

Graphical modelling of multivariate time series

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Received: 11 May 2006 / Revised: 14 January 2011 / Published online: 15 February 2011
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Abstract We introduce graphical time series models for the analysis of dynamic relationships among variables in multivariate time series. The modelling approach is based on the notion of strong Granger causality and can be applied to time series with non-linear dependences. The models are derived from ordinary time series models by imposing constraints that are encoded by mixed graphs. In these graphs each component series is represented by a single vertex and directed edges indicate possible Granger-causal relationships between variables while undirected edges are used to map the contemporaneous dependence structure. We introduce various notions of Granger-causal Markov properties and discuss the relationships among them and to other Markov properties that can be applied in this context. Examples for graphical time series models include nonlinear autoregressive models and multivariate ARCH models.

Keywords Graphical models · Multivariate time series · Granger causality · Global Markov property

Mathematics Subject Classification (2000) 60G10 · 62M10

1 Introduction

Graphical models have become an important tool for the statistical analysis of complex multivariate data sets, which are now increasingly available in many scientific fields. The key feature of these models is to merge the probabilistic concept of condi-

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tional independence with graph theory by representing possible dependences among the variables of a multivariate distribution in a graph. This has led to simple graphical criteria for identifying the conditional independence relations that are implied by a model associated with a given graph. Further important advantages of the graphical modelling approach are statistical efficiency due to parsimonious parameterisations of the joint distribution of the variables and the visualization of complex dependence structures, which allows an intuitive understanding of the interrelations among the variables and, thus, facilitates the communication of statistical results. For an introduction to graphical models we refer to the monographs by Whittaker [53], Edwards [21], and Cox and Wermuth [13]; a mathematically more rigorous treatment can be found in Lauritzen [39].

While graphical models originally have been developed for variables that are sampled with independent replications, they have been applied more recently also to the analysis of time dependent data. Some first general remarks concerning the potential use of graphical models in time series analysis can be found in Brillinger [10]; since then there has been an increasing interest in the use of graphical modelling techniques for analyzing multivariate time series (e.g., [14, 15, 23, 24, 44, 45, 48, 50]). However, all these works have been restricted to the analysis of linear interdependences among the variables whereas the recent trend in time series analysis has shifted towards non-linear parametric and non-parametric models (e.g., [28, 49, 52]). Moreover, in most of these approaches, the variables at different time points are represented by separate nodes, which leads to graphs with theoretically infinitely many vertices for which no rigorous theory exists so far.

In this paper, we present a general approach for graphical modelling of multivariate stationary time series, which is based on simple graphical representations of the dynamic dependences of a process. To this end, we utilize the concept of strong Granger causality (e.g., [29]), which is formulated in terms of conditional independences and, thus, can be applied to model arbitrary non-linear relationships among the variables. The concept of Granger causality originally has been introduced by Granger [34] and is commonly used for studying dynamic relationships among the variables in multivariate time series.

For the graphical representations, we consider mixed graphs in which each variable as a complete time series is represented by a single vertex and directed edges indicate possible Granger-causal relationships among the variables while undirected edges are used to map the contemporaneous dependence structure. We note that similar graphs have been used in Eichler [24] as path diagrams for the autoregressive structure of weakly stationary processes or—without undirected edges—in Didelez [18] for graphical modelling of time-continuous composable finite Markov processes based on the concept of local independence [1]. Formally, the graphical encoding of the dynamic structure of a time series is achieved by a new type of Markov properties, which we call Granger-causal Markov properties. We introduce various levels, namely the pairwise, the local, the block-recursive, and the global Granger-causal Markov property, and discuss the relationships among them. In particular, we give sufficient conditions under which the various Granger-causal Markov properties are equivalent; such conditions allow formulating models based on a simple Markov property while interpreting the associated graph by use of the global Granger-causal Markov property.

The paper is organized as follows. In Sect. 2, we introduce the concepts of Granger-causal Markov properties and graphical time series models; some examples of graphical time series models are presented in Sect. 3. In Sect. 4, we discuss global Markov properties, which relate certain separation properties of the graph to conditional independence or Granger noncausality relations among the variables of the process. Finally in Sect. 5, we compare the presented graphical modelling approach with other approaches in the literature and discuss possible extensions. The proofs are technical and put into the appendix.

2 Graphical time series models

In graphical modelling, the focus is on multivariate statistical models for which the possible dependences between the studied variables can be represented by a graph. In multivariate time series analysis, statistical models for a time series $X_V = (X_V(t))_{t \in \mathbb{Z}}$ are usually specified in terms of the conditional distribution of $X_V(t + 1)$ given its past $\mathbf{X}_V(t) = (X_V(s))_{s \leq t}$ in order to study the dynamic relationships over time among the series. Thus, a time series model may be described formally as a family of probability kernels P from $\mathbb{R}^{V \times \mathbb{N}}$ to \mathbb{R}^V , and we write $X_V \sim P$ if P is a version of the conditional probability of $X_V(t + 1)$ given $\mathbf{X}_V(t)$.

For modelling specific dependence structures, we utilize the concept of Granger (non-)causality, which has been introduced by Granger [34] and has proved to be particularly useful for studying dynamic relationships in multivariate time series. This probabilistic concept of noncausality from a process X_a to another process X_b is based on studying whether at time t the next value of X_b can be better predicted by using the entire information up to time t than by using the same information apart from the former series X_a . In practice, not all relevant variables may be available and, thus, the notion of Granger causality clearly depends on the used information set. In the sequel, we use the concept of strong Granger noncausality (e.g., [29]), which is defined in terms of conditional independence and σ -algebras and, thus, can be used also for non-linear time series models.

Let $X_V = (X_V(t))_{t \in \mathbb{Z}}$ with $X_V(t) = (X_v(t))_{v \in V} \in \mathbb{R}^V$ be a multivariate stationary stochastic process on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. For $A \subseteq V$, we denote by $X_A = (X_A(t))_{t \in \mathbb{Z}}$ the multivariate subprocess with components $X_a, a \in A$. The information provided by the past and present values of X_A at time $t \in \mathbb{Z}$ can be represented by the sub- σ -algebra $\mathcal{X}_A(t)$ of \mathcal{F} that is generated by $\mathbf{X}_A(t) = (X_A(s))_{s \leq t}$. We write $\mathcal{X}_A = (\mathcal{X}_A(t), t \in \mathbb{Z})$ for the filtration induced by X_A . This leads to the following definition of strong Granger noncausality in multivariate time series; for ease of notation, we subsequently usually drop the attribute “strong”.

Definition 2.1 Let A and B be disjoint subsets of V .

- (i) X_A is *strongly Granger-noncausal* for X_B with respect to the filtration \mathcal{X}_V if

$$\mathcal{X}_B(t + 1) \perp\!\!\!\perp \mathcal{X}_A(t) \mid \mathcal{X}_{V \setminus A}(t)$$

for all $t \in \mathbb{Z}$. This will be denoted by $X_A \nrightarrow X_B [\mathcal{X}_V]$.

- (ii) X_A and X_B are *contemporaneously conditionally independent* with respect to the filtration \mathcal{X}_V if

$$\mathcal{X}_A(t + 1) \perp\!\!\!\perp \mathcal{X}_B(t + 1) \mid \mathcal{X}_V(t) \vee \mathcal{X}_{V \setminus (A \cup B)}(t + 1)$$

for all $t \in \mathbb{Z}$. This will be denoted by $X_A \approx X_B \ [\mathcal{X}_V]$.

Intuitively, the dynamic relationships of a stationary multivariate time series X_V can be visualized by a mixed graph $G = (V, E)$ in which each vertex $v \in V$ represents one component X_v and two vertices a and b are joined by a directed edge $a \rightarrow b$ whenever X_a is Granger-causal for X_b or by an undirected edge $a - b$ whenever X_a and X_b are contemporaneously conditionally dependent. Conversely, for formulating models with specific dynamic dependences, a mixed graph G can be associated with a set of Granger noncausality and contemporaneous conditional independence constraints that are imposed on a time series model for X_V . Such a set of conditional independence relations encoded by a graph G is generally known as Markov property with respect to G . In the context of multivariate time series, graphs may encode different types of conditional independence relations, and we therefore speak of Granger-causal Markov properties when dealing with Granger noncausality and contemporaneous conditional independence relations. In the following definition, $\text{pa}(a) = \{v \in V \mid v \rightarrow a \in E\}$ denotes the set of parents of a vertex a , while $\text{ne}(a) = \{v \in V \mid v - a \in E\}$ is the set of neighbours of a ; furthermore, for $A \subseteq V$, we define $\text{pa}(A) = \cup_{a \in A} \text{pa}(a) \setminus A$ and $\text{ne}(A) = \cup_{a \in A} \text{ne}(a) \setminus A$.

Definition 2.2 (*Granger-causal Markov properties*) Let $G = (V, E)$ be a mixed graph. Then the stochastic process X_V satisfies

- (PC) the *pairwise Granger-causal Markov property* with respect to G if for all $a, b \in V$ with $a \neq b$
 - (i) $a \rightarrow b \notin E \Rightarrow X_a \not\rightarrow X_b \ [\mathcal{X}_V]$,
 - (ii) $a - b \notin E \Rightarrow X_a \approx X_b \ [\mathcal{X}_V]$;
- (LC) the *local Granger-causal Markov property* with respect to G if for all $a \in V$
 - (i) $X_{V \setminus (\text{pa}(a) \cup \{a\})} \not\rightarrow X_a \ [\mathcal{X}_V]$,
 - (ii) $X_{V \setminus (\text{ne}(a) \cup \{a\})} \approx X_a \ [\mathcal{X}_V]$;
- (BC) the *block-recursive Granger-causal Markov property* with respect to G if for all subsets A of V
 - (i) $X_{V \setminus (\text{pa}(A) \cup A)} \not\rightarrow X_A \ [\mathcal{X}_V]$,
 - (ii) $X_{V \setminus (\text{ne}(A) \cup A)} \approx X_A \ [\mathcal{X}_V]$.

Similarly, if P is a probability kernel from $\mathbb{R}^{V \times \mathbb{N}}$ to \mathbb{R}^V , we say that P satisfies the pairwise, the local, or the block-recursive Granger-causal Markov property with respect to a graph G whenever the same is true for every stationary process X_V with $X_V \sim P$.

Example 2.1 To illustrate the various Granger-causal Markov properties, we consider the graph G in Fig. 1. Suppose that a stationary process X_V satisfies the pairwise Granger-causal Markov property with respect to this graph G . Then the absence of

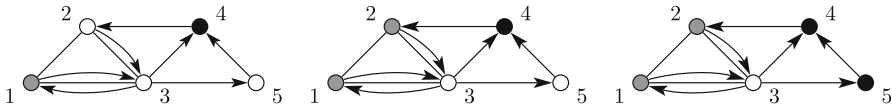


Fig. 1 Encoding of relations $X_A \rightarrow X_B$ [\mathcal{X}_X] by the **a** pairwise, **b** local, and **c** block-recursive Granger-causal Markov property (A and B are indicated by *grey* and *black* nodes, respectively)

the edge $1 \rightarrow 4$ in G implies that X_1 is Granger-noncausal for X_4 with respect to \mathcal{X}_V . Next, in the case of the local Granger-causal Markov property, we find that the bivariate subprocess $X_{\{1,2\}}$ is Granger-noncausal for X_4 with respect to \mathcal{X}_V since vertex 4 has parents 3 and 5. Similarly, if X_V obeys the block-recursive Granger-causal Markov property, the graph encodes that $X_{\{1,2\}}$ is Granger-noncausal for $X_{\{4,5\}}$ with respect to \mathcal{X}_V since $pa(4, 5) = \{3\}$.

The block-recursive Granger-causal Markov property obviously implies the other two Granger-causal Markov properties and, thus, is the strongest of the three Markov properties; similarly, the pairwise Granger-causal Markov property clearly is the weakest of the three properties. The question arises whether and under which conditions the three Granger-causal Markov properties are equivalent. In the case of random vectors $Y_V = (Y_v)_{v \in V}$ with values in \mathbb{R}^V , the various levels of Markov properties for graphical interaction models are equivalent if the distribution of Y_V satisfies

$$Y_A \perp\!\!\!\perp Y_B \mid Y_{C \cup D} \wedge Y_A \perp\!\!\!\perp Y_C \mid Y_{B \cup D} \Rightarrow Y_A \perp\!\!\!\perp Y_{B \cup C} \mid Y_D \tag{2.1}$$

for all disjoint subsets A, B, C , and D of V [47]. A necessary and sufficient condition for this intersection property is that the information common to $Y_{B \cup D}$ and $Y_{C \cup D}$ equals the information provided by Y_D . More precisely, let $(\Omega, \mathcal{F}, \mathbb{P})$ be the underlying probability space and let \mathcal{Y}_S be the sub- σ -algebra generated by $Y_S, S \subseteq V$. Furthermore, we denote the σ -algebra generated by \mathcal{Y}_S and the \mathbb{P} -null sets in \mathcal{F} by $\overline{\mathcal{Y}_S}$. Then the above intersection property holds if and only if $\overline{\mathcal{Y}_{C \cup D}} \cap \overline{\mathcal{Y}_{B \cup D}} = \overline{\mathcal{Y}_D}$ [17, 30]; we say that $\mathcal{Y}_{C \cup D}$ and $\mathcal{Y}_{B \cup D}$ are measurable separable conditionally on \mathcal{Y}_D . For more details on measurable separability we refer to Appendix A and the references therein.

In order to ensure validity of the intersection property in the time series case, we impose the following condition:

- (S) for all subsets A, B, C of $V, \mathcal{X}_A(t)$ and $\mathcal{X}_B(t)$ are measurably separable conditionally on $\mathcal{X}_{A \cap B}(t) \vee \mathcal{X}_C(t - k)$ for all $k \in \mathbb{N}$ and $t \in \mathbb{Z}$.

Here, $\mathcal{X}_{A \cap B}(t) \vee \mathcal{X}_C(t - k)$ denotes the smallest σ -algebra generated by $\mathcal{X}_{A \cap B}(t) \cup \mathcal{X}_C(t - k)$. The condition implies that for every \mathcal{F} -measurable random variable Z and all $t \in \mathbb{Z}$,

$$Z \perp\!\!\!\perp \mathcal{X}_A(t) \mid \mathcal{X}_{B \cup C}(t) \wedge Z \perp\!\!\!\perp \mathcal{X}_B(t) \mid \mathcal{X}_{A \cup C}(t) \Leftrightarrow Z \perp\!\!\!\perp \mathcal{X}_{A \cup B}(t) \mid \mathcal{X}_C(t). \tag{2.2}$$

In the case of random vectors Y_V , a commonly used sufficient condition for the intersection property and thus for conditional measurable separability is that the joint

distribution of Y_V is absolutely continuous with respect to some product measure and has a positive and continuous density (e.g., [39, Prop. 3.1]). The following result establishes a similar condition in terms of conditional distributions for the time series case; it requires an additional regularity condition on partial tail- σ -algebras [29,30].

Proposition 2.1 *Let $X_V = (X_V(t))_{t \in \mathbb{Z}}$ be a strictly stationary stochastic process on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$ taking values in \mathbb{R}^V and suppose the following two conditions hold:*

- (P) *the conditional distribution $\mathbb{P}^{X_V(t+1)|\mathcal{X}_V(t)}$, $t \in \mathbb{Z}$, has a regular version that is almost surely absolutely continuous with respect to some product measure ν on $\mathbb{R}^{|V|}$ with ν -a.e. positive and continuous density;*
- (M) *for all $A \subseteq V$ and $t \in \mathbb{Z}$*

$$\bigcap_{k \in \mathbb{N}} \left(\overline{\mathcal{X}_A(t)} \vee \overline{\mathcal{X}_{V \setminus A}(t-k)} \right) = \overline{\mathcal{X}_A(t)}.$$

Then the process X_V satisfies condition (S).

For an interpretation of condition (M), we note that it is equivalent to

$$\lim_{k \rightarrow \infty} \mathbb{E}(Z \mid \mathcal{X}_A(t) \vee \mathcal{X}_B(t-k)) = \mathbb{E}(Z \mid \mathcal{X}_A(t))$$

for all random variables Z and subsets $A, B \subseteq V$ [12]. Thus condition (M) implies that the process X_V is conditionally weakly mixing. For many types of non-linear time series stronger forms of mixing—but not conditional mixing—have been established (e.g., [19,28]). We believe that the above condition of conditional mixing is satisfied by most stationary time series models but a discussion of this is beyond the scope of this paper.

The intersection property now allows us to derive the following relations among the three Granger-causal Markov properties.

Theorem 2.1 *Suppose that X_V satisfies condition (S). Then the three Granger-causal Markov properties (BC), (LC), and (PC) are related by the following implications:*

$$(BC) \Rightarrow (LC) \Leftrightarrow (PC).$$

Furthermore, if X_V additionally satisfies the composition property

$$X_A \rightsquigarrow X_B \ [\mathcal{X}_V] \Leftrightarrow X_A \rightsquigarrow X_b \ [\mathcal{X}_V] \ \forall b \in B, \tag{2.3}$$

then the three Granger-causal Markov properties (BC), (LC), and (PC) are equivalent.

The theorem shows that, similarly as in the case of chain graph models with the Andersson–Madigan–Perlman (AMP) Markov property [2], the pairwise and the local Granger-causal Markov property are in general not sufficiently strong to encode all Granger-causal relationships that hold among the components of a multivariate time series with respect to full information \mathcal{X}_V . This suggests to specify graphical time series models in terms of the block-recursive Granger-causal Markov property.

Definition 2.3 (*Graphical time series model*) Let G be a mixed graph and let \mathcal{P}_G be a statistical time series model given by a family of probability kernels $P \in \mathcal{P}_G$ from $\mathbb{R}^{V \times \mathbb{N}}$ to \mathbb{R}^V . Then \mathcal{P}_G is said to be a *graphical time series model* associated with the graph G if, for all $P \in \mathcal{P}_G$, the distribution P satisfies the block-recursive Granger-causal Markov property with respect to G .

The three Granger-causal Markov properties considered so far encode only Granger noncausality relations with respect to the complete information \mathcal{X}_V . The discussion of phenomena such as spurious causality (e.g., [22, 35]), however, requires also the consideration of Granger-causal relationships with respect to partial information sets, that is, with respect to filtrations \mathcal{X}_S for subsets S of V . To this end, we introduce in Sect. 4 a global Granger-causal Markov property that more generally relates pathways in a graph to Granger-causal relations among the variables, and we establish, under condition (S), its equivalence to the block-recursive Granger-causal Markov property; this shows that the block-recursive Granger-causal Markov property is indeed sufficiently rich to describe the dynamic dependence structure in multivariate time series.

Before we continue our discussion of Markov properties in Sect. 4, we illustrate the introduced concept of graphical time series models by a few examples.

3 Examples

In the previous section, graphical time series models have been defined in terms of the block-recursive Granger-causal Markov property. For many time series models, however, condition (2.3) in Theorem 2.1 holds, and, hence, the pairwise, the local, and the block-recursive Granger-causal Markov property are equivalent. This enables us to derive the constraints on the parameters from the pairwise or the local Granger-causal Markov property.

There are no simple conditions known that are both necessary and sufficient for (2.3). The following proposition lists some sufficient conditions that cover many examples, as will be shown subsequently.

Proposition 3.1 *Suppose that X_V satisfies condition (S) and one of the following conditions:*

- (i) X_V is a Gaussian process;
- (ii) $X_v(t + 1), v \in V$, are mutually contemporaneously independent, that is, the joint conditional distribution factorizes as

$$\mathbb{P}^{X_V(t+1) | X_V(t)} = \otimes_{v \in V} \mathbb{P}^{X_v(t+1) | X_V(t)} \quad \forall t \in \mathbb{Z};$$

- (iii) $X_V(t + 1)$ depends on its past only in its conditional mean, that is,

$$X_V(t + 1) - \mathbb{E}[X_V(t + 1) | \mathcal{X}_V(t)] \perp\!\!\!\perp \mathcal{X}_V(t) \quad \forall t \in \mathbb{Z}.$$

Then the three Granger-causal Markov properties (BC), (LC), and (PC) are equivalent.

We note that processes satisfying condition (ii) can be described by directed graphs, that is, graphs without undirected edges. Thus the proposition implies that for directed graphs the pairwise and the block-recursive Granger-causal Markov property are always equivalent.

3.1 Nonlinear autoregressive models

As a first example, we consider the general class of multivariate nonlinear autoregressive models given by

$$X_V(t) = f_V(X_V(t - 1), \dots, X_V(t - p)) + \varepsilon_V(t),$$

where f_V is an \mathbb{R}^V -valued Borel measurable function on $\mathbb{R}^{p \times V}$ and $\varepsilon_V = (\varepsilon_V(t))_{t \in \mathbb{Z}}$ is a sequence of independent and identically distributed zero mean random vectors with density q_V and such that $\varepsilon_V(t)$ is independent of $\mathcal{X}_V(t - 1)$. Such models have been considered by many authors; in particular, conditions on f_V and q_V that guarantee geometric ergodicity and thus strong mixing of X_V have been established (e.g., [19, 41, 42]). We note, however, that currently there are no conditions known that ensure the conditional mixing condition (M). An exception are Gaussian autoregressive processes that will be briefly discussed below.

For the general class of multivariate nonlinear autoregressive models, the constraints imposed by a graph G are best formulated in terms of the local Granger-causal Markov property. More precisely, X_V satisfies the local Granger-causal Markov property with respect to G if for all $a \in V$

- (L1) $f_a(X_V(t - 1), \dots, X_V(t - p)) = f_a(X_{\text{pa}(a) \cup \{a\}}(t - 1), \dots, X_{\text{pa}(a) \cup \{a\}}(t - p))$;
- (L2) q_V factorizes as $q_V(z_V) = g_a(z_{\text{ne}(v) \cup \{a\}}) h_a(z_{V \setminus \{a\}})$.

The second condition implies $\varepsilon_a(t) \perp\!\!\!\perp \varepsilon_{V \setminus (\text{ne}(a) \cup \{a\})}(t) \mid \varepsilon_{\text{ne}(a)}(t)$, which is equivalent to X_a and $X_{V \setminus (\text{ne}(a) \cup \{a\})}$ being contemporaneously conditionally independent with respect to \mathcal{X}_V as required by the local Granger-causal Markov property. Since $X_V(t)$ depends on its past $\mathbf{X}_V(t - 1)$ only in its conditional mean, it follows from Theorem 2.1 and Proposition 3.1(iii) that the local and the block-recursive Granger-causal Markov properties are equivalent, that is, the above conditions on f_V and q_V define indeed a graphical nonlinear autoregressive model of order p associated with the graph G .

The general class of multivariate nonlinear autoregressive models covers many interesting and important models, of which we discuss only the following three.

- (a) *Vector autoregressive (VAR) model* Suppose that X_V is a stationary Gaussian process given by

$$X_V(t) = \sum_{u=1}^p \Phi(u) X_V(t - u) + \varepsilon_V(t), \quad \varepsilon_V(t) \stackrel{\text{iid}}{\sim} \mathcal{N}(0, \Sigma), \quad (3.1)$$

where $\Phi(u)$ are $V \times V$ matrices and the variance matrix Σ is non-singular with inverse $K = \Sigma^{-1}$. Then X_V satisfies the pairwise Granger-causal Markov property with respect to a graph $G = (V, E)$ if for all $a \neq b$

- (i) $a \rightarrow b \notin E \Rightarrow \Phi_{ba}(u) = 0 \quad \forall u = 1, \dots, p;$
- (ii) $a \dashrightarrow b \notin E \Rightarrow K_{ab} = K_{ba} = 0.$

Thus, the graphical VAR model of order p associated with the graph G , denoted by $\text{VAR}(p, G)$, is given by all stationary $\text{VAR}(p)$ processes whose parameters are constrained to zero according to the conditions (i) and (ii).

Furthermore, let $f(\lambda) = (2\pi)^{-1} \Phi(e^{-i\lambda})^{-1} \Sigma \Phi(e^{-i\lambda})'^{-1}$, $\lambda \in [-\pi, \pi]$, be the spectral density matrix of X_V , where $\Phi(z) = I_V - \Phi(1)z - \dots - \Phi(p)z^p$ and I_V is the $V \times V$ identity matrix. Then, if the eigenvalues of $f(\lambda)$ are bounded and bounded away from zero uniformly for all $\lambda \in [-\pi, \pi]$, the process X_V satisfies the separability condition (S) [24, Lemma A.2].

- (b) *Self-exciting threshold autoregressive (SETAR) model:* A stochastic process X_V is said to follow a multivariate SETAR model (e.g., [3, 52]) if for each $a \in V$

$$X_a(t) = \sum_{u=1}^p \sum_{b \in V} \phi_{ab}^{(n)}(u) X_b(t-u) + \varepsilon_a(t) \quad \text{if } X_a(t-d) \in I_{a,n},$$

where $\{I_{a,1}, \dots, I_{a,N}\}$ is a partition of \mathbb{R} , and $\varepsilon_V(t) \stackrel{\text{iid}}{\sim} Q_V$, say. Then X_V obeys the local Granger-causal Markov property with respect to a graph $G = (V, E)$ if, for $a \neq b$, $\phi_{ab}^{(n)}(u) = 0$ for all $n = 1, \dots, N$ and $u = 1, \dots, p$ whenever $b \rightarrow a \notin E$ and Q_V has density q_V satisfying condition (L2).

- (c) *Nonparametric additive autoregressive model:* A very useful class of nonparametric autoregressive models, which avoid the ‘‘curse of dimensionality’’, are the additive models given by

$$X_a(t) = \sum_{u=1}^p \sum_{b \in V} f_{ab}^{(u)}(X_b(t-u)) + \varepsilon_a(t), \quad a \in V, t \in \mathbb{Z},$$

where $f_{ab}^{(u)}$ are real-valued functions on \mathbb{R} . Here, condition (L1) obviously is equivalent to that the functions $f_{ab}^{(u)}$, $u = 1, \dots, p$, are constant whenever $a \neq b$ and the edge $b \rightarrow a$ is missing in the graph G .

3.2 Multivariate ARCH processes

Another important class of nonlinear time series models are the autoregressive conditional heteroscedasticity (ARCH) model and its various subsidiaries, which have been developed for modelling the time-varying volatility exhibited by many financial time series. A stationary stochastic process X_V is said to follow a multivariate ARCH(q) process if its conditional mean $\mathbb{E}(X_V(t) | \mathcal{X}_V(t-1))$ is zero and the conditional covariance matrix is of the form

$$\mathbb{E}(X_V(t)X_V(t)' | \mathcal{X}_V(t-1)) = \Sigma(t) = g_{VV}(X_V(t-1), \dots, X_V(t-p)).$$

For an overview of multivariate ARCH models we refer to Bollerslev et al. [7] and Gouriéroux [32]; sufficient conditions ensuring existence and strong mixing of such

processes can be found, for instance, in Lu and Jiang [42], Carrasco and Chen [11], and Liebscher [41].

One key issue in the specification of multivariate ARCH models is the restriction of the number of parameters involved, which in a general setting can be very large. Various parametrisations that allow different levels of complexity have been suggested. Here the graphical modelling approach can help to achieve a further reduction of the number of parameters.

In the following, we consider stochastic processes X_V with conditional distribution $\mathcal{N}(0, \Sigma(t))$ and formulate the constraints defining a graphical ARCH(q) model associated with a graph $G = (V, E)$ for three different parametrisations of $\Sigma(t)$.

- (i) *Constant conditional correlations:* The constant conditional correlation model of Bollerslev [6] provides the most parsimonious parametrisation of $\Sigma(t)$. The conditional variances are given by

$$\sigma_{aa}(t) = \sigma_{aa}^0 + \sum_{u=1}^q \sum_{k \in \text{pa}(a) \cup \{a\}} \alpha_k^a(u) X_k(t - u)^2,$$

whereas the conditional covariances are determined by the set of equations

$$\begin{aligned} \sigma_{ab}(t) &= \sigma_{aa}(t)^{1/2} \sigma_{bb}(t)^{1/2} \rho_{ab} && \text{if } a \text{ --- } b \in E, \\ K_{ab}(t) &= 0 && \text{if } a \text{ --- } b \notin E. \end{aligned}$$

Here $K(t) = \Sigma(t)^{-1}$ is the inverse conditional covariance matrix.

- (ii) *Constant conditional correlations with interaction:* In this parametrisation the conditional variance $\sigma_{aa}(t)$ additionally depends on interaction terms $X_k(t - u)X_l(t - u)$ if k and l are both parents of a . Thus the conditional variance can be written as

$$\sigma_{aa}(t) = \sigma_{aa}^0 + \sum_{u=1}^q \sum_{k, l \in \text{pa}(a) \cup \{a\}; k < l} \alpha_{kl}^a(u) X_k(t - u) X_l(t - u).$$

The entries $\sigma_{ab}(t)$ have the same form as in (i).

- (iii) *Vector ARCH model:* In the general vector ARCH model due to Kraft and Engle [38], also the correlation between the components of $X_V(t)$ may depend on the past values of X_V . This leads to conditional covariances $\sigma_{ab}(t)$, $a \leq b$, of the form

$$\sigma_{ab}(t) = \sigma_{ab}^0 + \sum_{u=1}^q \sum_{k, l \in P_{ab}; k < l} \alpha_{kl}^{ab}(u) X_k(t - u) X_l(t - u)$$

if $a = b$ or $a \text{ --- } b \in E$, where $P_{ab} = (\text{pa}(a) \cup \{a\}) \cap (\text{pa}(b) \cup \{b\})$, while the conditions $K_{ab}(t) = 0$ for $a \neq b$ and $a \text{ --- } b \notin E$ remain unchanged.

For the constant conditional correlation models it is easy to derive conditions to ensure that the conditional covariances are positive definite almost surely for all t . In contrast, such conditions are difficult to impose and verify for the vector ARCH model. Therefore Engle and Kroner [27] suggested an alternative representation for the multivariate ARCH(q) model in which $\Sigma(t)$ is guaranteed to be positive definite almost surely for all t . In this so-called BEKK representation,¹ the conditional covariances of a graphical ARCH model are parametrised by

$$\sigma_{ab}(t) = \sigma_{ab}^0 + \sum_{n=1}^N \sum_{u=1}^q \sum_{k,l \in P_{ab}: k < l} \alpha_{ka}^{(n)}(u) \alpha_{lb}^{(n)}(u) X_k(t-u) X_l(t-u).$$

In this form it is immediately clear that if $\sigma_{ab}(t)$ depends on the past of X_k then at least one of the conditional variances $\sigma_{aa}(t)$ and $\sigma_{bb}(t)$ must also depend on X_k . Although less obvious the same can be shown for the vector ARCH model in the original parametrisation noting that the conditional covariance matrix $\Sigma(t)$ must be positive definite. Hence graphical vector ARCH models fulfill condition (2.3). For the constant conditional correlation model condition (2.3) is trivially fulfilled.

3.3 A binary time series model

As an example with categorical data, we consider a binary time series model that has been used for the identification of neural interactions from neural spike train data [8, 9]. Suppose that the data consist of the recorded spike trains for a set of neurons, that is, of the sequences of firing times $(\tau_{v,n})_{n \in \mathbb{N}}$ for neurons $v \in V$, and let X_v be the binary time series obtained by setting $X_v(t) = 1$ if neuron v has fired in the interval $[t, t + 1)$ and $X_v(t) = 0$ otherwise. We assume that the hypothesized neural pathways between the observed neurons can be depicted by a purely directed graph G ; in particular, we thus exclude the possibility that the dependences among the observed neurons are affected by unmeasured confounders. Then the interactions between the neurons can be modelled by the conditional probabilities

$$\mathbb{P}(X_b(t) = 1 | \mathcal{X}_V(t-1)) = \Phi \left(\sum_{a \in \text{pa}(b)} U_{ba}(t) - \theta \right), \tag{3.2}$$

where $\Phi(x)$ denotes the normal cumulative function,

$$U_{ba}(t) = \sum_{u=1}^{\gamma_b(t)} g_{ba}(u) X_a(t-u) \tag{3.3}$$

¹ This is named after Baba, Engle, Kraft and Kroner, the authors of an earlier version of the paper (cf. Baba [4]).

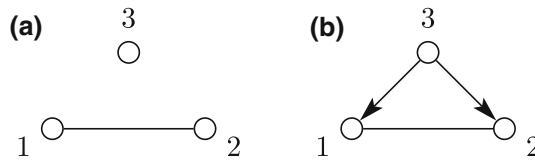


Fig. 2 Illustration of non-equivalence of pairwise and block-recursive Granger-causal Markov properties: the process with conditional variance (3.4) satisfies the pairwise Granger-causal Markov property with respect to the graphs in **a** and **b** whereas it satisfies the block-recursive Granger-causal Markov property only with respect to the graph in **b**

measures the influence of process a on process b , and

$$\gamma_b(t) = \min \{u \in \mathbb{N} \mid X_b(t - u) = 1\}$$

is the time elapsed since the last event of process X_b . Furthermore, we assume that the time unit has been chosen small enough such that there are no interactions among the neurons within one time interval, and that, consequently, the joint conditional probability factorizes as

$$\mathbb{P}(X_V(t) = x_V \mid \mathcal{X}_V(t - 1)) = \prod_{v \in V} \mathbb{P}(X_v(t) = x_v \mid \mathcal{X}_V(t - 1))$$

for all $x_V \in \{0, 1\}^V$. Then the pairwise and the block-recursive Granger-causal Markov property are equivalent by Proposition 3.1(ii) and, thus, we can use the former for modelling dependences between the processes. From (3.2) and (3.3), it follows that X_a is Granger-noncausal for X_b if and only if $g_{ba}(u) = 0$ for all $u \in \mathbb{N}$.

3.4 Two counter examples

Although condition (2.3) is satisfied by a wide variety of time series models it does not hold generally. As an example, we consider a simple nonlinear ARCH model X_V with conditional distributions $X_V(t) \mid \mathcal{X}_V(t - 1) \sim \mathcal{N}(0, \Sigma(t))$, where the conditional covariance matrix $\Sigma(t)$ is given by

$$\Sigma(t) = \begin{pmatrix} 1 & \rho(t) & 0 \\ \rho(t) & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{with} \quad \rho(t) = \begin{cases} \rho & \text{if } |X_3(t - 1)| > c \\ 0 & \text{otherwise} \end{cases} \quad (3.4)$$

for some constants ρ with $0 < |\rho| < 1$ and $c > 0$. Models of this type can be seen as a multivariate generalisation of the qualitative threshold ARCH(1) model of Gouriéroux and Monfort [33].

From the conditional covariance matrix, we find that, on the one hand, the marginal conditional distributions of $X_v(t)$ given $\mathcal{X}_V(t - 1)$ are standard normal and, thus, do not depend on $\mathcal{X}_V(t - 1)$. This implies that the process X_V satisfies the pairwise Granger-causal Markov property with respect to the graph (a) in Fig. 2. On the other

hand, X_k Granger-causes the subprocess (X_1, X_2) since the bivariate conditional distribution of $(X_1(t), X_2(t))$ depends on the value of $X_3(t - 1)$ through the conditional correlation $\rho(t)$. Thus X_V obeys the block-recursive Granger-causal Markov property with respect to the graph (b) in Fig. 2, but not with respect to the graph (a).

We note that the example can be easily generalized by considering models where the conditional variances $\text{var}(X_a(t) | \mathcal{X}_V(t - 1))$, $a \in V$, and the conditional correlation matrix $\text{corr}(X_V(t), X_V(t) | \mathcal{X}_V(t - 1))$ are modelled separately as functions of the past values $X_V(t - 1), \dots, X_V(t - p)$.

Next, consider the trivariate process X_V given by

$$X_1(t) = f(X_2(t - 1)) + \varepsilon(t), \quad X_2(t) = g(X_3(t - 1)), \quad X_3(t) = \eta(t),$$

where $\varepsilon(t)$ and $\eta(t)$ are independent sequences of *i.i.d.* random variables. Since

$$\mathcal{X}_1(t) \vee \mathcal{X}_2(t) = \mathcal{X}_1(t) \vee \mathcal{X}_3(t - 1),$$

condition (S) is violated. Indeed, we find that neither X_2 nor X_3 Granger-cause X_1 with respect to the full filtration \mathcal{X}_V whereas the bivariate process $(X_2, X_3)'$ is Granger-causal for X_1 . Therefore, the pairwise and the local Granger-causal Markov property are not equivalent for this process.

4 Global Markov properties

The interpretation of graphs describing the dependence structure of graphical models in general is enhanced by global Markov properties that merge the notion of conditional independence with a purely graph theoretical concept of separation allowing one to state whether two subsets of vertices are separated by a third subset of vertices. In this section, we show that the concept of p -separation introduced by Levitz et al. [40] for chain graph models with the AMP Markov property [2] can be used to obtain global Markov properties in the present context of graphical time series models. Throughout this section we assume that condition (S) in Sect. 2 holds.

4.1 The global AMP Markov property

We start with some further graphical terminology. Let $G = (V, E)$ be a mixed graph. Then a *path* π between two vertices a and b in G is a sequence $\pi = \langle e_1, \dots, e_n \rangle$ of edges $e_i \in E$ such that e_i is an edge between v_{i-1} and v_i for some sequence of vertices $v_0 = a, v_1, \dots, v_n = b$. The vertices a and b are the *endpoints* of the path, while v_1, \dots, v_{n-1} are the *intermediate points* on the path. Like [37] we do not require that the points v_j on a path π are distinct; this means that paths in general may be self-intersecting. A path π in G is called a *directed path* if it is of the form $a \rightarrow \dots \rightarrow b$ or $a \leftarrow \dots \leftarrow b$. Similarly, if π consists only of undirected edges it is called an *undirected path*. Furthermore, a path $\tilde{\pi}$ is a subpath of π if $\tilde{\pi} = \langle e_i, e_{i+1}, \dots, e_{j-1}, e_j \rangle$ for some $1 \leq i \leq j \leq n$.

An intermediate point c on a path π is said to be a p -collider on the path if the edges preceding and succeeding c on the path either have both an arrowhead at c or one has an arrowhead at c and the other is a line, i.e. $\rightarrow c \leftarrow$, $\rightarrow c \text{ ---}$, $\text{---} c \leftarrow$; otherwise the point c is said to be a p -noncollider on the path. Notice that this classification only applies to the intermediate points of a path π ; the endpoints are neither p -colliders nor p -noncolliders. We also note that a vertex can take different roles in different positions on a path: for example, on the path $1 \rightarrow 3 \leftarrow 2 \rightarrow 3 \rightarrow 4$ in Fig. 1, vertex 3 appears both as a p -collider and a p -noncollider.

A path π between vertices a and b is said to be p -connecting given a set S if

- (i) every p -noncollider on the path is not in S , and
- (ii) every p -collider on the path is in S ,

otherwise we say the path is p -blocked given S . In graphs encoding Markov properties of variables, p -connecting paths are exactly those paths inducing associations between the variables; conversely, if there are no p -connecting paths the corresponding variables are independent. This leads to the following definition.

Definition 4.1 (p -separation) Two vertices a and b in a mixed graph G are p -separated given a set S if all paths between a and b are p -blocked given S . Similarly, two sets A and B in G are said to be p -separated given S if, for every pair $a \in A$ and $b \in B$, a and b are p -separated given S . This will be denoted by $A \bowtie_p B \mid S$.

We note that the above conditions for p -separation are simpler than those in Levitz et al. [40] due to the fact that we consider the larger class of all possibly self-intersecting paths. The equivalence of the two notions of p -separation is shown in Appendix D. The following results show that the concept of p -separation can be applied to graphs encoding dynamic relationships in multivariate time series and allows reading off conditional independences among the stochastic processes that are represented by the vertices in the graph.

Lemma 4.1 Suppose that X_V satisfies the block-recursive Granger-causal Markov property with respect to the graph G . Then, for any disjoint subsets A , B , and S of V , we have

$$A \bowtie_p B \mid S \Rightarrow \mathcal{X}_A(t) \perp\!\!\!\perp \mathcal{X}_B(t) \mid \mathcal{X}_S(t) \quad \forall t \in \mathbb{Z}.$$

Letting t tend to infinity, we can translate p -separation in the graph into conditional independence statements for complete subprocesses. For this, we define $\mathcal{X}_S(\infty) = \bigvee_{t \in \mathbb{Z}} \mathcal{X}_S(t)$ as the σ -algebra generated by the subprocess X_S .

Theorem 4.1 Suppose X_V satisfies the block-recursive Granger-causal Markov property with respect to the graph G . Then, for any disjoint subsets A , B , and S of V , we have

$$A \bowtie_p B \mid S \Rightarrow \mathcal{X}_A(\infty) \perp\!\!\!\perp \mathcal{X}_B(\infty) \mid \mathcal{X}_S(\infty).$$

We say that X_V satisfies the global AMP Markov property (GA) with respect to G .

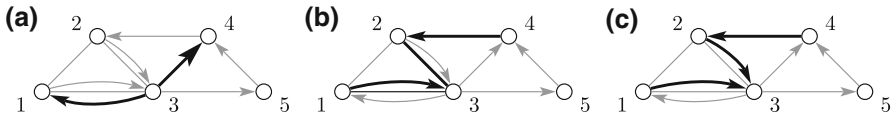


Fig. 3 Illustration of global AMP Markov property (paths are marked by bold lines): **a** path between 1 and 4 that is p -connecting given $S \subseteq \{2, 5\}$; **b** path between 1 and 4 that is p -connecting given $S = \{2, 3\}$ (or $\{2, 3, 5\}$); **c** path between 1 and 4 that is p -connecting given $S = \{3, 5\}$ (or $\{3\}$)

Example 4.1 For an illustration of the global AMP Markov property, we consider again the graph G in Fig. 1. In this graph, vertices 1 and 4 are not adjacent. Nevertheless, it can be shown that the two vertices cannot be p -separated by any set $S \subseteq \{2, 3, 5\}$: firstly, the path $1 \leftarrow 3 \rightarrow 4$ is p -connecting given a set S unless the set S contains the vertex 3 (Fig. 3a). Secondly, the path $1 \rightarrow 3 \rightarrow 2 \leftarrow 4$ is p -connecting given S whenever both intermediate points 2 and 3 belong to S (Fig. 3b). Finally, the path $1 \rightarrow 3 \leftarrow 2 \leftarrow 4$ is p -connecting given S if S contains vertex 3 but not 2 (Fig. 3c). Thus, if X_V is a stationary process that obeys the block-recursive Granger-causal Markov property with respect to G , then the graph G does not encode that X_1 and X_4 are conditionally independent given X_S regardless of the choice of $S \subseteq \{2, 3, 5\}$.

Similarly, it can be shown that vertices 1 and 5 are p -separated given $S = \{3, 4\}$: every path between 1 and 5 that contains the edge $3 \rightarrow 5$ or the subpath $3 \rightarrow 4 \leftarrow 5$ is p -blocked by vertex 3. All other paths between 1 and 5 contain the subpath $2 \leftarrow 4 \leftarrow 5$ and, thus, are blocked by vertex 4. It follows that for every process X_V that satisfies the block-recursive Granger-causal Markov property with respect to G the components X_1 and X_5 are conditionally independent given $X_{\{3,4\}}$.

4.2 The global Granger-causal Markov property

In this section, we apply the concept of pathwise separation to the problem of deriving general Granger noncausality relations from mixed graphs. To motivate the approach, we firstly consider the graphical VAR(1) model of all trivariate stationary processes $X_V = (X_1, X_2, X_3)$ given by

$$\begin{aligned}
 X_1(t) &= \phi_{11} X_1(t - 1) + \phi_{12} X_2(t - 1) + \varepsilon_1(t), \\
 X_2(t) &= \phi_{22} X_2(t - 1) + \phi_{23} X_3(t - 1) + \varepsilon_2(t), \\
 X_3(t) &= \phi_{33} X_3(t - 1) + \varepsilon_3(t)
 \end{aligned}
 \tag{4.1}$$

for $t \in \mathbb{Z}$ with independent and standard normally distributed errors $\varepsilon_V(t), t \in \mathbb{Z}$. The associated graph G that encodes the restrictions imposed on the model consists simply of the path $3 \rightarrow 2 \rightarrow 1$, which is p -connecting given the empty set. This indicates that the components X_1 and X_3 are, in general, not independent in a bivariate analysis. However, an intuitive interpretation of the directed path $3 \rightarrow 2 \rightarrow 1$ suggests that X_3 Granger-causes X_1 but not vice versa if only the bivariate process $X_{\{1,3\}}$ is considered. Indeed, the block-recursive Granger-causal Markov property implies that $X_3(t + 1) \perp\!\!\!\perp \mathcal{X}_{\{1,2\}}(t) \mid \mathcal{X}_{\{3\}}(t)$, from which it follows by decomposition (see

Appendix A) that X_1 is Granger-noncausal for X_3 with respect to $\mathcal{A}_{\{1,3\}}$. Obviously, the p -separation criterion is too strong for establishing this Granger-noncausality relationship between X_3 and X_1 since it requires that all paths between the two vertices are p -blocked whereas it seems sufficient that only certain paths, namely those ending with an arrowhead at vertex 3, are p -blocked.

This suggests the following definitions. A path π between two vertices a and b in G is said to be b -pointing² if it has an arrowhead at the endpoint b . More generally, a path π between two disjoint subsets A and B is said to be B -pointing if it is b -pointing for some $b \in B$.

For the derivation of contemporaneous conditional independences, we also need to consider paths with arrowheads at both endpoints; such paths π will be called bi -pointing. Furthermore, let $\pi = \langle \pi_1, \dots, \pi_n \rangle$ be a composition of paths π_i that are undirected or bi-pointing. Then π is said to be an *extended bi-pointing* path. In particular, this implies that any undirected or bi-pointing path is also an extended bi-pointing path; similarly, the composition $\pi = \langle \pi_1, \pi_2 \rangle$ of two extended bi-pointing paths π_i is again extended bi-pointing. Moreover, every extended bi-pointing path π is of the form $\pi = \langle u_1, \beta, u_2 \rangle$ for some paths u_1, u_2 , and β of possibly length zero, where u_1 and u_2 are undirected paths and β is a bi-pointing path (hence the term ‘extended bi-pointing’). With these definitions, we define the following global Granger-causal Markov property, which gives a path-oriented criterion for deriving general Granger noncausality relations from a mixed graph.

Definition 4.2 (*Global Granger-causal Markov property*) Let X_V be a stationary process and let $G = (V, E)$ be a mixed graph. Then X_V satisfies the *global Granger-causal Markov property* (GC) with respect to G if, for all disjoint subsets A, B , and S of V , the following conditions hold:

- (i) if every B -pointing path in G between A and B is p -blocked given $S \cup B$ then

$$X_A \not\leftrightarrow X_B \ [\mathcal{A}_{A \cup B \cup S}];$$

- (ii) if every extended bi-pointing path in G between A and B is p -blocked given $A \cup B \cup S$ then

$$X_A \approx X_B \ [\mathcal{A}_{A \cup B \cup S}].$$

From the definition, it is immediately clear by setting $S = V \setminus (A \cup B)$ that the global Granger-causal Markov property entails the block-recursive Granger-causal Markov property. The following theorem shows that in fact, under condition (S), the two Granger-causal Markov properties are equivalent; thus, the global Granger-causal Markov property may be employed to discuss the dynamic relationships implied by a graphical time series model defined in terms of the block-recursive Granger-causal Markov property.

² In the literature, a path with this property is also termed a path into b .

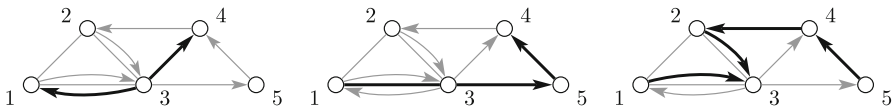


Fig. 4 Illustration of global Granger-causal Markov property: Three 4-pointing paths (solid lines) between 1 and 4 that are p -blocked by the set $\{3, 4\}$

Theorem 4.2 *Let X_V be a stationary process and let $G = (V, E)$ be a mixed graph. Then X_V satisfies the block-recursive Granger-causal Markov property with respect to G if and only if X_V satisfies the global Granger-causal Markov property with respect to G .*

As a consequence of the global Granger-causal Markov property, we find that p -separation in the graph implies Granger noncausality in both directions and contemporaneous conditional independence of the variables.

Corollary 4.1 *Suppose that the process X_V satisfies the block-recursive Granger-causal Markov property with respect to a mixed graph G . For disjoint subsets A, B , and S of V , if A and B are p -separated given S , then*

$$X_A \not\rightarrow X_B [\mathcal{X}_{A \cup B \cup S}], \quad X_B \not\rightarrow X_A [\mathcal{X}_{A \cup B \cup S}], \quad \text{and} \quad X_A \approx X_B [\mathcal{X}_{A \cup B \cup S}].$$

The following corollary summarizes the relationships between the various Markov properties for graphical time series models.

Corollary 4.2 *The various Granger-causal Markov properties are related as follows:*

$$(GC) \Leftrightarrow (BC) \Rightarrow (LC) \Leftrightarrow (PC).$$

Furthermore, we have $(BC) \Rightarrow (GA)$. If additionally condition (2.3) holds, then the four Granger-causal Markov properties (PC) , (LC) , (BC) , and (GC) are equivalent.

Proof The corollary summarizes Theorems 2.1, 4.1, and 4.2.

Example 4.2 For an illustration, we again consider a stationary time series X_V satisfying the block-recursive Granger-causal Markov property with respect to the graph G in Fig. 1. In Example 4.1, we have seen that vertices 1 and 4 are not p -separated given $S = \{3\}$, that is, X_1 and X_4 are in general not conditionally independent given X_3 . We now employ the global Granger-causal Markov property to examine the dynamic relationships between the components X_1 and X_4 further.

We start by examining the 4-pointing paths between 1 and 4. Straightforward considerations show that all 4-pointing paths end with either $3 \rightarrow 4$, $3 \rightarrow 5 \rightarrow 4$, or $2 \leftarrow 4 \leftarrow 5 \rightarrow 4$; three instances of such paths are depicted in Fig. 4. The paths ending with either $3 \rightarrow 4$ or $3 \rightarrow 5 \rightarrow 4$ are clearly p -blocked by vertex 3 whereas the paths ending with $2 \leftarrow 4 \leftarrow 5 \rightarrow 4$ are p -blocked by vertex 4. It follows that every 4-pointing paths between 1 and 4 is p -blocked by $\{3, 4\}$ and thus X_1 does not Granger-cause X_4 with respect to $\mathcal{X}_{\{1,3,4\}}$.

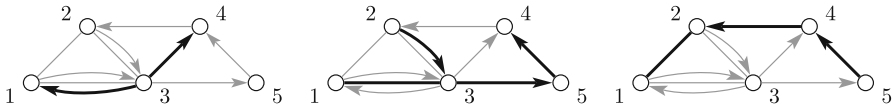


Fig. 5 Illustration of global Granger-causal Markov property: Three extended bi-pointing paths (solid lines) between 1 and 4 that are p -blocked by the set $\{3, 4\}$

Similarly, we can examine all extended bi-pointing paths between vertices 1 and 4 to show that X_1 and X_4 are contemporaneously conditionally independent with respect to $\mathcal{X}_{\{1,3,4\}}$. Figure 5 shows three examples of such paths: the first two are p -blocked by vertex 3 (notice that on the second path, the vertex 3 is once a p -collider and once a p -noncollider) whereas the last path is p -blocked by vertices 3 and 4. For similar reasons as above, these three paths are exemplary for all extended bi-pointing paths between 1 and 4, and we conclude that X_1 and X_4 are indeed contemporaneously conditionally independent with respect to $\mathcal{X}_{\{1,3,4\}}$.

Finally, we note that every 1-pointing path between 4 and 1 must end with the directed edge $3 \rightarrow 1$. Since this edge has a tail at vertex 3, every such path must be p -blocked given $S = \{1, 3\}$, which implies that X_4 does not Granger-cause X_1 with respect to $\mathcal{X}_{\{1,3,4\}}$.

5 Discussion

In this paper, we discussed a graphical modelling approach for multivariate time series that is based on mixed graphs in which each vertex represents one complete component series while the edges in the graph reflect possible dynamic interdependencies among the variables of the process. The constraints imposed by the graphs are formulated in terms of strong Granger noncausality and, thus, allow modelling arbitrary non-linear dependencies. The graphical modelling approach can help to reduce the number of parameters involved in modelling high-dimensional non-linear time series while encoding the constraints on the parameters in a simple graph, which is easy to visualize and allows an intuitive understanding of the dependencies in the model.

We have shown that the interpretation of these graphs, which for many models are built only from pairwise Granger noncausality relations, is enhanced by so-called global Markov properties, which relate separation properties of the graph to conditional independence or Granger noncausality statements about the process. In this paper, we have used the path-oriented concept of p -separation, which allows us to attribute Granger-causal relationships among the variables to certain pathways in the graphs.

Our objective has been to provide a general framework for modelling the dynamic interdependencies in multivariate time series; in particular, we focused on a simple graphical representation, which has been achieved by representing each component of a multivariate time series by a single vertex in the associated graph. The approach presented here, however, is not the only possible, and since the first papers on the application of graphical models in time series analysis [10, 43], there has been an increasing interest in the topic [14, 15, 23, 24, 44, 45, 48, 50]. All these approaches are basically restricted to the analysis of linear interdependencies, and most of them repre-

sent each variable at each time point by a separate vertex in the associated graph. In the following, we briefly compare our approach with alternative graphical representations and point out possible extensions.

Modelling processes of variables at separate time points

A more detailed modelling of dependencies among the components of a vector time series can be achieved by representing each random variable $X_{v_i}(t)$ by a different vertex v_i , say, in a graph G . This alternative approach has been discussed, for example, by Reale and Tunnicliffe Wilson [48], Dahlhaus and Eichler [15], and Moneta and Spirtes [44]. On the one hand, it leads to a more flexible class of graphical models and has the advantage that many of the concepts and methods that have been developed for the multivariate case carry over to the time series case. On the other hand, the increased flexibility leads to (sometimes much) larger graphs, which easily can become unwieldy and difficult to interpret, and it clearly also aggravates the model selection problem. Moreover, the underlying graph for such graphical time series models theoretically has infinitely many vertices, and it is not immediately clear how to prune this graph to a finite representation while preserving the Markov properties. In contrast, Lemma D.1 provides a simple local criterion that restricts the search for p -connecting paths in the type of graphs considered in this paper.

Apart from these theoretical and practical issues, we think that a high level of detail as provided by these models is not always wanted nor always appropriate. We give two examples. Firstly, Baccalá and Sameshima [5] proposed a frequency-domain approach for the discussion of Granger-causal relationships based on the concept of partial directed coherence. Although this approach still requires the fitting of VAR models, the identification of interactions is performed in the frequency-domain and hence only relations on the level of Granger noncausality can be identified. The results in Baccalá and Sameshima [5] were summarized by path diagrams associated with the identified VAR model as discussed in Eichler [24]. Our approach of representing each time series by one single vertex in the graph provides a theoretical framework for such frequency-domain based analyses.

Secondly, multivariate time series are often obtained by high-frequency sampling of continuous-time processes such as EEG-recordings or neural spike trains. Here, our approach yields a graphical representation of the interrelationships that does not depend (to some extent) on the sampling frequency (e.g., [22]). Moreover, many sophisticated models that have been proposed, for example, for analysing neural activity do not show a dependence on the past values only at specific lags. For instance, in the binary time series model discussed in Example 3.3, the conditional distribution of $X_b(t)$ given the past history $\mathcal{X}_V(t-1)$ depends on another process X_a through the past values $X_a(t-1), \dots, X_a(t-\gamma_b(t))$, where $\gamma_b(t)$ is the time elapsed since the last event of process X_b . In other words, the number of lagged variables $X_a(t-u)$ on which $X_b(t)$ depends varies over time depending on the past of X_b itself. Consequently, it seems inappropriate to break down the dependence of $X_a(t)$ on $\mathbf{X}_b(t)$ further into dependencies of $X_a(t)$ on $X_b(t-u)$ as required by the detailed modelling approach.

m-Separation versus *p*-separation

The contemporaneous dependence structure of a process X_V can also be described by conditional independencies of the form

$$\mathcal{X}_A(t + 1) \perp\!\!\!\perp \mathcal{X}_B(t + 1) \mid \mathcal{X}_V(t),$$

in which case X_A and X_B are said to be contemporaneously independent with respect to \mathcal{X}_V . This alternative approach, which is related to the concept of instantaneous causality by Granger [34], has been studied by Eichler [24] in the context of weakly stationary processes and linear dependencies.

The most important difference between these two approaches for defining graphical time series models is that the corresponding composition and decomposition property

$$\begin{aligned} &\mathcal{X}_A(t + 1) \perp\!\!\!\perp \mathcal{X}_B(t + 1) \mid \mathcal{X}_V(t) \\ \Leftrightarrow &\mathcal{X}_a(t + 1) \perp\!\!\!\perp \mathcal{X}_b(t + 1) \mid \mathcal{X}_V(t) \quad \forall a \in A, \forall b \in B \end{aligned} \tag{5.1}$$

does not follow from condition (S) but requires additional assumptions similar to condition (2.3). Furthermore, we note that only the first two conditions in Proposition 3.1 are sufficient for the above property (5.1). Consequently, the class of graphical time series models for which the pairwise and the block-recursive Granger-causal Markov properties are equivalent would be smaller under the alternative approach based on contemporaneous independence. Alternatively, if modelling is to be based on *m*-separation, one might consider use of an adapted variant of the connected set Markov property as in Drton and Richardson [20] instead of the pairwise Markov property.

Self-loops

In this paper, we have focused on modelling and analysing the interrelationships in multivariate time series. Therefore, we have not considered the possibility of directed self-loops $v \rightarrow v$, which could be used to impose additional constraints of the form $X_B(t + 1) \perp\!\!\!\perp \mathcal{X}_B(t) \mid \mathcal{X}_{V \setminus B}(t)$ on a model. We note that, for a discussion of the dynamic interrelationships among variables, these self-loops are irrelevant. In fact, it can be shown that two disjoint sets A and B are *p*-separated given S in a graph with self-loops if and only if they are also *p*-separated given S in the same graph with all self-loops removed. Similar statements can be formulated for pointing and extended bi-pointing paths.

Non-stationary time series

One of our main assumptions has been that the considered multivariate time series are stationary. This assumption, however, has been made mainly for the sake of simplicity, and the graphical modelling approach presented can be extended easily also to the case of non-stationary time series by requiring that the Granger noncausality

and contemporaneous conditional independence constraints encoded by a graph hold at all time points in an interval $T \subseteq \mathbb{Z}$, say; in that case, we say that the time series obeys a Granger-causal Markov property with respect to the graph over the time interval T . This allows us to consider non-stationary time series models in which the pattern of dependencies remains fixed whereas the strength of the dependencies may change over time. An interesting extension would be models where also the graphical structure changes at certain times. For instance, Talih and Hengartner [51] consider covariance selection models for multivariate time series where changes in the dependence structure occur at random times; this approach, however, does not model dynamic dependencies among the variables. Finally, we note that, despite their practical relevance, non-stationary models have attracted much less—particularly theoretical—interest than stationary models due to the involved inferential problems.

Two important issues have not been addressed in this paper. Firstly, in many applications there is little prior knowledge about the causal relationships between the variables, and empirical methods have to be used to find an appropriate graphical model. This step of model selection is hampered by the large number of possible models by which an exhaustive search becomes infeasible even for moderate dimensions. Therefore, model search strategies are required to lessen the computational burden.

A second issue, which is related to the problem of model selection, is the identification of causal effects. It is clear from the definition of Granger causality that we may conclude from Granger causality to the existence of a causal effect only if all relevant variables are included in a study, whereas the omission of important variables can lead to spurious causalities. However, Hsiao [35] noted that such spurious causalities may vanish if the information set is reduced. In other words, two processes that both satisfy the pairwise causal Markov property with respect to a graph G may exhibit different Granger noncausality relations with respect to partial information sets due to the presence or absence of spurious causalities. Some concepts as to how this observation could be exploited for causal inference have been discussed in Eichler [22, 25, 26].

Acknowledgments The author would like to thank two anonymous referees for their comments and suggestions, which greatly improved the paper.

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Appendix A: Conditional independence and stochastic processes

Throughout the paper we consider a fixed probability space $(\Omega, \mathcal{F}, \mathbb{P})$. For any sub- σ -algebra \mathcal{H} of \mathcal{F} , $\overline{\mathcal{H}}$ denotes the completed σ -algebra generated by \mathcal{H} and the \mathbb{P} -null sets in \mathcal{F} . Thus the sets in the completed σ -algebra $\overline{\mathcal{H}}$ are still measurable sets in \mathcal{F} . Next, let $\mathcal{F}_1, \mathcal{F}_2$, and \mathcal{F}_3 be sub- σ -algebras of \mathcal{F} . The smallest σ -algebra generated by $\mathcal{F}_i \cup \mathcal{F}_j$ is denoted as $\mathcal{F}_i \vee \mathcal{F}_j$. Then \mathcal{F}_1 and \mathcal{F}_2 are said to be independent conditionally on \mathcal{F}_3 if $\mathbb{E}(X|\mathcal{F}_2 \vee \mathcal{F}_3) = \mathbb{E}(X|\mathcal{F}_3)$ a.s. for all real-valued, bounded, \mathcal{F}_1 -measurable random variables X . Using the notation of Dawid [16] we write $\mathcal{F}_1 \perp\!\!\!\perp \mathcal{F}_2 | \mathcal{F}_3$ [\mathbb{P}] or $\mathcal{F}_1 \perp\!\!\!\perp \mathcal{F}_2 | \mathcal{F}_3$ if the reference to \mathbb{P} is clear.

Let $\mathcal{F}_i, i = 1, \dots, 4$ be sub- σ -algebras of \mathcal{F} . Then the basic properties of the conditional independence relation are:

- (CI1) $\mathcal{F}_1 \perp\!\!\!\perp \mathcal{F}_2 | \mathcal{F}_3 \Rightarrow \mathcal{F}_2 \perp\!\!\!\perp \mathcal{F}_1 | \mathcal{F}_3$ (symmetry)
- (CI2) $\mathcal{F}_1 \perp\!\!\!\perp \mathcal{F}_2 \vee \mathcal{F}_3 | \mathcal{F}_4 \Rightarrow \mathcal{F}_1 \perp\!\!\!\perp \mathcal{F}_2 | \mathcal{F}_4$ (decomposition)
- (CI3) $\mathcal{F}_1 \perp\!\!\!\perp \mathcal{F}_2 \vee \mathcal{F}_3 | \mathcal{F}_4 \Rightarrow \mathcal{F}_1 \perp\!\!\!\perp \mathcal{F}_2 \vee \mathcal{F}_3 | \mathcal{F}_3 \vee \mathcal{F}_4$ (weak union)
- (CI4) $\mathcal{F}_1 \perp\!\!\!\perp \mathcal{F}_2 | \mathcal{F}_4$ and $\mathcal{F}_1 \perp\!\!\!\perp \mathcal{F}_3 | \mathcal{F}_2 \vee \mathcal{F}_4 \Rightarrow \mathcal{F}_1 \perp\!\!\!\perp \mathcal{F}_2 \vee \mathcal{F}_3 | \mathcal{F}_4$ (contraction)

In some of the proofs in this paper, we make use of an additional property,

$$(CI5) \mathcal{F}_1 \perp\!\!\!\perp \mathcal{F}_2 | \mathcal{F}_3 \vee \mathcal{F}_4 \text{ and } \mathcal{F}_1 \perp\!\!\!\perp \mathcal{F}_3 | \mathcal{F}_2 \vee \mathcal{F}_4 \Leftrightarrow \mathcal{F}_1 \perp\!\!\!\perp \mathcal{F}_2 \vee \mathcal{F}_3 | \mathcal{F}_4,$$

which has been called *intersection property* by Pearl [46]. Unlike the other basic properties of conditional independence, this property does not hold in general. A sufficient and necessary condition for (CI5) is given by

$$\overline{\mathcal{F}_2 \vee \mathcal{F}_4} \cap \overline{\mathcal{F}_3 \vee \mathcal{F}_4} = \overline{\mathcal{F}_4}. \tag{A.1}$$

In that case, \mathcal{F}_2 and \mathcal{F}_3 are said to be measurably separated conditionally on \mathcal{F}_4 , denoted by $\mathcal{F}_2 \parallel \mathcal{F}_3 | \mathcal{F}_4$ [P] [30]. We note that the dependence on \mathbb{P} is only through the null sets of \mathbb{P} . For details on conditional measurable separability and its properties, we refer to Chapter 5.2 of Florens et al. [30].

If the σ -algebras \mathcal{F}_i are generated by random vectors X_i for $i = 1, \dots, 4$, in which case we write $\mathcal{F}_i = \sigma\{X_i\}$, a sufficient condition for conditional measurable separability of the X_i 's and, thus, of the \mathcal{F}_i 's is that the probability measure $\mathbb{P}^{X_1, \dots, X_4}$ is absolutely continuous with respect to a product measure μ and has a positive and continuous density. However, if each of the σ -algebras \mathcal{F}_i is generated by infinitely many random variables, the condition is obviously no longer valid. In the following we show that for strictly stationary processes X_V it is sufficient to assume the existence of a positive and continuous density for the conditional distribution of $X_V(t + 1)$ given its past $\mathbf{X}_V(t)$ at the cost of the additional regularity condition (M).

Lemma A.1 *Suppose that X_V is a stochastic process such that condition (P) holds, and let Y_1, Y_2 be finite disjoint subsets of $S(t) = \{X_v(s), s \leq t, v \in V\}$. Then*

$$Y_1 \parallel Y_2 | \sigma\{S(t) \setminus (Y_1 \cup Y_2)\} \text{ [P]}, \tag{A.2}$$

where $\sigma\{S(t) \setminus (Y_1 \cup Y_2)\}$ denotes the σ -algebra generated by $S(t) \setminus (Y_1 \cup Y_2)$.

Proof A sufficient condition for (A.2) [30, Corollary 5.2.11] is the existence of a probability measure \mathbb{P}' on $(\Omega, \mathcal{X}_V(t))$ such that \mathbb{P}' and $\mathbb{P}|_{\mathcal{X}_V(t)}$, the restriction of \mathbb{P} on $(\Omega, \mathcal{X}_V(t))$, are equivalent (i.e. have the same null sets) and

$$Y_1 \perp\!\!\!\perp Y_2 | \sigma\{S(t) \setminus (Y_1 \cup Y_2)\} \text{ [P]}. \tag{A.3}$$

Take $k \in \mathbb{N}$ such that $Y_1 \cup Y_2$ and $S(t - k)$ are disjoint, and let $Z_j = X_V(t - j)$ for $j = 0, \dots, k - 1$ and $\mathbf{Z}_k = S(t - k)$. Noting that by condition (P) the conditional

densities $f_{Z_{jv}|Z_k}$ exist and can be derived from the product of the conditional densities $f_{Z_j|Z_{j+1}, \dots, Z_{k-1}, Z_k}$, we define the probability kernel $Q(z_k, A)$ from $\mathbb{R}^{V \times \mathbb{N}}$ to $\mathbb{R}^{V \times k}$ by

$$Q(z_k, A_0 \times \dots \times A_{k-1}) = \int_{A_{k-1}} \dots \int_{A_0} \prod_{j=0}^{k-1} \prod_{v \in V} f_{Z_{jv}|Z_k}(z_{jv}|z_k) d\nu(z_0) \dots d\nu(z_{k-1}).$$

Then the probability \mathbb{P}' on $(\Omega, \mathcal{X}_V(t))$ defined by

$$\begin{aligned} \mathbb{P}'(Z_0 \in A_0, \dots, Z_{k-1} \in A_{k-1}, Z_k \in A_k) \\ = \int_{Z_k^{-1}(A_k)} \int_{A_{k-1}} \dots \int_{A_0} Q(Z_k(\omega), (dz_0, \dots, dz_{k-1})) d\mathbb{P}(\omega) \end{aligned}$$

is equivalent to $\mathbb{P}|_{\mathcal{X}_V(t)}$. Furthermore, the random variables Z_{jv} with $j = 0, \dots, k-1$ and $v \in V$ are mutually independent conditionally on Z_k under \mathbb{P}' , which implies (A.3) and hence (A.2).

The next result shows that this conditional measurable separability can also be extended to σ -algebras $\mathcal{X}_A(t)$ generated by the pasts $\mathbf{X}_A(t)$ provided the process X_V is conditionally mixing [in the sense of condition (M)].

Proposition A.1 *Suppose that X_V is a stochastic process such that conditions (M) and (P) hold. Then $\mathcal{X}_A(t)$ and $\mathcal{X}_B(t)$ are measurably separated conditionally on $\mathcal{X}_{V \setminus (A \cup B)}(t)$ for all disjoint subsets A and B of V and all $t \in \mathbb{Z}$.*

Proof Let A and B be disjoint subsets of V . We have to show that $\mathcal{X}_A(t)$, $\mathcal{X}_B(t)$, and $\mathcal{X}_{V \setminus (A \cup B)}(t)$ satisfy (A.1) and hence that

$$\overline{\mathcal{X}_{V \setminus B}(t) \cap \mathcal{X}_{V \setminus A}(t)} = \overline{\mathcal{X}_{V \setminus (A \cup B)}(t)} \tag{A.4}$$

for all $t \in \mathbb{Z}$. From Lemma A.1, it follows that, for all $t \in \mathbb{Z}$ and $k \in \mathbb{N}$, the σ -algebras $\sigma\{X_A(t), \dots, X_A(t-k+1)\}$ and $\sigma\{X_B(t), \dots, X_B(t-k+1)\}$ are measurably separable conditional on $\mathcal{X}_{V \setminus (A \cup B)}(t) \vee \mathcal{X}_V(t-k)$. Accordingly, we have by the definition of conditionally measurable separability

$$\overline{\mathcal{X}_{V \setminus B}(t) \vee \mathcal{X}_V(t-k)} \cap \overline{\mathcal{X}_{V \setminus A}(t) \vee \mathcal{X}_V(t-k)} = \overline{\mathcal{X}_{V \setminus (A \cup B)}(t) \vee \mathcal{X}_V(t-k)}$$

for all $t \in \mathbb{Z}$ and $k \in \mathbb{N}$. Since the σ -algebras on both sides are monotonically decreasing as k increases, this yields for $k \rightarrow \infty$

$$\bigcap_{k>0} \left[\overline{\mathcal{X}_{V \setminus B}(t) \vee \mathcal{X}_V(t-k)} \cap \overline{\mathcal{X}_{V \setminus A}(t) \vee \mathcal{X}_V(t-k)} \right] = \bigcap_{k>0} \overline{\mathcal{X}_{V \setminus (A \cup B)}(t) \vee \mathcal{X}_V(t-k)}$$

for all $t \in \mathbb{Z}$. Since by condition (M)

$$\bigcap_{k>0} \left[\overline{\mathcal{X}_S(t) \vee \mathcal{X}_V(t-k)} \right] = \overline{\mathcal{X}_S(t)}$$

for all subsets S of V , this establishes (A.4).

Proof of Proposition 2.1 The result follows directly from Lemma A.1 and Proposition A.1.

Appendix B: Graphical terminology

We firstly recall some basic graphical definitions used in this paper. In a graph $G = (V, E)$, if there is a directed edge $a \rightarrow b$, we say that a is a parent of b and b is a child of a ; similarly, if there is an undirected line $a - b$, the vertices a and b are called neighbours. The sets of parents, children and neighbours of a vertex a are denoted as $\text{pa}(a)$, $\text{ch}(a)$, and $\text{ne}(a)$, respectively. Furthermore, for $A \subseteq V$, let $\text{pa}(A) = \cup_{a \in A} \text{pa}(a) \setminus A$ be the set of all parents of vertices in A that are not themselves in A , and let $\text{ch}(A)$ and $\text{ne}(A)$ be defined similarly.

Next, as in Frydenberg [31], a vertex b is said to be an *ancestor* of a if either $b = a$ or there exists a directed path $b \rightarrow \dots \rightarrow a$ in G . The set of all ancestors of elements in A is denoted by $\text{an}(A)$. Notice that this definition differs from the one given in Lauritzen [39]. A subset A is called an *ancestral set* if it contains all its ancestors, that is, $\text{an}(A) = A$.

Finally, let $G = (V, E)$ and $G' = (V', E')$ be mixed graphs. Then G' is a *subgraph* of G if $V' \subseteq V$ and $E' \subseteq E$. If A is a subset of V it induces the subgraph $G_A = (A, E_A)$ where E_A contains all edges $e \in E$ that have both endpoints in A .

In the remainder of this section, we prove some auxiliary results that allow us to relate separation statements in the full graph to separation statement in so-called marginal graphs, which basically reflect the dynamic dependencies in appropriate subprocesses (see Lemma C.1).

Definition B.1 (*Marginal graph*) Let $G = (V, E)$ be a mixed graph and let A be an ancestral subset of V . Then the *marginal graph* $G_{[A]} = (A, E_{[A]})$ induced by A is obtained from the induced subgraph G_A by insertion of additional undirected edges $a - b$ whenever there exists an undirected path between a and b in G that does not intersect $\text{an}(A) \setminus \{a, b\}$.

Lemma B.1 Let $G = (V, E)$ be a mixed graph and A, B, S disjoint subsets of V . Then A and B are p -separated given S in G if and only if A and B are p -separated given S in $G_{[\text{an}(A \cup B \cup S)]}$.

Proof To show necessity, let $\pi = \langle e_1, \dots, e_n \rangle$ be a p -connecting path between A and B given S in $G_{[\text{an}(A \cup B \cup S)]}$. If all edges of π are edges in G , π is also p -connecting given S in G . Thus, we may assume that there exist edges in π , e_{j_1}, \dots, e_{j_m} say, that do not occur in G . These edges e_{j_k} are necessarily undirected since all directed edges in $G_{[\text{an}(A \cup B \cup S)]}$ also occur in G . Let $e_{j_k} = v_{j_k} - v_{j_k+1}$. Then by definition of the marginal graph there exists an undirected path ϕ_{j_k} between v_{j_k-1} and v_{j_k} which bypasses $\text{an}(A \cup B \cup S) \setminus \{v_{j_k-1}, v_{j_k}\}$ and therefore is p -connecting given S . Replacing all edges e_{j_k} in π by the corresponding paths ϕ_{j_k} we obtain a new path π' which connects A and B in G . This path π' is also p -connecting given S since the

replacement of e_{j_k} by the undirected and p -connecting path ϕ_{j_k} does not change the p -collider resp. p -noncollider status of the points $v_{j_{k-1}}$ and v_{j_k} .

Conversely for sufficiency, let $\pi = \langle e_1, \dots, e_n \rangle$ be a p -connecting path between A and B given S in G . Then all edges in π with both endpoints in $\text{an}(A \cup B \cup S)$ also occur in $G_{[\text{an}(A \cup B \cup S)]}$ since $G_{\text{an}(A \cup B \cup S)}$ is a subgraph of $G_{[\text{an}(A \cup B \cup S)]}$. We firstly show that the endpoints of any directed edge e_j in π are in $\text{an}(A \cup B \cup S)$. Let $e_j = v_j \rightarrow v_{j+1}$ (the case $e_j = v_j \leftarrow v_{j+1}$ is treated similarly). Then there exists a directed subpath $\langle e_j, \dots, e_{j+r} \rangle$ of maximal length such that either v_{j+r} is an endpoint of π and, thus, in $A \cup B$ or e_{j+r+1} is of the form $v_{j+r} \rightarrow v_{j+r+1}$ or $v_{j+r} \leftarrow v_{j+r+1}$. In the latter case v_{j+r} is a p -collider and, thus, in S since π is p -connecting given S . It follows that v_j and v_{j+1} are both in $\text{an}(A \cup B \cup S)$.

Next, if e_j is an edge in π that does not occur in $G_{[\text{an}(A \cup B \cup S)]}$, at least one of its endpoints v_{j-1} and v_j is not in $\text{an}(A \cup B \cup S)$. Thus, there exists an undirected subpath $\psi_{i,k} = \langle e_i, \dots, e_k \rangle$ with $i \leq j \leq k$ such that $v_{i-1}, v_k \in \text{an}(A \cup B \cup S)$ but all intermediate points are not in $\text{an}(A \cup B \cup S)$. In other words, v_{i-1} and v_k are not separated by $\text{an}(A \cup B \cup S) \setminus \{v_{j-1}, v_k\}$ in G which implies the presence of the undirected edge $f_{i,k} = v_{i-1} - v_k$ in $G_{[\text{an}(A \cup B \cup S)]}$. Replacing all undirected subpaths $\phi_{i,k}$ with intermediate points not in $\text{an}(A \cup B \cup S)$ by the corresponding edge $f_{i,k}$, we obtain a path between A and B in $G_{[\text{an}(A \cup B \cup S)]}$ which still has all its p -collider in S and all its p -noncolliders outside S and therefore is p -connecting given S .

The following lemma is an adapted version of Proposition 2 in Koster [36]. The proof is considerably shorter due to the fact that we allow paths to be self-intersecting.

Lemma B.2 *Let A, B, S be disjoint subsets of V . Then A and B are p -separated given S in $G_{[\text{an}(A \cup B \cup S)]}$ if and only there exist subsets A' and B' such that $A \subseteq A', B \subseteq B', A' \cup B' \cup S = \text{an}(A \cup B \cup S)$ and*

$$A' \not\propto_p B' \mid S \ [G_{[\text{an}(A \cup B \cup S)]}].$$

Proof By Lemma B.1 we may assume that $V = \text{an}(A \cup B \cup S)$. Let A' be the subset of vertices $v \in V \setminus (B \cup S)$ such that $v \not\propto_p B \mid S \ [G]$, and set $B' = V \setminus (A' \cup S)$. Then A' and B are obviously p -separated given S . Thus, we have to show that a and b' are p -separated given S whenever $a \in A'$ and $b' \in B' \setminus B$. Suppose to the contrary that there exists a p -connecting path π between some $a \in A'$ and $b' \in B' \setminus B$. Since A' contains all vertices in $V \setminus (B \cup S)$ that are p -separated from B given S , there exists a p -connecting path π' between b' and some $b \in B$. Furthermore, since $b' \in \text{an}(A \cup B \cup S) \setminus (A \cup B \cup S)$ there exists some vertex $u \in A \cup B \cup S$ and a directed path $\omega = b' \rightarrow \dots \rightarrow u$ with no intermediate points in $A \cup B \cup S$. Denoting by $\bar{\omega}$ the reverse path of ω , that is, $\bar{\omega} = u \leftarrow \dots \leftarrow b'$, we may compose a path ϕ between A and B by

- (i) $\phi = \langle \bar{\omega}, \pi' \rangle$ if $u \in A$,
- (ii) $\phi = \langle \pi, \omega \rangle$ if $u \in B$, and
- (iii) $\phi = \langle \pi, \omega, \bar{\omega}, \pi' \rangle$ if $u \in S$.

We note that the directed path ω is p -connecting given S since it has no intermediate points in S . Furthermore, $b' \notin S$ is a p -noncollider on ϕ in each of these cases and

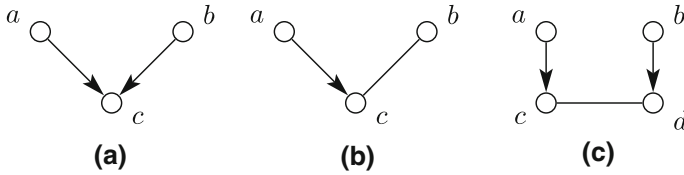


Fig. 6 Pure-collider paths between two vertices a and b

$v \in S$ is a p -collider on ϕ in case (iii). Hence ϕ is a p -connecting path between A and B given S which contradicts our assumption.

The opposite implication is obvious because of the elementwise definition of p -separation.

Because of Lemmas B.1 and B.2, it is often sufficient in the proofs to consider only the case of $A \not\propto_p B \mid S$ with $S = V \setminus (A \cup B)$. In this case, p -separation can be characterized in terms of pure-collider paths—paths on which every intermediate node is a collider—or in terms of local configurations (Fig. 6).

Lemma B.3 *Let G be a mixed graph and let A and B be two disjoint subsets of V . Then the following statements are equivalent:*

- (i) $A \not\propto_p B \mid V \setminus (A \cup B)$;
- (ii) A and B are not connected by a pure-collider path;
- (iii) $(A \cup \text{ch}(A)) \cap (B \cup \text{ch}(B)) = \emptyset$ and $\text{ne}(A \cup \text{ch}(A)) \cap (B \cup \text{ch}(B)) = \emptyset$.

Note that the second part of condition (iii) states that no two vertices $a \in A \cup \text{ch}(A)$ and $b \in B \cup \text{ch}(B)$ are adjacent; the condition thus is also symmetric in A and B .

Proof This observation follows directly from the definition of p -separation and pure-collider paths.

Appendix C: Proofs

Proof of Theorem 2.1 Setting $A = \{a\}$ in (BC), we obtain (LC). Conversely, since $\text{pa}(a) \cup \{a\} \subseteq \text{pa}(A) \cup A$, we have by (LC) together with (CI2) and (CI3)

$$X_{V \setminus \text{pa}(A) \cup A} \not\rightarrow X_a \quad \forall a \in A,$$

which, under condition (2.3), implies the first part of (BC). The second part is proved similarly.

To see that (LC) and (PC) are equivalent, we note that, under condition (S), the intersection property leads to the following composition and decomposition property for Granger noncausality relations:

$$X_A \not\rightarrow X_B \ [\mathcal{X}_V] \Leftrightarrow X_a \not\rightarrow X_B \ [\mathcal{X}_V] \quad \forall a \in A. \tag{C.1}$$

Similarly, we have for contemporaneous conditional independence relations

$$X_A \approx X_B [\mathcal{X}_V] \Leftrightarrow X_a \approx X_b [\mathcal{X}_V] \quad \forall a \in A, \forall b \in B. \tag{C.2}$$

Taking $A = V \setminus (B \cup \text{pa}(B))$ in (C.1) and $A = V \setminus (B \cup \text{pa}(B))$ in (C.2), we find that the pairwise and the local Granger-causal Markov properties are equivalent.

Proof of Proposition 3.1 By Theorem 2.1, it suffices to show that each of the three conditions (i), (ii), and (iii) implies

$$X_A \not\leftrightarrow X_b [\mathcal{X}_V] \quad \forall b \in B \Rightarrow X_A \not\leftrightarrow X_B [\mathcal{X}_V] \tag{C.3}$$

for any two disjoint subsets $A, B \subseteq V$.

For the first case, let H be the Hilbert space of all square integrable random variables on $(\Omega, \mathcal{F}, \mathbb{P})$. Furthermore, for $U \subseteq V$, let $H_U(t)$ be the closed subspace spanned by $\{X_u(s), u \in U, s \leq t\}$ and let $H_U^\perp(t)$ be its orthogonal complement. Then we have for any $Y \in H_{V \setminus A}^\perp(t)$

$$\text{cov}(X_B(t + 1), Y) = 0 \Leftrightarrow \text{cov}(X_b(t + 1), Y) = 0 \quad \forall b \in B,$$

which for a Gaussian process implies (C.3).

Next, suppose that condition (ii) holds and that X_A is Granger-noncausal for X_b with respect to \mathcal{X}_V for all $b \in B$. Then, the conditional distribution $\mathbb{P}^{X_B(t+1)|\mathbf{X}_V(t)}$ satisfies

$$\mathbb{P}^{X_B(t+1)|\mathbf{X}_V(t)} = \otimes_{b \in B} \mathbb{P}^{X_b(t+1)|\mathbf{X}_V(t)} = \otimes_{b \in B} \mathbb{P}^{X_b(t+1)|\mathbf{X}_{V \setminus A}(t)}$$

and, thus, is $\mathcal{X}_{V \setminus A}(t)$ -measurable, which proves (C.3).

Finally, if condition (iii) holds, we have

$$X_B(t + 1) - \mathbb{E}[X_B(t + 1) | \mathcal{X}_V(t)] \perp\!\!\!\perp \mathcal{X}_A(t) | \mathcal{X}_{V \setminus A}(t).$$

Since the left hand side of (C.3) implies that $\mathbb{E}[X_B(t + 1) | \mathcal{X}_V(t)]$ is $\mathcal{X}_{V \setminus A}(t)$ -measurable, we obtain $X_B(t + 1) \perp\!\!\!\perp \mathcal{X}_A(t) | \mathcal{X}_{V \setminus A}(t)$, which completes the proof.

For the proof of the equivalence of the block-recursive and the global Granger-causal Markov property, it will be convenient to restrict ourselves to mixed graphs for ancestral subsets. Due to the additional undirected edges inserted into the marginal graph $G_{[\text{an}(A)]}$, the subprocess $X_{\text{an}(A)}$ satisfies the pairwise Granger-causal Markov property with respect to $G_{[\text{an}(A)]}$ if X_V did so with respect to G . The following lemma shows that the same inheritance property also holds for the block-recursive Granger-causal Markov property.

Lemma C.1 *Suppose that X_V satisfies the block-recursive Granger-causal Markov property with respect to the mixed graph G , and let $U \subseteq V$. Then the subprocess $X_{\text{an}(U)}$ satisfies the block-recursive Granger-causal Markov property with respect to the marginal ancestral graph $G_{[\text{an}(U)]}$.*

Proof Let $H = G_{[\text{an}(U)]}$ and let A be a subset of $\text{an}(U)$. We first note that, since $\text{an}(U)$ is an ancestral set and, thus, contains the parents of all its subsets A , the parents of A in both graphs are the same, that is, $P = \text{pa}_G(A) = \text{pa}_H(A)$. By the block-recursive Granger-causal Markov property of X_V with respect to G , $X_{V \setminus (P \cup A)}$ does not Granger-cause X_A with respect to \mathcal{X}_V , which by (CI2) implies that $X_{\text{an}(U) \setminus (P \cup A)}$ is Granger-noncausal for X_A with respect to the smaller filtration $\mathcal{X}_{\text{an}(U)}$ as required by the block-recursive Granger-causal Markov property of $X_{\text{an}(U)}$ with respect to H .

Next, let $N = \text{ne}_H(A)$. Then A and $\text{an}(U) \setminus (N \cup A)$ are separated by N in H^u , that is, a and b are not adjacent in the undirected subgraph H^u whenever $a \in A$ and $b \in \text{an}(U) \setminus (N \cup A)$. By definition of H , this implies that A and $\text{an}(U) \setminus (N \cup A)$ are separated by N in G^u . By the block-recursive Granger-causal Markov property, it follows that

$$\mathcal{X}_A(t + 1) \perp\!\!\!\perp \mathcal{X}_{\text{an}(U) \setminus (N \cup A)}(t + 1) \mid \mathcal{X}_V(t) \vee \mathcal{X}_N(t + 1)$$

and, with (CI2) and (CI3),

$$\mathcal{X}_A(t + 1) \perp\!\!\!\perp \mathcal{X}_{V \setminus \text{an}(U)}(t) \mid \mathcal{X}_{\text{an}(U) \cup N}(t).$$

Combining these two relations by using (CI2) to (CI4), we find that $X_{\text{an}(U) \setminus (N \cup A)}$ and X_A are contemporaneously conditionally independent with respect to $\mathcal{X}_{\text{an}(U)}$ as required by the block-recursive Granger-causal Markov property of $X_{\text{an}(U)}$ with respect to the graph H .

Proof of Lemma 4.1 For notational convenience, we may assume in view of Lemma C.1 that $\text{an}(A \cup B \cup S) = V$ and, thus, $G_{[\text{an}(A \cup B \cup S)]} = G$. Furthermore, Lemma B.1 implies that, if $A \times_p B \mid S$ in the graph G , there exists a partition (A^*, B^*, S) of V such that $A \subseteq A^*$, $B \subseteq B^*$, and $A^* \times_p B^* \mid S$. Thus, without loss of generality, we may assume that $S = V \setminus (A \cup B)$.

With these simplifications, it suffices to show that $A \times_p B \mid V \setminus (A \cup B)$ implies

$$\mathcal{X}_{X_A}(t) \perp\!\!\!\perp \mathcal{X}_{X_B}(t) \mid \mathcal{X}_{V \setminus (A \cup B)}(t) \tag{C.4}$$

for all $t \in \mathbb{Z}$. To this end, we firstly show that

$$\mathcal{X}_A(t) \perp\!\!\!\perp \mathcal{X}_B(t) \mid \mathcal{X}_{V \setminus (A \cup B)}(t) \vee \mathcal{X}_{A \cup B}(t - k) \tag{C.5}$$

for all $t \in \mathbb{Z}$ and $k \in \mathbb{N}$.

We proceed by induction on k . For $k = 1$, we obtain (C.5) immediately from the block-recursive Granger-causal Markov property noting that $B \subseteq V \setminus (A \cup \text{ne}(A))$. For the induction step $k \rightarrow k + 1$ assume that

$$\mathcal{X}_A(t) \perp\!\!\!\perp \mathcal{X}_B(t) \mid \mathcal{X}_{V \setminus (A \cup B)}(t) \vee \mathcal{X}_{A \cup B}(t - k) \tag{C.6}$$

for all $t \in \mathbb{Z}$. Let $C_A = A \cup \text{ch}(A)$. Then, since by the block-recursive Granger-causal Markov property X_A is Granger-noncausal for $X_{V \setminus C_A}$ with respect to \mathcal{X}_V , we have

$$\mathcal{X}_A(t) \perp\!\!\!\perp \mathcal{X}_{V \setminus C_A}(t + 1) \mid \mathcal{X}_{V \setminus A}(t) \vee \mathcal{X}_{A \cup B}(t - k)$$

and further with (C.6) and (CI4)

$$\mathcal{X}_A(t) \perp\!\!\!\perp \mathcal{X}_B(t) \vee \mathcal{X}_{V \setminus C_A}(t + 1) \mid \mathcal{X}_{V \setminus (A \cup B)}(t) \vee \mathcal{X}_V(t - k).$$

With $N_A = \text{ne}(A \cup \text{ch}(A)) = \text{ne}(C_A)$, we obtain by (CI2) and (CI3)

$$\mathcal{X}_A(t) \perp\!\!\!\perp \mathcal{X}_B(t) \vee \mathcal{X}_{V \setminus (C_A \cup N_A)}(t + 1) \mid \mathcal{X}_{N_A}(t + 1) \vee \mathcal{X}_{V \setminus (A \cup B)}(t) \vee \mathcal{X}_V(t - k). \tag{C.7}$$

Next, we note that by Lemma B.3 $B \cup \text{ch}(B) \subseteq V \setminus (C_A \cup N_A)$ and thus

$$\mathcal{X}_{C_A}(t + 1) \perp\!\!\!\perp \mathcal{X}_B(t) \mid \mathcal{X}_{N_A}(t + 1) \vee \mathcal{X}_{V \setminus B}(t).$$

Furthermore, X_{C_A} and $X_{V \setminus (C_A \cup N_A)}$ are contemporaneously conditionally independent and thus

$$\mathcal{X}_{C_A}(t + 1) \perp\!\!\!\perp \mathcal{X}_{V \setminus (C_A \cup N_A)}(t + 1) \mid \mathcal{X}_{N_A}(t + 1) \vee \mathcal{X}_V(t).$$

Together with the previous relation, we obtain by (CI4)

$$\mathcal{X}_{C_A}(t + 1) \perp\!\!\!\perp \mathcal{X}_B(t) \vee \mathcal{X}_{V \setminus (C_A \cup N_A)}(t + 1) \mid \mathcal{X}_{N_A}(t + 1) \vee \mathcal{X}_{V \setminus B}(t).$$

By (C.7) together with properties (CI2), (CI3), and (CI5), this yields

$$\begin{aligned} \mathcal{X}_A(t) \vee \mathcal{X}_{C_A}(t + 1) &\perp\!\!\!\perp \mathcal{X}_B(t) \vee \mathcal{X}_{V \setminus (C_A \cup N_A)}(t + 1) \mid \mathcal{X}_{N_A}(t + 1) \\ &\vee \mathcal{X}_{V \setminus (A \cup B)}(t) \vee \mathcal{X}_V(t - k). \end{aligned}$$

Since this relation holds for all $t \in \mathbb{Z}$, we have by (CI2) and (CI3)

$$\mathcal{X}_A(t) \perp\!\!\!\perp \mathcal{X}_B(t) \mid \mathcal{X}_{V \setminus (A \cup B)}(t) \vee \mathcal{X}_{A \cup B}(t - k - 1),$$

which completes the induction step.

To show that (C.5) entails (C.4), we note that for $k \rightarrow \infty$ (C.5) yields

$$\mathcal{X}_A(t) \perp\!\!\!\perp \mathcal{X}_B(t) \mid \bigcap_{k>0} [\mathcal{X}_{V \setminus (A \cup B)}(t) \vee \mathcal{X}_{A \cup B}(t - k)]$$

for all $t \in \mathbb{Z}$. As in the proof of Proposition A.1, it follows that

$$\bigcap_{k>0} [\overline{\mathcal{X}_{V \setminus (A \cup B)}(t) \vee \mathcal{X}_{A \cup B}(t - k)}] = \overline{\mathcal{X}_{V \setminus (A \cup B)}(t)},$$

which concludes the proof of (C.4).

Proof of Theorem 4.1 Suppose that $A, B,$ and S are disjoint subsets of V such that $A \bowtie_p B \mid S$. Let ξ be any $\mathcal{X}_A(\infty)$ measurable random variable with $\mathbb{E}|\xi| < \infty$, where $\mathcal{X}_A(\infty) = \vee_{t \in \mathbb{Z}} \mathcal{X}_A(t)$ denotes the σ -algebra generated by X_A . Then $\xi(t) = \mathbb{E}(\xi | \mathcal{X}_A(t))$ is a martingale and converges to ξ in L^1 as t tends to infinity. Thus, we obtain on the one hand, as $t \rightarrow \infty$,

$$\mathbb{E}(\xi(t) | \mathcal{X}_{S \cup B}(t)) \rightarrow \mathbb{E}(\xi | \mathcal{X}_{S \cup B}(\infty)) \quad \text{in } L^1. \tag{C.8}$$

On the other hand, since $\xi(t) \perp\!\!\!\perp \mathcal{X}_B(t) \mid \mathcal{X}_S(t)$ by Lemma 4.1, we have, as $t \rightarrow \infty$,

$$\mathbb{E}(\xi(t) | \mathcal{X}_{S \cup B}(t)) = \mathbb{E}(\xi(t) | \mathcal{X}_S(t)) \rightarrow \mathbb{E}(\xi | \mathcal{X}_S(\infty)) \quad \text{in } L^1. \tag{C.9}$$

Since the limits in (C.8) and (C.9) must be equal in L^1 and, thus, also almost surely, this proves that $\mathcal{X}_A(\infty) \perp\!\!\!\perp \mathcal{X}_B(\infty) \mid \mathcal{X}_S(\infty)$. \square

Proof of Theorem 4.2 For the proof of the first part of the global Granger-causal Markov property, let A and B be subsets such that all B -pointing paths between A and B are p -blocked given $B \cup S$. We note that each B -pointing path π is of the form $\pi = \langle \tilde{\pi}, e \rangle$, where e is a directed edge $u \rightarrow b$ for some $b \in B$. Thus, π is p -blocked given $B \cup S$ if and only if $u \in B \cup S$ or $\tilde{\pi}$ is p -blocked given $B \cup S$. Therefore, if all B -pointing paths between A and B are p -blocked given $B \cup S$, then A and $\text{pa}(B) \setminus (B \cup S)$ are p -separated given $B \cup S$ and we obtain by Lemma 4.1

$$\mathcal{X}_{\text{pa}(B) \setminus (B \cup S)}(t) \perp\!\!\!\perp \mathcal{X}_A(t) \mid \mathcal{X}_{B \cup S}(t).$$

Since, in particular, every edge $a \rightarrow b$ for some $a \in A$ and $b \in B$ is p -connecting, it follows that A and $\text{pa}(B)$ are disjoint. Thus, we get by the block-recursive Granger-causal Markov property

$$\mathcal{X}_B(t + 1) \perp\!\!\!\perp \mathcal{X}_A(t) \mid \mathcal{X}_{\text{pa}(B) \cup S \cup B}(t).$$

Applying the contraction property to this and the previous relation, we find that X_A is Granger-noncausal for X_B with respect to $\mathcal{X}_{A \cup B \cup S}$.

For the proof of the second part, let $U = A \cup B \cup S$ and assume that every extended bi-pointing path between A and B is p -blocked given U . This includes in particular all bi-pointing paths π between $a \in A$ and $b \in B$, which are of the form $\pi = \langle e_1, \tilde{\pi}, e_n \rangle$, where e_1 and e_n are directed edges $a \leftarrow p_a$ and $p_b \rightarrow b$, respectively (Fig. 7a). Then π is p -blocked given U if and only if $p_a \in U, p_b \in U$, or $\tilde{\pi}$ is p -blocked given U . This implies that, if all bi-pointing paths between A and B are p -blocked given U , $\text{pa}(A) \setminus U$ and $\text{pa}(B) \setminus U$ are p -separated given U .

Next, we seek to find subsets S_A and S_B of S such that all extended bi-pointing paths between the enlarged sets $A \cup S_A$ and $B \cup S_B$ are still p -blocked given U . Then, by the same argument as above, $\text{pa}(A \cup S_A) \setminus U$ and $\text{pa}(B \cup S_B) \setminus U$ are p -separated given U . As an example, consider the extended bi-pointing path in Fig. 7b and suppose that

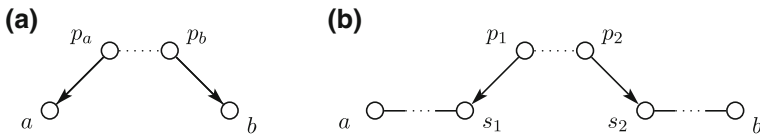


Fig. 7 **a** Bi-pointing path; **b** extended bi-pointing path

s_1 and s_2 are linked to a and b , respectively, by undirected paths that are p -connecting given S . Then the depicted extended bi-pointing path is p -blocked given U if and only if p_1 and p_2 are p -separated given U .

For a formal definition of the sets S_A and S_B , we first set $S_0 = \{s \in S \mid \text{pa}(s) \subseteq U\}$, which in particular includes all $s \in S$ that have no parents. Then adding any vertex in S_0 to either S_A or S_B to either $A \cup S_A$ or $B \cup S_B$ will not increase the sets $\text{pa}(A \cup S_A) \setminus U$ or $\text{pa}(B \cup S_B) \setminus U$. Therefore, we set For a formal argument, we need to define the sets S_A and S_B slightly differently. More precisely, let $S_0 = \{s \in S \mid \text{pa}(s) \subseteq U\}$, which in particular includes all $s \in S$ that have no parents. Furthermore, let S_A be the set of all $s \in S \setminus S_0$ such that every extended bi-pointing path between s and B is p -blocked given U and set $S_B = S \setminus (S_0 \cup S_A)$. Notice that for all $s \in S_B$ there exists an extended bi-pointing path between s and B that is p -connecting given U . We show that every extended bi-pointing path between $A \cup S_A$ and $B \cup S_B$ is p -blocked given U . Since all extended bi-pointing paths between $A \cup S_A$ and B must be p -blocked by assumption on A and B or by definition of S_A , we only have to show that all extended bi-pointing paths between $A \cup S_A$ and S_B are p -blocked given U . Suppose to the contrary that π is an extended bi-pointing path between $A \cup S_A$ and $s \in S_B$ that is p -connecting given U . Then, as mentioned above, there exists a p -connecting extended bi-pointing path π_s between s and B . If s is a p -collider on the composed extended bi-pointing path $\tilde{\pi} = \langle \pi, \pi_s \rangle$ then $\tilde{\pi}$ is p -connecting given U contradicting the assumption about A and B . Otherwise, if s is a p -noncollider, the two adjacent edges must be undirected (i.e. $— s —$) because extended bi-pointing paths never have a tail at either endpoint. Since $s \notin S_0$ there exists a path $\check{\pi} = \langle \pi, s \leftarrow v \rightarrow s, \pi_s \rangle$ with $v \in \text{pa}(s) \setminus U$. The two instances of s on $\check{\pi}$ that are adjacent to v are p -colliders and $\check{\pi}$ thus is p -connecting given U . Since $\check{\pi}$ is composed of extended bi-pointing paths, it is itself an extended bi-pointing path. Thus, by definition of S_A , $\check{\pi}$ must have endpoints in A and B , which contradicts again the assumption about A and B .

Since in particular all bi-pointing paths between $A \cup S_A$ and $B \cup S_B$ are p -blocked given U , we have

$$\text{pa}(A \cup S_A) \setminus U \not\propto_p \text{pa}(B \cup S_B) \setminus U \mid U. \tag{C.10}$$

Thus, we obtain by Lemma 4.1

$$\mathcal{X}_{\text{pa}(A \cup S_A) \setminus U}(t) \perp\!\!\!\perp \mathcal{X}_{\text{pa}(B \cup S_B) \setminus U}(t) \mid \mathcal{X}_U(t). \tag{C.11}$$

It also follows from (C.10) that the sets $\text{pa}(A \cup S_A) \setminus U$ and $\text{pa}(B \cup S_B)$ are disjoint and thus $\text{pa}(A \cup S_A) \setminus U \subseteq V \setminus \text{pa}(B \cup S_B)$, Noting furthermore that $\text{pa}(S_0) \subseteq U$ by

definition of S_0 , we obtain from the block-recursive Granger-causal Markov property

$$\mathcal{X}_{B \cup S_B \cup S_0}(t + 1) \perp\!\!\!\perp \mathcal{X}_{\text{pa}(A \cup S_A) \setminus U}(t) \mid \mathcal{X}_{U \cup \text{pa}(B \cup S_B)}(t). \tag{C.12}$$

Together with (C.11) this yields

$$\mathcal{X}_{B \cup S_B \cup S_0}(t + 1) \perp\!\!\!\perp \mathcal{X}_{\text{pa}(A \cup S_A) \setminus U}(t) \mid \mathcal{X}_U(t). \tag{C.13}$$

Moreover, since undirected paths are special cases of extended bi-pointing paths, we find that every undirected path between $A \cup S_A$ and $B \cup S_B$ intersects S_0 . Then, by a standard argument of graph theory (e.g., [53, Lemma 3.3.3]), there exists a partition (A^*, B^*, S_0) of V such that $A \cup S_A \subseteq A^*$, $B \cup S_B \subseteq B^*$, and every undirected path between A^* and B^* intersects S_0 ; in particular, this implies $\text{ne}(A \cup S_A) \subseteq S_0$. Thus, we obtain by the block-recursive Granger-causal Markov property

$$\mathcal{X}_{A \cup S_A}(t + 1) \perp\!\!\!\perp \mathcal{X}_{B \cup S_B}(t + 1) \mid \mathcal{X}_V(t) \vee \mathcal{X}_{S_0}(t + 1).$$

Together with

$$\mathcal{X}_{A \cup S_A \cup S_0}(t + 1) \perp\!\!\!\perp \mathcal{X}_{V \setminus (U \cup \text{pa}(A \cup S_A))}(t) \mid \mathcal{X}_{U \cup \text{pa}(A \cup S_A)}(t),$$

which, by $\text{pa}(S_0) \cup (A \cup S_A \cup S_0) \subseteq U$, also follows from the block-recursive Granger-causal Markov property, this implies

$$\mathcal{X}_{A \cup S_A}(t + 1) \perp\!\!\!\perp \mathcal{X}_{B \cup S_B}(t + 1) \mid \mathcal{X}_{U \cup \text{pa}(A \cup S_A)}(t), X_{S_0}(t + 1). \tag{C.14}$$

Applying (CI4) to (C.13) and (C.14), we finally obtain

$$\mathcal{X}_{A \cup S_A}(t + 1) \perp\!\!\!\perp \mathcal{X}_{B \cup S_B}(t + 1) \mid \mathcal{X}_U(t) \vee \mathcal{X}_{S_0}(t + 1),$$

from which the desired relation follows by (CI2).

Finally, to see that (GC) entails (BC), let $S = \text{pa}(B)$ and $A = V \setminus S$ for an arbitrary subset B of V . Then the first relation in (BC) follows directly from the global Granger-causal Markov property. The second relation in (BC) can be derived similarly.

Proof of Corollary 4.1 Suppose that all paths between A and B are p -blocked given S . We show that then all B -pointing paths between A and B are p -blocked given $S \cup B$, which implies by the global Granger-causal Markov property that X_A is Granger-non-causal for X_B with respect to $\mathcal{X}_{A \cup B \cup S}$.

We firstly note that, in particular, every B -pointing path π between A and B are p -blocked given S and, if π does not contain any intermediate points in B , also p -blocked given $S \cup B$. Now, suppose that π is a B -pointing path between A and B with some intermediate points in B . Then π can be partitioned as $\pi = \langle \pi_1, \pi_2 \rangle$ where π_1 is a path between A and some $b \in B$ with no intermediate points in B . Because of the assumption, the path π_1 is p -blocked given S and, since it has no intermediate

points in B , also given $S \cup B$. It follows that all B -pointing paths between A and B are p -blocked given $S \cup B$.

The other two cases $X_B \not\leftrightarrow X_A \ [\mathcal{X}_{AUBUS}]$ and $X_A \approx X_B \ [\mathcal{X}_{AUBUS}]$ can be derived similarly.

Appendix D: p -Separation in mixed graphs

The definition of p -separation presented in this paper is based on paths that may be self-intersecting. This leads to simpler conditions than in the original definition by Levitz et al. [40]. The latter is formulated in terms of paths on which all intermediate points are distinct, that is, these paths are not self-intersecting; such paths are called trails. According to Levitz et al. [40], a trail between vertices a and b is said to be p -active relative to S if

- (i) every p -collider (head-no-tail node) on π is in $\text{an}(S)$, and
- (ii) every p -noncollider v is either not in S or it has two adjacent undirected edges $(\text{---} v \text{---})$ and $\text{pa}(v) \setminus S \neq \emptyset$.

Otherwise the trail is p -blocked relative to S . Let A, B , and S be disjoint subsets of V . Then S p -separates A and B if all trails between A and B are p -blocked relative to S .

The following proposition shows that the two notions of p -separation are equivalent.

Proposition D.1 *Let $G = (V, E)$ be a mixed graph and A, B, S disjoint subsets of V . Then there exists a p -active trail between A and B relative to S if and only there exists a p -connecting path between A and B given S .*

Proof Suppose that π is a trail between two vertices a and b that is p -active relative to S . If all p -colliders on π are in S and all p -noncolliders are outside S , then π is also p -connecting given S . Otherwise, π is p -blocked by vertices u_{j_1}, \dots, u_{j_r} on the path. If u_{j_i} is a p -collider then $u_{j_i} \in \text{an}(S)$ since π is p -active. Hence there exists a directed path $\tau_i = \langle u_{j_i} \rightarrow \dots \rightarrow s_i \rangle$ for some $s_i \in S$ such that all intermediate points on τ_i are not in S and we set $\sigma_i = \langle \tau_i, \bar{\tau}_i \rangle$, where $\bar{\tau}_i$ denotes the reverse path of τ_i , that is, $\bar{\tau}_i = \langle s_i \leftarrow \dots \leftarrow u_{j_i} \rangle$. On the other hand, if u_{j_i} is a p -noncollider on π , then the two edges adjacent to u_{j_i} are undirected. Thus, there exists $w_i \in \text{pa}(u_{j_i}) \setminus S$ and we set $\sigma_i = \langle u_{j_i} \leftarrow w_i \rightarrow u_{j_i} \rangle$. Now, let π_i be the subpath of π between $u_{j_{i-1}}$ and u_{j_i} with $u_{j_0} = a$ and $u_{j_{r+1}} = b$ and set

$$\pi' = \langle \pi_0, \sigma_1, \pi_1, \sigma_2, \dots, \pi_{r-1}, \sigma_r, \pi_r \rangle.$$

Then all p -colliders on π' are in S and all p -noncolliders are not in S , which yields that π' is p -connecting given S .

Conversely, suppose that π is a p -connecting path between a and b given S . Let u_{j_1} be the first instance of a vertex that occurs more than once on the path. Then π can be partitioned as $\pi = \langle \pi'_0, \lambda_1, \pi_1 \rangle$ such that u_{j_1} is an endpoint, but not an intermediate

point of π'_0 and π_1 . Noting that π'_0 is already a trail, we continue to partition π_1 in the same way. After finitely many steps, we obtain the partition

$$\pi = \langle \pi'_0, \lambda_1, \pi'_1, \lambda_2, \dots, \pi'_{r-1}, \lambda_r, \pi'_r \rangle$$

such that the subpaths π'_j are all trails. Thus, the shortened path $\pi' = \langle \pi'_0, \dots, \pi'_r \rangle$ is also a trail. We show that π' is a p -active trail relative to S . We firstly note that all subtrails π'_j are p -connecting and hence p -active. We therefore have to show that the vertices u_{j_i} satisfy the conditions for a p -active trail.

Suppose that u_{j_i} is a p -collider that is not in S . Then at least one of the edges adjacent to u_{j_i} has an arrowhead at u_{j_i} and we may assume that π'_{i-1} is u_{j_i} -pointing (otherwise consider the reverse path). Since $u_{j_i} \notin S$, it must be a p -noncollider on π and hence λ_i starts with a tail at u_{j_i} . On the other hand, since u_{j_i} must be a p -noncollider on all its occurrences on π and π'_i does not start with a tail, the loop λ_i cannot be a directed path (otherwise u_{j_i} would not be a p -collider on $\langle \lambda_i, \pi'_i \rangle$). Consequently there exists an intermediate point w_i such that the subpath between u_{j_i} and w_i is directed and w_i is a p -collider. It follows that $w_i \in S$ and $u_{j_i} \in \text{an}(S)$.

Next, suppose that u_{j_i} is a p -noncollider on π' that is in S . Since u_{j_i} has been a p -collider on π , the two edges adjacent to u_{j_i} on π' must be undirected and λ_i must be a bi-pointing path. Hence λ_i is of the form $\lambda_i = \langle u_{j_i} \leftarrow w_i, \lambda'_i \rangle$ with $w_i \notin S$ (since w_i is a p -noncollider and π is p -connecting). Therefore, the set $\text{pa}(u_{j_i}) \setminus S$ is not empty and u_{j_i} satisfies the above condition (ii). Altogether it follows that π' is p -active relative to S .

In a remark on our simplified version of p -separation, Levitz et al. [40] argue that there are infinitely many possibly self-intersecting paths in a graph as opposed to finitely many trails. The following lemma shows that it is possible to restrict the search for p -connecting paths in G to a finite number of paths, namely all paths in which no edge occurs twice with the same orientation.

Lemma D.1 *Let $G = (V, E)$ be a mixed graph and suppose that π is a p -connecting path of the form $\pi = \langle \pi_1, e, \pi_2, e, \pi_3 \rangle$, where e is an oriented edge between some vertices u and v . Then the shortened path $\pi' = \langle \pi_1, e, \pi_3 \rangle$ is also p -connecting.*

Proof Since π is p -connecting, the two subpaths $\langle \pi_1, e \rangle$ and $\langle e, \pi_3 \rangle$ are p -connecting. This implies that also π' is p -connecting as every intermediate point has the same p -collider/noncollider status as in the corresponding subpath.

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