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GRAPHICAL REGULAR REPRESENTATIONS OF NON-ABELIAN GROUPS, I

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In this paper, all groups and graphs considered are finite and all graphs are simple (in the sense of Tutte [**8**, p. 50]). If X is such a graph with vertex set $V(X)$ and automorphism group $A(X)$, we say that X is a *graphical regular representation* (*GRR*) of a given abstract group G if

(I) $G \cong A(X)$, and

(II) $A(X)$ acts on $V(X)$ as a regular permutation group; that is, given $u, v \in V(X)$, there exists a unique $\varphi \in A(X)$ for which $\varphi(u) = v$.

That for any abstract group G there exists a graph X satisfying (I) is well-known (cf. [**3**]). The question of existence or non-existence of a *GRR* for a given abstract group G, however, has been settled to date only for relatively few classes of groups. This problem is the underlying motivation for this paper and its sequel.

In Section 1 we introduce some essential notation and attempt a summary of what is known to date (with references to the literature) about the foregoing problem.

In Section 2 some machinery involving at one time techniques from both group theory and graph theory is developed in order to facilitate proving, when true, that a given graph is indeed a *GRR*. These techniques are then used to prove the main result of the section:

THEOREM 1. *Let the non-abelian group G_1 be a cyclic extension of a group G such that $[G_1 : G] \geqq 5$. If G admits a GRR, then so does G_1.*

In Section 3, techniques from Section 2 are applied to resolve completely the question of existence of *GRR*'s for all groups which can be represented as the semi-direct product of a cyclic group by a cyclic group.

1. Preliminaries. If G is a finite group and H is a subset of G with $1 \notin H$, then the *Cayley graph $X_{G,H}$ of G with respect to H* is the graph with vertex set $V(X_{G,H}) = G$ and edge set

$$E(X_{G,H}) = \{[x, xh] | x \in G; h \in H\}.$$

We shall continue the notation of the above definition, reserving the symbol H for subsets of $G \backslash \{1\}$ having the property that $u \in H \Rightarrow u^{-1} \in H$. Clearly $X_{G,H}$ is a connected graph if and only if $G = \langle H \rangle$.

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1.1 (G. Sabidussi [**6**]). *A necessary and sufficient condition for a given graph X to be a Cayley graph $X_{G,H}$ is that $A(X)$ contain a subgroup G_0 which acts regularly on $V(X)$. In that case $G = G_0$. In particular, if $|V(X)| \geqq 3$ and X is a GRR of G, then $X = X_{G,H}$ where H generates G.*

We shall say a group G is in *Class* I *if G has a GRR.*
G is in *Class* II if for each H which generates G, there exists a non-identity group-automorphism α of G such that $\alpha(H) = H$.

1.2 (Watkins [**9**]). *Class* I *and Class* II *are disjoint.* It is conjectured that every finite group is in one or the other of these two classes.

1.3 (Chao [**1**] and Sabidussi [**7**]). *All abelian groups other than elementary abelian* 2*-groups are in Class* II.

1.4 (Imrich [**4**]). *The elementary abelian* 2*-group of order* 2^n *is in Class* II *for* $n = 2, 3, 4$ *and is in Class* I *otherwise.*

We write C_m for the cyclic group of order m.
A *generalized dicyclic group* G is a group generated by an abelian group A and an element $b \notin A$ with relations:

$$(1\text{-}1) \qquad b^4 = 1, b^2 \in A, \quad \text{and} \quad b^{-1}xb = x^{-1} \quad \text{for all } x \in A.$$

In particular, if $A = C_m$ where $4 \leqq m = 2^n$, then G is a generalized quaternion group.

1.5 [**5**; **9**]. *All generalized dicyclic groups are in Class* II.

1.6 [**9**]. *Let p be a prime and let* $G = \langle a, b, c | a^p = b^p = c^p = 1, ac = ca, ab = ba, ab = bac\rangle$. *If* $p \geqq 5$, *then G is in Class* I. *The case for* $p = 3$ *is undecided.*

1.7 [**9**]. *Class* I, *except for the group* C_2, *is closed under the operation of direct product.*

1.8. *The dihedral group* D_m *is in Class* II *for* $m = 3, 4, 5$ *and in Class* I *for* $m \geqq 6$.

1.9. *If p is an odd prime, then the group*

$$G = \langle a, b | a^{p^2} = b^p = 1, b^{-1}ab = a^{p+1}\rangle$$

is always in Class I.

From 1.3, 1.4, 1.5 and 1.8 it follows that

1.10. *Every group G in Class* I *other than* C_2 *has order* $\geqq 12$.

This section concludes with some additional terminology and notational conventions used in the sequel.
Let $X = X_{G,H}$ be a Cayley graph and let

$$A_x = \{\varphi \in A(X) | \varphi(x) = x\}.$$

The subgroup A_x is known as the *stabilizer* of x.

As mentioned in [**9**], showing that every element of A_1 fixes H point-wise is sufficient for proving that G is in *Class* I. Even stronger results will be obtained below.

The reader must take care to distinguish between (graph-) automorphisms of $X_{G,H}$ including those which fix 1, and (group-) automorphisms of G. We shall write $\mathrm{Aut}(G)$ for the group of automorphisms of G.

For $g \in G$, we define the vertex-map

$$\lambda_g : x \mapsto gx, \qquad x \in V(X_{G,H}),$$

and observe that $\lambda_g \in A(X_{G,H})$. Also $\lambda_g^{-1} = \lambda_{g^{-1}}$. On the other hand, right-multiplication $x \mapsto xg$ yields a graph-automorphism if and only if $g \in Z(G)$, the center of G. The following is immediate:

PROPOSITION 1.11. *For all* $x, g \in A(X_{G,H})$,

$$\lambda_g^{-1} A_{gx} \lambda_g = A_x.$$

This proposition is merely a particularly suitable formulation of the fact that point-stabilizers are conjugate subgroups of a transitive permutation group.

We shall always assume that $|G| > 2$.

If $K \subset G$, the subgroup of G generated by the elements of K will be denoted by $\langle K \rangle$. If $x \in G$, $Z(x)$ denoted the centralizer of x in G.

2. Machinery and a proof of Theorem 1. Let G be given. Let $H \subset G$ with $\langle H \rangle = G$, and let $X = X_{G,H}$.

PROPOSITION 2.1. *Let* $K \subset H$. *Suppose* $\varphi(K) = K$ *for all* $\varphi \in A_1$. *Then* $\varphi(\langle K \rangle) = \langle K \rangle$ *for all* $\varphi \in A_1$.

Proof. We first show that for all $x \in G$ and $\omega \in A_x$, $\omega(xK) = xK$. By Proposition 1.11, $\lambda_x^{-1}\omega\lambda_x \in A_1$. By hypothesis $\lambda_x^{-1}\omega\lambda_x(K) = K$, so $\omega(xK) = \omega\lambda_x(K) = \lambda_x(K) = xK$.

Let $\varphi \in A_1$ and let $k_1 \in K$. Let $k_2 = \varphi(k_1)$, and let $k \in K$. Then $\varphi(k_1 k_2) = k_2 h$ for some $h \in H$. Observe that

$$\lambda_{k_1 k_2^{-1}} \varphi \in A_{k_1}$$

and so $\lambda_{k_1 k_2^{-1}} \varphi(k_1 K) = k_1 K$. Thus

$$k_2 h = \varphi(k_1 k) \in (\lambda_{k_1 k_2^{-1}})^{-1}(k_1 K) = (k_2 k_1^{-1})(k_1 K) = k_2 K.$$

Thus $h \in K$.

The remainder of the proof is straightforward induction on t. Thus it is first shown that $\omega(xK^t) = xK^t$ for all $x \in G$, $\omega \in A_x$. Secondly one chooses $k \in K^t$, thereby showing that if

$$c = k_1 k_2 \ldots k_{t+1} \quad \text{where} \quad k_1, \ldots, k_{t+1} \in K, \quad \text{then} \quad \varphi(c) \in \langle K \rangle.$$

PROPOSITION 2.2. *Let $h \in H$ and suppose that $\psi \in A_1 \Rightarrow \psi(h) = h$ or h^{-1}. If $\varphi \in A_1$, then $\varphi(h) = h \Rightarrow \varphi(h^k) = h^k$ for all k, and $\varphi(h) = h^{-1} \Rightarrow \varphi(h^k) = h^{-k}$ for all k.*

Proof. We proceed by induction on k. The proposition is immediate for $k = 1$. Let m be a positive integer and suppose the proposition is true for all $k \leqq m$. We prove it for $k = m + 1$.

Let $\varphi \in A_1$. If also $\varphi \in A_h$, then $\lambda_h^{-1}\varphi\lambda_h \in A_1$ by Proposition 1.11. Applying the hypothesis, suppose on the one hand that $\lambda_h^{-1}\varphi\lambda_h(h) = h^{-1}$. Multiplication on the left by h then gives $1 = \varphi\lambda_h(h) = \varphi(h^2)$, or $h^2 = 1$, in which case the proposition holds trivially. Otherwise we have $\lambda_h^{-1}\varphi\lambda_h \in A_h$. By the induction hypothesis

$$h^m = \lambda_h^{-1}\varphi\lambda_h(h^m) = h^{-1}\varphi(h^{m+1}),$$

whence

$$\varphi(h^{m+1}) = h^{m+1}.$$

If $\varphi(h) = h^{-1}$, then clearly $\lambda_h\varphi\lambda_h \in A_1$. Also $\lambda_h\varphi\lambda_h(h^{-1}) = h\varphi(1) = h$. By the induction hypothesis applied to $\lambda_h\varphi\lambda_h$,

$$h^{-m} = \lambda_h\varphi\lambda_h(h^m) = h\varphi(h^{m+1}),$$

whence

$$\varphi(h^{m+1}) = h^{-(m+1)},$$

which proves the proposition.

PROPOSITION 2.3. *Let $K \subset H$. Suppose that $A_k = A_1$ for all $k \in K$. Then $A_k = A_1$ for all $k \in \langle K \rangle$.*

Proof. Let $\varphi \in A_1$ and $k_1, k_2 \in K$. For this proof write $\lambda_i = \lambda_{k_i}$ $(i = 1, 2)$.

Since $\lambda_i^{-1}\varphi\lambda_i \in A_1$ $(i = 1, 2)$, we have by hypothesis $\lambda_i^{-1}\varphi\lambda_i(k) = k$ for all $k \in K$. Hence

$$k_2 = \lambda_1^{-1}\varphi\lambda_1(k_2) = k_1^{-1}\varphi(k_1k_2),$$

whence

$$\varphi(k_1k_2) = k_1k_2. \tag{2-1}$$

By Proposition 2.2, each $\varphi \in A_1$ fixes each power of k for each $k \in K$, and by (2-1), all $\varphi \in A_1$ fix all powers of any product of any two elements of K. Now suppose $k_3, k_4 \in \langle K \rangle$ and that $\varphi(k_i) = k_i$ $(i = 3, 4)$ for all $\varphi \in A_1$. By an identical argument one obtains $\varphi(k_3k_4) = k_3k_4$, and hence that all $\varphi \in A_1$ fix $\langle K \rangle$ point-wise.

COROLLARY 2.4. *Let $K \subset H$ and suppose $\langle K \rangle = G$. Suppose $A_k = A_1$ for all $k \in K$. Then G is in Class* I.

Proof. The result is an immediate consequence of Proposition 2.3 and the definitions.

The following seemingly unmotivated result has application in Section 3.

PROPOSITION 2.5. *Let there exist some $a \in H$ such that $\langle a \rangle \lhd G$ and $A_1 = A_a$. Suppose further that for some $b \in H$ and all $\varphi \in A_1$ either $\varphi(b) = b$ or $\varphi(b) = b^{-1}$ while $\varphi_0(b) = b^{-1}$ holds for some $\varphi_0 \in A_1$. Then $b^2 \in Z(a)$.*

Proof. Let the hypothesis be satisfied. Since $\langle a \rangle \lhd G$, a relation of the form

$$b^{-1}ab = a^k \tag{2-2}$$

must hold for some integer k. If $a^r = 1$, let us assume $k^2 \not\equiv 1 \pmod r$ or else $b^{-2}ab^2 = a^{k^2} = a$, in which case $b^2 \in Z(a)$.

Since $\varphi_0(b) = b^{-1}$, observe that $\lambda_b\varphi_0\lambda_b \in A_1$ and so by the hypothesis and Proposition 2.2,

$$a^k = \lambda_b\varphi_0\lambda_b(a^k) = b\varphi_0(ba^k),$$

whence

$$\varphi_0(ba^k) = b^{-1}a^k. \tag{2-3}$$

Since $\lambda_a{}^{-1}\varphi_0\lambda_a \in A_1$, the hypothesis implies that $\lambda_a{}^{-1}\varphi_0\lambda_a(b) = b$ or b^{-1}. The one case gives

$$b = \lambda_a{}^{-1}\varphi_0\lambda_a(b) = a^{-1}\varphi_0(ab)$$

whence by (2-2) and (2-3),

$$ab = \varphi_0(ab) = \varphi_0(ba^k) = b^{-1}a^k,$$

and so $bab = a^k = b^{-1}ab$, or $b^2 = 1 \in Z(a)$. The other case gives $b^{-1} = \lambda_a{}^{-1}\varphi_0\lambda_a(b) = a^{-1}\varphi_0(ab)$, and so by (2-2) and (2-3),

$$ab^{-1} = \varphi_0(ab) = \varphi_0(ba^k) = b^{-1}a^k.$$

Thus

$$a = b^{-1}a^kb = a^{k^2},$$

contrary to assumption, and the proposition is proved.

We shall require the following result from [**10**].

LEMMA. *Let G be a finite non-abelian group in Class* I. *If the Cayley graph $X = X_{G,H}$ is a GRR of G, then*

$$3 \leqq |H| \leqq |G| - 4.$$

We are now prepared for the

Proof of Theorem 1. Let G be a non-abelian group in *Class* I and let G_1 be an extension of G by the cyclic group C_s. We shall represent C_s as the group $\{e_1, b, b^2, \ldots, b^{s-1}\}$ and let t be the smallest positive integer for which $b^t \in G$. Thus $t|s$, and by hypothesis, $t \geqq 5$. We mention that $G \lhd G_1$ and that G_1/G consists of cosets

$$e_1G, bG, b^2G, \ldots, b^{t-1}G. \tag{2-4}$$

The identities of C_s and G will be denoted by e_1 $(=b^0)$ and e_2, respectively.

The elements of G_1 have unique representation, therefore, as

$$b^i x, \qquad i = 0, 1, \ldots, t-1; x \in G. \tag{2-5}$$

There exists an automorphism $\beta \in \mathrm{Aut}(G)$ determined by G_1 where for $x \in G$, $\beta(x)$ is given by

$$b^{-1}xb = \beta(x).$$

Multiplication in G_1 is thus given by

$$(b^i x)(b^j y) = b^{i+j}\beta^j(x)y. \tag{2-6}$$

(While regretting that the notation here introduced is unconventional from the algebraists' standpoint, we feel that it is the least inconvenient for the proof at hand.)

We assert:

$$\text{there exists } g \in G \text{ such that } \beta(g) \neq g^{-1}. \tag{2-7}$$

Were (2-7) to fail, G would admit the inverting automorphism and would therefore be abelian. Since G is in *Class* I, G would be an elementary abelian 2-group, by 1.3 and 1.2, in which case β reduces to the identity automorphism. But then G_1 would be the direct product $G \times C_t$, contrary to the assumption that G_1 is non-abelian.

Let the Cayley graph $X = X_{G,H}$ be a *GRR* of G. If we let $H' = G \backslash (H \cup \{e_2\})$, then $X_{G,H'}$ is a *GRR* of G if and only if $X_{G,H}$ is. Without loss of generality we may therefore assume that

$$2|H| < |G|. \tag{2-8}$$

In terms of H and an element g assured by (2-7), form the Cayley graph $Y = X_{G_1,J}$ where J is given by

$$J = e_1H \cup (bG\backslash\{be_2, bg\}) \cup (b^{-1}G\backslash\{b^{-1}e_2, b^{-1}\beta^{-1}(g^{-1})\}) \cup \{b^2e_2, b^{-2}e_2\}.$$

Clearly $\langle J \rangle = G_1$. We shall demonstrate that Y is a *GRR* of G.

Let X_i denote the subgraph of Y induced by the vertices in the coset b^iG. Evidently X_i is isomorphic to X for each i. Let Y_0 be the subgraph of Y induced by the set of those vertices of Y identified with elements of J, that is, the set of vertices adjacent to (but not including) the identity e_1e_2 of the group G_1. Let $J_i = J \cap b^iG$. For $b^ix \in V(Y_0)$, let $\rho_0(b^ix)$ denote the number of vertices in Y_0 adjacent to it.

Let A_e denote the subgroup of $A(Y)$ which stabilizes the vertex e_1e_2. To prove the theorem, it will suffice to show that A_e is trivial. Let $\varphi \in A_e$. Then the restriction of φ to Y_0 belongs to $A(Y_0)$.

We first show that

$$\rho_0(b^{\pm 1}y) < \rho_0(e_1x), \quad \text{for all } e_1x, b^{\pm 1}y \in J. \tag{2-9}$$

Since e_1x is non-adjacent to precisely two vertices of X_1 and two vertices of

X_{-1} and since these four vertices could all lie in Y_0,

(2-10) $$\rho_0(e_1x) \geqq 2(|G| - 4).$$

The vertex by is adjacent to at most $|H|$ vertices in each of J_0 and J_1 and at most one vertex in each of J_{-1} and J_2. If $t = 5$, by might be adjacent to $b^{-2}e_2$. Since a symmetrical argument holds also for $b^{-1}y$, it follows from (2-8) that

(2-11) $$\rho_0(b^{\pm 1}y) \leqq 2|H| + 3 \leqq |G| + 2.$$

If (2-9) were false for some $e_1x, b^{\pm 1}y \in J$, then (2-10) and (2-11) would imply that $|G| + 2 \geqq 2|G| - 8$, or $|G| \leqq 10$. Since G is in *Class* I, we must then have $G \cong C_2$ by 1.10. But then G_1 would be abelian, contrary to hypothesis. This proves (2-9).

Observe that the neighbors of b^2e_2 in Y_0 include at most the $|G| - 2$ vertices of J_1 and, in case s $= 6$, the vertex $b^{-2}e_2$. Since a symmetric argument holds for $b^{-2}e_2$, we have

(2-12) $$\rho_0(b^{\pm 2}e_2) \leqq |G| - 1.$$

By comparison of (2-10) with (2-12), the foregoing argument implies, *a fortiori*, that $\rho_0(b^{\pm 2}e_2) < \rho_0(e_1x)$.

These inequalities imply that $\varphi(J_0) = J_0$. By Proposition 2.1, φ must fix $e_1G = \langle J_0 \rangle$ set-wise. Hence the restriction of φ to e_1G is a graph-automorphism of X_0 which fixes e_1e_2. Since X_0 is a *GRR* of G, it follows that φ restricted to X_0 is the identity; i.e.,

(2-13) $$\varphi(e_1x) = e_1x, \qquad x \in G, \varphi \in A_e.$$

Each element of $J_{\pm 1}$ is adjacent to at least $|H| - 2$ vertices in J_0, and $|H| - 2 > 0$ by the Lemma. Meanwhile b^2e_2 and $b^{-2}e_2$ are adjacent to no vertex in J_0. Thus φ either fixes b^2e_2 and $b^{-2}e_2$ or interchanges them. Moreover, φ fixes $J_1 \cup J_{-1}$ setwise.

If any one vertex of $X_1 \cup X_{-1}$ were fixed by all $\varphi \in A_e$, then because of (2-13), each $\varphi \in A_e$ would fix point-wise a generating set of G_1; the Theorem would then follow by Corollary 2.4. The four vertices

(2-14) $$be_2, b^{-1}e_2, bg, b^{-1}\beta^{-1}(g^{-1})$$

are the only vertices of Y simultaneously non-adjacent to the fixed-point e_1e_2 but adjacent, each, to $|G| - 2$ other vertices of X_0. Each $\varphi \in A_e$ acts as a permutation on the 4-set of vertices (2-14).

The vertex $e_1g \in V(X_0)$ is non-adjacent to $b^{-1}e_2$ since $(b^{-1}e_2)(bg) = e_1g$. But e_1g is adjacent to $be_2 = (e_1g)(b\beta(g^{-1}))$ since $bg \neq b\beta(g^{-1})$ by (2-7), and so $b\beta(g^{-1}) \in J$. Since e_1g is a fixed-point of A_e, we deduce

$$\varphi(be_2) \neq b^{-1}e_2; \varphi(b^{-1}e_2) \neq be_2; \text{ for all } \varphi \in A_e.$$

Since $(b^{-1}e_2)(b^2e_2) = be_2$, φ must map $\{be_2, b^{-1}e_2\}$ onto an adjacent pair from

(2-14). Writing $(b^{-1}\beta^{-1}(g^{-1}))w = bg$ and solving for w, we obtain $w = (bg)^2 = b^2\beta(g)g$, which is in J only if $\beta(g)g = e_2$, contrary to (2-7). Thus bg is not adjacent to $b^{-1}\beta(g^{-1})$. In summary,

$$\varphi \in A_e \backslash A_{be_2} \Rightarrow \varphi \in A_{b^{-1}e_2} \tag{2-15}$$

and

$$\psi \in A_e \backslash A_{b^{-1}e_2} \Rightarrow \psi \in A_{be_2}. \tag{2-16}$$

Since none of the vertices (2-14) is a fixed-point of A_e, there indeed exist $\varphi, \psi \in A_e$ such that (2-15) and (2-16), respectively, are satisfied non-vacuously. Consider the action of their composite $\psi\varphi \in A_e$. By (2-15) and (2-16), $\psi\varphi(b^{-1}e_2) = \psi(b^{-1}e_2) \neq b^{-1}e_2$. Hence $\psi\varphi(be_2) = be_2 = \psi(be_2)$ by (2-16). But this implies that $\varphi(be_2) = be_2$ contrary to assumption. This completes the proof of the Theorem.

Remark. In the foregoing proof $A(Y)$ is an imprimitive permutation group and the cosets (2-4) comprise a complete block system for $A(Y)$.

3. Classification of semi-direct products of cyclic groups. For some integers r, s, and k, we assume the group G is given by

$$G = \langle a, b | a^r = b^s = 1; b^{-1}ab = a^k \rangle \tag{3-1}$$

and understand that (3-1) gives the entire multiplication table for G. It is thus well-known that $|G| = rs$, that

$$(r, k) = 1, \tag{3-2}$$

and that

$$k^s \equiv 1 \pmod r. \tag{3-3}$$

THEOREM 2. *The group G given by* (3-1) *is in Class* II *under each of the following conditions:*

(i) $k \equiv 1 \pmod r$ (*abelian group*).
(ii) $s = 2$, $r = 3, 4, 5$, *and* $k \equiv -1 \pmod r$ (*dihedral groups* D_3, D_4, D_5).
(iii) $s = 4$ *and* $k \equiv -1 \pmod r$ (*generalized dicyclic group*).
(iv) *G is the group* $\langle a, b | a^8 = b^2 = 1, bab = a^5 \rangle$.

Otherwise G is in Class I.

The remainder of this section is devoted to demonstrating Theorem 2.

If either (i) or (ii) holds, it is known that G is in *Class* II. (See 1.8, 1.3, and 1.4 above.)

Under condition (iii), consider the abelian subgroup $A = \langle a, b^2 \rangle$ of G. Then (1-1) is surely satisfied. Thus G is a generalized dicyclic group and hence in *Class* II. (See 1.5 above.)

PROPOSITION 3.1. *The group* $G = \langle a, b; a^8 = b^2 = 1, bab = a^5 \rangle$ *is in Class* II.

Proof. We must show that for any arbitrary set H which generates G, there exists a non-identity automorphism $\varphi \in \mathrm{Aut}(G)$ such that $\varphi(H) = H$.

First observe that $Z(G) = \{1, a^2, a^4, a^6\}$. Since a^4 is the only central element of order 2, $\varphi(a^4) = a^4$ for all $\varphi \in \mathrm{Aut}(G)$. Also any automorphism of G either fixes a^2 or interchanges a^2 with its inverse a^6. So the presence or absence of a^2, a^4, a^6 in H is immaterial.

Note that ba^2 and ba^6 are inverses and are the only non-central elements of order 4 in G. Hence any $\varphi \in \mathrm{Aut}(G)$ either fixes ba^2 or interchanges ba^2 with ba^6 and their membership in H has no effect.

Consider the set

$$S = \{a, a^3, a^5, a^{-1}, ba, ba^3, ba^5, ba^{-1}\}$$

of all elements of order 8. We shall show that for any set $T \subset S$ with $T = T^{-1}$, there exists a non-identity $\varphi \in \mathrm{Aut}(G)$ such that $\varphi(T) = T$. Moreover, whereas b and ba^4 are the only non-central inversions in G, we will show that φ can be chosen so that

$$\varphi(b) = b \tag{3-4}$$

and that, therefore, the presence of b or ba^4 in H is also immaterial.

If $T = \{a, a^3, a^5, a^{-1}\}$, $\{a, a^{-1}\}$, or $\{a^3, a^5\}$, let

$$\varphi(a) = a^{-1}. \tag{3-5}$$

Then (3-4) and (3-5) determine $\varphi \in \mathrm{Aut}(G)$ by $\varphi(a^j) = a^{-j}$ for all j and $\varphi(ba^j) = ba^{-j} = \varphi(b)\varphi(a^j)$. Observe that $\varphi(T) = T$ and φ is not the identity.

If $T = \{a, a^{-1}, ba, ba^3\}$, let

$$\varphi(a) = ba. \tag{3-6}$$

Then (3-4) and (3-6) are extendable to an automorphism of G by $\varphi(a^j) = (ba)^j$. Hence $\varphi(ba) = \varphi(b)\varphi(a) = b(ba) = a$ and $a^{-1} = [\varphi(ba)]^{-1} = \varphi((ba)^{-1}) = \varphi(ba^3)$, whence $\varphi(T) = T$.

If $T = \{a, a^{-1}, ba^5, ba^{-1}\}$, define

$$\varphi(a) = ba^5. \tag{3-7}$$

Then (3-4) and (3-7) are extendable to an automorphism of G by $\varphi(a^j) = (ba^5)^j$. Then $\varphi(ba^5) = \varphi(b)\varphi(a^5) = b(ba^5)^5 = b(ba) = a$. And so φ interchanges a^{-1} and ba^{-1}. Thus $\varphi(T) = T$.

Finally, if $T = \{ba, ba^3\}$ or $\{ba^5, ba^{-1}\}$, define φ by

$$\varphi(ba) = (ba)^{-1}.$$

Then φ preserves T.

The above cases are all that need be considered, for if $\varphi(T) = T$, then $\varphi(S\backslash T) = S\backslash T$, and the proof of the proposition is complete.

Assuming none of (i)–(iv) to hold, we shall choose for each G given by (3-1) a generating set H and demonstrate that the corresponding graph $X = X_{G,H}$ is a *GRR* for G. As in the previous section, let X_1 always denote the subgraph of X induced by the vertex set H. Recall that the restriction of each automorphism in A_1 to H belongs to $A(X_1)$.

An edge $[x, z]$ belongs to X_1 if and only if some identity

$$xy = z \tag{3-8}$$

holds for some $x, y, z \in H$. However, (3-8) gives rise to two further identities:

$$z^{-1}x = y^{-1} \quad \text{and} \quad x^{-1}z = y,$$

and so $[z^{-1}, y^{-1}]$ and $[x^{-1}, y]$ are also in $E(X_1)$. An identity of the form

$$x^2 = y \tag{3-9}$$

for $x, y \in H$ implies $x \neq x^{-1}$ and gives rise to identities $x^{-2} = y^{-1}$ and $x^{-1}y = x$, and so edges $[x, y]$, $[x^{-1}, y^{-1}]$, and $[x^{-1}, x] \in E(X_1)$. Thus in general (that is, except when, for example, $y = x^{-1}$ in (3-9)), each identity (3-8) gives rise to three edges in X_1. One observes, moreover, that vertices x and x^{-1} always have the same valence in X_1. This valence will be denoted by $\rho_1(x)$.

The foregoing remarks are meant to assist the reader in checking the graphs X_1 in the various figures corresponding to various families of groups (3-1) presently to be considered. Solid lines indicate edges always present in X_1 Hatched lines represent additional edges present in particular cases to be indicated.

PROPOSITION 3.2. *If* $k^2 \not\equiv 1 \pmod r$, *then G is in Class* I.

Proof. The cases $k \equiv \pm 1 \pmod r$ are excluded here. Thus $r \geqq 5$. (All non-abelian groups (3-1) for which s is odd are covered by this proposition. Also covered are the K-metacyclic groups of Coxeter and Moser [**2**, p. 11].)

Case 1: $s \geqq 5$. Let H consist of elements a, b, b^2, ba, ba^{-k} and their respective inverses a^{-1}, b^{-1}, b^{-2}, $b^{-1}a^{-k^{s-1}}$, $b^{-1}a$.

$s \geqq 5,\ k^2 \not\equiv 1 \pmod r$

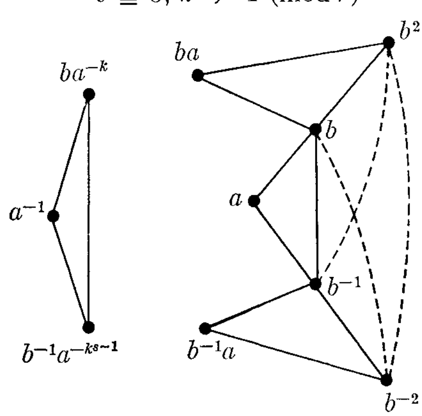

FIGURE 1

It is straightforward (though admittedly rather tedious) to verify that the graph of Figure 1 shows the entire graph X_1 with respect to a given set H (where the hatched lines represent edges present if and only if $s = 5$) except under the following two special conditions.

If

(3-10) $$2k \equiv -1 \pmod r$$

holds, then $a^{-1}(ba^{-k}) = ba^{-2k} = ba$. By the argument preceding this Proposition, X_1 contains three additional edges

(3-11) $$[a^{-1}, ba], [b^{-1}a, b^{-1}a^{-k^{s-1}}], \text{ and } [a, ba^{-k}].$$

If

(3-12) $$k \equiv -2 \pmod r,$$

then $(b^{-1}a)(ba) = (b^{-1}ab)a = a^{k+1} = a^{-1}$ and so X_1 contains three additional edges

(3-13) $$[b^{-1}a, a^{-1}], [a, b^{-1}a^{-k^{s-1}}], \text{ and } [ba, ba^{-k}].$$

(The lengthy but elementary demonstration that no other congruence (3-8) in H can obtain under the assumptions is omitted.)

Not both (3-10) and (3-12) can hold simultaneously since $r > 3$.

Referring back to Figure 1, vertices b and b^{-1} alone have valence 4 in X_1 if $s > 5$ (valence 5 if $s = 5$). The valence in X_1 of any other vertex is $\leqq 3$ if $s > 5$ ($\leqq 4$ if $s = 5$), independent of whether (3-10) or (3-12) holds. Thus if $\varphi \in A(X_1)$, φ either fixes or interchanges b and b^{-1}. Since a is the unique vertex adjacent to both b and b^{-1} if $s > 5$ (the vertex of least valence adjacent to b and b^{-1} if $s = 5$), $\varphi(a) = a$.

For any extension $\bar{\varphi} \in A_1$ of any $\varphi \in A(X_1)$, $\bar{\varphi}(a) = a$, and $\bar{\varphi}(b) = b$ or b^{-1}. Since $k^2 \not\equiv 1 \pmod r$, $b^2 \notin Z(a)$. Thus $\bar{\varphi}(b) = b$ for all $\bar{\varphi} \in A_1$, by Proposition 2.5. Since $G = \langle a, b\rangle$, G is in *Class* I by Corollary 2.4.

Case 2: $s = 3$. The procedure is not unlike that of Case 1. The generator b^2 is now redundant and so H is simply

$$H = \{a, a^{-1}, b, b^{-1}, ba, b^{-1}a^{-k^2}, ba^{-k}, b^{-1}a\}.$$

Elements of G not in $\langle a\rangle$, being of order 3, are now adjacent to their inverses in X. The edge $[ba^{-k}, b^{-1}a^{-k^{s-1}}]$ induced by b^2 in Case 1 is now induced by b^{-1} and so the additional edges $[b, b^{-1}a]$ and $[ba, b^{-1}]$ are also present. The graph of Figure 1 now assumes the form of Figure 2.

Observe in Figure 2 that the edge $[b, b^{-1}]$ lies on three triangles while no other edge lies on more than one. Even if one of the congruences (3-10) or (3-12) were to hold (and, as before, at most one of them can hold), no other edge of X_1 would lie on more than two triangles. If $\varphi \in A(X_1)$, then φ must fix this edge $[b, b^{-1}]$. Hence φ must permute the three vertices a, ba, $b^{-1}a$ (which are adjacent to both b and b^{-1}) among themselves. Whether or not

$s = 3,\ k^2 \not\equiv 1 \pmod r$

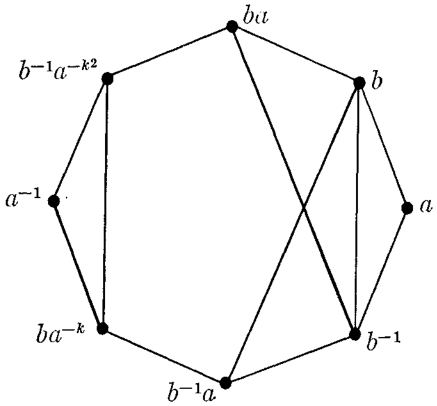

FIGURE 2

(3-10) or (3-12) holds, $\rho_1(a)$ is one less than $\rho_1(ba)$ and $\rho_1(b^{-1}a)$. Hence $\varphi(a) = a$. The final paragraph of Case 1 can now be applied to G.

Case 3: $s = 4$. If neither (3-10) nor (3-12) holds, let

$$H = \{a, a^{-1}, b, b^{-1}, ba, b^{-1}a^{-k^3}, ba^{-k}, b^{-1}a\}.$$

Then X_1 has the form of Figure 3(a) whence it is obvious for all $\varphi \in A(X_1)$ that $\varphi(a) = a$ and $\varphi(b) = b$ or b^{-1}.

$k^2 \not\equiv 1 \pmod r;\ s = 4$

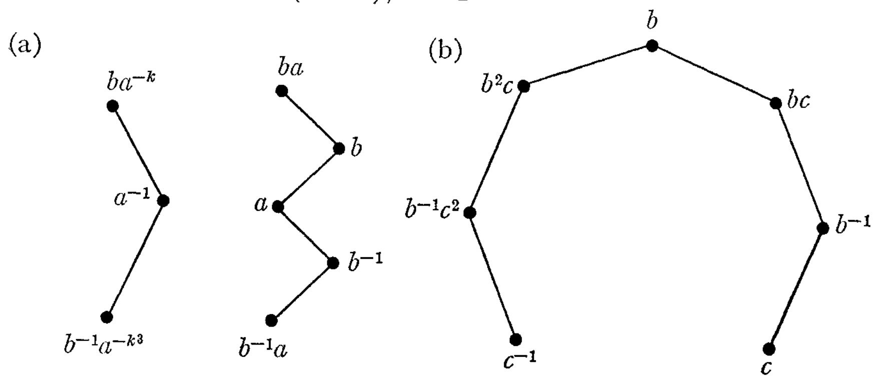

FIGURE 3

We continue then exactly as in Case 1.

If one of these congruences does hold, however, the method of Case 1 is unsuitable. Instead we begin by demonstrating that r must be a divisor of 15.

First assume that (3-12) holds. By (3-3), $16 = k^4 \equiv 1 \pmod r$, whence $r|15$. If (3-10) holds, then $k^4 \equiv \pm 2k \pmod r$, and by (3-2), $k^3 \equiv \pm 2 \pmod r$.

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Let $c = b^{-1}$. Then $G = \langle a, c\rangle$ with $a^r = c^4 = 1$ and $c^{-1}ac = bab^{-1} = a^{k^3} = a^{\pm 2}$. By (3-3), $16 = (\pm 2)^4 \equiv 1 \pmod r$, and again $r|15$. Since $k^2 \not\equiv 1 \pmod r$, either

(3-14) $$r = 5; k = 2$$

or

(3-15) $$r = 15; k = 2 \text{ or } 7.$$

First consider the K-metacyclic group (see [**2**, p. 11])

$$F = \langle c, b | c^5 = b^4 = 1; b^{-1}cb = c^2\rangle$$

for which (3-14) applies and choose generating set

(3-16) $$H = \{c, c^{-1}, b, b^{-1}, bc, b^{-1}c^2, b^2c\}.$$

(Note that $(b^2c)^2 = 1$.) The subgraph X_1 of $X_{G,H}$ thus formed is shown in Figure 3(b) to be an arc of even length. As the midpoint of the arc, b is fixed under $A(X_1)$. Hence b^{-1} is fixed under A_1, by Proposition 2.2. But then A_1 is trivial and F is in *Class* I.

Now let G be described by (3-15). Set $c = a^3$ and observe that F is the subgroup $\langle a^3, b\rangle$. To the set H of (3-16) adjoin elements a and a^{-1}. The effect on X_1 is to adjoin two isolated vertices but no new edges. If some $\varphi \in A_1$ interchanges a and a^{-1}, then φ interchanges c and c^{-1} by Proposition 2.2. However c is a fixed-point of A_1 by the foregoing argument, and G is in *Class* I, proving the proposition.

If $k^2 \equiv 1 \pmod r$, then (3-3) implies that s must be even and $b^2 \in Z(a)$.

PROPOSITION 3.3. *If* $k^2 \equiv 1 \pmod r$ *but* $k \not\equiv -1 \pmod r$, *then* G *is in Class* I *except when* G *is the group* $\langle a, b | a^8 = b^2 = 1; bab = a^5\rangle$.

Proof. By hypothesis, $r \geqq 8$ and $k \not\equiv \pm 1 \pmod r$. Assume that G is not the above-excluded group of order 16.

Case 1: $s = 2$. If one defines

$$H = \{a, a^{-1}, a^2, a^{-2}, b, ba^{-1}, ba^k\},$$

the graph X_1 assumes the form of Figure 4.

$$k^2 \equiv 1 \pmod r,\ k \not\equiv -1 \pmod r,\ s = 2$$

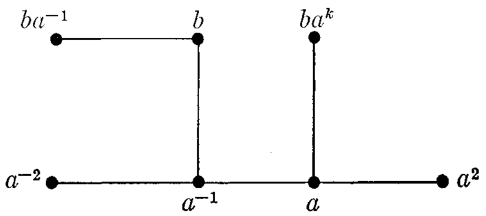

FIGURE 4

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Since a is the only vertex of X_1 adjacent to two vertices of valence 1, $\varphi(a) = a$ for all $\varphi \in A(X_1)$. Since b is the only vertex with $\rho_1(b) = 2$, $\varphi(b) = b$ for all $\varphi \in A(X_1)$. Thus any $\bar{\varphi} \in A_1$ fixes a and b, and G is in *Class* I by Corollary 2.4.

(Remarks. The semi-dihedral groups are covered by the present case. When G is the excluded group, $3k \equiv -1 \pmod r$ obtains, introducing additional identities such as $a^2(ba^k) = ba^{3k} = a^{-1}$, etc.)

Case 2: $s \geqq 8$. Let H consist of a, a^2, b, b^2, b^3, ba and their inverses. Note that $(ba)^{-1} = b^{-1}a^{-k}$. Then X_1 has the form of Figure 5, where the hatched lines represent edges present when $s = 8$.

$$k^2 \equiv 1 \pmod r;\ k \not\equiv -1 \pmod r,\ s \geqq 8$$

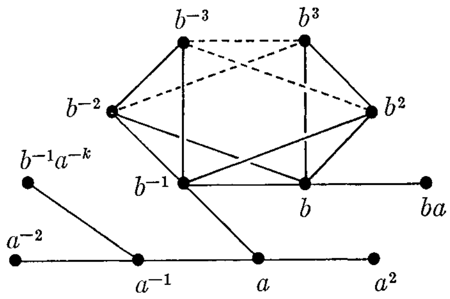

FIGURE 5

If $\varphi \in A(X_1)$, then $\varphi(a^{-1}) = a^{-1}$ since a^{-1} is the only vertex of X_1 adjacent to two vertices of valence 1. By Proposition 2.2, $\varphi(a) = a$ and so $\varphi(b^{-1}) = b^{-1}$. Since $\langle a, b^{-1}\rangle = G$, G is in *Class* I.

Case 3: $s = 6$. The procedure is identical to that of Case 2 except that the vertices b^3 and b^{-3} coalesce.

Case 4: $s = 4$. Take

$$H = \{a, a^{-1}, a^2, a^{-2}, b, b^{-1}, b^2, ba, b^{-1}a^{-k}\}.$$

Now X_1 has the form of Figure 6.

$$k^2 \equiv 1 \pmod r,\ k \not\equiv -1 \pmod r,\ s = 4$$

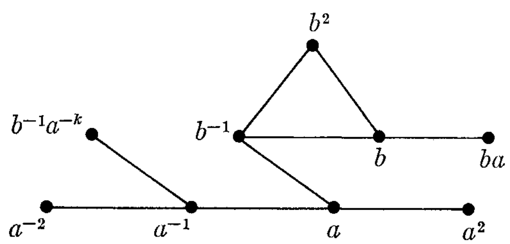

FIGURE 6

The argument is identical to that of Case 2.

The proof of the proposition is complete.

It remains to consider the case where $k \equiv -1 \pmod r$. If $s = 2$, then G is dihedral (see 1.8 above). If $s = 4$, then G is in *Class* II as already mentioned. Thus the following proposition is all that is required to complete the proof of Theorem 2.

PROPOSITION 3.4. *If* $k \equiv -1 \pmod r$ *and* $s \geqq 6$, *then* G *is in Class* I.

Remark. When $k \equiv -1 \pmod r$ and $s \equiv 2 \pmod 4$, then, as noted in [**2**, p. 10], $G = D_r \times C_{s/2}$. If $r \geqq 6$, then D_r is in *Class* I (see 1.8) and so when $s \geqq 10$, G is also in *Class* I by Theorem 1. Our method of proving Proposition 3.4 will be sufficiently general, however, so that this argument will not be required.

Proof. By assumption, $r \geqq 3$. If $s \geqq 8$, let H consist of a, b, b^3, ba, b^2a and their inverses a^{-1}, b^{-1}, b^{-3}, $b^{-1}a$, $b^{-2}a^{-1}$, respectively. Then X_1 has the form of Figure 7, where edge $[a, a^{-1}]$ is present if and only if $r = 3$.

$k \equiv -1 \pmod r$; $s \geqq 8$

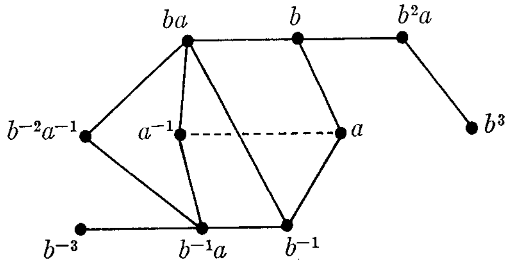

FIGURE 7

If $s = 6$, let H be the same with the understanding that b^3 and b^{-3} are identified. The effect on X_1 (Figure 7) is only to coalesce the two vertices so labeled. If both $r = 3$ and $s = 6$, then b^2a is adjacent to its inverse $b^{-2}a^{-1}$.

Since ba and $b^{-1}a$ are the only vertices of X_1 with valence 4, every $\varphi \in A(X_1)$ either fixes or interchanges them.

When $r > 3$, b^{-1} is the only vertex of valence 3 adjacent to both ba and $b^{-1}a$ and so b^{-1} is fixed under $A(X_1)$. By Proposition 2.2, so is b. Since a is the unique vertex of valence <4 adjacent to both b and b^{-1}, a is also a fixed-point of $A(X_1)$.

When $r = 3$ and $s > 6$, $b^{-2}a^{-1}$ is the unique vertex in X_1 of valence 2 adjacent to both ba and $b^{-1}a$, and so $b^{-2}a^{-1}$ is fixed by $A(X_1)$. By Proposition 2.2, b^2a is also a fixed-point. But then b is a fixed-point since it is the unique vertex of valence 3 adjacent to b^2a. Finally a is fixed, being the unique vertex of valence 3 adjacent to b.

If $s = 6$, then b^3 is fixed, being the unique vertex of valence 2. Its neighbors b^2a and $b^{-1}a$, being of different valences, are therefore each fixed. But $\{b^3, b^{-1}a\}$ generates G since $b = b^3(b^{-1}a)^2$ and $a = b(b^{-1}a)$.

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Thus a and b are fixed under $A(X_1)$, and the argument can proceed as in the previous cases, giving that G is in *Class* I.

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