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Graphs of Holomorphic Functions with Isolated Singularities Are Complete Pluripolar

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1. Introduction

In classical potential theory one encounters the notions of polar set and complete polar set. A set $E \subset \mathbb{R}^n$ is called *polar* if there exists a subharmonic u on a neighborhood of E such that $E \subset \{x : u(x) = -\infty\}$; E is called *complete polar* if one actually has $E = \{x : u(x) = -\infty\}$. (The function identically equal to $-\infty$ is not considered to be subharmonic.) It is well known that we may take u to be defined on all of $\mathbb{R}^n$ and also that E is complete polar if and only if E is polar and a G_δ (cf. [5]).

In pluripotential theory the situation is more complicated. A set E in a domain $D \subset \mathbb{C}^n$ is called *pluripolar in D* if there exists a plurisubharmonic function u on D such that $E \subset \{z : u(z) = -\infty\}$; E is called *complete pluripolar in D* if, for some plurisubharmonic function u on D, we have $E = \{z : u(z) = -\infty\}$. Although Josefson's theorem [4] asserts that E being pluripolar in D implies that E is pluripolar in $\mathbb{C}^n$, the corresponding assertion is false in the complete pluripolar setting. Also, a pluripolar G_δ need not be complete: the open unit disk Δ in the complex line $z_2 = 0$ in $\mathbb{C}^2$ is a G_δ but is not complete in $\mathbb{C}^2$. In fact, every plurisubharmonic function on $\mathbb{C}^2$ that equals $-\infty$ on Δ must equal $-\infty$ on the line $z_2 = 0$. Thus, it is reasonable to introduce the *pluripolar hull* of a pluripolar set $E \subset D$ as

$$E_D^* = \left\{z \in D : u\big|_E = -\infty \implies u(z) = -\infty \ \forall u \in \mathrm{PSH}(D)\right\},$$

where $\mathrm{PSH}(D)$ denotes the set of all plurisubharmonic functions on D. We also have use for the *negative pluripolar hull*,

$$E_D^- = \left\{z \in D : u\big|_E = -\infty \implies u(z) = -\infty \ \forall u \in \mathrm{PSH}(D),\ u \le 0\right\}.$$

If E is complete pluripolar in D then clearly E is a G_δ and $E_D^* = E$. A partial converse is Zeriahi's theorem [11].

Theorem 1. *Let E be a pluripolar subset of a pseudoconvex domain D in $\mathbb{C}^n$. If $E_D^* = E$ and E is a G_δ as well as an F_σ, then E is complete pluripolar in D.*

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Completeness and hulls of various pluripolar sets have been studied in [6; 7; 9; 10].

In [9], Sadullaev posed the following questions. Consider the sets

$$E_1 = \{(x, y) \in \mathbb{C}^2 : y = x^\alpha,\ x \in (0, 1)\},$$

with α irrational, and

$$E_2 = \{(x, y) \in \mathbb{C}^2 : y = e^{-1/x},\ x \in (0, 1)\}.$$

Sadullaev asked if there exists a plurisubharmonic function h_j on a neighborhood V of $\overline{E_j}$ such that

$$h_j(0) > \limsup_{\substack{(x,y)\to 0\\ (x,y)\in E_j}} h_j(x, y)$$

(see also Bedford's survey [1]). In [7], Levenberg and Poletsky gave a positive answer for E_1; in [10], the author gave a positive answer for E_2. In fact, it was shown in both cases that the pluripolar hulls $E^*_{\mathbb{C}^2}$ are equal to the graphs of the maximal analytic extension of $y = x^\alpha$ (respectively, $y = e^{-1/x}$).

In the present paper we generalize the results of [10] as follows.

THEOREM 2. *Suppose that D is a domain in $\mathbb{C}$ and that A is a sequence of points in D without density point in D. Let f be holomorphic on $D \setminus A$, and let E denote the graph of f in $(D \setminus A) \times \mathbb{C}$. Then E is complete pluripolar in $D \times \mathbb{C}$.*

Necessary preliminaries are dealt with in Section 2. After some preparations, we give in Section 3 the proof of the theorem. Surprisingly, it is more elementary than the special cases that were treated in [10]; part of it resembles proofs in [6] and [9].

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2. Preliminaries

2.1. Pluripolar Hulls

We will need some facts about the pluripolar hulls E^*_D and E^-_D, which were defined in Section 1. Of course, $E^*_D \subset E^-_D$. Moreover, these hulls are related as follows (see [7]),

THEOREM 3. *Let D be pseudoconvex in $\mathbb{C}^N$ and let $E \subset D$ be pluripolar. Suppose $D = \bigcup_j D_j$, where the D_j form an increasing sequence of relatively compact pseudoconvex subdomains of D. Then*

$$E^*_D = \bigcup_j (E \cap D_j)^-_{D_j}.$$

Moreover, if D is hyperconvex (*i.e., if D admits a bounded plurisubharmonic exhaustion function*) *then*

$$E_D^- = \bigcup_j (E \cap D_j)_{D_j}^-.$$

2.2. Pluriharmonic Measure

The notion of pluriharmonic measure was introduced in [2; 9]. Let E be a subset of a domain $D \subset \mathbb{C}^n$. The *pluriharmonic measure* at $z \in D$ of E relative to D is the number

$$\omega(z, E, D) = -\sup\{u(z) : u \in \mathrm{PSH}(D) \text{ and } u \leq -\chi_E\}. \tag{2.1}$$

Here χ_E is the characteristic function of E on D.

Notice that we do not regularize the supremum in (2.1). Doing so would yield 0 if E were pluripolar (the case we will consider), and all information would be lost. For $n = 1$, E compact in D, and $z \in D \setminus E$, this notion boils down to the usual concept of harmonic measure at z of the boundary of E in the domain $D \setminus E$.

The relation of pluriharmonic measure with pluripolar hulls is given by the following proposition. The proof may be found in [7].

PROPOSITION 4. *Let D be a hyperconvex domain in $\mathbb{C}^N$ and let $E \subset D$ be pluripolar. Then*

$$E_D^- = \{z \in D : \omega(z, E, D) > 0\}.$$

The following observation will be used in the proof of Theorem 2. Let E be the graph of some holomorphic function f on a domain $G \subset \mathbb{C}$, and let B be a domain in $\mathbb{C}$. Next, let K be a closed disk of positive radius in G such that $f(K) \subset B$, and let E_K denote the graph of f over K. Then

$$(E_K)^*_{G\times B} = (E \cap (G \times B))^*_{G\times B}, \qquad (E_K)^-_{G\times B} = (E \cap (G \times B))^-_{G\times B}. \tag{2.2}$$

One may simply observe that a plurisubharmonic function on $G \times B$ that equals $-\infty$ on E_K equals $-\infty$ on $E \cap (G \times B)$.

3. Proof of Theorem 2

3.1. Construction of a Plurisubharmonic Function

Our isolated singularities will of course be poles or essential singularities. The principal part at such a singularity will be considered as an infinite Laurent series, which may consist of only finitely many nonzero terms.

PROPOSITION 5. *Let f be holomorphic on a bounded domain D in $\mathbb{C}$ except for finitely many isolated singularities at $a_1, \dots, a_n \in D$. Let E denote the graph of f in $D \setminus \{a_1, \dots, a_n\} \times B$, where B is a disk about the origin. Then there is a negative plurisubharmonic function on $D \times B$ that equals $-\infty$ precisely at $E \cup \bigcup_{i=1}^n \{(z, w) \in D \times B : z = a_i\}$ and is continuous outside its $-\infty$ locus. In particular, $E \cup \bigcup_{i=1}^n \{(z, w) \in D \times B : z = a_i\}$ is complete pluripolar in $D \times B$.*

Proof. We write down the Mittag–Leffler decomposition of f as

$$f = f_0 + f_1 + \cdots + f_n,$$

where f_0 is holomorphic on D while, for $j = 1, \ldots, n$, the function f_j is holomorphic on $\mathbb{C} \setminus \{a_j\}$, vanishes at ∞, and has an isolated singularity at a_j. The functions f_j have the expansion

$$f_j = \sum_{m=1}^{\infty} c_{jm}(z - a_j)^{-m}$$

with

$$|c_{jm}|^{1/m} \to 0 \text{ as } m \to \infty. \tag{3.1}$$

After introducing the new coordinates

$$w' = w - f_0(z) \quad \text{and} \quad z' = z,$$

we may assume that $f_0 = 0$.

Let N be a positive integer, which later will be chosen sufficiently large to suit our purposes. Let

$$P_N(z, w) = \left(w - \sum_{j=1}^{n} \sum_{m=1}^{N} c_{jm}(z - a_j)^{-m} \right) \prod_{j=1}^{n} (z - a_j)^N$$

and let

$$h_N(z, w) = \frac{1}{N} \log|P_N(z, w)|.$$

Then h_N is a plurisubharmonic function on $D \times \mathbb{C}$.

We make some estimates of h_N. Let $\varepsilon_m = \max_{j=1,\ldots,n;\, k \geq m} |c_{jk}|^{1/k}$. Then ε_m decreases to 0 according to (3.1). Fix $\delta > 0$, and let K be the compact set consisting of those $z \in D$ with distance at least δ to the boundary of $D \setminus \{a_1, \ldots, a_n\}$. Let M be the diameter of D. On $(K \times B) \cap E$ we have, for $\varepsilon_N < \delta/2$,

$$\begin{aligned} h_N(z, f(z)) &= \frac{1}{N} \log\left| f(z) - \sum_{j=1}^{n} \sum_{m=1}^{N} c_{jm}(z - a_j)^{-m} \right| + \sum_{j=1}^{n} \log|z - a_j| \\ &= \frac{1}{N} \log\left| \sum_{j=1}^{n} \sum_{m=N+1}^{\infty} c_{jm}(z - a_j)^{-m} \right| + \sum_{j=1}^{n} \log|z - a_j| \\ &\leq \frac{1}{N} \log\left(\sum_{j=1}^{n} \sum_{m=N+1}^{\infty} \varepsilon_N^m |z - a_j|^{-m} \right) + \sum_{j=1}^{n} \log|z - a_j| \\ &\leq \left(1 + \frac{1}{N}\right) \log \varepsilon_N + \log 2n - \left(1 + \frac{1}{N}\right) \log \delta + n \log M \\ &\leq \log \varepsilon_N + C, \end{aligned} \tag{3.2}$$

where C depends on δ, M, n but not on ε_N. Next, let A be positive. For $z \in K$ and $|w - f(z)| > A > 0$, we have

$$
\begin{aligned}
h_N(z,w) &= \frac{1}{N}\log\left|w-\sum_{j=1}^{n}\sum_{m=1}^{N}c_{jm}(z-a_j)^{-m}\right|+\sum_{j=1}^{n}\log|z-a_j| \\
&\geq \frac{1}{N}\log\left||w-f(z)|-\sum_{j=1}^{n}\sum_{m=N+1}^{\infty}\varepsilon_N^m|z-a_j|^{-m}\right|+n\log|\delta| \\
&\geq \frac{1}{N}\log\left(A-2n\left(\frac{\varepsilon_N}{\delta}\right)^{N+1}\right)+n\log|\delta|\geq -C', \qquad (3.3)
\end{aligned}
$$

for some positive constant C', if N is sufficiently large.

The final estimate is that, for large enough C_0 and C'', the inequality

$$h_N(z,w) < \frac{C_0}{N}\log(|w|+1)+C_0 \leq C'' \qquad (3.4)$$

holds on $D \times B$.

Now we introduce the negative plurisubharmonic functions

$$u_N = \max(h_N - C'', \log\varepsilon_N).$$

Choose a sequence of positive integers N_i and a sequence of positive numbers d_i with the following property: $\sum d_i$ converges, but $\sum d_i \log\varepsilon_{N_i}$ diverges to $-\infty$. This is possible since $\varepsilon_N \downarrow 0$. We form the series

$$u(z,w) = \sum_{i=1}^{\infty} d_i u_{N_i}(z,w). \qquad (3.5)$$

This is, on $D \times B$, a limit of a decreasing sequence of plurisubharmonic functions. On $E \cap (K \times B)$ we use (3.2) and find that $u = -\infty$ on $E \cap (K \times B)$ and hence also on E. By (3.3) we obtain that $u > -\infty$ if $w \neq f(z)$. The convergence properties of (3.5) are independent of δ; we conclude that u represents a negative plurisubharmonic function on $D \times B$ that satisfies

$$E \subset \{(z,w) : u(z,w) = -\infty\} \subset E \cup \bigcup_{1\leq j\leq n}\{z=a_j\}.$$

Moreover, the convergence of (3.5) is uniform on compact sets in the complement of $E \cup \bigcup_{1\leq j\leq n}\{z=a_j\}$. Hence, u is continuous in this complement. Finally, the function

$$u(z,w)+\sum_{i=1}^{n}\log|z-a_i| - n\log M$$

satisfies all our conditions. □

3.2. Estimates for Harmonic Measure

The next one-variable proposition will allow us to estimate pluriharmonic measure in the proof of the main theorem. The proposition is a small variation on a classical result concerning the existence of barriers (cf. [3]). We will denote classical harmonic measure by ω_0.

PROPOSITION 6. *Let G be a Dirichlet domain in $\mathbb{C}$, let K be a closed disk contained in G, and let a be a point in the boundary of G. Assume that there is an arc $\gamma : [0,1] \to \mathbb{C}$ contained in the complement of G with $\gamma(0) = a$. Then, for every $\varepsilon > 0$, there exist a $\delta > 0$ and a negative subharmonic function h on $G_\delta = G \cup \{z : |z-a| < \delta\}$ such that $h|_K = -1$ and $h(a) \geq -\varepsilon$.*

Proof. Without loss of generality we can take $a = 0$. Applying a Möbius transformation if necessary, we can assume that $\infty = \gamma(1)$. For every $0 < \delta < 1$ we let γ_δ be the component of $\gamma \setminus B(0,\delta)$ that contains ∞. Abusing notation, let $\gamma_\delta(\delta)$ denote the other endpoint of γ_δ, so that $|\gamma_\delta(\delta)| = \delta$.

Let $\Omega_0 = \mathbb{C}^* \setminus \gamma_\delta$. Choose an analytic branch f_δ of $\log(z - \gamma_\delta(\delta))$ on Ω_0. Let B_δ be the disk $\{z : |z - \gamma_\delta(\delta)| < 1\}$ and C_δ its boundary. The image of $G_\delta \cap B_\delta$ under f_δ is contained in the half-plane $H = \{\Re w < 0\}$ and X, the intersection of its boundary with the imaginary axis, has length $< 2\pi$. Set $u(z) = -\omega_0(z, X, H)$. Then $u \circ f_\delta$ is a negative harmonic function on $G_\delta \cap B_\delta$ that equals -1 on $G_\delta \cap C_\delta$. Moreover, $u(f_\delta(0)) \geq 2/\log\delta$. Now let $h(z) = -\omega_0(z, K, G_\delta)$. Then $h(z) > -1$ on $G_\delta \cap C_\delta$ and $h(z) = 0$ on $\partial G_\delta \cap B_\delta$. Hence $h(z) \geq u(z)$ on $\partial(G_\delta \cap B_\delta)$. Therefore, $h(0) \geq u(0) \geq 2/\log\delta$, so that $h(0) > -\varepsilon$ if δ is sufficiently small. Also, h is negative subharmonic on G_δ and $h = -1$ on K. □

3.3. *Proof of Theorem 2*

Let $D'' \subset D'$ be subdomains of D with $D'' \subset\subset D' \subset\subset D$. Assume also that D' is a Dirichlet domain. Let K be a (small) closed disk in D' that does not meet A, and denote by E_K the graph of f over K. Let M be the maximum of $|f|$ on K and let $B'' \subset B'$ be disks about the origin with different radii, each bigger than M. Let $\Omega' = D' \times B'$ and $\Omega'' = D'' \times B''$. We put $Z = E \cup \{(z,w) : z \in A\}$.

By Proposition 5, there exists a negative continuous plurisubharmonic function u on Ω' such that u equals $-\infty$ precisely on $\Omega' \cap Z$. It follows that

$$\omega((z,w), E_K, \Omega'') = 0 \quad \text{for } (z,w) \in \Omega'' \setminus Z.$$

We next estimate $\omega((a,w), E_K, \Omega'')$ for $a \in A \cap D''$. Let $\varepsilon > 0$ and let G denote the projection of $E \cap \Omega'$ on the first coordinate. If a is a pole of f, then a is an interior point of the complement of G. The function h defined by -1 on G and $-\varepsilon$ on a small neighborhood of a is harmonic on $G \cup B(a,\delta)$ for sufficiently small δ. If a is an essential singularity, then a is a boundary point of G. Since there exists a curve ending in a along which f tends to ∞, it is clear that the conditions of Proposition 6 are met. We thus find a small $\delta > 0$ and a negative subharmonic function h on $G \cup B(a,\delta)$ with $h(a) > -\varepsilon$ and $h\big|_K = -1$.

In either case we view the function h as a plurisubharmonic function on $(G \cup B(a,\delta)) \times B'$. It is plurisubharmonic in a neighborhood of $\{(z,w) \in \Omega' : u(z,w) = -\infty\}$. There exists an $\eta > 0$ such that this neighborhood contains $X = \{(z,w) \in \Omega'' : \operatorname{dist}((z,w), Z) = \eta\}$. Now the supremum of h on X is negative and u is continuous. Thus, for sufficiently small (positive) λ we find that $\lambda u > h$ on X.

The function h can be extended to a negative plurisubharmonic function on Ω'' as follows. Set

$$\tilde{h}(z,w) = \begin{cases} \max\{(h(z,w), \lambda u(z,w)\} & \text{if } \operatorname{dist}((z,w),Z) < \eta, \\ \lambda u(z,w) & \text{otherwise.} \end{cases}$$

The function $\tilde{h}$ competes for the supremum in the definition of $\omega((a,w),E_K,\Omega'')$. We conclude that $\omega((a,w),E_K,\Omega'') = 0$.

It now follows from Proposition 4 that $(E_K)^-_{\Omega''} = E \cap \Omega''$. Application of (2.2) and Theorem 3 shows that

$$E^*_{D\times\mathbb{C}} = (E_K)^*_{D\times\mathbb{C}} = E.$$

Finally, observing that E is a G_δ as well as an F_σ, we infer from Theorem 1 that E is complete pluripolar in $D \times \mathbb{C}$. □

References

[1] E. Bedford, *Survey of pluripotential theory,* Several complex variables: Proceedings of the Mittag–Leffler Inst. 1987–1988 (J.-E. Fornæss, ed.), Math. Notes, 38, Princeton University Press, Princeton, NJ, 1993.

[2] E. Bedford and B. A. Taylor, *A new capacity for plurisubharmonic functions,* Acta Math. 149 (1982), 1–40.

[3] M. Heins, *Selected topics in the classical theory of functions of a complex variable,* Holt, Rinehart & Winston, New York, 1962.

[4] B. Josefson, *On the equivalence between locally polar and globally polar sets for plurisubharmonic functions on* $\mathbb{C}^n$, Ark. Mat. 16 (1978), 109–115.

[5] N. S. Landkof, *Foundations of modern potential theory,* Grundlehren Math. Wiss., 180, Springer-Verlag, Berlin, 1972.

[6] N. Levenberg, G. Martin, and E. A. Poletsky, *Analytic disks and pluripolar sets,* Indiana Univ. Math. J. 41 (1992), 515–532.

[7] N. Levenberg and E. A. Poletsky, *Pluripolar hulls,* Michigan. Math. J. 46 (1999), 151–162.

[8] E. A. Poletsky, *Holomorphic currents,* Indiana Univ. Math. J. 42 (1993), 85–144.

[9] A. Sadullaev, *Plurisubharmonic measures and capacities on complex manifolds,* Uspekhi Mat. Nauk 36 (1981), 53–105; translation in Russian Math. Surveys 36 (1981), 61–119.

[10] J. Wiegerinck, *The pluripolar hull of* $\{w = e^{-1/z}\}$, Ark. Mat. 38 (2000), 201–208.

[11] A. Zeriahi, *Ensembles pluripolaires exceptionnels pour la croissance partielle des fonctions holomorphes,* Ann. Polon. Math. 50 (1989), 81–91.

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