GRAPHICAL PROOF THEORY I: Multiplicative Linear Logic Beyond Cographs

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Abstract

Cographs are a class of (undirected) graphs, characterized by the absence of induced subgraphs isomorphic to the four-vertices path, showing an intuitive one-to-one correspondence with classical propositional formulas. In this paper we study sequent calculi operating on graphs, as a generalization of sequent calculi operating on formulas – therefore on cographs.

We mostly focus on sequent systems with multiplicative rules (in the sense of linear logic, that is, linear and context-free rules) extending multiplicative linear logic with connectives allowing us to represent modular decomposition of graphs by formulas, therefore obtaining a representation of a graph with linear size with respect to the number of its vertices. We show that these proof systems satisfy basic proof theoretical properties such as initial coherence, cut-elimination and analyticity of proof search. We prove that the system conservatively extend multiplicative linear logic with and without mix, and that the system extending the former derives the same graphs which are derivable in the deep inference system GS from the literature.

We provide a syntax for proof nets for our systems by extending the syntax of Retoré's RB-structures to represent graphical connectives. A topological characterization of those structures encoding correct proofs is given, as well as a sequentialization procedure to construct a derivation from a correct structure.

We conclude the paper by discussing how to extend those linear systems with the structural rules of weakening and contraction, providing a sequent system for an extension of classical propositional logic beyond cographs.

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1 INTRODUCTION

In theoretical computer science, formulas are used to describe complex structures using elementary operators such as logical connectives and modalities. In particular, the proof theory of propositional logic typically considers formulas built from a very limited palette of binary (connectives) and unary (modalities) operators. Beside the restriction on the basic operators does not generally limit the expressiveness of the language, as soon as proof theory is used to define paradigms as "formulas-as-types", "formulas-asprograms", or "formulas-as-processes", this limitation leads to a payout in term of efficiency whenever we aim at providing efficient implementations: in order to describe complex interaction, ad-hoc encodings need to be put in place. As a consequence, automated tools relying on formula-based proof systems are either sub-optimal, because of the blow-up in computational complexity due to the use of encodings, or sacrifice the quality of information, by reducing their scope to only considering simpler configurations. This latter possibility may lead to information loss, potentially causing, among others, security issues or imprecise results in AI for decision systems.

For this reason, graphs are often used in computer science practice from abstract definitions to practical implementation to describe systems with complex interactions: it is often the case that "a picture is worth a thousand words"¹. By means of example, consider a system consisting of four processes *a*, *b*, *c* and *d* racing to access shared resources, and assume that the pairs of processes *a* and *b*, *b* and *c*, and *c* and *d* share the access to a same resource. This configuration can be represented by the graph below on the left (called P_4) where vertices represent processes and an edge is drawn whenever two processes share the access to a same resource.

$$\begin{array}{c|c}
b & d \\
a & c \\
\end{array} \qquad b & d \\
a & c \\
\end{array}$$
(1)

Similarly, we could consider a dependency relation (e.g., causality) in a system with where *a* depends from *b*, and *c* depends from both *b* and *d*. In this case, again, the binary relation of "non-causal dependency" can be represented by a graph with similar shape (see the graph above on the right). It is well-known that the graph P_4 cannot be expressed by a formula containing only binary connectives with a one-to-one correspondence between atoms and vertices of the graph [34, 55]. Beside the simplicity of the patter P_4 , it occurs in a graph representing a relation as soon as we consider non series-parallel ones, which are ubiquitous in distributed systems (see, e.g., non-transitive conflict of interest relations in control access models such as [18], in dependency graphs, or in producer-consumer queues).

It is worthy notice that the use of graph-based syntaxes is not new in logic and proof theory for this exact reason: a same object may admit multiple representations, but graphs allows us to provide more canonical ones. By means of example, graphs are largely used in defining semantics (see, e.g., Kripke semantics for modal logics [17]), in proof systems capturing semantical structures (see, e.g., nested sequents [58, 20, 70]), in proof systems capturing proof nets [42] or combinatorial proofs [52, 52]).

However, proof theory has rarely considered graphs as primitive terms to reason on: prior to [7, 8, 5] we cannot find proof systems conceived to handle graphs as terms of an inference system defined with proof-theoretical purposes.² In these works, the authors move from the well-known correspondence between classical propositional formulas and cographs (graphs containing no induced subgraph isomorphic to a P₄) [55] to generalize proof theoretical methodologies for inference systems on formulas to graphs. In fact, we could say that inference systems operating on formulas can be seen as inference systems operating on cographs, that is, on graphs with "less complex" structure where no induced subgraph isomorphic to P₄ occurs³. In these works, the authors consider only *deep inference* [48, 12] formalism to design proof systems operating on graphs. Such unconventional choice with respect to, e.g., sequent calculi or natural deduction, pays off in [5], where a proof system operating on graphs with both symmetric and non-symmetric edges defines a conservative extension of the non-commutative logic BV⁴, for which a cut-free sequent calculus cannot exist [76].

¹To be more precise, we should instead say "a picture is worth an exponential number of words"...

²Another line of works [23, 78, 24, 31, 32] explored the extensions of the semantics of boolean logic from cographs enconding formulas to graphs. However, in these works graphical logic is investigated from a semantical viewpoint rather than under the lens of proof systems.

³Note that several NP-hard optimization problems on graphs become solvable in polynomial time if restricted to cographs [57].

⁴The logic BV is a NP-time decidable fragment of Pomset logic [69, 68]. This logic is sound and complete with respect to seriesparallel order refinements: if ϕ and ψ are formulas encoding series-parallel orders, then the order encoded by ϕ is a refinement of the order encoded by ψ iff $\vdash_{BV} \phi \rightarrow \psi$.



Figure 1: The lattice of the logics on formulas studied in this paper and the lattice of graphical logics defined by the interpretation of graphical connectives as prime graphs. The logics below the dotted line contain formulas where only the binary connectives for conjunction and disjunction occurs; similarly, the graphical logics below the dotted line are families of cographs.

1.1 MAIN CONTRIBUTIONS

This paper aims at extending methodologies from proof theory from formulas to graphs.

For this purpose, I define the notion of *graphical connectives* to define formulas whose purpose is to represent graphs via the *graph modular decompositions*, that is, abstract syntax trees uniquely describing graphs with a term of linear size with respect to the number of vertices of the graph. This provides foundation to the methodologies used in [7, 8, 24] to design proof systems operating on graphs by handling their modular decomposition trees. Note that in this paper I only discuss graphical connectives designed on undirected graphs generalizing the well-known correspondence between classical propositional formulas and cographs. However, the proposed methodology scales to more general graphs such as the mixed graphs used in [5].

Using graphical connectives I then define *multiplicative* proof systems (in the sense of [29, 44]) operating on formulas which can naturally be interpreted as graphs (see Figure 1), proving basic proof theoretical properties for these systems such as cut-elimination, initial coherence and a weaker notion of the analyticity condition taking into account the richer structure of non-binary connectives. I prove that the logic MPL is a conservative extensions of *multiplicative linear logic* and that the logic MPL° is a conservative extensions of *multiplicative linear logic with mix*. Moreover, I prove that MPL° is sound with respect to the semantics interpreting formulas as graphs, that is, if two formulas are interpreted as isomorphic graphs, then we can prove that they are logically equivalent.

I prove that one of these sequent systems internalizes the notion of graph isomorphism as a logical equivalence between formulas, that is, two formulas encoding isomorphic graphs are logically equivalent, and that such a system is sound and complete with respect to the set of graphs in the graphical logic GS from my previous works [7, 8]. This latter result indirectly provides a proof of analiticity and transitiveness of implication for the logic GS using more standard techniques⁵.

Then in the second part of the paper I provide a formalism for proof nets for these substructural graphical logics extending Retoré's syntax of RB-proof nets [73]. In particular, this syntax is defined by generalizing the gates for the binary connectives \Im and \otimes in order to represent the graphical connectives I introduce. Then I provide a correctness criterion for these proof nets with respect to the logics MPL and MPL°, together with a sequentialization procedure. This criterion is obtained by refining Retoré's criterion by weakening the condition of acyclicity, therefore ruling out only specific cycles⁶, and including an order relation on gates, derivable by the topological structure of the RB-structure, providing constraints on the order in which connectives can be sequentialized, as in *C-nets* [33].

 $^{^{5}}$ Note that the full proof of the admissibility of the rule simulating the cut in deep inference systems in the system GS, as well as the proof that GS is a conservative extension of multiplicative linear logic with mix, are quite convoluted and takes several pages in the Appendix of [8].

⁶In [67] the authors theorized the possibility of the existence of logics satisfying weaker correctness criterion on RB-structures with respect to the one from [73].



Figure 2: A graph, its modular decomposition and a derivation in MPL° of the formula encoding it.



Figure 3: The RB-proof net encoding the derivation in Figure 2.

1.2 OUTLINE OF THE PAPER

In Section 2 we recall definitions and results in graph theory and the notion of modular decomposition. We then use these notions to extend the correspondence between classical propositional formulas and cographs to formulas containing *graphical connectives* and any graph. In Section 3 we define sequent calculi operating on formulas containing graphical connectives, proving their proof-theroretical properties, as well as the fact that they are conservative extensions of multiplicative linear logic with and without mix. In Section 4 we recall the graphical proof system on undirected graphs from [7, 8] and we prove it recognize the same set of graphs recognized by the extension of multiplicative linear logic with mix with graphical connectives.

In the second part of the paper we provide a way to represent proofs via RB-proof nets [71, 73]. In Section 5 we recall the original definitions and we explain how to generalize RB-proof nets for multiplicative linear logic to represent proofs in our logics, and we explain why the criterion for multiplicative linear logic with mix does not extends to our logics. We then extend this criterion for the graphical logic GS in Section 7, providing a sequentialization procedure for correct proof nets. To conclude, in Section 8 we show how our multiplicative graphical logics can be extended with structural rules, and we summarize in Section 9 some of the possible the research directions opened by this work.

2 FROM FORMULAS TO GRAPHS

In this section we recall standard results from the literature on graphs, and how the notion of graphs modular decomposition allows us to extend the connection between *cographs*, i.e., the class of graphs containing no four vertices whose induced subgraph is isomoprhic to the four-vertices paths, and classical propositional formulas to general graphs and a class of formulas built using certain new *n*-ary connectives we introduce in this paper called *graphical connectives*.

2.1 GRAPHS AND MODULAR DECOMPOSITION

In this work are interested in using graphs to model patterns of interactions, describing such patterns by means of the binary relations (edges) between its components (vertices). For this reason we define at the same time the graphs and the corresponding notion of *identity* allowing us to consider patterns differing for their syntactic description as the same graph.

Definition 1. A *L-labeled graph* (or simply *graph*) $G = \langle V_G, \ell_G, \stackrel{G}{\frown} \rangle$ is given by a finite set of *vertices* V_G , a partial *labeling function* $\ell_G \colon V_G \to \mathcal{L}$ associating a label $\ell(v)$ from a given set of labels \mathcal{L} to each vertex $v \in V_G$ (we denote by \emptyset the empty function), and a non-reflexive symmetric edge relation $\stackrel{G}{\frown} \subset V_G \times V_G$ whose elements, called *edges*, may be denoted *vw* instead of (v, w). A graph is *empty* (denoted $G = \emptyset$) if $V_G = \emptyset$.

A symmetry between two graphs G and G' is a bijection $f: V_G \to V_{G'}$ such that $x \cap y$ iff $f(x) \cap f(y)$ for any $x, y \in V_G$. An *isomorphism* is a symmetry f such that $\ell(v) = \ell(f(v))$ for any $x, y \in V_G$.

Two graphs G and G' are symmetric (denoted $G \sim G'$) if there is an symmetry between G and G'. They are *isomorphic* if there is a isomorphism between G and G'. From now on, we consider isomorphic graphs to be *the same* graph (denoted G = G').

Two vertices v and w in G are **connected** if there is a sequence $v = u_0, ..., u_n = w$ of vertices in G (called **path**) such that $u_{i-1} \stackrel{G}{\frown} u_i$ for all $i \in \{1, ..., n\}$. A **connected component** of G is a maximal set of connected vertices in G.

A graph G is a *clique* (resp. a *stable set*) iff $\stackrel{G}{\leftarrow} = \emptyset$ (resp. $\stackrel{G}{\frown} = \emptyset$).

Observation. The problem of graph isomorphism is a standard **NP**-problem. That is, verify that a given bijection between the sets of vertices of two graphs is an isomorphism can be checked in polynomial time, while finding a graph isomorphism is a problem admitting no polynomial time algorithm. For this reason, whenever we say that two graphs are the same, either we assume they share the same set of vertices, therefore implicitly assuming the isomorphism to be given. This allows us to verify whether two graphs are the same in polynomial time.

Notation 2. When drawing a graph or an unlabeled graph we draw v - w whenever v - w, we draw no edge at all whenever v + w. We may represent a vertex of a graph by using its label instead of its name. For example, the single-vertex graph $G = \langle \{v\}, \ell_G, \emptyset \rangle$ may be represented either by a the vertex name v or by the vertex label $\ell(v)$. Note that because of our notion of identity of graphs, whenever there are no ambiguity because of two vertices with a same label, the representation of a graph provides us the same information of its definition as the triple containing the set of vertices, the label function and the set of edges.

Example 3. Consider the following graphs:

 $F = \langle \{u_1, u_2, u_3, u_4\}, \{\ell(u_1) = a, \ell(u_2) = b, \ell(u_3) = c, \ell(u_4) = d\}, \{u_1u_2, u_2u_3, u_3u_4\} \rangle$ $G = \langle \{v_1, v_2, v_3, v_4\}, \{\ell(v_1) = b, \ell(v_2) = a, \ell(v_3) = c, \ell(v_4) = d\}, \{v_1v_2, v_1v_3, v_3v_4\} \rangle$ $H = \langle \{w_1, w_2, w_3, w_4\}, \{\ell(w_1) = a, \ell(w_2) = b, \ell(w_3) = c, \ell(w_4) = d\}, \{w_1w_2, w_1w_3, w_3w_4\} \rangle$

They are all symmetric, that is $F \sim G \sim H$, but $F = G \neq H$ as can easily be verified using their

$$F = a - b - c - d = G$$
 and $H = b - a - c - d$

In order to use proof theoretical methodologies on graphs, we need a suitable notion of subgraphs to be used in the same way sub-formulas are used in proof systems, that is, to state properties of the calculus or to define the behavior of rules. For this purpose, we use for a notion of *module* to identify subgraph allowing us to decompose a graph using abstract syntax trees similar to the ones underlying formulas. Intuitively, a module is a subset of vertices of a graph having the same edge-relation with any vertex outside the subset. This generalize what we observe in formulas, where any propositional atom of a subformula has the same relation (the one given by the least common ancestor node in the formula tree) with a given propositional atom not in the subformula with a propositional atom .

Definition 4. Let $G = \langle V_G, \ell_G, E_G \rangle$ be a graph and $W \subseteq V_G$. The *graph induced* by W is the graph $G|_W := \langle W, \ell_G|_W, \stackrel{G}{\frown} \cap (W \times W) \rangle$ where $\ell_G|_W(v) := \ell_G(v)$ for all $v \in W$. A *module* of a graph G is a subset M of V_G such that $x \frown z$ iff $y \frown z$ for any $x, y \in M, z \in V_G \setminus M$.

A *module* of a graph *G* is a subset *M* of V_G such that $x \frown z$ iff $y \frown z$ for any $x, y \in M, z \in V_G \setminus M$. A module *M* is *trivial* if $M = \emptyset$, $M = V_G$, or $M = \{x\}$ for some $x \in V_G$. From now on, we identify a module *M* of a graph *G* with the induced subgraph $G|_M$.

Remark 5. A connected component of a graph G is a module of G.

representations:

Using modules we can optimize the way we represent graphs reducing the number of edges drawn without losing information, relying on the fact that all vertices of a module has the same edge-relation with any vertex outside the module.

Notation 6. In order to improve reading, we may border vertices of a same module by a closed line and draw edges connecting those closed lines to denote the existence of an edge between each vertex inside it. By means of example, consider the following graph and its more compact modular representation.

$$a \xrightarrow{b} e = (a-b) - (c-d) - e$$
(2)

The notion of module is related to a notion of context, which can be intuitively formulated as a graph with a special vertex playing the role of a hole in which we can plug in a module.

Definition 7. A *context* $C[\Box]$ is a (non-empty) graph containing a single occurrence of a special vertex \Box . It is *trivial* if $C[\Box] = \Box$. If $C[\Box]$ is a context and G a graph, we define C[G] as the graph obtained by replacing \Box by G. Formally,

$$C[G] := \left\langle \left(V_{C[\Box]} \setminus \{\Box\} \right) \uplus V_G, \left\{ vw \mid v, w \in V_{C[\Box]} \setminus \{\Box\}, v \overset{C[\Box]}{\frown} w \right\} \cup \left\{ vw \mid v \in V_{C[\Box]} \setminus \{\Box\}, w \in V_G, v \overset{C[\Box]}{\frown} \Box \right\} \right\rangle$$

Remark 8. A set of vertices *M* is a module of a graph *G* iff there is a context $C[\Box]$ such that G = C[M].

This idea of plugging a graph inside another graph can be generalized, providing the definition of a *composition-via* a graph, allowing to compose multiple graphs in a "modular way" using a graph itself as an operation.

Definition 9. Let *G* be a graph with $V_G = \{v_1, \ldots, v_n\}$ and let H_1, \ldots, H_n be *n* graphs. We define the *composition of* H_1, \ldots, H_n *via G* as the graph $G(H_1, \ldots, H_n)$ obtained by replacing each vertex v_i of

G with the graph H_i for all $i \in \{1, ..., n\}$. Formally,

$$G(H_1,\ldots,H_n) = \left(\bigcup_{i=1}^n V_{H_i}, \left(\bigcup_{i=1}^n \stackrel{H_i}{\frown} \right) \cup \left\{ (x,y) \mid x \in V_{H_i}, y \in V_{H_j}, v_i \stackrel{G}{\frown} v_j \right\} \right)$$
(3)

The subgraphs H_1, \ldots, H_n are called *factors* of $G(H_1, \ldots, H_n)$ and are (possibly not maximal) modules of $G(H_1, \ldots, H_n)$.

Observation. By definition, H_1, \ldots, H_n are (possibly not maximal) modules of $G(H_1, \ldots, H_n)$.

Remark 10. The information about the labels of the graph *G* used to define the composition-via operation is lost. In particular, if *G* is a graph with $V_G = \{v_1, \ldots, v_n\}$ and σ a permutation over the set $\{1, \ldots, n\}$ such that the map $f_{\sigma} : V_G \to V_G$ mapping v_i in $f_{\sigma}(v_i) = v_{\sigma(i)}$ for all $i \in \{1, \ldots, n\}$ is an isomorphism from *G* to *G*, then $G(H_1, \ldots, H_n) = G'(H_1, \ldots, H_n)$.

In order to establish a connection between graphs and formulas, from now on we only consider graphs whose set of labels belong to the set $\mathcal{L} = \{a, a^{\perp} \mid a \in \mathcal{A}\}$ where \mathcal{A} is a fixed set of propositional variables. We then define the *dual* of a graphs.

Definition 11. Let $G = \langle V_G, \ell_G, E_G \rangle$ be a graph. We define the edge relation $\not\leftarrow^G := \{(v, w) \mid v \neq w \text{ and } vw \notin \stackrel{G}{\frown} \}$ and we define the *dual* graph of G as the graph $G^{\perp} := \langle V_G, \not\leftarrow^G, \ell_{G^{\perp}} \rangle$ with $\ell_{G^{\perp}}(v) = (\ell_G(v))^{\perp}$.

Remark 12. By definition, each module of a graph corresponds to a module of its dual graph. It follows that a connected component of G^{\perp} is a module of G.

Notation 13. If \mathcal{G} is the representation of a graph G, then we may represent the graph G^{\perp} by bordering the representation of G with a closed line and with the symbol for negation on the upper-right corner, that is, $\widehat{\mathcal{G}}^{\perp}$.

2.2 CLASSICAL PROPOSITIONAL FORMULAS AND COGRAPHS

The set of *classical (propositional) formulas* is generated from a set of propositional variable \mathcal{A} using the *negation* $(\cdot)^{\perp}$, the *disjunction* \vee and the *conjunction* \wedge using the following grammar:

$$\phi, \psi \coloneqq a \mid \phi \lor \psi \mid \phi \land \psi \mid \phi^{\perp} \qquad \text{with } a \in \mathcal{A}.$$
(4)

We consider the following equivalence laws over classical formulas:

$$\phi \lor \psi \equiv \psi \lor \phi \qquad \phi \lor (\psi \lor \chi) \equiv (\phi \lor \psi) \lor \chi
 \phi \land \psi \equiv \psi \land \phi \qquad \phi \land (\psi \land \chi) \equiv (\phi \land \psi) \land \chi
 (5)$$

and with the following *De-Morgan laws*:

$$(\phi^{\perp})^{\perp} \equiv \phi \qquad (\phi \land \psi)^{\perp} \equiv \phi^{\perp} \lor \psi^{\perp} \tag{6}$$

We denote by \equiv the equivalence relation generated by equivalence and De-Morgan laws.

We define a map from literals to single-vertex graphs, which extends to formulas via the compositionvia a two-vertices stable set with S_2 (for formulas which are disjunctions) and a two-vertices clique K_2 (for formulas which are conjunctions). **Definition 14.** Let ϕ be a classical formula, then $[\![\phi]\!]$ is the graph inductively defined as follows:

 $\llbracket a \rrbracket = a \qquad \llbracket \phi^{\perp} \rrbracket = \llbracket \phi \rrbracket^{\perp} \qquad \llbracket \phi \lor \psi \rrbracket = \mathsf{S}_2(\llbracket \phi \rrbracket, \llbracket \psi \rrbracket) \qquad \llbracket \phi \land \psi \rrbracket = \mathsf{K}_2(\llbracket \phi \rrbracket, \llbracket \psi \rrbracket)$

where K_2 is a given clique with 2 vertices and where we denote by *a* the single-vertex graph, whose vertex is labeled by *a*.

We can easily observe that the map $[\cdot]$ well-behaves with respect to the equivalence over formulas \equiv , that is, equivalent formulas are mapped to the symmetric graphs.

Proposition 15. Let ϕ and ψ be classical formulas. Then $\phi \equiv \psi$ iff $[\![\phi]\!] = [\![\psi]\!]$.

We finally recall the definition of *cographs*, and the theorem establishing the relation between cographs and classical formulas, i.e., providing an alternative definition of cographs as graphs generated by single-vertex graphs using the composition-via a two-vertices no-edge graph and a two-vertices one-edge graph.

Definition 16. A *cograph* is a graph G such that for any four distinct vertices $v_1, v_2, v_3, v_4 \in V_G$ the induced subgraph $G|_{\{v_1, v_2, v_3, v_4\}}$ is not symmetric to the graph $\langle \{a, b, c, d\}, \emptyset, \{ab, bc, cd\} \rangle$ (i.e., a - b - c - d).

Theorem 17 ([39]). A graph is a cograph iff there is a formula ϕ such that $G \sim [\![\phi]\!]$.

2.3 MODULAR DECOMPOSITION OF GRAPHS

We can now introduce the notion of *prime graph* which plays a special role in graphs *modular decomposition*, that is, in the possibility of inductively define graphs from single-vertices graphs using the operation of composition-via a graph restrained specific graphs (see e.g., [39, 55, 51, 60, 63, 35]).

Definition 18. A graph G is *prime* if $|V_G| > 1$ and all its modules are trivial. A graph G is *quasi-prime* if it is prime, a clique or a stable set.

We recall the following standard result from the literature.

Theorem 19 ([55]). Let G be a graph with at least two vertices. Then there are non-empty modules M_1, \ldots, M_n of G and a prime graph P such that $G = P(M_1, \ldots, M_n)$.

This result enforces the existence of the possibility of inductively describe graphs using single-vertex graphs and the operation of composition-via prime graphs. More precisely, we can define the notion of *modular decomposition* of a graph composition-via quasi-prime graphs to provide a more canonical representation.

Definition 20. Let G be a non-empty graph. A *modular decomposition* of G is a way to write G using single-vertex graphs and the operation of composition-via quasi-prime graphs:

- if G is a graph with a single vertex x labeled by a, then G = a (i.e. $G = \langle \{x\}, \ell(x) = a, \emptyset \rangle$);
- if G is disconnected with connected components H_1, \ldots, H_n , then $G = S(|H_1, \ldots, H_n|)$ for a stable set S with $|V_S| = n$;
- if G^{\perp} is disconnected with connected components H_1, \ldots, H_n , then $G = C(H_1, \ldots, H_n)$ for a clique *C* with $|V_C| = n$;
- if both G and G[⊥] are connected and H₁,..., H_n are maximal modules of G, then there is a unique prime graph P (with |V_P| > 2) such that G = P(|H₁,..., H_n).

A *spurious modular decomposition* of G is a modular decomposition of G in which we allow occurrences of the empty graph \emptyset occur as leaves of the abstract syntactic tree.

Observation. Modular decomposition does not provide a unique way to write graphs.

In fact, whenever we have two isomorphic graphs (not even symmetric), this provide multiple ways to define a graph using the composition-via. By means of example, if we consider the non-isomorphic but symmetric graphs P = a - b - c - d and P' = a - c - b - d, then we have that P(a', b', c', d') = a' - b' - c' - d' = P'(a', c', b', d').

Even considering isomorphic graphs, we may have permutations allowing us to write a graph using the same composition-via a graph *G* changing the order of its factors, that is, $G(H_1, \ldots, H_n)$ may be the same graph of $G(H_{\sigma(1),\ldots,H_{\sigma(n)}})$ for some permutations σ over the set $\{1,\ldots,n\}$. Note that if *G* is a clique or a stable set, then σ can be any permutation.

Moreover, the associativity of cliques and stable sets creates additional ambiguity. By means of example, consider two cliques K_3 and K_2 with respectively three and two vertices, then $K_2(|a, b, c|) = K_2(|a, K_2(|b, c|)) = K_2(|K_2(|a, b|), c|)$.

In order to limit the proliferation of operation of composition-via graphs, we introduce the notion of *base* of *graphical connectives*, allowing us to provide more canonical modular decomposition of graphs.

Definition 21. A graphical connective $C(v_1, \ldots, v_n) = \langle V_C, \stackrel{C}{\frown} \rangle$ is given by a finite list of vertices $V_C = \langle v_1, \ldots, v_n \rangle$ and a non-reflexive symmetric edge relation $\stackrel{C}{\frown}$ over the set of vertices occurring in V_C . We define the *composition-via* a graphical connective similarly to the composition-via a graph $G_C = \langle \bigcup_{v \in V_C} \{v\}, \emptyset, \stackrel{C}{\frown} \rangle$ for a labeling function ℓ (see Definition 9).

A graphical connective is *prime* (resp. a *clique* and a *stable set*) if $C(a_1, \ldots, a_n)$ is a prime graph (resp. a clique and a stable set) for any a_1, \ldots, a_n single-vertex graphs.

The *group of symmetries* and the *set of dualizing symmetries* of a graphical connective *C* are respectively defined the following subset of the set \mathfrak{S}_n of permutations over the set $\{1, \ldots, n\}$: ^{*a*}

$$\begin{aligned} \mathfrak{S}(C) &\coloneqq \{\sigma \in \mathfrak{S}_{|V_G|} \mid C(|a_1, \dots, a_{|V_G|}|) = C(|a_{\sigma(1)}, \dots, a_{\sigma(|V_G|}|))\} \\ \mathfrak{S}^{\perp}(C) &\coloneqq \{\sigma \in \mathfrak{S}_{|V_G|} \mid (C(|a_1, \dots, a_{|V_G|}|))^{\perp} = C(|a_{\sigma(1)}^{\perp}, \dots, a_{\sigma(|V_G|}^{\perp}|))\} \end{aligned}$$
(7)

for any set of single-vertex graphs $\{a_1, \ldots, a_{|V_G|}\}$.

A set of graphical connectives Q is a *base* (resp. *prime base*) if for each quasi-prime graph (resp. for each prime graph) Q with $V_Q = \{w_1, \ldots, w_n\}$ there is a unique $C \in Q$ such that $Q = C(w_{\sigma(1)}, \ldots, w_{\sigma(n)})$ for some permutations $\sigma \in \mathfrak{S}_n$.

 $\mathcal{P}_{n}(v_{1},\ldots,v_{n}) \coloneqq \langle \langle v_{1},\ldots,v_{n} \rangle, \emptyset \rangle \qquad \otimes_{n}(v_{1},\ldots,v_{n}) \coloneqq \langle \langle v_{1},\ldots,v_{n} \rangle, \{v_{i}v_{j} \mid i \neq j\} \rangle$ $\mathsf{Bull}(v_{1},\ldots,v_{5}) \coloneqq \langle \langle v_{1},\ldots,v_{5} \rangle, \{(v_{1}v_{2},v_{2}v_{3},v_{3}v_{4},v_{5}v_{2},v_{5}v_{3})\} \rangle \qquad (8)$ $\mathsf{P}_{n}(v_{1},\ldots,v_{n}) \coloneqq \langle \langle v_{1},\ldots,v_{n} \rangle, \{v_{i}v_{i+1} \mid i \in \{1,\ldots,n-1\}\} \rangle$

and we denote by $\mathfrak{N} := \mathfrak{N}_2$ and by $\otimes := \otimes_2 = \mathsf{P}_2$. That is,

Notation 22. We define the following graphical connectives (with n > 1):

We use the following notation for the composition-via the graphical connectives \Re and \otimes :

 $H_1 \ \mathfrak{N} \ H_2 = \mathfrak{N}(H_1, H_2) \qquad H_1 \otimes H_2 = \otimes (H_1, H_2)$

From now on, we assume only bases containing the graphical connectives in Equation (8).

^{*a*}More precisely, \mathfrak{S}_n provided with the operation of composition is a group whose neutral element the identity permutation (denoted id).

Example 23. Consider the following graph

We can write G as $P_4(a \otimes b, c \otimes d, e \otimes f, \otimes_3(g, h, i))$ or $P_4(\otimes_3(g, i, h), e \otimes f, d \otimes c, a \otimes b)$ (or $P_4(a \otimes b, c \otimes d, e \otimes f, g \otimes (h \otimes i))$) if we only use prime connectives). The dual graph of G is defined as the graph

$$G^{\perp} = \begin{array}{c} c^{\perp} & \overset{d^{\perp}}{\overbrace{b^{\perp}}} & \overset{e^{\perp}}{\overbrace{b^{\perp}}} & \overset{f^{\perp}}{\overbrace{b^{\perp}}} & = \underbrace{e^{\perp} \quad f^{\perp}}{\overbrace{b^{\perp}}} & = \underbrace{e^{\perp} \quad f^{\perp}}{\overbrace{b^{\perp}}} & \underbrace{g^{\perp} \quad h^{\perp} \quad i^{\perp}}{\overbrace{b^{\perp}}} & \underbrace{e^{\perp} \quad d^{\perp}}{\overbrace{b^{\perp}}} & \overset{d^{\perp}}{\overbrace{b^{\perp}}} & \overset{d^{$$

and can be written as $G^{\perp} = \mathsf{P}_{4}^{\perp} \left(a^{\perp} \otimes b^{\perp}, c^{\perp} \Im d^{\perp}, e^{\perp} \Im f^{\perp}, \Im_{3} \left(g^{\perp}, h^{\perp}, i^{\perp} \right) \right)$.

We can reformulate the standard result on modular decomposition as follows.

Theorem 24. Let G be a non-empty graph. Then then there is a unique way (up to symmetries of graphical connectives) to write G using single-vertex graphs and the graphical connectives in a given base Q.

Proof. The proof follows by Definition 20 and the unique way, up to connective symmetries, to write a quasi-prime graph using the operation of composition-via a graphical connective of a base. \Box

Corollary 25. Two graphs are symmetric iff they admit a same modular decomposition.

3 MULTIPLICATIVE AND ISOMIX GRAPHICAL LOGIC

In this section we define connectives with a one-to-one correspondence with a graphical connectives, and a set of formulas constructed using these connectives which we can interpret (semantically) as graphs. We then provide two sequent calculi using linear and context-free sequent rules and we prove their proof-theoretical properties.

3.1 GENERALIZED FORMULAS

In order to represent graphs as formulas using graph modular decomposition, we need to define new connectives beyond conjunction and disjunction in order to have a correspondence between the graphs of our base and the connectives of our logic. For this purpose, we define a set of formulas whose connectives are in one-to-one correspondence with the graphical connectives in a prime base \mathcal{P} .

Definition 26. Assume a prime base \mathcal{P} to be fixed. The set of *formulas* is generated by the set of propositional atoms \mathcal{A} , a *unit* \circ , and the set of *(graphical) connectives* $\mathfrak{C} = \{\kappa_Q \mid Q \in Q\}$ using the following syntax:

$$\phi_1, \dots, \phi_n \coloneqq \circ \mid a \mid a^{\perp} \mid \kappa_P(\phi_1, \dots, \phi_{|P|}) \qquad \text{with } a \in \mathcal{A} \text{ and } \kappa \in \mathfrak{C}$$
(9)

The *arity* of the connective κ_Q is defined as $|\kappa_Q| := |V_Q|$. We may denote by \mathfrak{R} (resp. \otimes) the binary connective $\kappa_{\mathfrak{R}}$ (resp. κ_{\otimes}) and we may write $\phi \mathfrak{R} \psi$ (resp. $\phi \otimes \psi$) instead of $\kappa_{\mathfrak{R}} (\phi, \psi)$ (resp. $\kappa_{\otimes} (\phi, \psi)$).

A *literal* is a formula of the form a or a^{\perp} for an atom $a \in \mathcal{A}$. The set of literals is denoted \mathcal{L} . A formula is *unit-free* if it contains no occurrences of \circ . A formula containing no literal is said *vacuous*. A **MLL**-*formula* is a formula containing only occurrences of \mathfrak{P} and \otimes connectives.

A formula $\kappa(\phi_1, \ldots, \phi_n)$ is called a κ -formula and we say that κ is its main connective. A formula is compact if it contains no subformu-

 $\underset{\otimes_{n}(\phi_{1},\ldots,\phi_{k},\otimes_{m}(\phi_{k+1},\ldots,\phi_{k+m}),\phi_{k+m+1},\ldots,\phi_{k},\Im_{m}(\phi_{k+1},\ldots,\phi_{k+m}),\phi_{k+m+1},\ldots,\ldots,\phi_{n-m+1}) \text{ or } \\ \underset{\otimes_{n}(\phi_{1},\ldots,\phi_{k},\otimes_{m}(\phi_{k+1},\ldots,\phi_{k+m}),\phi_{k+m+1},\ldots,\ldots,\phi_{n-m+1}) \text{ for any } n,m \in \mathbb{N}.$

The *size* (resp. *energy*) of a formula ϕ is the number $|\phi|$ of (resp. the multiset of) literals, units, and connectives occurring in it.

We consider the following *equivalence laws*:

$$\kappa_{Q}(\phi_{1},\ldots,\phi_{|Q|}) \equiv \kappa_{Q}(\phi_{\sigma(1)},\ldots,\phi_{\sigma(|V_{Q}|)}) \quad \text{for each } \sigma \in \mathfrak{S}(Q)$$

$$\phi \otimes (\psi \otimes \chi) \equiv (\phi \otimes \psi) \otimes \chi$$

$$\phi \mathfrak{N}(\psi \otimes \chi) \equiv (\phi \mathfrak{N} \psi) \mathfrak{N} \chi$$
(10)

and the following *De-Morgan laws*:

$$\circ^{\perp} \equiv \circ \qquad \phi^{\perp\perp} \equiv \phi$$

only if $\mathfrak{S}^{\perp}(Q) = \varnothing : \quad (\kappa_{Q} \langle \phi_{1}, \dots, \phi_{|Q|} \rangle)^{\perp} \equiv \kappa_{Q^{\perp}} \langle \phi_{\sigma(1)}^{\perp}, \dots, \phi_{\sigma(|V_{Q}|)}^{\perp} \rangle$
only if $\mathfrak{S}^{\perp}(Q) \neq \varnothing : \quad (\kappa_{Q} \langle \phi_{1}, \dots, \phi_{|Q|} \rangle)^{\perp} \equiv \kappa_{Q} \langle \phi_{\rho(1)}^{\perp}, \dots, \phi_{\rho(|V_{Q}|)}^{\perp} \rangle$ for each $\rho \in \mathfrak{S}^{\perp}(Q)$ (11)

We denote by \equiv the equivalence relation generated by equivalence and De-Morgan laws.

A *context formula* (or simply *context*) $\zeta[\Box]$ is a formula containing an *hole* \Box taking the place of an atom. Given a context $\zeta[\Box]$, the formula $\zeta[\phi]$ is defined by simply replacing the atom \Box with the formula ϕ . For example, if $\zeta[\Box] = \psi \Re (\Box \otimes \chi)$, then $\zeta[\phi] = \psi \Re (\phi \otimes \chi)$.

Each formula ϕ with set of occurrences of literals x_1, \ldots, x_n can be considered as a *synthetic connective*, that is, given ψ_1, \ldots, ψ_n formulas we denote by $\phi(\psi_1, \ldots, \psi_n)$ the formula obtained by replacing x_i with ψ_i for all $i \in \{1, \ldots, n\}$. Therefore we define the set of *symmetries* of ϕ as the set $\mathfrak{S}(\phi)$ of permutations σ over $\{1, \ldots, n\}$ such that $\phi(\psi_1, \ldots, \psi_n) \equiv \phi(\psi_{\sigma(1)}, \ldots, \psi_{\sigma(n)})$ for any formulas ψ_1, \ldots, ψ_n .

The *linear implication* $\phi \to \psi$ is defined as $\phi^{\perp} \Im \psi$. We write $\phi \to \psi$ as a shortcut for " $\phi \to \psi$ and $\psi \to \phi$ ".

Observation. In multiplicative linear logic a De Morgan law linking the connectives \Re and \otimes are considered commutative because the following implications are provable

 $\phi \ \mathfrak{P} \psi \multimap \psi \ \mathfrak{P} \phi$ and $\phi \otimes \psi \multimap \psi \otimes \phi$ for all ϕ and ψ .

For this reason, the laws $\phi \ \mathfrak{N} \psi \equiv \psi \ \mathfrak{N} \phi$ and $\phi \otimes \psi \equiv \psi \otimes \phi$ are in some way subsumed and the De-Morgan law establishing a relation between \mathfrak{N} and \otimes is usually written in the form $(\phi \otimes \psi)^{\perp} = \phi^{\perp} \mathfrak{N} \psi^{\perp}$ establishing a relation between the connectives \otimes and \mathfrak{N} , similar to the one in the second line of Equation (11) with σ being the identity.

However, in [1, 2], where authors consider non-commutative versions of linear logic where sequents are considered as lists of formulas and the exchange rule is removed or restricted. In this logic both \otimes and \Re connectives are non-commutative and the De-Morgan law establishing the duality between \Re and \otimes written as $(\phi \otimes \psi)^{\perp} = \psi^{\perp} \Re \phi^{\perp}$, that is, is of the form $(\kappa (\phi_1, \phi_2))^{\perp} \equiv \kappa^{\perp} (\phi_{\sigma(1)}^{\perp}, \phi_{\sigma(2)}^{\perp})$ as in the second line of Equation (11) with σ the permutation exchanging 1 an 2.

These two cases covers all the possible way to define De-Morgan laws between pairs of binary connectives. However, we can consider cases of connectives such that only one connective occurs in the laws. The non-commutative connective logic logic Pomset [71] provides an example of this case, where the non-commutative binary connective \triangleleft is *self-dual*, that is, it satisfy the De-Morgan law $(\phi \triangleleft \psi)^{\perp} = \phi^{\perp} \triangleleft \psi^{\perp}$, as in the third line of Equation (11) with σ being the identity.

A fourth and last way to define a De-Morgan law for binary connectives would be law of the form $(\kappa(\phi, \psi))^{\perp} \equiv \kappa^{\perp}(\psi^{\perp}, \phi^{\perp})$. Note that this writing is closer to the law defining the inverse of the product of two elements in a group.

For graphical connectives we observe new behaviors similar to this latter (but more complex), which cannot be properly called *self-duality*, as in the case of \triangleleft but rather a duality "up to isomorphism". Consider the connective κ_{P_4} whose the dual connective is κ_{P_4} itself; the De-Morgan law establishing this duality ($\kappa_{P_4}(a, b, c, d)$)^{\perp} $\equiv \kappa_{P_4}(b^{\perp}, d^{\perp}, a^{\perp}, c^{\perp})$ is not simply expressed by negating the subformulas of a κ_{P_4} -formula, as in the case of \triangleleft , but also changing their order in a more complex

 $\mathrm{ax} \frac{}{\vdash a, a^{\perp}} \qquad \sqrt[\infty]{} \frac{\vdash \Gamma, \phi, \psi}{\vdash \Gamma, \phi \, ^{\infty} \psi} \qquad \otimes \frac{\vdash \Gamma, \phi \vdash \psi, \Delta}{\vdash \Gamma, \phi \otimes \psi, \Delta}$ $d_{-\kappa} \stackrel{\vdash \Gamma_1, \phi_{\sigma(1)}, \psi_{\tau(1)}}{\vdash \Gamma_1, \dots, \Gamma_n, \kappa([\phi_1, \dots, \phi_n]), \kappa^{\perp}([\psi_1, \dots, \psi_n])} \begin{cases} \sigma \in \mathfrak{S}(\kappa) \\ \tau \in \mathfrak{S}(\kappa^{\perp}) \end{cases}$ $\operatorname{ax} \frac{}{\vdash a, a^{\perp}} \qquad \operatorname{mix} \frac{\vdash \Gamma_1 \vdash \Gamma_2}{\vdash \Gamma_1, \Gamma_2} \qquad \operatorname{d}_{\kappa} \frac{\vdash \Gamma_1, \phi_{\sigma(1)}, \psi_{\tau(1)} \qquad \cdots \qquad \vdash \Gamma_n, \phi_{\sigma(n)}, \psi_{\tau(n)}}{\vdash \Gamma_1, \dots, \Gamma_n, \kappa(\phi_1, \dots, \phi_n), \kappa^{\perp}(\psi_1, \dots, \psi_n)} \begin{cases} \sigma \in \mathfrak{S}(\kappa) \\ \tau \in \mathfrak{S}(\kappa^{\perp}) \end{cases}$, ⊢ o $\mathfrak{N}_{n} \frac{\vdash \Gamma, \phi_{1}, \dots, \phi_{n}}{\vdash \Gamma, \mathfrak{N}_{n}(\phi_{1}, \dots, \phi_{n})} \xrightarrow{n > 1} \qquad \mathfrak{N}_{n} \frac{\vdash \Gamma_{1}, \phi_{1} \cdots \vdash \Gamma_{n}, \phi_{n}}{\vdash \Gamma_{1}, \dots, \Gamma_{n}, \mathfrak{N}_{n}(\phi_{1}, \dots, \phi_{n})} \xrightarrow{n > 1}$ $\mathsf{wd}_{\otimes} \frac{\vdash \Gamma, \psi \quad \vdash \Delta, \chi(\!\!|\phi_{\sigma(1)}, \dots, \phi_{\sigma(n)}|\!\!)}{\vdash \Gamma, \Delta, \kappa(\!\!|\phi_1, \dots, \phi_k, \psi, \phi_{k+1}, \dots, \phi_n|\!\!)} \begin{cases} n > 1 \\ \llbracket \kappa(\!\!|\phi_1, \dots, \phi_k, \circ, \phi_{k+1}, \dots, \phi_n|\!\!) \rrbracket = \llbracket \chi(\!\!|\phi_{\sigma(1)}, \dots, \phi_{\sigma(n)}|\!\!) \rrbracket \end{cases}$ $\mathsf{AX}_{\frac{\vdash \phi, \phi^{\perp}}{\vdash \phi, \phi^{\perp}}} \qquad \mathsf{cut}_{\frac{\vdash \Gamma_{1}, \phi \vdash \Gamma_{2}, \phi^{\perp}}{\vdash \Gamma_{1}, \Gamma_{2}}} \qquad \mathsf{cxt}_{\cdot \otimes \frac{\vdash \Gamma_{1}, \phi \vdash \Gamma_{2}, \zeta[\circ]}{\vdash \Gamma_{1}, \Gamma_{2}, \zeta[\phi]}} \qquad \mathsf{cxt}_{\cdot \Im}_{\frac{\vdash \Gamma, \zeta'[\phi]}{\vdash \Gamma, \zeta'[\circ], \phi}}$ $\vdash \Gamma_1, \phi_{\sigma(1)}, \psi_{\tau(1)}$ $\vdash \Gamma_n, \phi_{\sigma(n)}, \psi_{\tau(n)} (|\chi| > 1$ •••

$$\overset{\mathsf{d}}{\longrightarrow} \overset{\mathsf{d}}{\longrightarrow} \Gamma_1, \dots, \Gamma_n, \chi(\phi_1, \dots, \phi_n), \chi^{\perp}(\psi_1, \dots, \psi_n) \qquad \begin{cases} \sigma \in \tilde{\mathfrak{S}}(\chi) \\ \tau \in \tilde{\mathfrak{S}}(\chi^{\perp}) \end{cases}$$

Figure 4: Linear sequent calculus rules. In the first line the rules for MPL, in the second line the rules for MPL°. Below the admissible rules in MPL°.

way that just by "inverting the order of subformulas" as in the aforementioned fourth possible way to define De-Morgan laws for a binary connective. -

Remark 27. As explained in [8] (Section 9), the so-called generalized multiplicative connectives from the literature in linear logic [29, 44, 62, 9] are different from the ones discussed here. In fact, the unique 4-ary graphical connectives P_4 is iso-dual and has symmetry group {id, (1, 4)(2, 3)}, while the unique pair of dual generalize multiplicative connectives (which are not iso-dual) G_4 and G_4^{\perp} have symmetry group $\{id, (1, 2), (3, 4), (1, 2)(3, 4), (1, 2, 3, 4), (1, 3)(2, 4), (1, 4, 2, 3), (1, 4)(2, 3)\}$.

Definition 28. If ϕ is a formula, we define the graph $\llbracket \phi \rrbracket$ as follows:

 $\llbracket \phi^{\perp} \rrbracket = \llbracket \phi \rrbracket^{\perp} \qquad \llbracket \kappa_Q(\phi_1, \dots, \phi_n) \rrbracket = Q\left(\llbracket \phi_1 \rrbracket, \dots, \llbracket \phi_n \rrbracket\right)$ **[**○]] = Ø $\llbracket a \rrbracket = a$

where we denote by *a* the single-vertex graph, whose vertex is labeled by *a*. Conversely given a (possibly spurious) modular decomposition (via graphical connectives) of a graph G, we define $[G]^{-1}$ as the formula whose abstract syntax trees are has a one-to-one one-toone correspondence (respecting the parenthood relation) between laves labeled by a literal x, leaves labeled by units (\emptyset and \circ respectively), and between nodes labeled by the graphical connective Q and nodes labeled by connectives κ_Q .

For each formula $\phi = \psi(x_1, \dots, x_n)$ (where x_1, \dots, x_n are literals), we define $\mathfrak{S}(\phi) := \mathfrak{S}(\llbracket \phi \rrbracket)$.

Observation. Intuitively, compact and unit-free formulas are the representation of graphs modular decomposition via graphical connectives, providing a one-to-one correspondence between graphical connectives in the abstract syntax trees in the two syntexes. We have the following immediate results.

Proposition 29. Let ϕ and ψ be formulas. If $\phi \equiv \psi$, then $\llbracket \phi \rrbracket = \llbracket \psi \rrbracket$. Moreover, if ϕ and ψ are unit-free, then $\phi \equiv \psi$ iff $\llbracket \phi \rrbracket = \llbracket \psi \rrbracket$.

However, for expected stronger statements such as connections between implication between formulas whose interpretation is the same graph, we need the results in the next sections.

3.2 EXTENDING MULTIPLICATIVE LINEAR LOGIC WITH GRAPHICAL CONNECTIVE

We assume the reader to be familiar with the definition of sequent calculus derivations as trees of sequents (see, e.g., [77]) but we recall here some definitions.

Definition 30. We define a *sequent* is a set of occurrences of formulas.

A sequent system S is a set of sequent rules as the ones in Figure 4. In a sequent rule ρ , we say that a formula is *active* if it occurs in one of its premises (the sequents above the horizontal line) but not in its conclusion (the sequent below the horizontal line), and *principal* if it occurs in its conclusion but in none of its premises.

A *proof* of a sequent Γ is a derivation with no open premises, denoted $\begin{bmatrix} \pi \\ \Gamma \end{bmatrix}$. We denote by $\pi' \\ \begin{bmatrix} s \\ \Gamma \end{bmatrix}$ and Γ

(*open*) *derivation* of Γ from Γ' , that is, is a proof tree having exactly one open premise Γ' .

A rule is *admissible* if given provable premises, then its conclusion is is derivable without using the rule itself. A rule is *derivable* from a set of rules S, if it is possible to define an open derivation having the same premises and the same conclusion of the rule using only rules in S.

We then use the sequent rules in Figure 4 to define two logic over formulas.

Notation 31. In this paper, as in the tradition of linear logic, we use the same notation to denote a proof system S and the *logic* it identifies, that is, the set of formulas admitting a proof in S.

We generalize the *multiplicative linear logic* (*with mix*) [40] to the following logics operating on (more general) formulas constructed using graphical connectives beyond \Re and \otimes .

Definition 32. We define the following logics via their sequent systems:

 $\begin{aligned} \textbf{Multiplicative Prime Logic} &: \mathsf{MPL} = \{\mathsf{ax}, \mathfrak{N}, \otimes, \mathsf{d}\text{-}\kappa \mid \kappa \in \mathfrak{C} \text{ prime} \} \\ \textbf{Multiplicative Prime Logic with mix} : \mathsf{MPL}^\circ = \{\mathsf{ax}, \circ, \mathfrak{N}_n, \mathsf{mix}, \otimes_n, \mathsf{d}\text{-}\kappa, \mathsf{wd}_\otimes \mid \kappa \in \mathfrak{C} \} \end{aligned}$ (12)

Observation (Rules Exegesis). The rules ax, \mathcal{D} , \otimes , cut, mix and \circ the standard rules from multiplicative linear logic with mix. In particular, the ax is the restriction of the general axiom rule AX to atomic formulas. The rule \mathcal{D} can be read as the rule making explicit the meta-connective "comma" we use in sequents to separate formulas. The "true" commutativity of \mathcal{D} , that is the fact that we consider the formulas $\phi \mathcal{D} \psi$ and $\psi \mathcal{D} \phi$ to be the same formula is natural consequence of on the fact that we consider the sequents ϕ, ψ and ψ, ϕ to be the same sequent (as soon as we do not consider sequents as lists of formulas). Similarly, the rule \otimes can be read as the rule making the meta-connective "parallel branches" in derivation trees a concrete one, an then applying two occurrences of the weak-distributivity law (i.e. $\phi \otimes (\psi \mathcal{D} \chi) \rightarrow_{w.d.} (\phi \otimes \psi) \mathcal{D} \chi$) in the following way:

$$\overset{\Gamma,\phi}{=} \overset{\psi,\Delta}{\Gamma,\phi\otimes\psi,\Delta} \qquad \longleftrightarrow \qquad \underset{\text{same interpretation of }}{\overset{\text{the premises }(\Gamma,\phi) \text{ and }(\psi,\Delta) \text{ are valid}}{\overset{\text{weak distr.}}{\underset{\text{same interpretation of }}{\overset{(\Gamma \stackrel{\mathcal{R}}{\to} \phi)\otimes(\psi \stackrel{\mathcal{R}}{\to} \Delta)}}}_{\text{weak distr.} \underbrace{\frac{(\Gamma \stackrel{\mathcal{R}}{\to} \phi)\otimes(\psi \stackrel{\mathcal{R}}{\to} \Delta)}{\Gamma \stackrel{\mathcal{R}}{\to} (\phi\otimes\psi) \stackrel{\mathcal{R}}{\to} \Delta}}_{\text{same interpretation of } \underbrace{\frac{\Gamma \stackrel{\mathcal{R}}{\to} (\phi\otimes\psi) \stackrel{\mathcal{R}}{\to} \Delta}{\Gamma, \phi\otimes\psi, \Delta}}$$

To simplify proofs, in MPL° we generalize the rules for \mathfrak{P} and \otimes to their *n*-ary versions, proving that

the *n*-ary versions of these two connectives are derivable using the associativity of the binary ones.

The (*double*) *dual connectives* rule d- κ introduces a pair of dual connectives at the same time.^{*a*} This rule is a reformulation in sequent calculi of the $p\downarrow$ from the logic GS (see Section 4), where the rule takes two dual graphical connectives, and creating a graph where the edges of these two connectives have been merged (i.e. a clique) and where modules in the same "position" of the graphical connectives are gathered in a same module. To have an intuition about how this rule behaves, consider the equivalent two-sided formulation of our sequents and rules, where formulas can move across the turnstile (\vdash) modulo negation. In setting, the rule d- \otimes below introduces a \otimes on the right-hand side, together with a \otimes on the left-hand side (that is, a \Im). As mentioned above, the premises of this rule should be considered in a \otimes relation, as in the regular \otimes (see below).

$$\underset{\mathsf{d}}{\overset{\bullet}{\otimes}} \underbrace{\underbrace{\left(\Gamma_{1}, \phi_{1} \vdash \psi_{1}, \Delta_{1} \right)}_{\Gamma_{1}, \Gamma_{2}, \phi_{1} \otimes \phi_{2} \vdash \psi_{1} \otimes \psi_{2}, \Delta_{1}, \Delta_{2}}}_{\Gamma_{1}, \Gamma_{2}, \phi_{1} \otimes \phi_{2} \vdash \psi_{1} \otimes \psi_{2}, \Delta_{1}, \Delta_{2}} \quad \text{equivalent to} \quad \overset{\otimes}{\otimes} \underbrace{\frac{\vdash \Gamma_{1}^{\perp}, \phi_{1}^{\perp}, \psi_{1}, \Delta_{1} \sqcup \vdash \Gamma_{2}^{\perp}, \phi_{2}^{\perp}, \psi_{2}, \Delta_{2}, \Delta_{2}}}_{\overset{\otimes}{\otimes} \frac{\vdash \Gamma_{1}^{\perp}, \Gamma_{2}^{\perp}, \phi_{1}^{\perp}, \psi_{1}, \Delta_{1} \sqcup \vdash \Gamma_{2}^{\perp}, \phi_{2}^{\perp}, \psi_{2}, \Delta_{2}, \Delta_{2}}$$

Similarly, we should consider that the premises of a $d-\kappa_Q$ to be in a Q-shaped relation. For example, for $Q = P_4$ we should have consider the first and the second premises, the second and the third premises, and the third and fourth premises in such a \otimes relation, as shown below.

$$\underbrace{(\overline{\Gamma_1,\phi_1+\psi_1,\Delta_1})}_{\Gamma_1,\Gamma_2,\Gamma_3,\Gamma_4,\kappa_{\mathsf{P}_4}(\![\phi_1,\phi_2,\phi_3,\phi_4]\!] \vdash \kappa_{\mathsf{P}_4}(\![\psi_1,\psi_2,\psi_3,\psi_4]\!],\Delta_1,\Delta_2,\Delta_3,\Delta_4} (\overline{\Gamma_4,\phi_4\vdash\psi_4,\Delta_4})$$

With this intuition, it appears clear that during proof construction (top-down interpretation of a derivation) the d- κ simply makes explicit the relation between the premises of the rule (which could be thought as a meta-connective) by "keeping track" of this relation by introducing a copy of the connective capturing this same pattern in each side of the sequent. At the same time, in proof search (i.e., bottom-up interpretation of a derivation) in single-sided sequents, the rule d- κ could be thought as a rule merging a κ -formula $\kappa(\phi_1, \ldots, \phi_n)$ with a κ^{\perp} -formula dual $\kappa^{\perp}(\psi_1, \ldots, \psi_n)$ into the formula $(\phi_1 \stackrel{\mathcal{R}}{\rightarrow} \psi_1) \otimes \cdots \otimes (\phi_n \stackrel{\mathcal{R}}{\rightarrow} \psi_n)$, followed by a \otimes_n splitting the context, and some $\stackrel{\mathcal{R}}{\rightarrow}$.

Note that having a rule introducing only one of the two dual connectives inevitably leads to the same problems of the rules for generalized multiplicative connectives introduced in the early works on linear logic [29, 44], where *initial coherence* (i.e. the possibility of having only atomic axioms in a cut-free system, [14]) is ruled out because of the so-called *packaging problem*. This problem, due to the fact that some rules would require to introduce a new connective between formulas from a same sequent and from different sequents (therefore enforcing strong constraints in proof search), does not occur in the syntax of proof nets with generalized connectives, where the rigid structure imposed by derivation branching is removed. An extensive discussion of such a single-connective rule and the results on it is provided in Appendix C.

The rule wd_{\otimes} allows us construct derivations where we can simulate the possibility of applying certain deep inference rules to subformulas of a sequent system, in the style of deep inference systems [47]. It is a generalization of the *weak-distribution law* in symmetric monoidal categories (see, e.g., [61, 3])

$$\phi \otimes (\psi \,^{\mathfrak{N}} \chi) \longrightarrow (\phi \otimes \psi) \,^{\mathfrak{N}} \chi \tag{13}$$

distributing the \otimes over the other connectives, that is,

d-

$$\chi \otimes \kappa(\phi_1, \dots, \phi_k, \psi, \phi_{k+1}, \dots, \phi_n) \longrightarrow \kappa(\phi_1, \dots, \phi_k, \psi \otimes \chi, \phi_{k+1}, \dots, \phi_n)$$
(14)

Note this law toghether with the "dual" weak-distributive law

$$\kappa(\phi_1, \dots, \phi_k, \psi \ \mathcal{X} \chi, \phi_{k+1}, \dots, \phi_n) \longrightarrow \kappa(\phi_1, \dots, \phi_k, \psi, \phi_{k+1}, \dots, \phi_n) \ \mathcal{X} \chi \tag{15}$$

distributing the connective \Re over the other connectives usually collapse on a single law (the standard one in Equation (13)) whenever we consider only the two connectives \otimes and \Re . This law for the \Re is captured by the admissible rule *context-par* (cxt- \Re) generalizing it using the unit \circ , while the admissible rule *context-tensor* (cxt- \otimes) generalizes wd $_{\otimes}$.

The other derivable rules in the last row are the generalization of the axiom to any formulas (AX), the generalizing of the rule $d-\kappa$ to synthetic connectives $(d-\chi)$ and the standard cut-rule.

Notation 33. Unless needed for sake of clarity, we omit to the permutations over the indices of the subformulas in rules.

Remark 34. If we consider only MLL-formulas, then the rule wd_{\otimes} is admissible.

3.3 Properties of the Logics MPL and MPL°

We start by observing that these systems are *initial coherent* [14, 64], that is, we can derive the implication $\phi \rightarrow \phi$ for any formula ϕ only using atomic axioms. This allows us to prove that in IsoMix the unit \circ is, in some sense, the unit for all connectives. We conclude by proving the admissibility of cut via cutelimination, together with the admissibility of certain rules which are useful to prove the results in the next sections.

Theorem 35. *The logics* MPL *and* MPL° *are initial coherent.*

Proof. By induction on the structure of a formula ϕ :

- if φ = ∘, since ∘[⊥] = ∘, then we have a derivation of ⊢ ∘, ∘ by applying mix to the conclusion of two ∘-rules;
- if $\phi = a$ is a literal, then there is a derivation a, a^{\perp} made of a single ax occurrence;
- if $\phi = \kappa (|\psi_1, \dots, \psi_n|)$, then we can apply a d- κ to the sequent ϕ, ϕ^{\perp} and obtaining sequents ψ_i, ψ_i^{\perp} for all $i \in \{1, \dots, |\kappa|\}$. We conclude by inductive hypothesis.

Note that if ϕ is unit-free, then we only need the rules ax and p to prove $\vdash \phi, \phi^{\perp}$.

The derivability of the general axiom rule and the general $d-\chi$ immediately follows by a similar argument.

Lemma 36. Let χ be a formula such that $|\chi| > 1$. Then rule $d-\chi$ is derivable.

Proof. By induction on the structure of ϕ using the rule d- κ .

Corollary 37. The rule AX is derivable in MPL and in MPL°.

To prove cut-elimination in MPL°, we rely on the admissibility of the rule $cxt-\Re$ in MPL°.

Lemma 38. The rule cxt - \Re is admissible in MPL°.

Proof. To prove the admissibility of cxt-2 we show that $\vdash_{\mathsf{MPL}^\circ} \Gamma, \zeta[\phi]$, then $\vdash_{\mathsf{MPL}^\circ} \Gamma, \zeta[\circ], \phi$:

• If $\zeta[\Box] = \Box$ is trivial, then $\zeta[\phi] = \phi$ and we conclude since $\inf_{\min} \frac{\varphi}{\varphi} \mapsto \varphi = \varphi$.

• If, w.l.o.g.,
$$\zeta[\Box] = \Re_n(\zeta'[\Box], \psi_2, \dots, \psi_n)$$
, then there is a derivation

$$\frac{\mathbb{I}}{\mathfrak{P}_{n} \underbrace{\vdash \Gamma, \zeta'[\phi], \psi_{2}, \dots, \psi_{n}}{\vdash \Gamma, \mathfrak{P}_{n}(\zeta'[\phi], \psi_{2}, \dots, \psi_{n})} , \text{ thus a derivation } \mathfrak{P}_{n} \underbrace{\vdash \Gamma, \zeta'[\circ], \psi_{2}, \dots, \psi_{n}, \phi}{\vdash \Gamma, \mathfrak{P}_{n}(\zeta'[\circ], \psi_{2}, \dots, \psi_{n}), \phi}$$

^{*a*}The existence of rules introducing two (or more than two) operators at the same time is not a novelty in structural proof theory. Similar rules can be observed in focused proof systems (ee, e.g.[13, 65, 64]), where a rule can handle multiple connectives of a same formula, or in modal logic and linear logic (more precisely, variants of linear logic with functorial promotion rule), where rules for modalities often introduces multiple modalities in a single application (see, e.g., [43, 17, 21, 59]). In recent works on display calculi [25], the authors use rules for two sided sequent systems where rules introduce a modality (which could be of any arity) on one side of the sequent together with their associated weak modality, internalizing the introduction of the dual connective on the other side of the sequent.

• If, w.l.o.g., $\zeta[\Box] = \phi \otimes C'[\Box]$, then there is a derivation

$$\underset{\otimes_{n}}{\overset{\vdash \Gamma_{1}, \zeta'[\phi]}{\vdash \Gamma_{1}, \Gamma_{2}, \otimes_{n}(\zeta'[\phi], \psi_{2}, \dots, \psi_{n})}} \prod_{\text{, thus a derivation}} \underset{\otimes_{n}}{\overset{\prod_{n} \underset{\mapsto}{\sqcup} \prod_{n} \underset{\mapsto}{\sqcup}}{\underset{\vdash \Gamma_{1}, \zeta'[\circ], \phi}{\vdash \Gamma_{2}, \psi_{2}}} \xrightarrow{\underset{\mapsto}{\sqcup} \underset{\vdash \Gamma_{n}, \psi_{n}}{\underset{\vdash \Gamma_{n}, \psi_{n}}{\vdash \Gamma_{n}, \Gamma_{2}, \psi \otimes \zeta'[\circ], \phi}}$$

- If, w.l.o.g., $\zeta[\Box] = \kappa \langle \zeta'[\Box], \psi_2, \dots, \psi_n \rangle$, then:
 - either there is a derivation

$$d_{\kappa} \frac{\prod_{i=1}^{n} \prod_{j=1}^{n} \prod_{j=1}^$$

thus a derivation

$$d_{\kappa} \frac{ \prod_{i \in I} H \prod_{i \in I} H P \prod_{i \in I} H P }{ + \Gamma_1, \zeta'[\circ], \phi, \chi_1 + \Gamma_2, \psi_2, \chi_2} \cdots + \Gamma_n, \psi_n, \chi_n } + \Gamma_n, \psi_n, \chi_n$$

- or there is a derivation

$$\mathsf{wd}_{\otimes} \frac{ \prod_{\substack{\ell \in \Gamma_1, \psi' \in \Gamma_2, \chi(\zeta'[\phi], \psi_2, \dots, \psi_n) \\ \ell \in \Gamma_1, \Gamma_2, \kappa(\zeta'[\phi], \psi_2, \dots, \psi_k, \psi', \psi_{k+1}, \dots, \psi_n] }$$

thus a derivation

$$\mathsf{wd}_{\otimes} \frac{\prod_{i=1}^{n} \prod_{j=1}^{n} \prod_{i=1}^{n} \prod_{j=1}^{n} \prod_{$$

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The proof of admissibility of cut is proven by providing a cut-elimination procedure.

Theorem 39 (Cut-elimination). Let $X \in \{MPL, MPL^\circ\}$. The rule cut is admissible in X.

Proof. For MPL, it suffices to define the *weight* of an instance of a cut-rule as the maximum length of a branch above one of its premises and the *weight* of of a derivation as the sum of the weights of the cuts. To conclude it suffices to remark that each *cut-elimination step* from Figure 5 reduces the weight of a derivation.

For MPL°, we also have to define the *energy* of an instance of a cut-rule as the (multiset) union of the energies of its cut-formulas and the *energy* of a derivation as the multiset of the energies its cuts. We then consider the order over multisets of units, literals and connectives defined in such a way $\kappa < \kappa'$ whenever $|\kappa| < |\kappa'|$ and $\circ < x$ for any x literal. According to this order, each non-commutative cut-elimination step reduces the energy of a derivation. The only non-trivial case is the case in which we cut a principal formula of a wd_{\otimes} against a principal formula of another wd_{\otimes} where the two wd_{\otimes}rules are applied to principal subformulas with different indices (more precisely, whose indices are not related by a permutation in $\mathfrak{S}(\kappa)$). In this case, the cut-elimination step introduces three new cuts, all of which with smaller energy. To conclude it suffices to remark that the lexicographic order over the pairs give by the energy and the weight of a derivation reduce at each step because each commutative cut-elimination step does not change the energy but reduces the weight.

The admissibility of cut implies analyticity via the sub-formula property for MPL.

$$\begin{split} & \frac{a_{1}^{n} \frac{b_{2} - a_{1}^{n} \frac{b_{1}}{h_{1} - h_{1}}}{a_{1} - h_{1}} \longrightarrow b_{1}^{n} \frac{b_{1} + b_{1}^{n} \frac{b_{1}}{h_{1} + h_{1}}}{a_{1} + h_{1} +$$

Figure 5: Cut-elimination steps. The two rules in the bottom are called *commutative cut-elimination steps*.

Corollary 40 (Analyticity of MPL). Let Γ be a sequent. If $\vdash_{\mathsf{MPL}} \Gamma$ then there is a proof of Γ in MPL only containing occurrences of sub-formulas of formulas Γ .

Proof. It suffices to remark that the rules in MPL satisfy the subformula property, that is, all formulas occurring in a premise of a rule are subformulas of the formulas occurring in the conclusion. \Box

The same result cannot be immediately stated for MPL° because of the rule unitor_{κ}. This because, as already observed in the previous works on graphical logic [7, 8, 5], having more-than-binary connectives implies the possibility of having *sub-connectives*, that is, graphical connectives with smaller arity corresponding to the synthetic connective obtained by fixing certain of the entries of a connective to be units.

Definition 41. A graphical connective κ_Q is a *sub-connective* of $\kappa_{Q'}$ if Q is an induced (quasiprime) subgraph of Q'. We may denote $\kappa_Q = \kappa_{Q'|i_1,...,i_k}$ with $i_1,...,i_k \in \{1,...,n\}$ such that $i_i < \cdots < i_k$ if $Q(0,...,0,v_{i_1},0,...,0,v_{i_k},0,...,0) \sim Q'(v_1,...,v_n)$ for any single-vertex graphs $v_1,...,v_n$. A *quasi-subformula* of a formula $\phi = \zeta[\kappa_{Q'}|\psi_1,...,\psi_n]$ is a literal in ϕ or is a formula $\kappa_{Q'|i_1,...,i_k}(\psi'_{i_1},...,\psi'_{i_k})$ with ψ'_{i_i} a quasi-subformula of ψ_j for all $j \in \{1,...,k\}$.

Corollary 42 (Analyticity of MPL°). Let Γ be a sequent. If $\vdash_{\mathsf{MPL}^\circ} \Gamma$ then there is a proof of Γ in MPL° only containing occurrences of quasi-subformula of formulas in Γ .

Corollary 43 (Conservativity). *The logic* MPL *is a conservative extension of* MLL. *The logic* MPL[°] *is a conservative extension of* MLL[°].

Proof. For MPL it is consequence of the subformula property. For MPL^{\circ} it suffices to remark that \Re and \otimes have no sub-connectives, therefore quasi-subformula are simply sub-formulas.

For both MPL and MPL° we have the following result which takes the name of *splitting* in the deep inference literature (see, e.g, [11, 49, 50]). This lemma states that is always possible, during proof search, to apply a rule removing a connective after having applied certain rules in the context. Note that in the linear logic literature, the term splitting is usually used as adjective for an instance of a \otimes on which a rule can be applied splitting the context into two premises, that is, as a specific instance of this more general formulation.

Lemma 44 (Splitting). Let $\Gamma, \kappa(\phi_1, \ldots, \phi_n)$ be a sequent and let $X \in \{MPL, MPL^\circ\}$. If $\vdash_X \Gamma, \kappa(\phi_1, \ldots, \phi_n)$, then there are sequents $\Delta_1, \ldots, \Delta_n, \Gamma'$ such that

 $\rho \frac{ \stackrel{\pi_{1} \prod \qquad \pi_{n} \prod$

for some proofs π_1, \ldots, π_n and an open derivation π_0 .

Proof. By case analysis of the last rule occurring in a proof π of Γ , $\kappa(\phi_1, \ldots, \phi_n)$:

- the last rule cannot be a ax or \circ since the formula $\kappa(\phi_1, \ldots, \phi_n)$ occurs in the conclusion.
- if the last rule is a \mathcal{P}_n , then we conclude by inductive hypothesis on its premise.
- if the last rule is a mix, then we conclude by inductive hypothesis on the premise containing the formula κ(φ₁,...,φ_n);
- if the last rule is in $\{\otimes_n, d-\kappa, wd_{\otimes}\}$ then:
 - either this is the desired rule ρ ;

- or one of the (provable) premises of this rule is of the shape $\Gamma', \kappa(\phi_1, \ldots, \phi_n)$, allowing us to conclude by inductive hypothesis.

We conclude this section proving the admissibility of the rule cxt- \otimes in MPL°.

Lemma 45. The rule $cxt \cdot \otimes$ is admissible in MPL°.

Proof. We proceed by induction on $\zeta[\Box]$:

- If $\zeta[\Box] = [\Box]$, then cxt- \otimes is an instance of mix.
- If $\zeta[\Box] = \zeta'[\Box] \Im \psi$, then cxt- \otimes can be replaced by a mix followed by a \Im .
- If, w.l.o.g., $\zeta[\Box] = \kappa (\zeta'[\Box], \psi_2, \dots, \psi_n)$, for a $\kappa \neq \Im$, then we can apply Lemma 44 and conclude since we have a derivation

∏ін	ΠHP		HP
$\vdash \Gamma_1', \Delta_1, \zeta'[\phi]$	$\vdash \Delta_2, \psi_2$	•••	$\vdash \Delta_n, \psi_n$
$^{\rho}$ \vdash Γ'_1, Γ'_2, I	$\kappa(\zeta'[\Box],\psi_2)$, · · · ,	ψ_n)
	π_0 HP		
$\vdash \Gamma_1, \Gamma_2, \Gamma_3$	$\kappa(\zeta'[\Box],\psi_2)$	····	ψ_n)

3.4 Soundness of Logical Equivalence in MPL°

In order to prove that two formulas ϕ and ψ interpreted by a same graph $\llbracket \phi \rrbracket = \llbracket \phi \rrbracket$ are logically equivalent (i.e., $\phi \multimap \psi$), we here provide intermediate results allowing to decompose this equivalence in smaller steps.

We first prove that connectives symmetries are derivable in MPL, therefore in MPL°.

Lemma 46. The following rules are admissible in MPL. $sym_{\kappa} \xrightarrow{\vdash \Gamma, \kappa([\phi_1, \dots, \phi_n])}_{\vdash \Gamma, \kappa([\phi_{\sigma(1)}, \dots, \phi_{\sigma(n)}])} \xrightarrow{\sigma \in \mathfrak{S}(Q)} dsym_{\kappa} \xrightarrow{\vdash \Gamma, \kappa^{\perp}([\phi_1, \dots, \phi_n])}_{\vdash \Gamma, \kappa([\phi_{\rho(1)}, \dots, \phi_{\rho(n)}])} \xrightarrow{\rho \in \mathfrak{S}^{\perp}(Q)} (16)$

Proof. By Theorem 39, it suffices to prove that the following implications are derivable.

$$\underbrace{\begin{array}{c} \kappa(\phi_1,\ldots,\phi_n) \multimap \kappa(\phi_{\sigma(1)},\ldots,\phi_{\sigma(n)}) \\ \kappa(\phi_{\sigma(1)},\ldots,\phi_{\sigma(n)}) \multimap \kappa(\phi_1,\ldots,\phi_n) \\ \text{for all } \sigma \in \tilde{\mathbb{Z}}(Q) \end{array}}_{\text{for all } \tau \in \tilde{\mathbb{Z}}^1(Q)} \text{ and } \underbrace{\begin{array}{c} \kappa(\phi_1,\ldots,\phi_n) \lor \kappa^{\perp}(\phi_{\rho(1)},\ldots,\phi_{\rho(n)}) \\ \kappa^{\perp}(\phi_{\rho(1)},\ldots,\phi_{\rho(n)}) \lor \kappa(\phi_1,\ldots,\phi_n) \\ \text{for all } \tau \in \tilde{\mathbb{Z}}^1(Q) \end{array}}_{\text{for all } \tau \in \tilde{\mathbb{Z}}^1(Q)}$$

These are easily derivable using an instance of $d-\kappa$ and AX-rules.

Remark 47. The rule sym- \Re is derivable directly because sequents are sets if occurrences of formulas, therefore the order of the occurrences of the formulas in a sequent is not relevant, and we can permute this order before applying the rule \Re . This because the interpretation of the meta-connective comma we use to separate formulas in a sequent is the same of \Re .

Similarly, the rule sym- \otimes is derivable because in our sequent system, as in standard sequent calculus, the order of the premises of the rules is not relevant. Said differently, the space between branches in a derivation is a commutative meta-connective which is internalized by the \otimes .

As a consequence of the analyticity MPL and MPL[°], we could consider the connectives *multi-par* ($\kappa_{\mathfrak{N}_n}$) and *multi-tensor* ($\kappa_{\mathfrak{N}_n}$) superfluous in our syntax for formulas since they are synthetic connectives definable via the binary \mathfrak{N} and \mathfrak{S} . In particular, this allows us restrain our reason on compact formulas only since rules expressing the associativity of \mathfrak{N}_n and \mathfrak{S}_n are derivable.

Lemma 48. The following rules are admissible.

$$\frac{\vdash \Gamma, \mathcal{N}_{n-m+1}(k_{\mathcal{N}_m}(\phi_1, \dots, \phi_m), \phi_{m+1}, \dots, \phi_n)}{\vdash \Gamma, \mathcal{N}_n(\phi_1, \dots, \phi_n)} \underset{m < n}{\overset{m < n}{\longrightarrow}} \frac{\vdash \Gamma, \otimes_{n-m+1}(k_{\otimes_m}(\phi_1, \dots, \phi_m), \phi_{m+1}, \dots, \phi_n)}{\vdash \Gamma, \otimes_n(\phi_1, \dots, \phi_n)} \underset{m < n}{\overset{m < n}{\longrightarrow}} (17)$$

Therefore, the connectives \Re and \otimes are associative and any formula admits an equivalent compact formula.

Proof. We only prove the associativity result for \mathfrak{P}_n , since the proof for \otimes_n is similar. The result follows by Theorem 39 after proving that following implications hold for any $n, m \in \mathbb{N}$

 $\mathcal{P}_{n}(\phi_{1},\ldots,\phi_{n}) \sim \mathcal{P}_{n-m+1}(\kappa_{\mathcal{P}_{m}}(\phi_{1},\ldots,\phi_{m}),\phi_{m+1},\ldots,\phi_{n}) \\ \mathcal{P}_{n-m+1}(\kappa_{\mathcal{P}_{m}}(\phi_{1},\ldots,\phi_{m}),\phi_{m+1},\ldots,\phi_{n}) \sim \mathcal{P}_{n}(\phi_{1},\ldots,\phi_{n})$

We can therefore immediately conclude that MPL is sound and complete with respect to graph isomorphism if we consider unit-free formulas.

Proposition 49. Let ϕ and ψ be unit-free formulas. If $\phi \equiv \psi$, then $\phi \multimap \psi$ and $\psi \multimap \phi$.

Proof. By induction on the formulas ϕ and ψ using Lemmas 46 and 48.

For a stronger result on general formulas, we need to show that for any two formulas ϕ and ψ are interpreted (via $[\cdot]$) by the same non-empty graph, both these formulas are equivalent to a unit-free formula χ representing the modular decomposition of this graph via graphical connectives.

Lemma 50. The following rule, is derivable in MPL°.

 $\underset{\kappa}{\mathsf{unitor}_{\kappa}} \xrightarrow{\vdash \Gamma, \chi(\phi_{\sigma(1)}, \dots, \phi_{\sigma(n)})}_{\vdash \Gamma, \kappa(\phi_{1}, \dots, \phi_{k}, \circ, \phi_{k+1}, \dots, \phi_{n})} \begin{cases} n > 1\\ \chi \text{ compact formula}\\ [\kappa(\phi_{1}, \dots, \phi_{k}, \circ, \phi_{k+1}, \dots, \phi_{n})]] = [\chi(\phi_{\sigma(1)}, \dots, \phi_{\sigma(n)})]] \end{cases}$ (18)

Proof. It suffices to consider the derivation

$$\mathsf{wd}_{\approx} \frac{\stackrel{\circ}{\vdash \circ} \vdash \Gamma, \chi(\phi_{\sigma(1)}, \dots, \phi_{\sigma(n)})}{\vdash \Gamma, \kappa(\phi_{1}, \dots, \phi_{k}, \circ, \phi_{k+1}, \dots, \phi_{n})}$$

Theorem 51. Let ϕ and ψ be formulas. If $\llbracket \phi \rrbracket = \llbracket \psi \rrbracket \neq \emptyset$, then ϕ and ψ are equivalent in MPL°, that is, $\phi \multimap \psi$ is valid in MPL°.

Proof. Given any formula ϕ , we can define by induction on the number of units \circ occurring in a unit-free compact formula ϕ' such that $\phi \leadsto \phi'$.

- if ϕ is a literal, then $\phi' = \phi$;
- if $\phi = \kappa(\phi_1, \dots, \phi_n)$ and $\phi_i \neq \circ$ for all $i \in \{1, \dots, n\}$, then $\phi' = \kappa(\phi'_1, \dots, \phi'_n)$. Otherwise, w.l.o.g., we assume $\phi_i = \circ$ and we let $\phi' = \chi(\phi'_2, \dots, \phi'_n)$ for a compact formula χ such that $[\kappa(\circ, \phi_2, \dots, \phi_n)] = [\chi(\phi_2, \dots, \phi_n)]$ and we conclude by inductive hypothesis since we the

Figure 6: Inference rules for the system GS, where P is a prime graph and $M_i \neq \emptyset \neq M'_i$ for all $i \in \{1, ..., n\}$.

following derivations:

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Therefore we can construct unit-free compact formulas ϕ' and ψ' such that $\phi \multimap \phi'$ and $\psi \multimap \psi'$. Moreover, by definition of $\llbracket \cdot \rrbracket$ and the rule unitor_{κ} we have $\llbracket \phi \rrbracket' = \llbracket \phi \rrbracket = \llbracket \psi \rrbracket = \llbracket \psi \rrbracket'$. Because of the unicity of the modular decomposition via graphical connectives of the graph $\llbracket \phi' \rrbracket = \llbracket \psi' \rrbracket$ modulo symmetries of connectives, and their correspondence with unit-free compact formulas, then we must have $\phi' \equiv \psi'$. We conclude using the transitivity of \multimap and Proposition 49, by letting χ be the formula ϕ' .

4 The Graphical Logic GS is a Model for MPL°

In this section we prove that the logic on graphs defined by the deep inference proof system GS from [7, 8] is the same logic identified by the graph corresponding to formulas which are provable in MPL°.

In this paper we define deep inference system $GS = \{ai\downarrow, s_{\Im}, s_{\otimes}, p\downarrow\}$ using the rules in Figure 6. The definition of derivations in deep inference systems operating on graphs are provided in Appendix A.

Remark 52. At the syntactical level, the system GS operates on graphs by manipulating their spurious modular decompositions via graphical connectives. Therefore, for any derivation in GS we can assume to be given a spurious modular decomposition of each graph *G* occurring in a derivation, therefore a unique formula $[[G]]^{-1}$ (defined as in Definition 28) to be given.

Remark 53. We here provide a slightly different formulation of with respect to [7] and [8]. In particular, we consider a p-rules with stronger side condition which is balanced by the presence of s_{\otimes} in the system. However, it can be easily shown that the systems are equivalent (see Appendix A.1).

We can easily prove that each sequent provable in MPL° is interpreted by $[\cdot]$ as a graph which is admitting a proof in GS.

Lemma 54. Let Γ be a sequent. If $\vdash_{\mathsf{MPL}^\circ} \Gamma$, then $\vdash_{\mathsf{GS}} \llbracket \Gamma \rrbracket$. *Proof.* Let π be a proof of Γ in MPL°, we define a derivation $\begin{bmatrix} \varnothing \\ \llbracket \pi \rrbracket \rrbracket \mathsf{GS} \\ \llbracket \Gamma \rrbracket \end{bmatrix}$ by induction on the last rule r $\llbracket \Gamma \rrbracket$ in π according to Figure 7.

To prove the converse, we use the admissibility of cxt- \Im to prove in a more concise way that every time there is a rule in GS with premise *H* and conclusion *G*, then there are formulas ϕ and ψ such that $[\![\phi]\!]$ and $[\![\psi]\!]$, and such that $\psi \to \phi$.

Figure 7: Rules to translate derivations in MPL° into derivations in GS.

Lemma 55. Let $r \frac{H}{G} \in GS$ and $C[\Box]$ be a context. Then there are formulas ϕ and ψ such that $\vdash_{\mathsf{MPL}^{\circ}} \psi^{\perp}, \phi$, and $\llbracket \phi \rrbracket = C[G]$ and $\llbracket \psi \rrbracket = C[H]$.

Proof. We can prove by case analysis on the rule r that $H \multimap G$. If $C[\Box] = \Box$, then:

- if an isomorphism is applied, then G = H and we conclude by Theorem 51, letting $\phi = \llbracket G \rrbracket^{-1}$ and $\psi = \llbracket H \rrbracket^{-1}$ (see Remark 52);
- if $\mathbf{r} = \mathbf{ai}\downarrow$, then $\phi = a \Re a^{\perp}$ and $\psi = \circ$ and a derivation

$$\underset{\text{mix}}{\circ} \underbrace{ \circ}_{\mu} \underbrace{ \overset{ax}{} + a, a^{\perp}}_{\mu} \underbrace{ \overset{ax}{} + a \overset{ax}{} a^{\perp}}_{\mu} \underbrace{ \overset{ax}{} + a \overset{ax}{} a^{\perp}}_{\mu} \underbrace{ \overset{ax}{} a^{\perp}}_{\mu} \underbrace{ \overset{ax}{} a \overset{ax}{} a \overset{ax}{} a^{\perp}}_{\mu} \underbrace{ \overset{ax}{} a \overset{ax$$

• if $\mathbf{r} = \mathbf{s}_{\mathfrak{P}}$, then $\phi = \mu_i \mathfrak{P} \kappa(\mu_1, \dots, \mu_{i-1}, \circ \mathfrak{P} \nu, \mu_{i+1}, \dots, \mu_n)$ and $\psi = \kappa(\mu_1, \dots, \mu_{i-1}, \mu_i \mathfrak{P} \nu, \mu_{i+1}, \dots, \mu_n)$ for some formulas μ_1, \dots, μ_n, ν such that $[\mu_i]] = M_i$ for all $i \in \{1, \dots, n\}$ and $[\nu_i] = N$. We conclude by Lemma 38 since we have the following derivation

• if $\mathbf{r} = \mathbf{s}_{\otimes}$ then $\phi = \kappa(\mu_1, \dots, \mu_{i-1}, \mu_i \otimes \nu, \mu_{i+1}, \dots, \mu_n)$ and $\psi = \mu_i \otimes \kappa(\mu_1, \dots, \mu_{i-1}, \circ \otimes \nu, \mu_{i+1}, \dots, \mu_n)$ for some formulas μ_1, \dots, μ_n, ν such that $[\![\mu_i]\!] = M_i$ for all $i \in \{1, \dots, n\}$ and $[\![\nu]\!] = N$. We conclude by Lemma 38 since we have the following derivation

$$\overset{\mathsf{AX}}{\underset{\mathfrak{T}}{\vdash} \kappa^{\perp}(\mu_{1}^{\perp},\ldots,\mu_{i-1}^{\perp},\mu_{i}^{\perp} \overset{\mathfrak{T}}{\to} \nu^{\perp},\mu_{i+1}^{\perp},\ldots,\mu_{n}^{\perp}),\phi} }_{\mathfrak{T}} \underbrace{\overset{\mathsf{AX}}{\vdash} \mu_{i}^{\perp},\kappa^{\perp}(\mu_{1}^{\perp},\ldots,\mu_{i-1}^{\perp},\circ \overset{\mathfrak{T}}{\to} \nu^{\perp},\mu_{i+1}^{\perp},\ldots,\mu_{n}^{\perp}),\phi}_{\vdash \psi^{\perp},\phi}$$

• if $\mathbf{r} = \mathbf{p} \downarrow$ then $\phi = \kappa_{P^{\perp}}(\mu_1, \dots, \mu_n) \Im \kappa_P(\nu_1, \dots, \nu_n)$ and $\psi^{\perp} = (\mu_1^{\perp} \otimes \nu_1^{\perp}) \Im \cdots \Im (\mu_n^{\perp} \otimes \nu_n^{\perp})$ for some formulas $\mu_1, \dots, \mu_n, \nu_1, \dots, \nu_n$ such that $[\![\mu_i]\!] = M_i \neq \emptyset$ and $[\![\nu_i]\!] = N_i \neq \emptyset$ for all $i \in \{1, \dots, n\}$. We conclude since we have the following derivation

$$\begin{array}{c} \mathsf{AX} \underbrace{\overbrace{\vdash \mu_{1}, \mu_{1}^{\perp}}_{\mathsf{d} - \kappa} \underbrace{\mathsf{AX}}_{\vdash \nu_{1}, \mu_{1}^{\perp}}_{\mathsf{d} - \kappa} \underbrace{\mathsf{AX}}_{\vdash \nu_{1}, \mu_{1}^{\perp}}_{\mathsf{d} - \kappa} \underbrace{\mathsf{AX}}_{\vdash \nu_{1}, \mu_{1}, \nu_{1}}_{\mathsf{d} - \kappa} \underbrace{\mathsf{AX}}_{\vdash \nu_{1}, \mu_{1}, \nu_{1}}_{\mathsf{d} - \kappa} \underbrace{\mathsf{AX}}_{\vdash \mu_{n}, \mu_{n}^{\perp}}_{\mathsf{d} - \kappa} \underbrace{\mathsf{AX}}_{\vdash \nu_{n}, \nu_{n}^{\perp}}_{\mathsf{d} - \kappa} \underbrace{\mathsf{AX}}_{\vdash \nu_{n}, \nu_{n}^{\perp}}_{\mathsf{d} - \kappa} \underbrace{\mathsf{AX}}_{\vdash \nu_{n}, \nu_{n}}_{\mathsf{d} - \kappa} \underbrace{\mathsf{AX}}_{\vdash \nu_{n}, \nu_{n}^{\perp}}_{\mathsf{d} - \kappa} \underbrace{\mathsf{AX}}_{\mathsf{d} - \kappa}$$

If $C[\Box] \neq \Box$, then by Theorem 24 that $C[\Box] = \kappa_Q (C'[\Box], M_1, ..., M_n)$ for a quasi-prime graph Q and non-empty graph $M_1, ..., M_n$. In this case we let $\zeta[\Box] = \kappa_P (\zeta'[\Box], \mu_1, ..., \mu_n)$ for some formulas μ_i such that $[\![\mu_i]\!]$ for all $i \in \{1, ..., n\}$ and a context $\zeta'[\Box]$ such that $[\![\zeta'[\Box]]\!] = C'[\Box]$. We conclude since, w.l.o.g., there is a derivation of the following forms:

$$d^{-\kappa} \frac{ \underset{\vdash}{\overset{\vdash}{}} (\zeta'[\psi'])^{\perp}, \zeta'[\phi']}{ \underset{\vdash}{\overset{\leftarrow}{}} \kappa_{P^{\perp}} \left((\zeta'[\psi'])^{\perp}, \mu_1^{\perp}, \dots, \mu_n^{\perp} \right), \kappa_P \left(\zeta'[\phi'], \mu_1, \dots, \mu_n \right) }$$

We are now able to prove the main result of this section, that is, establishing a correspondence between graphs provable in GS and graphs which are interpretation via $[\cdot]$ of formulas provable in MPL°.

Definition 56. We define the following graphical logics (i.e. sets of graphs):

$$Graphical Multiplicative Logic: GML = \{ \llbracket \phi \rrbracket \mid \phi \text{ formula such that } \vdash_{\mathsf{MPL}} \phi \}$$

$$Graphical IsoMix Logic : GML^{\circ} = \{ \llbracket \phi \rrbracket \mid \phi \text{ formula such that } \vdash_{\mathsf{MPL}^{\circ}} \phi \}$$
(19)

We say that G is *provable* in $X \in \{GML, GML^\circ\}$ (denoted $\vdash_X G$) if there is a formula ϕ such that $\vdash_X \phi$ and $\llbracket \phi \rrbracket = G$.

Theorem 57. Let G be a graph. Then $\vdash_{GS} G$ iff $\vdash_{GML} G$.

Proof. If $\vdash_{GML} G$, then by there is a (compact unit-free) formula ϕ and such that $\vdash_{MPL^{\circ}} \phi$. We conclude by applying Lemma 54 to a given proof of ϕ in MPL^{\circ}.

To prove the converse, let \mathcal{D} be a proof of G in GS. We define a proof $\pi_{\mathcal{D}}$ and a formula $\phi = \llbracket G \rrbracket^{-1}$ (see Remark 52) by induction on the number n of rules in \mathcal{D} .

- If n = 0, then $G = \emptyset$ and $\pi_{\mathcal{D}} = \circ \frac{1}{1 + 0}$.
- If n = 1, then $G = a \Re a^{\perp}$ and $\pi_{\mathcal{D}} = \frac{}{\Re a^{\perp}} \frac{}{+ a \Re a^{\perp}}$.
- If n > 1, then by inductive hypothesis we have a proof π_D of a formula ψ such that [[ψ]] is the premise graph of the last rule r in π_D (which may be applied deep inside a context). By Lemma 55 we can define a proof of φ in MPL° ∪ {cut} as the one below

$$\begin{array}{c} \prod \left| \mathsf{IH} \right| & \prod \left| \mathsf{Lemma 5} \right| \\ \psi & \vdash \psi^{\perp}, \phi \\ \hline \psi & \vdash \phi \end{array} \\ \mathsf{cut} \\ \hline \psi \\ \vdash \phi \\ \end{array}$$

and conclude by Theorem 39.

5 **RB-Proof** Nets

In this section we present a way to encode proofs in MPL and MPL° by means of graphs with two kind of edges. We then provide a calculus operating on these graphs in the style of sequent calculus which is sound and complete with respect to graphs encoding proofs.

For this purpose, we extend the syntax of RB-proof nets introduced by Retoré in his PhD thesis for MLL-proof nets [71, 75]. ⁷The main difference between the syntax for proof nets commonly used in the literature (that is, as the ones used in, e.g., [40, 29, 42] or any of their reformulation which can be found in the literature) and RB-proof nets, is there in the former nodes are labeled by connectives and the edges (called wires) by formulas, while in the latter each connective is represented by a small graph keeping track of the relations between the inputs and the output of the gate (see Figure 8), while wires are represented by a different kind of edges.



Figure 8: The same proof net in the original Girard's syntax and Retoré's one.

However, the idea behind the correctness criterion for MLL-proof nets can be found almost unchanged in the syntax of RB-proof nets. In fact, this correctness criterion for MLL-proof nets checks the absence of elementary cycles in any possible graph obtained by pruning one of the two input wires of each \Im -gate. In RB-proof nets this is captured by simply having no edge connecting the two inputs of a \Im -gate, preventing the existence of an alternating elementary path passing from one input to another. This elegant change in the syntax allows us to still have a criterion based on checking the absence of cycles in a graph, but avoiding the need of checking an exponential number of graphs with respect to the \Im -gates (one for each possible combination of pruning of the inputs of the \Im -gates).

⁷More precisely, in these works the author defines proof nets for Pomset logic including not only undirected edges, but also directed edges connecting the two inputs of a gate representing the non-commutative connective \triangleleft internalizing the pomset order by a logical connective.

The idea for designing RB-proof nets for multiplicative prime logic (with and without mix) comes from the remark that in RB-proof nets the graph induced by the inputs of a graph representing a \mathscr{P} -gate (\otimes -gate) is isomorphic to the prime graph \mathscr{P} (respectively \otimes). We define *G*-gates for any graph *G* by mimicking this construction: consider the vertices of a graph as inputs of a gate where an output and an outgoing wire are attached (see Figure 9).



Figure 9: A \Im -gate, a \otimes -gate, a P₄-gate and an intuitive depiction of a *G*-gate in RB-structures.

5.1 FROM GRAPHS TO **RB**-FORESTS

We start by recalling the definition of *RB-graph*, which are graphs with two kind of (undirected) edges we use to represent RB-structures, and which we use to represent both the decomposition trees of a graph, and encoding of proofs.

Definition 58. An **RB**-graph $G = \langle V_G, \ell_G, \stackrel{G}{\frown}, \stackrel{G}{\bot} \rangle$ is given by a set of vertices V_G , a (partial) labeling function ℓ_G (we denote \emptyset the *empty function*), and two symmetric and non-reflexive edge relations $\stackrel{G}{\frown}$ and $\stackrel{G}{\bot}$ over V_G such that $\stackrel{G}{\bot}$ is a *perfect matching* in the graph $\langle V_G, \ell_G, \stackrel{G}{\frown} \cup \stackrel{G}{\bot} \rangle$ (that is, every vertex in V_G belongs to exactly one edge in $\stackrel{G}{\bot}$), and such that if $v \stackrel{G}{\bot} w$, then $\ell(v) = (\ell(w))^{\perp}$. The edges in $\stackrel{G}{\frown}$ are called R-edges (for red or regular) and the edges in $\stackrel{G}{\frown}$ are called B-edges (for blue or bold). We denote by \emptyset the *empty* RB-graph $\langle \emptyset, \emptyset, \emptyset, \emptyset \rangle$ and we extend to notion of *induced* subgraph to RB-graphs.

Notation 59. When drawing a RB-graph we draw red/regular edges v - w wherever v - w, and blue/bold edges v - w whenever $v \perp w$.

In order to represent tree-structure of the formula tree of a formula or the modular decomposition of a graph, we define *gates* encoding graphical connectives.

Definition 60. Let $G(\{i\})v_1, \ldots, v_n$ be a connective. A *G*-gate (or simply gate) is a RB-graph of the form $\mathfrak{G} = \langle V_{\mathfrak{G}}, \emptyset, \overset{\mathfrak{G}}{\frown}, \overset{\mathfrak{G}}{\bot} \rangle$ with a vertex $i_{\mathfrak{G}}^i$ (its *i*-th *input*) for each vertex in $v_i \in V_G$ plus a vertex $r_{\mathfrak{G}}$ (its *root*) and a vertex $o_{\mathfrak{G}}$ (its *output*), and having a R-edge between the *i*-th and the *j*-th inputs iff $v_i \overset{\mathfrak{G}}{\frown} v_j$, a R-edge between each input and the output and a B-edge between the output and the root. Formally

$$\mathfrak{G} = \left\langle \left\{ \mathbf{i}_{\mathfrak{G}}^{i} \mid v_{i} \in V_{G} \right\} \cup \{\mathbf{o}_{\mathfrak{G}}, \mathbf{r}_{\mathfrak{G}}\} , \left\{ \mathbf{i}_{\mathfrak{G}}^{i} \mathbf{i}_{\mathfrak{G}}^{j} \mid v_{i} \stackrel{G}{\frown} v_{j} \right\} \cup \left\{ \mathbf{i}_{\mathfrak{G}}^{i} \mathbf{o}_{\mathfrak{G}} \mid v_{i} \in V_{G} \right\} , \left\{ \mathbf{o}_{\mathfrak{G}} \mathbf{r}_{\mathfrak{G}} \right\} \right\rangle$$

We denote $ln(\mathfrak{G})$ the set of inputs of \mathfrak{G} and we call a R-edge connecting two inputs of a gate a *connector edge* and the B-edge connecting the output to the root of a gate a *wire*. We say that \mathfrak{G} has *type* of *G* (denoted \mathfrak{G} : *G*) or that \mathfrak{G} is a *G-gate* whenever $|_{\mathfrak{G}} ln(\mathfrak{G}) \sim G$.

we say that \mathbb{G} has *type* of \mathcal{G} (denoted \mathbb{G} . \mathcal{G}) of that \mathbb{G} is a *G*-gate whenever $|_{\mathbb{G}}$ in $(\mathbb{G}) \sim \mathbb{G}$

We define the operation of *gluing* two graphs by identifying some of their vertices.

Definition 61. Let *G* and *H* be RB-graphs with disjoint set of vertices. An *interface X* is a set of pairs $(x, y) \in V_G \times V_H$ such that if $\ell(x)$ and $\ell(y)$ are both defined, then $\ell(x) = \ell(y)$ and such that $x \neq x'$ and $y \neq y'$ for any $(x, y), (x', y') \in X$.

The *gluing* of G and H via an interface X as the RB-graphs $G \bowtie_X H$ obtained by identifying the

vertices occurring in a same pair in in X. Formally $G \bowtie_X H$ has vertices $V_G \cup (V_H \setminus \{y \mid (x, y) \in X\})$, the labeling function defined by the union of the two labeling functions, and a R-edge (resp. B-edge) an edge uv whenever either u and v are both in V_G or both in V_H and $u \frown v$ (resp. $u \perp v$), or $u \in V_G$,

$$v \in V_H$$
 and there is $(v', v) \in X$ such that $u \stackrel{\frown}{} v'$ (resp. $u \stackrel{\frown}{} v'$).
The *disjoint union* of G and H is defined as $G \uplus H := G \bowtie_{\mathcal{D}} H$.

We use the operation of gluing to construct tree-like RB-graph reproducing the abstract syntax tree of a graph (i.e., its modular decomposition via graphical connectives) or of a formula (i.e., its formula tree).

Definition 62. Let G be a graph described by a (possibly spurious) modular decomposition via graphical connectives of a given base Q. The **RB**-tree of G is the RB-graph $\{\!\!\{G\}\!\!\}$ define as follow by induction on the modular decomposition of G using graphical connectives of a base:

• if $G = \circ$, then $\{\!\!\{G\}\!\!\} = \emptyset$;

- if G = x is a single-vertex graph with label x, then {{\$\delta\$}} is the RB-graph made of a single vertex v labeled by x. We say that v is at the same time the *leaf* and a *root* of {{G}};
- if $G = Q(G_1, \ldots, G_n)$, then (G) is the RB-graph obtained by gluing the root of (G_i) with the *i*-th input of a fresh new Q-gate for all $i \in \{1, \ldots, n\}$. That is,

$$\{\!\!\{\phi\}\!\!\} = \mathfrak{G}_Q \bowtie_X \left(\bigcup_{1 \le i \le n} \{\!\!\{\phi_i\}\!\!\}\right)$$

where $X = \{(i_Q^i, r_{\{\!\{\phi_i\}\!\}}) \mid i \in \{1, \dots, n\}\}$. In this case the *root* of $\{\!\{G\}\!\}$ is the vertex r_Q (also denoted r_G). The set of *leaves* of $\{\!\{G\}\!\}$ is given by the union of the set of leaves of $\{\!\{G_1\}\!\}, \dots, \{\!\{G_n\}\!\}$. The set of *gates* of *G* is the set $\mathfrak{Gates}(G) = \{\mathfrak{G}_Q\} \cup (\bigcup_{1 \le i \le n} \mathfrak{Gates}(G_i))$.

The **RB-tree** $\{\!\{\phi\}\!\}$ of a formula ϕ is defined analogously, by considering the formula tree instead of the spurious modular decomposition describing a graph. That is, $\{\!\{\phi\}\!\}$ is the RB-tree of a (spurious modulate decomposition of a) graph G such that $[\![G]\!]^{-1} = \phi$.

A **RB**-forest is a disjoint union of RB-trees. The RB-forest a sequent $\Gamma = \phi_1, \ldots, \phi_n$ is defined as the RB-graph $\{\!\{\Gamma\}\!\} = \{\!\{\phi_1\}\!\} \oplus \cdots \oplus \{\!\{\phi_n\}\!\}$. The set $\Re oot(\{\!\{\Gamma\}\!\})$ of **roots** of $\{\!\{\Gamma\}\!\}$ is the set containing all the roots of $\{\!\{\phi_1\}\!\}, \ldots, \{\!\{\phi_n\}\!\}$. The set of **gates** of $\{\!\{\Gamma\}\!\}$ is the set $\Re otes(\{\!\{\Gamma\}\!\}) = \bigcup_{1 \le i \le n} \Re otes(\{\!\{G\}\!\})$ and we denote by \Re - $\Re otes(\{\!\{\Gamma\}\!\})$ the subset of **root gates** of $\{\!\{\Gamma\}\!\}$, that is, gates whose roots are in $\Re oot(\{\!\{\Gamma\}\!\})$.

Notation 63. When referring to gates we may intend them as the induced subgraphs in a RB-tree, as we do for moldules. Note that gluing identifies roots with inputs of gates, but we may still denote such a vertex using any two of the identified vertices names in order to simplify certain definitions.

Observation. We naturally have correspondences between leaves of $\{\!\{\Gamma\}\!\}\)$ (leaves in $\{\!\{G\}\!\}\)$ and occurrences of literals in Γ (vertices in V_G), between gates of $\{\!\{\Gamma\}\!\}\)$ and occurrences of quasi-prime graphs in the modular decomposition of $[\![\Gamma]\!]\)$ by means of quasi-prime graphs (respectively occurrence of connectives in Γ), and between roots of $\{\!\{\Gamma\}\!\}\)$ and occurrences of formulas in Γ .

In the following section we need to consider RB-forest obtained by pruning certain leaves in a given RB-forest. For this purpose we introduce the following notation.

Notation 64. Let *G* be a RB-tree. If *W* is a subset of leaves of *G*, then the RB-tree $G \clubsuit W$ is defined as the RB-forest of the graph $G(x_1, \ldots, x_n)$ where $x_i = \emptyset$ if $v_i \in W$ and $x_i = v_i$ otherwise.

^aIn the literature of *graphs* and *hypergraphs with interfaces*, interfaces are defined as pairs of bijections from a set of *n* elements to the sets of vertices of two distinct graphs. This definition is equivalent to our definition of interface, as well as our definition of gluing is equivalent to the one of graph composition by pushout.

Example 65. Let $G = P_4(a \otimes b, c \otimes d, e \otimes f, \otimes_3(g, h, i))$ be the graph from Example 23. Its RB-forest is shown below on the left.



The RB-forest above on the right is the RB-forest $\{\!\{G\}\!\} \triangleq \{a, b, c, e, f, i\}$.

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5.2 **RB-GRAPHS REPRESENTING PROOFS**

Intuitively, MLL-proof nets of a formula ϕ are encoded as the formula tree of ϕ decorated with a function pairing occurrences of literals occurring in ϕ satisfying certain topological properties. In RB-graphs, such pairing function is encoded by means of B-edges (called *wires*).

Definition 66. Let Γ be a \mathfrak{P} -sequent. An *axiom linking* Link for Γ is a (total) bijection between occurrences of literals in Γ such that if x is an occurrence of a literal a in Γ , then Link(x) is an occurrence of a literal a^{\perp} . We denote by \sqcap_{Link} the set two-vertices sets $\{v_a, v_{fa}\}$ containing leaves of $\{\!\{\Gamma\}\!\}\)$ paired by Link.

A *RB-structure* of Γ is a **RB-graph** *G* of the form

$$G = \left\langle V_{\{\!\!\{\Gamma\}\!\!\}}, \ell_{\Gamma}, \stackrel{(\!\!\{\Gamma\}\!\!\}}{\frown}, \begin{pmatrix}^{(\!\!\{\Gamma\}\!\!\}}_{\bot} \cup \sqcap_{\mathsf{Link}} \end{pmatrix} \right\rangle \coloneqq \{\!\!\{\Gamma\}\!\!\} \sqcup \mathsf{Link}$$

Gates, leaves, roots and *wires* of *G* are the ones of $\{\!\{\Gamma\}\!\}$. The *(axiom) links* of *G* are B-edges in $\sqcap_{\text{Link.}}$. Let $X \in \{\text{MLL}, \text{MLL}^\circ, \text{MPL}, \text{MPL}^\circ\}$ and let π be a derivation of Γ in X. The *axiom linking of* π (denoted Link_{π}) is defined by the set of pairs of dual literals matched by the ax-rules in π , that is, Link(x) is the unique occurrence of literal such that there is a ax-rule with conclusion $\vdash x$, Link(x) in π . A **X**-*net* is a RB-structure of the form $\{\!\{\pi\}\!\} \coloneqq \{\!\{\Gamma\}\!\}\} \sqcup \text{Link}_{\pi}$ for a proof π of Γ in X.

We define two inference systems for RB-structures in order to characterize families of RB-graph.

Definition 67. We define the following inference systems for RB-graphs using the rules in Figure 10:

$$\begin{aligned} \mathsf{RB}_{\mathcal{P}} \colon \left\{ \mathsf{ax}^{\mathsf{RB}}, \mathfrak{N}^{\mathsf{RB}}, \otimes^{\mathsf{RB}}, \mathsf{d}\text{-}\kappa_{P}^{\mathsf{RB}} \mid P \in \mathcal{P} \setminus \{\mathfrak{N}, \otimes\} \right\} \\ \mathsf{RB}_{\mathcal{Q}}^{\circ} \colon \left\{ \circ^{\mathsf{RB}}, \mathsf{ax}^{\mathsf{RB}}, \mathfrak{N}^{\mathsf{RB}}, \mathsf{mix}^{\mathsf{RB}}, \otimes^{\mathsf{RB}}, \mathsf{d}\text{-}\kappa_{\mathcal{Q}}^{\mathsf{RB}}, \mathsf{s}_{\mathcal{R}}^{\mathsf{RB}} \mid Q \in \mathcal{Q} \setminus \{\mathfrak{N}, \otimes\} \right\} \end{aligned}$$
(20)

We say that a RB-graph G is *provable* in $X \in \{RB_{\mathcal{P}}, RB_{\mathcal{Q}}^{\circ}\}$ (denoted $\vdash_X G$), or simply is *in* X, if there is proof (i.e. a derivation with no open premises) of G in X.

Remark 68. Rules $\mathfrak{P}^{\mathsf{RB}}$, d - κ_Q^{RB} and $\mathsf{s}_{\otimes}^{\mathsf{RB}}$ glue roots of RB-structures to a gate. Thus they subsume the non-emptiness of each of their premises.

We establish a correspondence between proofs in MPL° of a sequent Γ and proofs of RB-structures of the form $\{\!\{\Gamma\}\!\} \sqcup \text{Link in } \mathsf{RB}^\circ_{\mathcal{O}}$.

Theorem 69. Let Γ be a \mathfrak{P} -sequent. Then

1. $\vdash_{\mathsf{MPL}} \llbracket \Gamma \rrbracket \iff \vdash_{\mathsf{RB}_{\mathcal{P}}} \llbracket \Gamma \rrbracket \sqcup \mathsf{Link} \text{ for an axiom link Link for } \Gamma$

2. $\vdash_{\mathsf{MPL}^{\circ}} \llbracket \Gamma \rrbracket \iff \vdash_{\mathsf{RB}^{\circ}_{\mathcal{Q}}} \llbracket \Gamma \rrbracket \sqcup \mathsf{Link} \text{ for an axiom link Link for } \Gamma$

$$s \cdot \kappa_G^{\mathsf{RB}} \xrightarrow{\vdash G_1 \cdots \vdash G_n} \mathfrak{K}_{X_{\mathsf{S}}} (G_1 \uplus \cdots \uplus G_n) \mathfrak{K} : G$$

$$\begin{split} X_{\mathfrak{B}} &:= \left\{ \begin{pmatrix} \mathfrak{i}_{6}^{i}, \mathfrak{r}_{i} \end{pmatrix} \mid i \in \{1, \dots, n\} \text{ and } \mathfrak{r}_{1}, \dots, \mathfrak{r}_{n} \text{ distinct roots in } \mathfrak{Root}(G) \right\} \\ X_{\mathsf{s}} &:= \left\{ \begin{pmatrix} \mathfrak{i}_{6}^{i}, \mathfrak{r}_{i} \end{pmatrix} \mid i \in \{1, \dots, n\} \text{ and } \mathfrak{r}_{i} \in \mathfrak{Root}(G_{i}) \right\} \\ X_{\mathsf{d}} &:= \left\{ \begin{pmatrix} \mathfrak{i}_{6}^{i}, \mathfrak{r}_{i}^{1} \end{pmatrix}, \begin{pmatrix} \mathfrak{i}_{5}^{i}, \mathfrak{r}_{i}^{2} \end{pmatrix} \mid i \in \{1, \dots, n\} \text{ and } \mathfrak{r}_{i}^{1}, \mathfrak{r}_{i}^{2} \text{ distinct roots in } \mathfrak{Root}(G_{i}) \right\} \\ X^{+} &:= \left\{ \begin{pmatrix} \mathfrak{i}_{6_{\mathcal{O}}}^{i}, \mathfrak{r}_{i}^{1} \end{pmatrix}, \begin{pmatrix} \mathfrak{i}_{5}^{i}, \mathfrak{r}_{i}^{2} \end{pmatrix} \mid i \in \{1, \dots, n\} \text{ with } \mathfrak{r}_{k} \in \mathfrak{Root}(G_{2}) \text{ and } \mathfrak{r}_{1}, \dots, \mathfrak{r}_{k-1}, \mathfrak{r}_{k+1}, \dots, \mathfrak{r}_{m} \text{ distinct roots in } \mathfrak{Root}(G_{2}) \right\} \\ X^{-} &:= \left\{ (\mathfrak{x}_{i}, \mathfrak{r}_{i}) \mid i \in \{1, \dots, n\} \setminus \{k\} \text{ with } \begin{pmatrix} \mathfrak{i}_{6_{\mathcal{O}}}^{i}, \mathfrak{r}_{i} \end{pmatrix} \in X^{+} \text{ and } \mathfrak{x}_{i} \text{ the leaf of } \mathfrak{G}_{\mathcal{O}} \bigstar[\mathfrak{i}_{6_{\mathcal{O}}}^{k}] \text{ corresponding to } \mathfrak{v}_{i} \in V_{\mathcal{O}(\mathfrak{v}_{1}, \dots, \mathfrak{v}_{n})} \right\} \end{split}$$

Figure 10: Inference rules of the system RB_Q° and the derivable rule \mathbf{s} - κ_G^{RB} , where *P* is a prime graph.

Sketch of proof. Both implication are proven by induction on the size of a derivation using the correspondence between rules in MPL° and RB^o_Q. We here highlight some relevant details of the proof:

- (⇒) The (implicit) non-emptiness condition on the rules in RB^o_Q observed in Remark 68 requires particular care in the inductive definition in presence of units (thus only for the translation of MPL°). For this purpose, the rule s_{\otimes}^{RB} plays a crucial role since it allows to simulate specific instances of wd_⊗.
- (\Leftarrow) Each instance of $s_{\otimes}^{\mathsf{RB}}$ corresponds to an instance of wd_{\otimes} , which is admissible in MPL°. For the other rules there is a one-to-one correspondence between the two systems.

Details of the proof are provided in Appendix B.

6 A Correctness Criterion for MLL°-nets

In this section we recall Retoré's topological characterization RB-graphs encoding proofs in MLL and MLL° and we show where where this criterion fails for MPL°-nets.

6.1 Æ-Connectedness in **RB**-graphs

We recall here the topological notions required to formulate the correctness criterion for RB-structures encoding proofs in MLL and MLL° in terms of connectness and acyclicity of RB-structures with respect to *alternating-elementary paths*.

Definition 70. Let $G = \langle V, \ell, \frown, \bot \rangle$ be a RB-graph. An *alternating path* is a path $p = v_0 \cdots v_n$ in the graph $\langle V, \ell, \frown \cup \bot \rangle$ such that $v_i \frown v_{i+1}$ iff $v_{i+1} \bot v_{i+2}$ and such that $v_i \bot v_{i+1}$ iff $v_{i+1} \frown v_{i+2}$ for all $i \in \{0, \ldots, n\}$. We say that such an æ-path *connects* v_0 with v_n , and that it *covers* the vertices v_i for all $i \in \{1, \ldots, n\}$ and the edges $v_i v_{i+1}$ for all $i \in \{0, \ldots, n-1\}$.

An *æ*-path is an alternating path which is also *elementary*, that is, such that a vertex occurs at most once in the *æ*-path. If X, Y \in {R, B}, a **XY**-path is an *æ*-path $v_0 \cdots v_n$ such that v_0v_1 is a X-edge and $v_{n-1}v_n$ is a Y-edge. We say that two vertices v and w of G are *æ*-connected if there is an *æ*-path connecting them, a RB-graph is *æ*-connected if any two of its vertices are *æ*-connected.

An *æ-cycle* is an *æ*-path $c = v_0 \cdots v_{2n}$ such that $v_0 = v_{2n}$. Note that we consider *æ*-cycles modulo cyclic permutations of the indices, that is, we identify the *æ*-cycle $v_0 \cdots v_{2n-1} \cdot v_0$ with the *æ*-cycle $v_i \cdots v_{2n-1} \cdot v_0$ with the *æ*-cycle $v_i \cdots v_{2n-1} \cdot v_0 \cdots v_i$ for any $i \in \{0, \dots, 2n-1\}$. A *chord* of *c* is a R-edge $v_h v_k$ with $h, k \in \{0, \dots, 2n\}$ with k > h + 1. It is a *shortcut* if there is a BB-path from v_k to v_h which is a sub-sequence of *c* and such that $v_h \cdots v_k \cdot v_h$ is an *æ*-cycle. The set of *æ*-cycles of *G* is denoted *Æ*(*G*).

Notation 71. When drawing a RB-graph we draw v - w if there is an æ-path between v and w. Whenever we want to point at a specific path or an induced subgraph, we highlight the vertices and the edges it covers as follow v - w.

Remark 72. An æ-cycle of a RB-structure covering a connector edge and the output of a gate has a shortcut. This can be observed in the example below on the left where the non-connector edge between the rightmost input and the output of the gate is a shortcut (dotted lines represent possible R-edges).



Note that also the æ-cycle above on the right has a shortcut $i_{6}^{J} i_{6}^{h}$

We then formalize in this framework two intuitive notions we use in the next sections. The first is simply a formalization of the idea that in RB-forests we represent roots are at the bottom of our forests.

Definition 73. Let $G = \langle V, \ell, \frown, \bot \rangle$ be a RB-forest or a RB-structure. We say that a vertex *v* is *above* a vertex *w* (*w* is *below v*) if there is a æ-path from *w* to a leaf of *T* passing through no connector edges covering *v*. Similarly, a gate \mathfrak{G} is *above* (*below*) a gate \mathfrak{G}' if its output is below the output of \mathfrak{G}' .

We then define the notion of *g*-*path*, allowing us to define a notion of connectess for RB-structures similar to the one used in standard MLL-proof nets, that is, where paths are sequences of wires connected by a gate.

Definition 74. We write $v \nabla w$ if $v \neq w$ are vertices in a same gate, that is, if there is a gate $\mathfrak{G} \in \mathfrak{Gates}(G)$ such that $v, w \in \mathsf{ln}(\mathfrak{G}) \cup \{\mathsf{o}_{\mathfrak{G}}\}$.

A *g-path* from *v* to *w* in *G* an alternating elementary path $p = v_0 \cdots v_n$ in the RB-graph $\langle V, \ell, \nabla, \bot \rangle$ with n > 2 such that there are at most two $i, j \in \{1, ..., n\}$ with $i \neq j$ such that $v_i \nabla v_j$. The notions of *g-connectness* and of *g-connected component* are defined in the standard way (see Definition 1), using g-paths instead of paths (or x-paths).

Lemma 75. Let G be a RB-structure. Then $G = G_1 \uplus G_2$ iff there are no g-paths connecting vertices in G_1 with vertices in G_2 .

Proof. The non-trivial implication follows by definition: the absence of such g-paths implies the absence of B-edge or gates containing at the same time vertices in G_1 and in in G_2 .

6.2 A TOPOLOGICAL CHARACTERIZATION OF MLL AND MLL°

We recall Retoré's characterization of those RB-structure which are encoding of proofs in MLL° via {{-}}.

Theorem 76 ([75]). Let $G = \{\!\!\{\Gamma\}\!\!\} \sqcup \text{Link be a RB-structure of a sequent } \Gamma \text{ of MLL-formulas. Then }$

- 1. *G* is a MLL-net iff *G* is α -connected and $\mathcal{E}(G) = \emptyset$;
- 2. *G* is a MLL°-net iff $\mathcal{E}(G) = \emptyset$.

The proof of this theorem can be reconstructed by the one using the Danos-Regnier switching criterion [30] for standard MLL-proof nets. For the reader familiar with the terminology of *Danos-Regnier* switching for MLL proof nets [30], the idea is that æ-paths in a RB-structure of a MLL-formula are exactly the paths which may be observed in a *test* of the proof net. In fact, æ-paths can only pass at most once through each \mathcal{P} - or \otimes -gate, thus an æ-path may only pass through one input of a \mathcal{P} -gate to its output; this can be interpreted as if a switch has been applied a switching selecting the input occurring in the path. Details can be found in [72, 36, 66].

Remark 77. In a RB-structure G such that any gate is a \Im - or a \otimes -gate, any æ-cycle is chordless because gates have only two inputs and one output.

However, it is easy to find graphs which are provable in GML, but not satisfying this criterion: any graph of the shape $P \multimap P$ for a prime graph $P \notin \{\Re, \otimes\}$ is provable in GML (see Theorem 35) but the RB-structure representing such a proof has æ-cycles.



More in general, we remark that during the construction of a MPL°-nets using the rules in RB_Q° , æ-cycles can be introduced only by an instance of d- κ_Q^{RB} and those æ-cycles always cover the P₄'s over the inputs of the gates in its conclusion which are not in its premises.

7 Generalizing the Correctness Criterion to MPL and MPL°

In this section identify a topological characterization of those RB-structures which are MPL°-nets by means of a *correctness criterion*, and we define a *sequentialization procedure* allowing us to reconstruct a proof in MPL°. For this purpose, we isolate a family of æ-cycles allowing us to retrieve all the information witnessing the correct application of d- κ -rules. This result is possible thanks to results on the *primeval* decomposition of graphs [56] allowing us to further characterize prime graphs by specific topological properties we recall in the next subsection.

We then provide a method inspired by the *sequential edges* introduced in *C-nets* [33] in order to recover partial information about possible order in which the connectives (or more precisely, the rules introducing them) can be sequentialized. The correctness criterion is obtained by combining this order with a refinement of Retoré's criterion (via the absence of specific æ-cycles, as theorized in [67] for *coherent interaction graphs*).

7.1 CONNECTEDNESS AND P-CONNECTNESS IN GRAPHS

The notion of modular decomposition have been refined in [15] by underlying the importance of the induced subgraphs isomorphic to P_4 . Due to their importance, we introduce the following convention.

Notation 79. We say that a quadruple $\langle a, b, c, d \rangle$ of four vertices of a graph *G* is *a* **P**₄ of *G* if $G|_{\{a,b,c,d\}} = \mathsf{P}_4(\{a,b,c,d\})$. A P₄ of a RB-graph is a P₄ containing only R-edges.

Definition 80. In a graph $P_4(a, b, c, d)$ we call a and d its *end-points*, b and c its *mid-points*, the edge bc is its *mid-edge*, and the edges ab and cd are its *end-edges*.

 $^{^{}a}$ We here put emphasis on the presence of chords because the absence of chordless æ-cycles allows us to provide a correctness criterion for the different encoding of MLL-nets via the RB-structures discussed in Section 9.

We recall that a graph G is *connected* if there is a path connecting any pair of vertices. This definition is equivalent to require that for any partition of V_G into disjoint two non-empty sets V_1 and V_2 there is a *crossing* P_2 , that is, an induced subgraph isomorphic to P_2 with vertices in both V_1 and V_2 . In this paper we are interested in the generalization of this alternative definition using P_4 instead of P_2 .

Definition 81 ([56]). A graph G is **p**-connected if for any partition of V_G into disjoint two non-empty sets V_1 and V_2 there is a crossing P₄, that is, there is a P₄ of G with vertices in both V_1 and V_2 . A **p**-component of G is a maximal p-connected subset of V. A p-component V' of G is separable if there is a partition of V' in two disjoint subsets V_1 and V_2 such that every P₄ in G has middle points in V_1 and end-points in V_2 . Such a partition is denoted $\langle V_1 | V_2 \rangle$ and is called a separation of V'.

Notation 82. As for modules, we may identify a p-component with its induced subgraph. Moreover, we may identify a separable p-component with its separation.

Proposition 83 ([56]). Let G be a graph. Then G is p-connected iff any two vertices $v, w \in V_G$ admit a p-chain from v to w, that is, a path $u = v_1, \ldots, v_n = w$ such that $\langle v_i, v_{i+1}, v_{i+2}, v_{i+3} \rangle$ is a P₄ of G for all $i \in \{1, \ldots, n-3\}$.

The following result is a consequence of a more general result known as *Structure Theorem* [56] and results on separable p-connected graphs (for a survey on the topic see [15]).

Theorem 84. Let P be a prime graph which is not a clique or a stable set. If P is not p-connected, then P has a unique separable p-component $V' = \langle K | S \rangle$ such that $V_P = V' \uplus \{w_P\}$ and such that

- *if* $v \in K$, then v is a mid-point of a P_4 in P and $v \frown w_P$;
- *if* $v \in S$, *then* v *is a end-point of a* P_4 *in* P *and* $v \not\frown w_P$.

The vertex w_P is called the **weak vertex** of *P*, and the set of vertices *K* and *S* are called the **strong** component and the stable component of *P* respectively.

We conclude this section by providing some additional lemmas required for the proofs in the rest of this section.

Lemma 85. Let P be a non p-connected prime graph with weak vertex w_P and let G be a graph obtained by removing from P some edges (at least one) containing w_P from P. If G is connected, then G is p-connected.

Proof. We first observe that if vw are p-connected in P, then they are in G. Moreover, if G is connected, then there is a $v \in S$ connected to w_P , thus there is a P_4 of the form $\langle w, v, u, t \rangle$ in P. We conclude by Proposition 83 since for any $u \in S$ either $\{w_P, v, u\}$ induces a P_3 in G, or there is a $v_u \in K \setminus \{v\}$ such that $\langle w_P, v, v_u, u \rangle$ is a P_4 of G.

Remark 86. If $\langle a, b, c, d \rangle$ is a P₄ of a graph G, then $\langle c^{\perp}, a^{\perp}, d^{\perp}, b^{\perp} \rangle$ is a P₄ of G^{\perp} .

Lemma 87. Let $P = P(v_1, \ldots, v_n)$ and $P' = P'(v_1^{\perp}, \ldots, v_n^{\perp})$ be prime graph such that $\langle v_i, v_j, v_h, v_k \rangle$ is a P_4 of P iff $\langle v_j^{\perp}, v_k^{\perp}, v_i^{\perp}, v_h^{\perp} \rangle$ is a P_4 of P'. Then $P' = P^{\perp}$.

Proof. If *P* is p-connected, then for any $u, v \in V_P$ such that $u \stackrel{P}{\frown} v$ there is a P₄ in *P* of the form, w.l.o.g., $\langle u, v, w, t \rangle$ or $\langle w, u, v, t \rangle$. Then $u^{\perp} \stackrel{P'}{\frown} v^{\perp}$ since, by hypothesis, there is a P₄ in *P'* of the form $\langle w^{\perp}, u^{\perp}, t^{\perp}, v^{\perp} \rangle$ or $\langle w, u, v, t \rangle$ respectively.

Otherwise by Theorem 84 know that *P* is not p-connected and we repeat the same argument above for the vertices of its p-component. Moreover, we know that the weak vertex w_P of *P* is connected to each vertex in the strong component of *P*. If the same does not hold for the vertex w_P^{\perp} in P^{\perp} , then by

Lemma 85 P^{\perp} admits a P₄ containing w_P . This is impossible (see Remark 86) since P admits no P₄ containing w_P .

7.2 A Correctness Criterion for MPL

From the results on p-connectedness of prime graphs in the previous subsection, we deduce that any prime graph different from \Re and \otimes is "tiled" by P₄'s except for at most one vertex (the weak vertex of a non p-connected prime graph). This allows us to isolate a family of æ-cycles we can use to check whether two gates can be sequentialized by a same d- κ_Q^{RB} -rule (i.e. if they are gates with dual type). Moreover, we prove that from these æ-cycles we can also extract information about whether two gates of dual type can eventually be sequentialized at the same time.

Definition 88. Let *G* be a RB-structure, $\mathfrak{G} \in \mathfrak{Gates}(G)$, and $\mathfrak{c} \in \mathcal{E}(G)$. The RB-subgraph *induced by* \mathfrak{c} in *G* (in \mathfrak{G}) is defined as the RB-graph $G|_{\mathfrak{c}}(\mathfrak{G}|_{\mathfrak{c}})$ induced by the vertices of *G* (of \mathfrak{G}) covered by \mathfrak{c} . A vertex v of *G* is **p**-covered if there is a $\mathfrak{c} \in \mathcal{E}^m(G)$ such that v belongs to a P₄ in $G|_{\mathfrak{c}}$.

The \mathfrak{E} -cycle \mathfrak{c} is *minimal* if it contains no shortcuts and if for any $\mathfrak{G} \in \mathfrak{Gates}(G)$ the graph $\mathfrak{G}|_{\mathfrak{c}}$ is isomorphic to \emptyset or \otimes or \mathbb{P}_4 . The set of minimal \mathfrak{E} -cycles of G is denoted $\mathfrak{E}^m(G)$.



In order to ensure that gates whose type is not \Im or \otimes can eventually be sequentialized using a d- κ -rule, we need have a criterion ensuring us the possibility of pairing those gates in such a way paired gates not only have dual types, and their inputs are connected in a proper way with respect of this duality.

Definition 90. Let *G* be a RB-structure. An *entailing relation* is a bijection \heartsuit over the set of inputs of gates of *G* whose type is not \Im or \bigotimes such that the following hold:

- 1. \heartsuit is an involution, that is, $\heartsuit(\heartsuit(v)) = v$;
- 2. if $v \in \mathfrak{G}_1$ and $\mathfrak{P}(v) \in \mathfrak{G}_2$, then $\mathfrak{G}_1 \neq \mathfrak{G}_2$;
- 3. for any P_4 over the inputs of \mathfrak{G}_1 , there is an *entangling* $\mathfrak{c} \in \mathbb{A}^m(G)$ covering it, that is, a minimal æ-cycle \mathfrak{c} such that $G|_{\mathfrak{c}}$ is of the following form

$$\begin{array}{c} w_{1} - v_{1} & v_{2} - w_{2} \\ w_{3} - v_{3} & v_{4} - w_{4} \end{array} \quad \text{where} \quad \begin{cases} \langle v_{1}, v_{2}, v_{3}, v_{4} \rangle & \text{is a } \mathsf{P}_{4} \text{ in } \mathfrak{G}_{1}, \\ \langle w_{3}, w_{1}, w_{4}, w_{2} \rangle & \text{is a } \mathsf{P}_{4} \text{ in } \mathfrak{G}_{2}, \\ \heartsuit(v_{i}) = w_{i} & \text{for all } i \in \{1, 2, 3, 4\}. \end{cases}$$

$$\begin{array}{c} \langle (24) \\ (24)$$

We say that two gates \mathfrak{G}_1 and \mathfrak{G}_2 are *entangled* (denoted $\mathfrak{G}_1 \diamond \mathfrak{G}_2$) iff all their inputs are. We denote by $\mathbb{A}^{\diamond}(G)$ the set of minimal entangling cycles in $\mathbb{A}^{\mathsf{m}}(G)$. We say that \diamond is *simply entangling* iff $\mathbb{A}^{\mathsf{m}}(G) = \mathbb{A}^{\diamond}(G)$ and for all $\mathfrak{c} \in \mathbb{A}^{\diamond}(G)$ the graph $G|_{\mathfrak{c}}$ contains extactly two P_4 's.

Remark 91. If \mathfrak{G}_1 and \mathfrak{G}_2 are two entangled gates of a RB-structure G, then, as consequence of Lemma 87, their types are dual. Said differently, for each $v, v' \in \mathfrak{G}_1$, if $v \frown v'$ (resp. $v \not\frown v'$) in $G|_{\mathsf{ln}(\mathfrak{G}_1)}$, then $\heartsuit(v) \not\frown \heartsuit(w')$ (resp. $v \frown v'$) in $G|_{\mathsf{ln}(\mathfrak{G}_2)}$.

Example 92. Consider the RB-structures in Figure 11. The P₄-gate in the RB-structure on the left is not covered and the unique æ-cycle in $\mathbb{A}^{m}(G)$ which is not in $\mathbb{A}^{\heartsuit}(G)$. In the the right, the P₄'s containing the vertex w^{\perp} in the P₅-gate are not p-covered.



Figure 11: Examples of RB-structures whose gates cannot be entangled. The highlighted P_4 's are not p-covered



Figure 12: An example of a RB-structure where \prec_{\heartsuit} is not well-founded: $\mathfrak{G}_1 \prec_{\heartsuit} \mathfrak{G}_4$ and $\mathfrak{G}_4 \prec_{\heartsuit} \mathfrak{G}_1$.

However, this condition is not enough tu guarantee sequentializability (see Figure 13). In fact, since the rule $d_{\kappa}^{\mathsf{RB}}_{Q}$ sequentializes two gates at a time, we need to check whether a pair of entangled gates can eventually both occurs as root gates during the sequentialization procedure. This condition is equivalent of checking whether we can formulate a version of the splitting lemma (Lemma 44) for the system $\mathsf{RB}_{\mathcal{P}}$.

For this purpose, we enrich the partial order over gates given by the *below* relation between vertices in a RB-structure with the minimal set of constraints on the order in which cycles could be removed during proof search. This order provides the same information of a minimal set of *sequential edges* in a *C-nets* (see [33]), but without modifying the structure of our RB-graphs to accommodate additional edges. In fact, the information of this order can be solely defined in function of the æ-paths in the RB-structure.

Definition 93. Let *G* be a RB-structure, \heartsuit is a entangling relation for *G*, and $\mathfrak{G}_1, \mathfrak{G}_2 \in \mathfrak{Gates}(G)$. We say that \mathfrak{G}_1 is *precede* \mathfrak{G}_2 (denoted $\mathfrak{G}_1 \prec_{\heartsuit} \mathfrak{G}_2$) whenever \mathfrak{G}_1 is below \mathfrak{G}_2 or below a vertex of an æ-cycle in $\mathcal{E}^m(G)$ covering a P_4 of \mathfrak{G}_2 .

The transitive closure of the relation \prec_{\heartsuit} is a (strict) order over $\mathfrak{Gates}(G)$ (see Figure 12).

Using \prec_{\heartsuit} wa are able to characterize "statically", i.e., without performing an attempt of sequentialization, those pairs of entangled gates which can eventually be removed in such a way the premises of a sound application of a d- κ_Q^{RB} , that is, in such way if we remove both the two entangled gates we split the RB-structure into a set of disjoint RB-structures.

Definition 94. A RB-structure *G* is **MPL**-*correct* iff *G* is æ-connected, there is a simply entangling relation \heartsuit over $\mathfrak{Gates}(G)$ such that \prec_{\heartsuit} is well-founded.

Remark 95. The correctness criterion for MPL subsumes Retoré's criterion for MLL since $\heartsuit = \emptyset$ and the RB-structure of a sequent of MLL-formulas contains no P₄'s.

Example 96. The RB-structure in the left-hand side of Figure 13 contains a unique æ-cycle which should entangle a P₄ and a Bull. This is impossible since $|V_{P_4^{\perp}}| = 4 \neq 5 = |V_{Bull}|$. Similarly in the RB-structure on the right-hand side of Figure 11 we cannot have entangling relations.

The RB-structure in the right-hand side of Figure 13 contains a unique æ-cycle, but it induces four P_4 's, therefore it cannot be simply entangling.

The RB-structure in Figure 12 has an entangling relation which is simply entangling.



Figure 13: Examples of non-correct RB-structures: the entangling relation defined by Link is not simply entangling.

Theorem 97. Let Γ be a sequent. Then

 $\vdash_{\mathsf{MPL}} \Gamma \iff there is a \mathsf{Link} such that G = \{\!\!\{\Gamma\}\!\!\} \sqcup \mathsf{Link} is \mathsf{MPL}\text{-}correct$

Proof. If $\vdash_{\mathsf{MPL}} \Gamma$, then there is a derivation π of Γ and we can define a derivation of $G = \{\!\{\Gamma\}\!\} \sqcup \mathsf{Link}_{\pi}$ in $\mathsf{RB}_{\mathcal{P}}$ by induction on the rules in a derivation in MPL. To conclude it suffices to check that each rule preserves correctness, that is, if all its premises are MPL-correct, then also its conclusion is.

- the conclusion of a rule ax^{RB} is MPL-correct;
- the conclusion G of a rule 𝔅^{RB} contains no new æ-cycles with respect to its premise G₁, that is, Æ^m(G) = Æ^m(G₁). We conclude since the order ≺_♡ over 𝔅ates(G) is well-defined iff it also is over 𝔅ates(G₁);
- the conclusion *G* of a rule \otimes^{RB} contains no new \mathfrak{E} -cycles with respect to its premises G_1 and G_2 , that is, $\mathcal{E}^{\mathsf{m}}(G) = \mathcal{E}^{\mathsf{m}}(G_1) \cup \mathcal{E}^{\mathsf{m}}(G_2)$. We conclude since the order $<_{\heartsuit}$ over $\mathfrak{Gates}(G)$ is well-defined iff it also is over $\mathfrak{Gates}(G_1)$ and $\mathcal{E}^{\mathsf{m}}(G) = \mathcal{E}^{\heartsuit}(G)$ iff both $\mathcal{E}^{\mathsf{m}}(G_1) = \mathcal{E}^{\heartsuit}(G_1)$ and $\mathcal{E}^{\mathsf{m}}(G_2) = \mathcal{E}^{\heartsuit}(G_2)$;
- the conclusion G of a rule d-κ_Q^{RB} contains new æ-cycles with respect to its premises G₁,...,G_n and G₂, but they are all simply entangling. We conclude similarly to the previous case, since both new gates can only be below of some gates in ∪_{i∈{1,...,n}} Gates(G_i);

To prove the converse we provide a sequentialization procedure returning a derivation π_G in MPL defined by induction on number of gates in $\mathfrak{Gates}(G)$ and links in \sqcap_G : we apply ax^{RB} and $\mathfrak{P}^{\mathsf{RB}}$ whenever possible, and \otimes^{RB} whenever there is a \otimes -gate in $\mathfrak{Root}(G)$ whose inputs are not covered by any cycle in $\mathfrak{K}^{\mathsf{m}}(G)$, and a d- κ_Q^{RB} -rules whenever the correctness ensures the presence of two entangled root-gates.

More precisely, we define a proof π_G from roots to leaves as follows:

- 1. if $\mathfrak{Gates}(G) = \emptyset$, then $G = \langle \{a, a^{\perp}\}, \emptyset, \{aa^{\perp}\} \rangle$ and π_G is an instance ax^{RB} ;
- 2. if \Re -Gates(G) contains a \Re_n -gate, then $G = \Im_{\Re_n} \bowtie_{X_{\Re}} G_1$ and

$$\pi_G = \frac{\pi_1 \prod H}{\Pr G_1}$$

$$= \frac{F_G}{\Pr G_1} \xrightarrow{F_G} F_G$$

where π_1 is defined by inductive hypothesis since $|\Box_G| = |\Box_{G_1}|$ but $|(\mathfrak{Gates}(G_1)| < |\mathfrak{Gates}(G)|$, and the correctness of G_1 is guaranteed by the fact that $\mathcal{A}^m(G) = \mathcal{A}^m(G_1)$;

3. if no gate in \Re - $\mathfrak{Gates}(G) \neq \emptyset$ is a \mathfrak{N}_n -gate and at there is a \otimes -gate in \Re - $\mathfrak{Gates}(G)$ which is not covered by a cycle in $\mathcal{R}^m(G)$, then there is a $r_1 \in \mathfrak{Root}(G_1)$ and a $r_2 \in \mathfrak{Root}(G_2)$ such that

$$\pi_{G} = \underset{\otimes^{\mathsf{RB}}}{\overset{\pi_{1} \prod ||\mathcal{H}|}{\vdash} G_{1}} \underset{\overset{\pi_{2} \prod ||\mathcal{H}|}{\vdash} G_{2}}{\overset{\pi_{1} \prod ||\mathcal{H}|}{\vdash} G_{2}}$$



Figure 14: A proof in MPL° where the connective κ_{P_9} has been introduced by a s_{\otimes} , and the corresponding derivation in RB^o_Q where we highlighted the two residual components of the P₉-gate.

where the two proofs π_1 and π_2 are defined by inductive hypothesis since $\Box_G = \Box_{G_1} \uplus \Box_{G_2}$ and the correctness of G_1 and G_2 is guaranteed by the fact that $\mathscr{E}^m(G) = \mathscr{E}^m(G_1) \uplus \mathscr{E}^m(G_2)$;

Otherwise, we can assume ℜ-֍ates(G) contains no ℜ-gates, or ⊗-gates whose inputs are not covered by a cycle in Æ[♡](G). Therefore we must have two entangled gates 𝔅, 𝔅' ∈ ℜ-𝔅ates(G) in ℜ-𝔅ates(G) because <_𝔅 is well-founded.

By Remark 91, the types of these gates are connectives. and $G = (\mathfrak{G}_P \uplus \mathfrak{G}_{P^\perp}) \bowtie_X G'$. In order to be sure that such a RB-structure is the conclusion of a $d \cdot \kappa_Q^{\mathsf{RB}}$, it suffices to prove that $G' = G_1 \uplus \cdots \uplus G_n$. This is equivalent to check that there are no \mathfrak{x} -paths from $r_{G_i}^1$ to any $r_{G_j}^1$ or $r_{G_j}^2$ whenever $i \neq j$; However, if such a path existed, then we should have a cycle shortcut for one of the \mathfrak{x} -cycles in $\mathcal{A}^{\mathsf{m}}(G)$ covering a P_4 in \mathfrak{G}_1 or in \mathfrak{G}_2 .

Thus we conclude since we have

$$\pi_{G} = \frac{\pi_{1} \| \mathbb{H}}{\operatorname{d-}\kappa_{Q}^{\operatorname{RB}}} \frac{G_{1}}{\vdash (\mathfrak{G}_{P} \uplus \mathfrak{G}_{P^{\perp}}) \bowtie_{Y_{n}} (G_{1} \uplus \cdots \uplus G_{n})}$$

where π_1, \ldots, π_n are defined by inductive hypothesis since $\Box_G = \bigcup_{i=1}^n \Box_{G_i}$ and the correctness of G_1, \ldots, G_n is guaranteed by the fact that $\mathscr{R}^m(G) \supseteq \bigcup_{i=1}^n \mathscr{R}^m(G_i)$.

7.3 A Correctness Criterion for MPL°

As shown in Figure 14, in MPL° the rule s_{\otimes} could introduce (top-down) an occurrence of a new connective from a formula containing smaller ones. This prevent us to define a correctness criterion reasoning directly on the gates of a RB-structure since some of them could be deconstructed by a s_{\otimes}^{RB} , splitting a gate into smaller ones. For this purpose, we define the residual components allowing us to spot in a RB-structure the connectives originally introduced by d- κ .



 \mathfrak{G}_1 is the P_5 -gate containing *a* \mathfrak{G}_2 is the P₄-gate containing c \mathfrak{G}_3 is the P₅-gate containing e^{\perp} \mathfrak{G}_4 is the \mathfrak{P} -gate containing g \mathfrak{G}_5 is the \mathfrak{P}_4 -gate containing $\mathbf{r}_{\mathsf{P}_4}$

 \mathfrak{G}_6 is the \mathfrak{P} -gate containing h

 \mathfrak{G}_7 is the bottommost \mathfrak{P} -gate

- $\mathfrak{G}_5 \prec_{\heartsuit} \mathfrak{G}_1$ and $\mathfrak{G}_5 \prec_{\heartsuit} \mathfrak{G}_2$ because of \prec $\mathfrak{G}_3 \prec_{\heartsuit} \mathfrak{G}_4$ because of \prec
- $\mathfrak{G}_7 \prec_{\heartsuit} \mathfrak{G}_3$ because of \prec
- $\mathfrak{G}_3 \prec_{\heartsuit} \mathfrak{G}_5$ because of \prec
- - $\mathfrak{G}_7 \prec_{\heartsuit} \mathfrak{G}_3$ and $\mathfrak{G}_7 \prec_{\heartsuit} \mathfrak{G}_6$ because of \prec
 - $\mathfrak{G}_7 \prec_{\mathfrak{V}} \mathfrak{G}_1$ because $\mathfrak{G}_3 \prec \mathfrak{G}_5$ and there is a cycle in $\mathcal{\mathbb{R}}^{\mathsf{m}}(G)$ covering \mathfrak{G}_1 and \mathfrak{G}_5
 - $\mathfrak{G}_7 \prec_{\mathfrak{T}} \mathfrak{G}_2$ because $\mathfrak{G}_3 \prec \mathfrak{G}_5$ and there is a cycle in $\mathcal{\mathbb{A}}^{\mathsf{m}}(G)$ covering \mathfrak{G}_2 and \mathfrak{G}_5

Figure 15: A RB-structure where the residual components are highlighted and the lattice of the \prec_{\circ} relation between gates.

Definition 98. Let $G = \{\!\{\Gamma\}\!\} \sqcup \text{Link be a RB-structure. A input } v \text{ of a gate } \mathfrak{G} \in \mathfrak{Gates}(G) \text{ is a graft if }$ there is no BB-path from v to any other input of the same gate.

The *residual* of G is the graph induced by the inputs of the gates in G which are not grafts and the R-edges of G (we assume the labelling function to be empty). We denote by $\Re \mathfrak{es}(G)$ the set of *residual components* of G, that is, the set of connected component of the residual of G.

Remark 99. Intuitively, grafts witness the an application of $S_{\otimes}^{\mathsf{RB}}$ (or \otimes^{RB}) while the residual of *G* identify the type of the gates which have been introduced by a d- κ_Q^{RB} -rule. Therefore, removing grafts we are able to split those gates which may have been "merged" by a S_{\otimes}^{RB} .

For an example, see Figure 14 where the P_9 -gate in the RB-proof net in the conclusion of the derivation in RB_Q° has been introduced by a s_{\otimes}^{RB} -rule merging two P_4 -gates in the RB-structure (this because $\Re es(\mathfrak{G}_{\mathsf{P}_9} \clubsuit \left\{ i_{\mathfrak{G}_{\mathsf{P}_9}}^5 \right\}) \sim (\mathsf{P}_4 \ \mathfrak{P} \oslash \ \mathfrak{P}_4)).$

Remark 100. Since in MLL°-nets we only have \Re - and \otimes -gates, then by æ-acyclicity each input of a ⊗-gate must be a graft.

We refine the notion of *entanglement* defining it on residual components instead of gates, in order to ensure sequentializability of a residual component which, during sequentialization, eventually become a gate which should have been introduced by a $d - \kappa_Q^{\mathsf{RB}}$ (or a \otimes^{RB}).

Definition 101. Two residual components R_1 and R_2 of a RB-structure G are *entangled* (denoted $R_1 \heartsuit R_2$) iff there is a total bijection \heartsuit between the vertices in R_1 and R_2 (we denote $v \heartsuit w$ if $v = \heartsuit(w)$) such that it satisfies conditions (1)-(3) in Definition 90 formulated on residual components instead of on gates. An entangling relation is *fully entangling* iff $\mathbb{A}^{\heartsuit}(G) = \mathbb{A}^{\mathsf{m}}(G)$.

Given $R_1, R_2 \in \Re \mathfrak{es}(G)$, we write $R_1 < R_2$ if there is a vertex of R_2 below a vertex of R_2 . We write $R_2 \prec_{\heartsuit} R_2$ if there are $R'_1, R'_2 \in \Re \mathfrak{es}(G)$ such that $R'_1 \prec R'_2$ and such that there are \mathfrak{E} -cycles in $\mathbb{E}^m(G)$ covering R_1 and R_2 and covering R_2 and R'_2 (see Figure 15 for an example).

Remark 102. As in Remark 91, two entangled residual components induces graphs which are dual modulo symmetries, that is, such that $G|_{R_1} \sim \heartsuit(G|_{R_2})^{\perp}$.

Definition 103. The RB-structure *G* is **GS**-correct iff there is a fully entangled relation \heartsuit over $\Re \mathfrak{es}(G)$ such that \prec_{\heartsuit} is well-founded.

Remark 104. Our correctness criteria subsume Retoré's ones for MLL and MLL° since in a RB-structure of a sequent of MLL-formulas there are no P_4 's.

Theorem 105. Let $G = \{\!\{\Gamma\}\!\} \sqcup \text{Link be a RB-structure. Then }_{\mathsf{MPL}^\circ} [\![\Gamma]\!] \iff G \text{ is GS-correct}$

Proof. The proof is similar to the one of Theorem 97.

To construct the RB-proof net encoding a derivation in RB_Q° we have to consider the following additional cases to handle the additiona rules in the systems RB_Q° which are not in $RB_{\mathcal{P}}$:

- if $\rho = \min_{i=1}^{RB}$, then the gates in the conclusion are the same gates in the premises, $\mathcal{E}^{m}(G) = \bigcup_{i=1}^{n} \mathcal{R}^{m}(G_{i})$, and $\Re es(G) = \bigcup_{i=1}^{n} \Re es(G_{i})$. Note that the conclusion is not æ-connected nor g-connected;
- if $\rho = s_{\otimes}^{\mathsf{RB}}$, then (using the same convention of Figure 10) the *k*-th input $\mathsf{i}_{\mathfrak{G}_Q}^k$ of the active gate \mathfrak{G}_Q is a graft and no new æ-cycle has been created; that is, $\mathscr{E}^m(G) = \mathscr{E}^m(G_1) \uplus \mathscr{E}^m(G_2)$ and $\mathfrak{Res}(G) = \mathfrak{Res}(G_1) \cup \mathfrak{Res}(G_2)$. In fact, even if \mathfrak{G}_Q contains a P₄ not occurring in its premises, it must contain the vertex $\mathsf{i}_{\mathfrak{G}_Q}^k$, which is a graft, therefore not occurring in any residual component of *G*.
- if $\rho = d \kappa_Q^{\text{RB}}$, then the only difference with the case analysis of in the proof of Theorem 97 is that we are assuming the weaker condition that \heartsuit is fully entangling.

Note that an instance of a $\mathfrak{P}^{\mathsf{RB}}$ could have conclusion a g-connected RB-structure but a premise which is not g-connected.

Conversely, the sequentialization procedure is refined as follows:

- 1. if $G = \emptyset$ then $\Gamma = \circ$ and π_G is an instance of \circ^{RB} ;
- 2. if $G \neq \emptyset$ and $\mathfrak{Gates}(G) = \emptyset$, then $G = \langle \{a, a^{\perp}\}, \emptyset, \{aa^{\perp}\} \rangle$ and π_G is an instance ax^{RB} ;
- 3. if $G \neq \emptyset$ is not g-connected, then $G = G_1 \uplus \cdots \uplus G_n$ and

$$\pi_G = \frac{\pi_1 \prod H}{\min^{\mathsf{RB}} + G_1} \cdots + G_n$$

where the proofs π_1, \ldots, π_n are inductively defined since we have that $\square_G = \bigcup_{i \in \{1,\ldots,n\}} \square_{G_i}$, $\Re es(G) = \bigcup_{i \in \{1,\ldots,n\}} \Re es(G_i)$ and the correctness of G_1, \ldots, G_n is guaranteed by the fact that $\mathscr{E}^{\mathsf{m}}(G) = \bigcup_{i \in \{1,\ldots,n\}} \mathscr{E}^{\mathsf{m}}(G_i)$;

4. if G is g-connected and \Re - $\mathfrak{Gates}(G)$ contains a \mathfrak{P}_n -gate, then $G = \mathfrak{G}_{\mathfrak{P}_n} \bowtie_{X_{\mathfrak{P}}} G_1$ and

$$\pi_G = \frac{\pi_1 \prod_{i \in G} \prod_{i \in G} \prod_{j \in G} \prod_{i \in G} \prod_{i \in G} \prod_{i \in G} \prod_{j \in G} \prod_{i \in G} \prod_{i \in G} \prod_{j \in G} \prod_{i \in G} \prod_$$

where π_1 is defined by inductive hypothesis since $|\Box_G| = |\Box_{G_1}|$ but $|\operatorname{Gates}(G_1)| < |\operatorname{Gates}(G)|$ and the correctness of G_1 is guaranteed by the fact that $\mathcal{X}^m(G) = \mathcal{X}^m(G_1)$ and $\operatorname{Res}(G) = \operatorname{Res}(G_1)$;

5. if $G \neq \emptyset$ is g-connected and no gate in \Re -Gates $(G) \neq \emptyset$ is a \Re_n -gate and at least one input of a $\mathfrak{G} \in \Re$ -Gates(G) is a graft, then

• either \mathfrak{G} is a \otimes_n -gate whose inputs are all graft, therefore there are G_1, \ldots, G_n and $\mathfrak{r}_i \in \mathfrak{Root}(G_i)$ for all $i \in \{1, \ldots, n\}$ such that

$$\pi_{G} = \mathop{\otimes}_{\otimes^{\mathsf{RB}}} \frac{\pi_{1} \prod H}{\vdash G_{1} \cdots \vdash G_{n}} \prod H}{\vdash G_{1} \cdots \vdash G_{n}} \left(\bigcup_{i \in \{1, \dots, n\}} \left(\bigcup_{i \in \{1, \dots, n\}} G_{i} \right) \right)$$

• or \mathfrak{G} is a *Q*-gate with *Q* quasi-prime and $|V_Q| = n + 1 > 2$ and, w.l.o.g., there is am interface $X^+ = X^- \cup \{(i_Q^1, r_2)\}$ where $r_2 \in \mathfrak{Root}(G_2)$ such that

$$\pi_{G} = \underset{\mathbb{S}_{\otimes}^{\mathsf{RB}}}{\overset{\vdash}{\underbrace{\|\mathcal{Q}\|\emptyset, v_{2}, \dots, v_{n+1}\|}}_{\vdash} \underbrace{\mathbb{S}_{\otimes}^{\pi_{1}} G_{1}}_{\vdash} \underbrace{\mathbb{S}_{2}}_{\vdash} \underbrace{\mathbb{S}_{2$$

In both cases, the two proofs of the premises are defined by inductive hypothesis;

Otherwise G ≠ Ø is g-connected, no gate in ℜ-𝔅ates(G) ≠ Ø is a ℜ_n-gate and no input of a gate in ℜ-𝔅ates(G) is a graft.

We conclude as in Case 4 in the proof of Theorem 97.

(25)

7.4 A TOPOLOGICAL CHARACTERIZATION OF GS

An *axiom linking* Link for a graph G is a (total) bijection between atoms mapping each atom to one of its dual. If \mathcal{D} is a proof of G in GS we define a set of B-edges $\sqcap_{\mathcal{D}}$ by pairing the vertices matched by the applications of the ai \downarrow -rules in \mathcal{D} .

Theorem 106. Let G be a graph. Then there is a proof \mathcal{D} of G in GS iff the RB-structure $\langle V_{\|G\|}, \ell_G, \overset{\|G\|}{\frown}, \overset{T}{\bot} \cup \sqcap_{\mathcal{D}} \rangle$ is GS-correct.

Proof. It follows Theorems 57 and 105 and the fact that the translation between GS and MPL° preserve the pairs of atoms matched by an ax-rule and vertices matched by an $ai\downarrow$ -rule.

8 CLASSICAL LOGIC BEYOND COGRAPHS

We conclude this paper by providing a simple extension of MPL with structural rules allowing us to generalize classical logic beyond cographs.

Definition 107. We define the following logics of formulas (defined via a proof system) and graphs respectively:

Classical Prime Logic: $PLK = MPL \cup \{w, c\}$ Classical Graphical Logic: $GLK = \{[\![\phi]\!] \mid \phi \text{ formula such that } \vdash_{PLK} \phi\}$

We say that a graph G is *provable* in GLK (denoted $\vdash_{GLK} G$) if $G \in GLK$.

For PLK we can prove the admissibility of the cut-rule via cut-elimination.

Theorem 108 (Cut-elimination). The rule cut is admissible in PLK.

Proof. Consider the *cut-elimination steps* from Figure 5 and Figure 16 and the definition of weight from the proof of Theorem 39. We conclude by remarking that applying cut-elimination steps to any one of the top-most cut in the derivation, then we obtain a derivation with smaller weight. \Box

Figure 16: Structural rules for sequent calculi, the corresponding rules in deep inference, the atomic contraction and the generalized medial rule, and cut-elimination steps to handle them.

Moreover, we can define a sequent system containing the deep inference version of the structural rules (see Figure 16) to obtain the following decomposition result.

Theorem 109 (Decomposition). Let ϕ be a formula such that $\vdash_{\mathsf{PLK}} \phi$. Then there is a formula ϕ' such that $\vdash_{\mathsf{MPL}} \phi'$ and $\phi' \vdash_{\{\mathsf{W},\mathsf{W},\mathsf{L},\mathsf{L}\}} \phi$.

Proof. The proof is immediate by applying rule permutations. For a reference, see [10]. \Box

This result can be refined using a *generalized medial* rule proposed proposed in [24] to restrict the instances of contraction rules to atomic ones.

Lemma 110. *The* (*deep*) *contraction rule* $c \downarrow$ *is derivable using atomic contraction* ($ac \downarrow$) *and medial rule* (m).

Proof. By induction on the contracted formula ϕ . If $\phi = a$ is an atom, then an instance of $c \downarrow$ can be replaced by an instance of $ac \downarrow$. Otherwise, $\phi = \kappa (\psi_1, \dots, \psi_n)$ and we conclude since we can replace each application of $c \downarrow$ with a derivation of the following form

$$\underset{\kappa}{\overset{\kappa}{(\psi_1,\ldots,\psi_n) \xrightarrow{\gamma} \kappa}{(\psi_1,\ldots,\psi_n)}}{\overset{\kappa}{(\psi_1,\ldots,\psi_n)}} \longrightarrow$$

by applying inductive hypothesis.

This would allow to provide a stronger result similar to the one for in deep inference systems for classical graphical logic [19, 22] formulated as follows.

Corollary 111 (Decomposition). Let ϕ be a formula such that $\vdash_{\mathsf{PLK}} \phi$. Then there are formulas ϕ', ψ' and ψ such that

 $\vdash_{\mathsf{PLK}} \phi \iff \vdash_{\mathsf{MPL}} \phi' \vdash_{\mathsf{m}} \psi' \vdash_{\mathsf{ac}\downarrow} \psi \vdash_{\mathsf{w}\downarrow} \phi$

Proof. By Theorem 109 we find the desired ϕ' . Using rules permutations, we can push occurrences of $w \downarrow$ down in a derivation, finding the desired ψ . We then apply 110 and replace all instances of $c \downarrow$ -rules with derivations containing only m and $ac \downarrow$. We conclude by applying rule permutations to move all ac-rules below the instances of m-rules.

To conclude this section, we recall that classical graphical logic, beside being a conservative extension of classical propositional logic is not the same logic of the *boolean graphical logic* (denoted GBL) defined in [24] (an inference systems on graphs by extending the semantics of read-once boolean relations from cographs to general graphs). As shown in [8] the following graph expected to be provable in GBL, but is not provable in GS nor in GLK.



9 CONCLUSION AND FUTURE WORKS

In this paper we provide the definition of the notion of graphical connectives, and we defined a class of formulas generated by a signature of logical connectives corresponding to the graphical ones. We have provided three proof systems operating on these generalized formulas (MPL, MPL° and PLK), proving that these systems satisfy cut-elimination and are conservative extensions of multiplicative linear logic with and without mix, and of classical logic respectively. We proved that the logic GS form [7, 6] provides a model for the system extending the multiplicative linear logic with mix (MPL°).

We then provide proof nets for the substructural logics MPL and MPL° by extending the syntax of RBproof net with additional types of gates whose design is based on the structure of prime graphs, cliques and stable sets. We e provided a topological characterization of formulas and graphs which are provable in both these logics by extending the syntax of RB-proof nets for multiplicative logic.

9.1 FUTURE WORKS

Categorical Semantics. We are interested in defining the categorical structures for the multiplicative prime logic (with and without mix). We conjecture that such categories are extensions of *star-autonomous* and *IsoMix* [27, 28] categories respectively, with additional (*n*-ary) monoidal products. In particular, the categorical structure for MPL[°] should be a quotient of a free multi-monoidal category whose products share the same unit (for each of their entries) and whose unitors are defined according to the unitor_k defined in Lemma 50.

Digraphs, Games and Event Structures. In this work we started our investigation from the correspondence between classical formulas (and multiplicative linear logic formulas) and cographs. However, a different approach could be considered by considering the correspondence between intuitionistic propositional formulas and the Hyland-Ong *arenas* [54] (directed graphs). We foresee interesting connections with game semantic, concurrent games and event structures [79]. In this setting, graphs generalizing the connectives from *additive linear logic* [26] could allow express non-transitive conflict relations, as well as to handle the general setting where the conflict relation # could define patterns which cannot be expressed via linear formulas (i.e., without repetition of events) constructed using with binary connectives only.

A new framework for GoI. In Section 5 we defined a general setting for generalized proof nets. The current models for the *geometry of interaction* (or *GoI*) [41, 46, 67] and Girard's transcendental syntax [45, 46, 37, 38] are constructed using the Danos-Regnier correctness criterion for multiplicative proof nets to define *tests*. The atomic component used to build of these proof structures, as well as the tests used to define the correctness criterion, are based on a paradigm which could be named *connectives-as-permutations* (if we follow the approach from, e.g., [29, 37]) or *connectives-as-partitions* (if we follow the approach from, e.g., [44, 9]). As explained in Section 9 of [8], connectives-as-partitions are distinct from the graphical connectives used in this paper. Thus we foresee the possibility of exploring entire new models of GoI over the RB-structures defined in this paper.

Relational RB-Nets. In Section 7 we provided a topological characterization of RB-structures encoding correct derivations in RB_Q° , relying on the encoding of a graph (or a squent) with an RB-forest. However, RB-cographs⁸ provide another possible encoding of RB-graphs proofs in MLL [72, 74, 73]. This encoding is obtained by directly enriching a cograph (whose edges are R-edges) encoding a MLL-formula with B-edges pairing the vertices corresponding to the atoms matched by the axiom rules. The correctness criterion for these RB-graph crucially rely on the presence of chords in an æ-cycle. Intuitively, chords identify (non-elementary) alternating cycles in $\{\!(\Gamma)\!\} \sqcup \text{Link}$ covering the connector edge of a \otimes . It follows that chords are indeed paired, inducing "bow-tie" subgraphs as the one below on the left.



The criterion for MLL fails for MPL because of the presence of P_4 's as shown in the example above on the right. We conjecture the possibility of reproducing the correctness criterion for RB-structure directly on these RB-graph.

Moreover, a correctness criterion on RB-nets could be used to extended this syntax to include *exponentials* by reformulating the correctness criterion provided in [4]. **On Graphical Classical Logic**.

⁸Also known as handsome proof nets, relational RB-prenets [69], RB-cographs [68], or closed coherent spaces [36].

In [24] the authors investigate the possibility of extending boolean logic beyond cographs. Beside this logic have been proved to be incompatible with extensions of the multiplicative graphical logic, it is still interesting to see the exact relation between our graphical classical logic and the *boolean graphical logic* presented in the aforementioned paper.

The decomposition result for GLK suggests the possibility of defining combinatorial proofs *combina-torial proofs* [53, 52] for this logic and study its proof equivalence which would present some non-trivial derivations (see the left-most RB-structure in Figure 13 for an example)

Moreover, we foresee the possibility of extending the results in [16] to classical graphical logic.

A correctness criterion for BV. The technique of sequentialize and entail connectives could provide insights on the correctness criterion BV [47, 69, 68] and its extensions GV and GV^{sl} from [5].

Automated Theorem Provers for Graphical Logic. The introduction of graphical connectives provides a representation of graphs which is linear with respect to the number of its vertices. Such a representation could be used to implement efficient in automated tools to implement the current results in graphical logic, as well as to provide new automated tools to address challenging problems in mathematics and computer science where the graphical syntax improve usability and efficiency.

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A DEEP INFERENCE AND THE OPEN DEDUCTION FORMALISM

Open deduction [48] is a proof formalism based on deep inference [12]. It has originally been defined for formulas, but it is abstract enough such that it can equally well be used for graphs, as already done in [6].

Definition 112. An *inference system* S is a set of inference rules (as for example shown in Figure 4).

G

A *derivation* \mathcal{D} in **S** with premise *G* and conclusion *H* is denoted \mathcal{D} and is defined inductively as follows:

- Every graph G is a (*trivial*) derivation with premise G and conclusion G (also denoted G).
- An instance of a rule $r\frac{G}{H}$ in S is a derivation with premise G and conclusion H.
- If \mathcal{D}_1 is a derivation with premise G_1 and conclusion H_1 , and \mathcal{D}_2 is a derivation with premise G_2 and conclusion H_2 , and $H_1 = G_2$, then the composition of \mathcal{D}_1 and \mathcal{D}_2 is a derivation \mathcal{D}_2 ; \mathcal{D}_1 denoted as below.

G_1		G_1		G_1 \mathcal{D}_1 s		G_1		G_1
	or	or $\begin{array}{c} \mathcal{D}_1 \parallel S \\ H_1 \\ \dots \\ G_2 \\ \hline \mathcal{D}_2 \parallel S \\ H_2 \end{array} $	or	$\frac{H_1}{G_2}$ or	or	$\mathcal{D}_1 \parallel S \\ \mathcal{O}_2 \mathcal{O}_2$	or	$\mathcal{D}_1 \parallel S$ H_1
$\mathcal{D}_2 \parallel S H_2$				$\mathcal{D}_2 \parallel S H_2$		$\mathcal{D}_2 \parallel S H_2$		$\mathcal{D}_2 \ S$ H_2

Note that even if the symmetry between G_2 and H_1 is not written, we always assume it is part of the derivation and explicitly given.

• If G is a graph with n vertices and $\mathcal{D}_1, \ldots, \mathcal{D}_n$ are derivations with premise G_i and conclusion H_i for each $i \in \{1, \ldots, n\}$, then $G(\mathcal{D}_1, \ldots, \mathcal{D}_n)$ is a derivation with premise $G(G_1, \ldots, G_n)$ and conclusion $G(H_1, \ldots, H_n)$ denoted as below on the left.

(G_1	G_n		G_1		G_1
G	$\mathcal{D}_1 \ S , \dots$., <i>D</i> _n S	2	D_1	\star	\mathcal{D}_1
	H_1	H_n		H_1		H_1

If $G = \star \in \{\mathcal{V}, \otimes\}$ we may write the derivations as above on the right.

Therefore, $C\begin{bmatrix}G\\ \mathcal{D} \parallel S\\ H\end{bmatrix} := \begin{bmatrix}C[G]\\ C[\mathcal{D}] \parallel S\\ C[H]\end{bmatrix}$ is well-defined for any context $C[\Box]$ and any derivation $\begin{bmatrix}G\\ \mathcal{D} \parallel S\\ H\end{bmatrix}$. A *proof* in S is a derivation in S whose premise is \emptyset . A graph G is *provable* in S (denoted $\vdash_S G$) iff there is a proof in S with conclusion G.

A.1 EQUIVALENT DEFINITIONS OF GS

We here show that the formulation of the system GS provided in this paper is equivalent to one provided in [7, 8]. In particular, in the previous these papers the rule s_{\otimes} was not included in the system. However, as shown in [8] this rule plays a crucial role in the proof that GS is a conservative extension of MLL° and in [5] it is shown that this rule cannot be admissible in the proof systems operating on mixed graphs. Moreover, we here give a weaker side condition on the p-rule with respect to the rules below:

p↓ in [8]	p↓ in [7]	
$p_{1}\downarrow \frac{(M_{1} \Im N_{1}) \otimes \cdots \otimes (M_{n} \Im N_{n})}{P^{\perp} (M_{1}, \ldots, M_{n}) \Im P (N_{1}, \ldots, N_{n})} \star$	$p_{P} \downarrow \frac{(M_1 {}^{\mathfrak{N}}N_1) \otimes \cdots \otimes (M_n {}^{\mathfrak{N}}N_n)}{P^{\perp}(M_1,\ldots,M_n) {}^{\mathfrak{N}}P(N_1,\ldots,N_n)} ^{\dagger}$	(27)
★ := $P \notin \{\Re, \otimes\}$ prime $M_i \neq \emptyset$ for all $i \in \{1,, n\}$	<i>†</i> := <i>P</i> ∉ {𝔅, ⊗} prime $M_i 𝔅 N_i \neq \emptyset$ for all <i>i</i> ∈ {1,, <i>n</i> }	

In order to prove the equivalence between our system and the ones in [7, 8] we recall the following lemma allowing us to prove that in GS we can derive any graph of the shape $G \multimap G$.

Lemma 113. Let $M_1, \ldots, M_n, N_1, \ldots, N_n$ and G be graphs such that $|V_G| = n$. Then there is a derivation $(M_1 \stackrel{\mathcal{D}}{\rightarrow} N_1) \otimes \cdots \otimes (M_n \stackrel{\mathcal{D}}{\rightarrow} N_n)$

$$\|_{\{\mathbf{s}_{\otimes},\mathsf{p}_{\downarrow}\}}$$

$$G^{\perp}(M_{1},\ldots,M_{n}) \stackrel{\otimes}{\mathcal{T}} G(N_{1},\ldots,N_{n})$$

Proof. By induction on the modular decomposition of *G*.

Thanks to this lemma, we can therefore prove the admissibility of the weaker

Proposition 114. The following rule, which is a version of $p \downarrow$ with weaker side conditions, is admissible in GS $(M_1 \stackrel{\infty}{\to} N_1) \otimes \cdots \otimes (M_n \stackrel{\infty}{\to} N_n)$

$$\mathsf{p}_1 \downarrow \frac{(M_1 \circ \mathcal{N}_1) \otimes \cdots \otimes (M_n \circ \mathcal{N}_n)}{P^{\perp}(M_1, \dots, M_n) \otimes \mathcal{P}(N_1, \dots, N_n)}$$

where *P* is prime and $M_i \neq \emptyset$ for all $i \in \{1, ..., n\}$.

Proof. Note that we may have $N_i = \emptyset$ for some $i \in \{1, ..., n\}$. Thus, if $N_i \neq \emptyset$ for all $i \in \{1, ..., n\}$, then $p_1 \downarrow$ is an occurrence of $p \downarrow$. Otherwise, w.l.o.g., $N_1 = \emptyset$, thus we have a derivation

$$M_{1} \otimes \underbrace{ \begin{array}{c} (M_{2} \ \Im \ N_{2}) \otimes \cdots \otimes (M_{n} \ \Im \ N_{n}) \\ \| \text{Lemma 113} \\ H^{\perp}(M_{2}, \ldots, M_{n}) \ \Im \ H(N_{2}, \ldots, N_{n}) \end{array}}_{\mathfrak{s}_{\otimes} \underbrace{ \begin{array}{c} M_{1} \otimes P^{\perp}(\emptyset, M_{2}, \ldots, M_{n}) \\ P^{\perp}(M_{1}, M_{2}, \ldots, M_{n}) \end{array}} \ \Im \ P(\emptyset, N_{2}, \ldots, N_{n})$$

Theorem 115. Let G be a graph. Then

 $\vdash_{\mathsf{GS}} G \Leftrightarrow \vdash_{\{\mathsf{ai} \downarrow, \mathsf{s}_{\mathfrak{V}}, \mathsf{s}_{\otimes}, \mathsf{p}_{1} \downarrow\}} G \Leftrightarrow \vdash_{\{\mathsf{ai} \downarrow, \mathsf{s}_{\mathfrak{V}}, \mathsf{p}_{1} \downarrow\}} G \Leftrightarrow \vdash_{\{\mathsf{ai} \downarrow, \mathsf{s}_{\mathfrak{V}}, \mathsf{p}_{2} \downarrow\}} G$

Proof. The first equivalence follows from Proposition 114. The other has been proved in [8].

B Soundness and Completeness of **RB**-nets

In order to prove Theorem 69, we use the following technical lemmas.

Lemma 116. Let *H* be a graph. Then the following rule is admissible in $\mathsf{RB}_{\mathcal{P}}$.

$$d \kappa_{G}^{\mathsf{RB}} \xrightarrow{\vdash G_{1}} \cdots \xrightarrow{\vdash G_{n}} K_{X_{d}} (G_{1} \uplus \cdots \uplus G_{2})$$

where G is a graph with $|V_G| = n$ and

$$X_{d} = \left\{ \left(\mathsf{i}_{\mathfrak{G}_{G}}^{i}, \mathsf{r}_{i}^{1} \right), \left(\mathsf{i}_{\mathfrak{G}_{G^{\perp}}}^{i}, \mathsf{r}_{i}^{2} \right) \mid i \in \{1, \ldots, n\} \text{ and } \mathsf{r}_{i}^{1}, \mathsf{r}_{i}^{2} \in \mathfrak{Root}(G_{i}) \right\}$$

for some $\mathbf{r}_i^1, \mathbf{r}_i^2 \in \mathfrak{Root}(G_i)$ with $\mathbf{r}_i^1 \neq \mathbf{r}_i^2$ for all $i \in \{1, \ldots, n\}$.

Proof. By induction on the modular decomposition-via quasi-prime graphs of G:

• if $G = P(G_1, \ldots, G_m)$ with $P \notin \{2, 8\}$ prime, then

$${}_{\mathsf{d}\text{-}\kappa_{P}^{\mathsf{RB}}} \frac{ \begin{matrix} \mathsf{I} \text{ } \mathsf{H} & \mathsf{I} \\ \mathsf{H} & \mathsf{G}_{1}' & \cdots & \mathsf{H} \\ \hline \mathsf{G}_{p} \uplus \mathfrak{G}_{P^{\perp}} \end{matrix} \bowtie_{X_{\mathsf{d}}} \left(G_{1}' \uplus \cdots \uplus \mathfrak{G}_{m}' \right)$$

for a given X_d ;

• otherwise $G = Q(G_1, \ldots, G_m)$ with, w.l.o.g., $Q = \otimes_m$, then

for given X_d and $X_{\mathfrak{F}_n}$. Note that the conclusion can also be written as $(\mathfrak{G}_{\mathfrak{F}_m} \uplus \mathfrak{G}_{\otimes_m}) \bowtie_{X_d} (G'_1 \uplus \cdots \uplus G'_m)$ with $X_d = X_s \cup X_{\mathfrak{F}}$.

We can now provide a proof of Theorem 69.

Theorem (Theorem 69). Let Γ be a \mathfrak{P} -sequent. Then

- 1. $\vdash_{\mathsf{MPL}} \llbracket \Gamma \rrbracket \iff \vdash_{\mathsf{RB}_{\mathcal{P}}} \llbracket \Gamma \rrbracket \sqcup \mathsf{Link} \text{ for an axiom link Link for } \Gamma$
- 2. $\vdash_{\mathsf{MPL}^\circ} \llbracket \Gamma \rrbracket \iff \vdash_{\mathsf{RB}^\circ_{\mathcal{Q}}} {\!\!\!\!\!{\{\!\!\!\ \ \!\!\!\!\}}} \sqcup \mathsf{Link} \text{ for an axiom link Link for } \Gamma$

Proof. We prove (2) since (1) immediately follows.

Given a proof π of Γ in MPL°, we define a proof RB(π) of the RB-structure $\{\!\{\pi\}\!\} = \{\!\{\Gamma\}\!\} \sqcup \text{Link in } RB_{\rho}^{\circ}$ by induction on the last rule ρ in π :

• if $\rho = \circ$, then

• if $\rho = ax$, then

$$\operatorname{ax} \frac{}{\vdash a, a^{\perp}} \rightsquigarrow \operatorname{ax}^{\operatorname{RB}} \frac{}{\vdash a - a^{\perp}}$$

- if $\rho = \Re_n$, then
 - if all ϕ_1, \ldots, ϕ_n and ψ are not vacuous, then

$$\gamma \frac{ \begin{matrix} \pi_1 \\ \vdash \Delta, \phi_1, \dots, \phi_n \\ \vdash \Delta, \Im_n(\phi_1, \dots, \phi_n) \end{matrix} \rightsquigarrow \gamma^{\text{RB}} \frac{ \begin{matrix} \Pi \\ \vdash \Re \\ \vdash \Re \\ \vdash \Re \end{matrix} \underset{ \vdash \Re \\ \vdash \Re \\ \vdash \Re \\ \vdash \Re \end{matrix} \underset{ \begin{matrix} \Pi \\ \vdash \Re \\ \vdash \Re \end{matrix} \underset{ \mid \{ [i_{\mathcal{D}}, f_{\{ \{ \phi_i \} \}}] | i \in \{1, \dots, n\} \\ \end{matrix} }{ \begin{matrix} \Pi \\ \blacksquare \\ \vdash \Re \end{matrix} }$$

- if, w.l.o.g. $\phi_{k+1}, \ldots, \phi_n$ are vacuous and

If k = 1, then we now consider the root of $\{\!\{\phi_1\}\!\}$ as if it is the root of $\{\!\{\Im_n \| \phi_1, \dots, \phi_n\}\!\}$.

• if $\rho = \otimes_n$, then

- if all ϕ_1, \ldots, ϕ_n are not vacuous, then

- if, w.l.o.g. $\phi_{k+1}, \ldots, \phi_n$ are vacuous and

$$\approx \frac{ \prod_{\substack{n \in \mathcal{M}, \\ \vdash \Delta, \\ \vdash \Delta, \\ \leftarrow A, \\ \leftarrow A,$$

If k = 1, then we now consider the root of $\{\!\{\phi_1\}\!\}$ as if it is the root of $\{\!\{\phi_1, \dots, \phi_n\}\!\}$.

• if $\rho = \mathsf{d}$ - κ and if, w.l.o.g., $\llbracket \phi_j \rrbracket = \emptyset$ for all $k \in \{k + 1, \dots, l\}$, $\llbracket \psi_i \rrbracket = \emptyset$ for all $i \in \{l + 1, \dots, m\}$, and $\llbracket \phi_i \rrbracket = \emptyset = \llbracket \psi_i \rrbracket$ for all $i \in \{m + 1, \dots, n\}$ then

where

$$\begin{aligned} &- G'' = \{\!\!\{H\}\!\} \uplus \{\!\!\{H^{\perp}\}\!\} \bowtie_{Y} (\{\!\!\{\pi_{1}\}\!\} \uplus \cdots \uplus \{\!\!\{\pi_{l}\}\!\}) \\ &- H(\!\!\{v_{1}, \ldots, v_{l}\!\}) = P(\!\!\{v_{1}, \ldots, v_{l}, \varnothing, \ldots, \varnothing\!\}) \\ &- Y = \left\{\!\!\left(\mathsf{i}_{\{\!\{H\}\!\}}^{i}, \mathsf{r}_{\{\!\{\phi_{i}\}\!\}}\right), \left(\mathsf{i}_{\{\!\{H^{\perp}\}\!\}}^{i}, \mathsf{r}_{\{\!\{\psi_{i}\}\!\}}\right) \mid i \in \{1, \ldots, n\}\!\right\} \\ &- G' = \{\!\!\{K\}\!\} \uplus \{\!\!\{K^{\perp}\}\!\} \bowtie_{X} (\{\!\!\{\pi_{n+1}\}\!\} \uplus \cdots \uplus \{\!\!\{\pi_{m}\}\!\}) \\ &- K(\!\!\{v_{1}, \ldots, v_{m}\!\}) = P(\!\!\{v_{1}, \ldots, v_{m}, \varnothing, \ldots, \varnothing\!\}) \\ &- X = Y \cup \left\{\!\!\left(\mathsf{i}_{\{\!\{K\}\!\}}^{i}, \mathsf{r}_{\{\!\{\phi_{i}\}\!\}}\right), \left(\mathsf{i}_{\{\!\{K^{\perp}\}\!\}}^{i}, \mathsf{r}_{\{\!\{\psi_{i}\}\!\}}\right) \mid i \in \{n+1, \ldots, m\}\!\right\} \end{aligned}$$

In this case, we now consider the root of $\{\!\{\kappa_P(\phi_1, \ldots, \phi_n)\}\!\}$ and of $\{\!\{\kappa_{P^{\perp}}(\psi_1, \ldots, \psi_n)\}\!\}$ to respectively be the root of $\{\!\{\kappa_P(\phi_1, \ldots, \phi_k, \circ)\}\!\} \in \Reoot(\{\!\{K\}\!\})$ and the root of $\{\!\{\kappa_{P^{\perp}}(\psi_1, \ldots, \psi_k, \circ)\}\!\} \in \Reoot(\{\!\{K^{\perp}\}\!\})$.

• if $\rho = mix$, then

- if, w.l.o.g., Δ_2 only contains vacuous formulas, then

$$\underset{\mathsf{mix}}{\overset{\pi_1 \prod \quad \pi_2 \prod \quad }{\vdash \Delta_1 \quad \vdash \Delta_2}} \leadsto \overset{\pi_1 \mathbb{I}}{\vdash} \overset{\pi_2 \prod \quad }{\vdash \Delta_1 \mathbb{I}}$$

- otherwise

$$\min \frac{ \prod_{i=1}^{n_1} \prod_{i=1}^{n_2} \prod_{i=1}^{n_2}}{\vdash \Delta_1, \Delta_2} \rightsquigarrow \min_{i=1}^{n_1} \frac{\prod_{i=1}^{n_1} \prod_{i=1}^{n_1} \prod_{i=1}^{n_2} \prod_{i=1$$

• if $\rho = wd_{\otimes}$, then

$$\mathsf{wd}_{\otimes} \frac{ \stackrel{\pi_{1} \prod}{\vdash} \quad \pi_{1} \prod}{\vdash} \quad \mu_{k-1}, \phi_{k} \vdash \Delta_{2}, \chi(\phi_{1}, \dots, \phi_{k-1}, \phi_{k+1}, \dots, \phi_{n}) }{\vdash} \quad \Delta_{1}, \Delta_{2}, \kappa(\phi_{1}, \dots, \phi_{n})$$

since $\{\!\{\Delta, \kappa \| \phi_1, \ldots, \phi_{k-1}, \circ, \phi_{k+1}, \ldots, \phi_n \}\!\} = \{\!\{\Delta, \chi \| \phi_1, \ldots, \phi_n \}\!\}$ by definition of the rule wd_{\otimes}.

- if $\chi(\phi_1, \ldots, \phi_{k-1}, \phi_{k+1}, \ldots, \phi_n)$ is vacuous, then we also let $\{\!\{\pi\}\!\} = \{\!\{\pi_1\}\!\} \uplus \{\!\{\pi_2\}\!\}$ but now we consider the root of $\{\!\{\phi_k\}\!\}$ to be the root of $\{\!\{\kappa(\phi_1, \ldots, \phi_n)\}\!\}$;

We let the root of $\kappa(\phi_1, \ldots, \phi_n)$ to be the root of $\chi(\phi_1, \ldots, \phi_{k-1}, \phi_{k+1}, \ldots, \phi_n)$ in $\{\pi\}$.

Conversely, given a proof π of $G = \{\!\!\{\Gamma\}\!\!\} \sqcup \text{Link in } \mathsf{RB}_Q^\circ$, we define a proof $\pi(G)$ of Γ in MPL° by induction on the last rule ρ in π :

- if $\rho = \circ^{\mathsf{RB}}$, then $G = \emptyset$ and $\pi' = \circ_{F \circ \Theta}$
- if $\rho = ax^{\text{RB}}$, then $G = a a^{\perp}$ and $\pi' = ax a^{\perp} + a a^{\perp}$
- if $\rho = \mathfrak{P}^{\mathsf{RB}}$, then $G = \mathfrak{G} \bowtie_{X_{\mathfrak{P}}} G_1$ with $G_1 = \{\!\!\{\Gamma, \phi_1, \dots, \phi_n\}\!\} \sqcup \mathsf{Link}$ and $\mathfrak{G} \colon \mathfrak{P}_n$. Thus

$$\pi(G) = \frac{\pi(G_1) \| \mathbb{H}}{\vdash \Gamma, \phi_1, \dots, \phi_n}$$
$$\frac{\vdash \Gamma, \mathfrak{P}_n(\phi_1, \dots, \phi_n)}{\vdash \Gamma, \mathfrak{P}_n(\phi_1, \dots, \phi_n)}$$

• if $\rho = \otimes^{\mathsf{RB}}$, then $G = \mathfrak{G}_{\otimes_n} \bowtie_{X_s} (\biguplus_{i=1}^n G_i)$ with $G_i = \{\{\Gamma_i, \phi_i\}\} \sqcup \mathsf{Link}_i \text{ for all } i \in \{1, \ldots, n\}$. Thus $\pi(G)$ is of the shape

$$\approx \frac{\pi(G_1) \prod \mathbb{H}}{\vdash \Gamma_1, \phi_1 \cdots \vdash \Gamma_n, \phi_n}$$

• if $\rho = \mathsf{d} - \kappa_Q^{\mathsf{RB}}$ then $G = (\mathfrak{G}_P \uplus \mathfrak{G}_{P^{\perp}}) \bowtie_{X_{\mathsf{d}}} (\biguplus_{i=1}^n G_i)$ with $G_i = \{ \{\Gamma_i, \phi_i, \psi_i \} \} \sqcup \mathsf{Link}_i$ for all $i \in \{1, \ldots, n\}$. Thus

$$\pi(G) = \frac{\pi(G_1) \prod H}{\vdash \Gamma_1, \phi_1, \psi_1} \cdots \frac{\pi(G_n) \prod H}{\vdash \Gamma_n, \phi_n, \psi_n}$$

• if $\rho = \min^{\mathsf{RB}}$, then $G = \bigcup_{i=1}^{n} G_i$ with $G_i = \{\{\Gamma_i\}\} \sqcup \operatorname{Link}_i$ for all $i \in \{1, \ldots, n\}$. Thus

$$\pi(G) = \frac{\pi(G_1) \| \mathbb{I} \mathbb{H}}{\min \frac{\vdash \Gamma_1}{\vdash \Gamma_1} \cdots \vdash \Gamma_n}$$

• if $\rho = \mathbf{s}_{\otimes}^{\mathsf{RB}}$, then $G = \mathfrak{G}_Q \bowtie_{X^+} (G_1 \uplus G_2)$ with

1.
$$G_1 = \{\!\!\{\Gamma_1, \phi_k\}\!\!\} \sqcup \mathsf{Link}_1;$$

2.
$$G_2 = \{\!\!\{Q(v_1, \ldots, v_{k-1}, \emptyset, v_{k+1}, \ldots, v_n)\}\!\} \bowtie_{X^-} \{\!\!\{\Gamma_2, \phi_1, \ldots, \phi_{k-1}, \phi_{k+1}, \ldots, \phi_n\}\!\} \sqcup \mathsf{Link}_2.$$

Thus

$$\pi(G) = \frac{\pi(G_1) \prod H}{\mathsf{wd}_{\otimes}} \frac{\vdash \Gamma_1, \phi_k \vdash \Gamma_2, \chi(\phi_1, \dots, \phi_{k-1}, \phi_{k+1}, \dots, \phi_n)}{\vdash \Gamma_2, \kappa_Q(\phi_1, \dots, \phi_n)}$$

with $[\![\chi]\!] \sim Q(\![v_1, \ldots, v_{k-1}, \emptyset, v_{k+1}, \ldots, v_n]\!).$

C ON RULES INTRODUCING A CONNECTIVE AT A TIME

In this appendix we discuss the results about the system extending multiplicative linear logic with the rule $s-\kappa$, that is, the system.

 $\mathsf{MLL}^{\mathsf{s}\cdot\kappa} := \{\mathsf{ax}, \mathfrak{N}, \otimes, \mathsf{mix}, \mathsf{s}\cdot\kappa \mid \kappa_P \in \mathfrak{C} \text{ with } P \notin \{\mathfrak{N}, \otimes\} \text{ prime}\}$

where we consider the following rule introducing a prime graphical connective different from \mathfrak{P}_n and \otimes at a time instead of d- κ .

$$s \cdot \kappa \frac{\vdash \Gamma_1, \phi_1 \cdots \vdash \Gamma_n, \phi_n}{\vdash \Gamma_1, \dots, \Gamma_n, \kappa \langle \phi_1, \dots, \phi_n \rangle}$$

We first observe that in the system does not satisfy anymore initial coherence (it suffices to consider the formula $\kappa_{P_4}(a, b, c, d) \rightarrow \kappa_{P_4}(a, b, c, d)$) even if the system still satisfies cut-elimination. In fact, for a proof of cut-elimination it suffices to include in the proof of Theorem 39 the following cut-elimination step.

$$s \cdot \kappa \frac{\vdash \Gamma_{1}, \phi_{1} \cdots \vdash \Gamma_{n}, \phi_{n}}{\underset{cut}{\vdash} \Gamma_{1}, \dots, \Gamma_{n}, \kappa_{P}(\phi_{1}, \dots, \phi_{n})} s \cdot \kappa \frac{\vdash \Delta_{1}, \phi_{1}^{\perp} \cdots \vdash \Delta_{n}, \phi_{n}^{\perp}}{\vdash \Delta_{1}, \dots, \Delta_{n}, \kappa_{P^{\perp}}(\phi_{1}^{\perp}, \dots, \phi_{n}^{\perp})} \xrightarrow{\qquad \text{out}} \frac{\vdash \Gamma_{1}, \phi_{1} \vdash \Delta_{1}, \phi_{1}^{\perp}}{\underset{mix}{\vdash} \Gamma_{1}, \Delta_{1}, \chi_{1}, \psi_{1}} \cdots \frac{\vdash \Gamma_{n}, \phi_{n}^{\perp} \vdash \Delta_{n}, \phi_{n}^{\perp}}{\vdash \Gamma_{n}, \chi_{n}, \psi_{n}}$$

Note that \mathbf{s} - κ is derivable in MPL° using \mathbf{d} - κ and unitor_ κ .

Lemma 117. The rule s- κ is derivable in MPL°.

Proof. It suffices to consider the following derivation

We can directly prove the derivability of the corresponding rule in the system RB°_Q .

Lemma 118. The rule \mathbf{s} - κ_G^{RB} is derivable in RB_Q° .

Proof. We proceed by induction on the modular decomposition of G via quasi-prime graphs:

• if $G = \Re_n$, then

- if $G = \otimes_n$, then $\mathbf{S} \kappa_Q^{\mathsf{RB}} = \otimes^{\mathsf{RB}}$.
- if G = P, for a prime graph $P \notin \{\mathfrak{N}, \otimes\}$, then there is $T = \mathfrak{G}_{\mathcal{Q}} \clubsuit \{\mathbf{i}_{\mathfrak{G}_{\mathcal{Q}}}^{k}\}$ such that

$$\mathbf{s}_{\otimes}^{\mathsf{RB}} \xrightarrow{\vdash \ \mathfrak{G}_{T} \bowtie_{X'_{\$}} (G_{1} \uplus \cdots \uplus G_{k-1} \uplus G_{k+1} \uplus \cdots \uplus G_{n})} \xrightarrow{\llbracket \mathsf{II}}_{\vdash \ \mathfrak{G}_{k}}$$

• otherwise $G = Q(M_1, ..., M_n)$ and we can apply inductive hypothesis.