\phi^{|E|/3} \pm \phi^{-|E|/3}) E^{\circlearrowleft}(G;1)$$
$$= \sum_{\text{proper edge } Z_5\text{-colourings}} (-1)^{\#\text{anticlockwise}} (\phi^{\#\text{ nonconsecutive}} \pm (-\phi)^{-\#\text{ nonconsecutive}}).$$

By considering separately the cases $|E|$ even and $|E|$ odd and using the definition of the Lucas and Fibonnaci sequences, we obtain the following summations over integer-weighted proper edge 5-colourings:

Corollary 6.3.6 *Let G be a cubic graph embedded on an orientable surface. Then*

$$E^{\circlearrowleft}(G;1) = \frac{1}{5^{\frac{|E|}{2}} L(\frac{|E|}{3})} \sum_{\text{proper edge } Z_5\text{-colourings of } G} (-1)^{\#\text{anticlockwise}} L(\#\text{nonconsecutive})$$

$$= \frac{1}{5^{\frac{|E|}{2}} F(\frac{|E|}{3})} \sum_{\text{proper edge } Z_5\text{-colourings of } G} (-1)^{\#\text{anticlockwise}} F(\#\text{nonconsecutive})$$

if $|E|$ is even, and

$$E^{\circlearrowleft}(G;1) = -\frac{1}{5^{\frac{|E|+1}{2}} F(\frac{|E|}{3})} \sum_{\text{proper edge } Z_5\text{-colourings of } G} (-1)^{\#\text{anticlockwise}} L(\#\text{nonconsecutive})$$

$$= -\frac{1}{5^{\frac{|E|-1}{2}} L(\frac{|E|}{3})} \sum_{\text{proper edge } Z_5\text{-colourings of } G} (-1)^{\#\text{anticlockwise}} F(\#\text{nonconsecutive})$$

if $|E|$ is odd.

By entirely similar calculations we find that for $k = 5$ and $\rho_{\{1,2\}}$ in Lemma 6.3.2 we have the following expressions for $E^{\circlearrowleft}(G;2)$:

Theorem 6.3.7 *Let G be a cubic graph embedded on an orientable surface. Then*

$$E^{\circlearrowleft}(G;2) = (\sqrt{5})^{-|E|}(-\phi)^{|E|} \sum_{\text{proper edge } Z_5\text{-colourings of } G} (-1)^{\#\text{anticlockwise}} (\phi^{-3})^{\#\text{nonconsecutive}},$$

where $\phi = \frac{1+\sqrt{5}}{2}$. Also,

$$E^{\circlearrowleft}(G;2) = (\sqrt{5})^{-|E|}(-\phi)^{-|E|} \sum_{\text{proper edge } Z_5\text{-colourings of } G} (-1)^{\#\text{anticlockwise}} (-\phi^{3})^{\#\text{nonconsecutive}}.$$

Similarly, we can combine the equations of Theorem 6.3.7 to obtain a sum over integer-weighted proper edge 5-colourings of G.

Corollary 6.3.8 *Let G be a cubic graph embedded on an orientable surface. Then*

$$E^{\circlearrowleft}(G;2) = \frac{1}{5^{\frac{|E|}{2}} L(|E|)} \sum_{\text{proper edge } Z_5\text{-colourings of } G} (-1)^{\#\text{anticlockwise}+\#\text{nonconsecutive}} L(3\#\text{nonconsecutive})$$

$$= \frac{1}{5^{\frac{|E|}{2}} F(|E|)} \sum_{\text{proper edge } Z_5\text{-colourings of } G} (-1)^{\#\text{anticlockwise}+\#\text{nonconsecutive}} F(3\#\text{nonconsecutive})$$

if $|E|$ is even, and

$$E^{\circlearrowleft}(G;2) = -\frac{1}{5^{\frac{|E|+1}{2}} F(|E|)} \sum_{\text{proper edge } Z_5\text{-colourings of } G} (-1)^{\#\text{anticlockwise}+\#\text{nonconsecutive}} L(3\#\text{nonconsecutive})$$

$$= -\frac{1}{5^{\frac{|E|-1}{2}} L(|E|)} \sum_{\text{proper edge } Z_5\text{-colourings of } G} (-1)^{\#\text{anticlockwise}+\#\text{nonconsecutive}} F(3\#\text{nonconsecutive})$$

if $|E|$ is odd.

Both Theorem 6.3.5 and 6.3.7 give expressions for $E^{\circlearrowleft}(G;1)$ and $E^{\circlearrowleft}(G;2)$ in terms of two different evaluations of the polynomial $Q(G;x)$ defined by [2]

$$Q(G;x) =: Q(x) = \sum_{\text{proper edge } Z_5\text{-colourings of } G} (-1)^{\#\text{anticlockwise}} x^{\#\text{nonconsecutive}}.$$

We conclude this chapter by showing that $Q(G;x)$ satisfies an identity which makes one of the equations in each of Theorem 6.3.5 and 6.3.7 redundant.

For a given cubic graph G embedded on an orientable surface, we write

$$Q(G;x) = \sum_{0 \le n \le |V|} q_n x^n,$$

where $q_n = q_n(G)$ is given by

$$q_n = \#\{\text{proper edge 5-colourings of } G:\ \#\text{anticlockwise} \equiv 0 \bmod 2,\ \#\,\text{nonconsecutive} = n\}$$

$$-\#\{\text{proper edge 5-colourings of } G:\ \#\text{anticlockwise} \equiv 1 \bmod 2,\ \#\,\text{nonconsecutive} = n\}.$$

Then, for example, the first equation of Corollary 6.3.6 and the first equation of Corollary 6.3.8 say that when $|E|$ is even we have

$$5^{\frac{|E|}{2}} L(\frac{|E|}{3})\ E^{\circlearrowleft}(G;1) = \sum_{0 \le n \le |V|} q_n L(n),$$

[2]For $G = I_3$ the graph with three parallel edges $Q(I_3;x) = -30(1+x^2)$. After some computer-aided calculations, $Q(K_4;x) = 30(3-4x+12x^2+4x^3+3x^4)$, $Q(K_2 \times K_3;x) = -150(2-6x+15x^2+15x^4+6x^5+2x^6)$, and $Q(K_{3,3};x) = 0$.

and

$$5^{\frac{|E|}{2}} L(|E|)\ E^{\circlearrowleft}(G;2) = \sum_{0 \le n \le |V|} (-1)^n q_n L(3n).$$

The equations in Theorem 6.3.5 and Theorem 6.3.7 are

$$E^{\circlearrowleft}(G;1) = (\sqrt{5})^{-|E|}(1-\phi)^{|E|/3} Q(\phi) = (\sqrt{5})^{-|E|}(-\phi)^{|E|/3} Q(1-\phi),$$

and

$$E^{\circlearrowleft}(G;2) = (\sqrt{5})^{-|E|}(-\phi)^{|E|} Q(\phi^{-3}) = (\sqrt{5})^{-|E|}(-\phi)^{-|E|} Q(-\phi^3),$$

where $\phi = \frac{1+\sqrt{5}}{2}$ is the golden ratio. Therefore, with $2|E| = 3|V|$ for a cubic graph $G = (V, E)$ and $1 - \phi = -\phi^{-1}$,

$$Q(\phi) = \phi^{|V|} Q(-\phi^{-1}), \qquad Q(-\phi^3) = (\phi^3)^{|V|} Q(\phi^{-3}).$$

We will show that the equation $Q(x) = x^{|V|} Q(-x^{-1})$ is in fact an identity for $x \neq 0$.

Since $|V|$ is even, we have, for $x \neq 0$,

$$x^{|V|} Q(-x^{-1}) = (-x)^{|V|} Q(-x^{-1}) = \sum_{\text{proper edge } Z_5\text{-colourings of } G} (-1)^{\#\text{anticlockwise}} (-x)^{\#\text{consecutive}}.$$

Thus, the equation $Q(x) = x^{|V|} Q(-x^{-1})$ is equivalent to the equation

$$\sum_{0 \le n \le |V|} q_n x^n = \sum_{0 \le n \le |V|} (-1)^{|V|-n} q_n x^{|V|-n}$$

$$= \sum_{0 \le n \le |V|} (-1)^n q_{|V|-n} x^n.$$

Theorem 6.3.9 *Let G be a cubic graph embedded on an orientable surface and the polynomial $Q(G; x)$ and its coefficients $q_n(G)$ as defined previously. Then,*

$$Q(G; x) = x^{|V|} Q(G; -x^{-1}).$$

Equivalently, for $0 \le n \le |V|$,

$$q_{|V|-n}(G) = (-1)^n q_n(G).$$

Proof. We will prove first statement, which says that

$$\sum_{\text{proper edge } Z_5\text{-colourings of } G} (-1)^{\#\text{anticlockwise}} x^{\#\text{nonconsecutive}}$$

$$= \sum_{\text{proper edge } Z_5\text{-colourings of } G} (-1)^{\#\text{anticlockwise}}(-x)^{\#\text{consecutive}}.$$

To do this we will use the definitions of nonconsecutive and anticlockwise in terms of the difference multiset of a proper triple.

Given a proper triple $\ell = (l_0, l_1, l_2) \in Z_5^3$ with difference multiset $\{l_0 - l_1, l_1 - l_2, l_2 - l_0\} = \{d_0, d_1, d_2\}$, we recall that ℓ is nonconsecutive if and only if $\{d_0, d_1, d_2\}$ is either $\{-1, -2, -2\}$ or $\{1, 2, 2\}$ and that ℓ is anticlockwise if and only if $\{d_0, d_1, d_2\}$ is either $\{1, 1, -2\}$ or $\{1, 2, 2\}$. Equivalently, ℓ is nonconsecutive if and only if the product $d_0d_1d_2$ belongs to $\{-1, +1\}$ and anticlockwise if and only if $d_0d_1d_2 \in \{-1, -2\}$.

For a given proper edge Z_5-colouring of G, we write $\#\{-1, -2\}$ for the number of vertices whose incident edges in clockwise order are coloured with a proper triple $(l_0, l_1, l_2) \in Z_5^3$ such that the product of differences $(l_0-l_1)(l_1-l_2)(l_2-l_0)$ belongs to $\{-1, -2\}$. Thus $\#\{-1, -2\} = \#\,$anticlockwise. A similar meaning is given to $\#\{-1, +1\} = \#\,$nonconsecutive and $\#\{-2, +2\} = \#\,$consecutive.

For fixed m (mod 2) and $0 \le n \le |V|$, we will establish a bijection between the set

$$\{\text{proper edge } Z_5\text{-colourings of } G \,:\, \#\{-1, -2\} \equiv m \bmod 2,\;\; \#\{-1, +1\} = n\}$$

and the set

$$\{\text{proper edge } Z_5\text{-colourings of } G \,:\, \#\{-1, -2\} + \#\{-2, +2\} \equiv m \bmod 2,\;\; \#\{-1, +1\} = |V| - n\}.$$

This is equivalent to showing that the contributions of $(-1)^m x^n$ to the sum defining $Q(G; x)$ are equal in number to the contributions of $(-1)^m x^n$ to $x^{|V|}Q(G; -x^{-1})$, for we have $\#\{-2, +2\} = |V| - \#\{-1, +1\}$,

$$\sum_{\text{proper edge } Z_5\text{-colourings of } G} (-1)^{\#\text{anticlockwise}} x^{\#\text{nonconsecutive}} = \sum_{\text{proper edge } Z_5\text{-colourings}} (-1)^{\#\{-1,-2\}} x^{\#\{-1,+1\}},$$

and

$$\sum_{\text{proper edge } Z_5\text{-colourings of } G} (-1)^{\#\text{anticlockwise}}(-x)^{\#\text{consecutive}}$$

$$= \sum_{\text{proper edge } Z_5\text{-colourings of } G} (-1)^{\#\{-1,-2\}+\#\{-2,+2\}} x^{\#\{-2,+2\}}.$$

We use the bijection $\lambda \mapsto -2\lambda$ on the set of proper edge Z_5-colourings of G, considering its effect on the triple of colours of the edges incident with any given vertex of G. If a given vertex of G is incident with edges of colours (l_0, l_1, l_2) in clockwise order, then the map $\lambda \mapsto -2\lambda$ changes this triple of colours to the triple $(-2l_0, -2l_1, -2l_2)$. The product $(l_0 - l_1)(l_1 - l_2)(l_2 - l_0)$ is multiplied by $-2^3 = 2$ (mod 5) in order to obtain the product $(2l_0 - 2l_1)(2l_1 - 2l_2)(2l_2 - 2l_0)$.

We observe that $\#\{-1,-2\} + \#\{-2,+2\} \equiv \#\{-1,2\} \bmod 2$, since the element -2 is double-counted. Multiplying the elements in $\{-1,2\}$ by 2 gives the set $\{-1,-2\}$, and multiplying the elements in $\{-2,+2\}$ by 2 gives the set $\{-1,+1\}$. Thus the map $\lambda \mapsto -2\lambda$ provides the bijection we wished to establish between the set of proper edge Z_5-colourings of G with the property that both $\#\{-1,-2\} + \#\{-2,+2\} \equiv m \bmod 2$ and $\#\{-1,+1\} = |V| - n$ and the set of proper edge Z_5-colourings of G with the property that both $\#\{-1,-2\} \equiv m \bmod 2$ and $\#\{-1,+1\} = n$. $\square$

Chapter 7

Conclusion

In this concluding section we indicate in general terms some possible directions for future work and highlight a few of the many questions which arise from the results of Chapters 2-6.

We begin with the following:

Problem 1 *Give combinatorial proofs of theorems obtained by the discrete Fourier transform such as those in Chapter 6.*

We use "combinatorial proof" in a similar sense to Stanley [57, §1.1] as meaning a bijective proof or, more vaguely, any proof more explanatory of the reasons *why* theorems obtained by use of the discrete Fourier transform are true. For example, Tarsi's proof of his Theorem 1.2 in [62] would count as combinatorial.

The classical discrete Fourier transform fits into a more general theory of group characters (see [24, appendix] for a succinct introduction). A character of an arbitrary group is a homomorphism from the group into the multiplicative group $\mathbb{C}^\times$ of the field of complex numbers. When the group is finite and Abelian, it is a standard theorem that the characters of A form a group isomorphic to A itself. The character used by the usual discrete Fourier transform is the homomorphism $\chi : Z_k \to \mathbb{C}^\times$ from the additive group Z_k into the multiplicative group $\mathbb{C}^\times$ defined for each $m \in Z_k$ by $\chi(m) = e^{2\pi i m/k}$. The other characters of the cyclic additive group Z_k are given by $\chi^l : Z_k \to \mathbb{C}^\times,\ \chi^l(m) = \chi(lm) = e^{2\pi i l m/k}$. In Chapter 6, by taking the weight ρ_l instead of ρ_1 we effectively changed the character χ to χ^l. The (usual) discrete Fourier transform of ρ_l relative to the character χ is then the discrete Fourier transform of ρ_1 relative to the character χ^l.

Although we imposed a linear order on Z_k in Chapters 3 and 5, it is the cyclic structure of the additive group of Z_k which relates it to the local cyclic rotation at each vertex in a surface embedding of a graph. In particular, for a cubic graph G with half-edge set H, a half-edge Z_3-colouring of G which is proper on each block of the partition $\mathcal{V}$ of H by vertices defines a local rotation at each vertex v according to the rotational sense in which the cyclic order of colours appear in $H(v)$.

The number of proper vertex k-colourings of a graph is independent of any structure put on the k colours. However, for any Abelian group A of order k we can represent the set of proper vertex k-colourings as a set of half-edge A-colourings with the property that each block of $\mathcal{V}$ is monochrome and each block of $\mathcal{E}$ is proper.

Problem 2 *What happens when we define weights on half-edge A-colourings for different Abelian groups A of the same order? For different Abelian groups A, do the group characters of A yield expressions for the number of proper vertex $|A|$-colourings of a graph with different combinatorial interpretations?*

The *transitions* of Jaeger [36] defined for a 4-regular graph G suggest using the group $Z_2 \times Z_2$ to colour the half-edges of G rather than Z_4, the latter more suitable when we are interested in the local vertex rotations in an orientable embedding of G. The "white", "black" and "crossing" transitions of Jaeger are each invariant under a group of actions isomorphic to $Z_2 \times Z_2$ and not the cyclic group Z_4.

Problem 3 *Develop the similarities between Jaeger's transitions and weight functions* [36] *and the vertex weights of Chapter 3 on the set of half-edge colourings.*

It may also be interesting to use characters of such non-Abelian groups as S_3, the group of symmetries of the triangle, for example when colouring the half-edges of a cubic graph G embedded in a surface. Rotational symmetries of the three half-edges at a vertex of G preserve the topology of the surface in which G is embedded while reflections correspond to "twisting" a pair of edges and changes the embedding of G. This approach is also suggested by the topological treatment of cubic graphs in [17] and the interpretation of the Matiyasevich polynomial as a knot invariant in [21].

Theorems 5.4.3 and 5.4.5 imply that the correlation $\Pr(\text{Equivalent} \mid \text{Same Parity}) - \Pr(\text{Equivalent})$ and the probability $\Pr(\text{Equivalent})$ are both *Tutte-Grothendieck invariants* if $k = 3$, that is, they satisfy a recursive deletion-contraction formula (see [68] for definitions). However, for $k \geq 5$ it is easily checked by using the results of §5.2 that this correlation and probability are not Tutte-Grothendieck invariants.

For $k = 2$, the indegree minus the outdegree of a vertex in any orientation of a graph is always equal to the degree. However, by considering just the indegree modulo 2 the methods used to prove Theorems 5.4.3 and 5.4.5 indicate that the following problem has an interesting answer. We pose it as a problem since, in the spirit of Problem 1, we are more interested explaining the answers to this and similar problems.

Problem 4 *What are the probability* $\Pr(\text{Equivalent})$ *and the correlation* $\Pr(\text{Equivalent} \mid \text{Same Parity}) - \Pr(\text{Equivalent})$ *when we induce vertex* 4*-colourings by taking the indegree minus the outdegree modulo* 4*? Equivalently, what happens when we induce vertex* 2*-colourings by taking the indegree modulo* 2*?*

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