Discrete Comput Geom 32:623–637 (2004)
DOI: 10.1007/s00454-004-1141-3

Discrete & Computational
Geometry

Graphs, Syzygies, and Multivariate Splines*

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Abstract. The module of splines on a polyhedral complex can be viewed as the syzygy module of its dual graph with edges weighted by powers of linear forms. When the assignment of linear forms to edges meets certain conditions, we can decompose the graph into disjoint cycles without changing the isomorphism class of the syzygy module. Thus we can use this decomposition to compute the homological dimension and the Hilbert series of the module. We provide alternate proofs of some results of Schenck and Stillman, extending those results to the polyhedral case. We also provide examples which illustrate the role that geometry plays in determining the syzygy module.

1. Introduction

Let Δ be a polyhedral subdivision of a region in $\mathbf{R}^d$, and let $C^r_m(\Delta)$ denote the vector space of r-differentiable piecewise polynomials on Δ of degree at most m. A classical problem in approximation theory is to find dimensions of and bases for these vector spaces. (See for example [1], [2], [14], and [15].) Billera pioneered the use of algebraic and homological techniques in this study, and received the Fulkerson prize for his work on this topic. Motivated by the questions above, he introduced $C^r(\Delta)$, the algebra of r-differentiable piecewise polynomials on Δ, in [3]. Although $C^r(\Delta)$ is an infinite-dimensional vector space over $\mathbf{R}$, it is a module of finite rank over $S = \mathbf{R}[x_1, \ldots, x_d]$, the polynomial ring in d variables over S, via pointwise multiplication. The elements of $C^r(\Delta)$ are called *splines* or *r-splines*. Billera's idea was that one could obtain information about the $C^r_m(\Delta)$'s by looking at the algebraic structure of $C^r(\Delta)$. This idea led to the development of an algebraic theory of spline modules, which has had subsequent applications to hyperplane arrangements and face rings of simplicial complexes. The use

* This work was partially supported by the Bunting Institute Science Scholar Fellowship of Radcliffe College.

of algebro-geometric methods to study $C^r(\Delta)$ can be found in [11]–[13] in the simplicial case, and in the work of Yuzvinsky [17], who provides a criterion for determining the projective dimension of spline modules on polyhedral complexes, using a sheaf-theoretic approach.

In [6] we showed that when $C^r(\Delta)$ is free, a basis for the module $C^r(\Delta)$ which is also a Gröbner basis will yield bases simultaneously for each $C^r_m(\Delta)$. In [4] and [5] we showed that the Hilbert Series of $C^r(\hat{\Delta})$ is the generating function of the dimensions of the $C^r_m(\Delta)$'s, where $\hat{\Delta}$ is the join of Δ with a point in $\mathbf{R}^{d+1}$ outside the affine span of Δ, i.e., the homogenization of Δ. In [6] we were concerned with finding combinatorial and topological conditions on Δ for $C^r(\Delta)$ to be a free module, and in [9] this study was extended to finding the homological dimension of $C^r(\Delta)$. Since $C^r(\Delta)$ is in general neither combinatorially nor topologically determined, one of the motivations of this work is to explore how the particular embedding of Δ affects the algebraic structure of $C^r(\Delta)$.

Instead of working with $C^r(\Delta)$, we focus on $B^r(\Delta)$, the syzygy module of the dual graph of Δ. The use of syzygies to describe splines first appeared in [14], and a vector space analog of $B^r(\Delta)$ appears in [16]. We introduced $B^r(\Delta)$ in [9] as an alternative way of representing elements of $C^r(\Delta)$, and we used this syzygy module to characterize the homological dimension of $C^r(\Delta)$ when the dual graph of Δ has a basis of disjoint cycles.

We can extend the definition of $B^r(\Delta)$ to arbitrary graphs G with an assignment L of linear forms to edges. The main result is that when G decomposes into cycles with respect to L and r, the homological dimension of the syzygy module is completely determined by the one cycle case. When $B^r(\Delta)$ is graded this decomposition also gives a straightforward way of computing its Hilbert Series. We describe conditions on G, L, and r for such a decomposition to occur, providing alternate proofs for some results of Schenck and Stillman from [12] and [13], and extending those results to polyhedral complexes. Schenk and Stillman claim that these are the best possible results using only local data. In the case where $B^0(\Delta)$ is free but $B^r(\Delta)$ is not free for some r, we describe a process for subdividing Δ to get Δ', so that $B^s(\Delta')$ will be free for all $s \leq r$.

2. Definition of the Syzygy Module

Let $S = K[x_1, \dots, x_d]$ be the polynomial ring in d variables over the field K. Let G be a graph with a fixed orientation of its m edges, and let $L = \{\ell_1, \dots, \ell_m\}$ be an assignment of homogeneous linear forms in S to the edges $\{e_1, \dots, e_m\}$ of G. For each i, the linear form associated to $-e_i$ is $-\ell_i$. Let $\mathcal{C}$ denote the set of cycles of G, and fix an integer $r \geq 0$.

Definition 2.1.

$$Syz^r(G) = \left\{ (\alpha_1, \dots, \alpha_m) \in S^m : \text{for all } C \in \mathcal{C}, \sum_{e \in C} \alpha_e {\ell_e}^{r+1} 0 \right\}.$$

We call $Syz^r(G)$ the **syzygy module** of G weighted by (L, r).

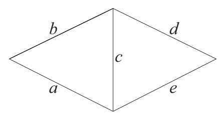

Fig. 1. A graph weighted by $\{a, b, c, d, e\}$.

We can represent $Syz^r(G)$ as the kernel of the map given by a matrix M as follows. Fix a basis $\mathcal{B}$ for the cycle space of G. The rows of M are indexed by the elements of $\mathcal{B}$, and the columns are indexed by the edges of G. The (i, j)th entry of M is 0 if edge e_j is not contained in cycle C_i and $\pm\ell_j^{r+1}$ depending on the orientation of edge e_j in cycle C_i. If G is acyclic we can take M to be the row matrix $(0, \ldots, 0)$ with one entry for each edge of G. Note that a given matrix M can correspond to a number of different graphs. For example, the row matrix $(0, \ldots, 0)$ with m entries corresponds to any acyclic graph G on m edges, regardless of the number of connected components of G.

Example 2.2. Let G be the graph in Fig. 1, where $L = \{a, b, c, d, e\}$ and $r = 0$. Then

$$M = \begin{pmatrix} a & b & c & 0 & 0 \\ 0 & 0 & c & d & e \end{pmatrix}.$$

Example 2.3. Consider the graphs in Fig. 2, where $L = \{a, b, c, d, e, f\}$ and $r = 2$. Since they have the same cycles, they also have the same matrix,

$$M = \begin{pmatrix} a^3 & b^3 & c^3 & 0 & 0 & 0 \\ 0 & 0 & 0 & d^3 & e^3 & f^3 \end{pmatrix}.$$

Proposition 2.4. *$Syz^r(G)$ is a torsion-free graded S-module of rank $n - c$, where n is the number of vertices of G, and c denotes the number of connected components of G.*

Proof. $Syz^r(G)$ is clearly a torsion-free S-module, as it is a submodule of the free module S^m. It is easily seen to be graded, since it is the kernel of a matrix with homogeneous entries all of the same degree $r + 1$. The rank of $Syz^r(G)$ will be $m - \dim H_1(G, K)$, where m is the number of edges of G and $H_1(G, K)$ is the first homology group of G over the field K. This is true because each cycle in a basis for the cycle space of G will lower the rank of $Syz^r(G)$ by 1. Also, $\dim H_1(G, K) - \dim H_0(G, K) = m - n$, so the rank of $Syz^r(G) = m - \dim H_1(G, K) = n - \dim H_0(G, K) = n - c$. $\square$

Definition 2.5. The **rank** of a cycle C in G, denoted $\mathrm{rk}(C)$, is the dimension of the linear span of the ℓ_i's.

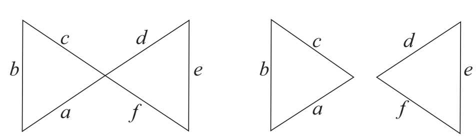

Fig. 2. Two graphs with the same matrix.

The following result is Theorem 4.2 in [9]. Let $\mathrm{hd}(A)$ denote the homological dimension of the S-module A.

Proposition 2.6. *If G contains only one cycle C, then*

$$\mathrm{hd}(Syz(G)) = \mathrm{rk}(C) - 2.$$

Thus in the case of a single cycle, $Syz^r(G)$ is free if and only if the cycle has rank 2. If G contains no cycles, then $M = (0, \ldots, 0)$, so $Syz^r(G)$ will be a free S-module of rank m, the number of edges of G. Moreover, a module basis for $Syz^r(G)$ consists of the standard basis elements of S^m.

Note that when the graph contains at most one cycle, the homological dimension of $Syz^r(G)$ is the same for every r. This is false in general. In fact, freeness is often lost when going from $r = 0$ to $r = 1$, and the homological dimension is likely to increase as r increases. However, it is possible for the homological dimension to decrease, as shown by Dalbec and Schenck in [7]. For computations and related results see [9].

Example 2.7. Let G be a cycle weighted by $L = \{x, y, x + y\}$. Since $x + y$ is in the span of x and y, this cycle has rank 2. Thus $\mathrm{hd}(Syz^r(G)) = 0$, i.e., $Syz^r(G)$ is free of rank 2 for every r.

3. Syzygies and Spline Modules

Let Δ be a pure polyhedral complex in $\mathbf{R}^d$, i.e., a convex polyhedral subdivision of a region in $\mathbf{R}^d$. For details and definitions, see [8]. In this section we introduce $C^r(\Delta)$, the module of r-differentiable piecewise polynomial functions on Δ. The elements of $C^r(\Delta)$ are called *splines* or *r-splines*.

Definition 3.1. The **dual graph** of Δ, G_Δ, is defined as follows. The vertices of G_Δ correspond to d-polytopes of Δ, and two vertices share an edge whenever the corresponding polytopes meet in a $(d-1)$-dimensional face.

Recall that if $\sigma \in \Delta$, then the **star** of σ in Δ, denoted $st(\sigma)$, is the complex generated by all faces of Δ containing σ.

Definition 3.2. Δ is **strongly connected** if G_Δ is connected, and **hereditary** if for every face σ of Δ, $G(st(\sigma))$ is connected.

We now give a formal definition of $C^r(\Delta)$. Let $S = \mathbf{R}[x_1, \ldots, x_d]$.

Definition 3.3. For a non-negative integer r and a d-complex Δ, $C^r(\Delta)$ is the set of r-differentiable functions $F : \Delta \to \mathbf{R}$ such that for every d-face σ, F restricted to σ is given by a polynomial in S.

In this work we only consider hereditary complexes. In this case there is an algebraic criterion for whether a piecewise polynomial function is in $C^r(\Delta)$. Choose an ordering $\sigma_1, \ldots, \sigma_n$ of the d-faces of Δ. With respect to this ordering, a piecewise polynomial function can be represented as an n-tuple of polynomials, $(f_1, \ldots, f_n)$, where f_i is the restriction of F to the face σ_i. If σ_i is adjacent to σ_j, i.e., $\sigma_i \cap \sigma_j$ has dimension $d-1$, then $I(\sigma_i \cap \sigma_j)$, the ideal of polynomials which vanish on $\sigma_i \cap \sigma_j$, is generated by an affine form, denoted ℓ_{ij}. Note that ℓ_{ij} is unique up to constant multiple. Another way to think of this is as follows: the affine span of $\sigma_i \cap \sigma_j$ is a hyperplane in $\mathbf{R}^d$, and ℓ_{ij} is an affine form whose kernel is that hyperplane. The following proposition is proved in [5] in a more general setting.

Proposition 3.4. *If Δ is hereditary and $F = (f_1, \ldots, f_n)$ is a piecewise polynomial function on Δ, then F is in $C^r(\Delta)$ if and only if whenever σ_i is adjacent to σ_j, ℓ_{ij}^{r+1} divides $f_i - f_j$.*

Choose an ordering of the vertices of G_Δ. This induces an orientation on the edges of G_Δ. If e is a directed edge in G_Δ, let $\ell_{-e} = -\ell_e$. Let $e_1, \ldots, e_m$ be an ordering of the positively oriented edges in G_Δ, with corresponding affine forms $\ell_1, \ldots, \ell_m$.

We focus primarily on polyhedral complexes Δ that are stars of vertices, i.e., there is a single vertex v that is an element of every maximal face. By shifting Δ so that v is the origin, we may then assume that all of the affine forms ℓ_e are homogeneous linear forms, so that $C^r(\Delta)$ will be a graded module. If Δ is not the star of a vertex, we can *homogenize* it by viewing Δ in the $x_{d+1} = 1$ plane of $\mathbf{R}^{d+1}$, and joining Δ with the vertex $v = (0, 0, \ldots, 0)$. We call this new complex $\hat{\Delta}$. It is easy to see that $G_{\hat{\Delta}} = G_\Delta$, and that $L_{\hat{\Delta}}$ is obtained from L_Δ by homogenizing each affine form $\{\ell_e\}$ with the variable x_{d+1}.

Definition 3.5. Let $G = G_\Delta$ and let $L_\Delta = \{\ell_e : e \text{ is an edge of } G\}$. We define $B^r(\Delta)$ to be $Syz^r(G)$.

The following theorem, proved in [9], connects the S-modules $B^r(\Delta)$ and $C^r(\Delta)$.

Theorem 3.6 [9]. *If Δ is hereditary and the star of a vertex, then*

$$C^r(\Delta) \cong B^r(\Delta) \oplus S.$$

Moreover, this is a graded isomorphism with a degree shift in $B^r(\Delta)$ of $r+1$.

See [5] and [10] for Hilbert series computations and their connection to the dimension problem for spline spaces. In the next section we describe a process for decomposing G with respect to (L, r), and in Section 5 we apply these results to spline modules.

4. Decompositions of G Weighted by (L, r)

We begin by fixing a basis $\mathcal{B} = \{C_1, \ldots, C_k\}$ for the cycle space of G. Let e be an edge contained in two or more cycles of $\mathcal{B}$. Our goal is to find conditions for deleting e in one

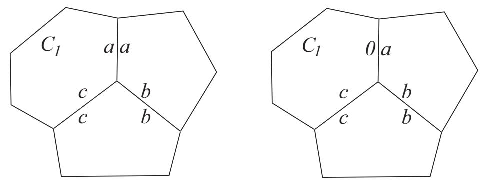

Fig. 3. Removing an edge from C_1.

of the cycles, without changing the isomorphism class of $Syz^r(G)$. This will have the effect of disconnecting one cycle from the others. We will then look for graphs that we can split completely into disjoint cycles. In order to describe this process geometrically, we consider a finer weighting of G so that we can place a weight of 0 on an edge in some of its cycles without changing the underlying graph.

When we refer to the graph G, we are referring not only to the underlying graph but also to an assignment of linear forms L to the edges of G. For a given r, each edge e_j will be weighted by ℓ_j^{r+1}. We can also think of e_j as weighted by the ordered k-tuple $(w_1, w_2, \ldots, w_k)$ where $w_i = \ell^{r+1}$ if e is an edge of C_i, and 0 otherwise. In fact, these weight vectors are precisely the columns of the matrix M defined in Section 2. This notation is ideal if we want to remove an edge e_j from a cycle C_i. We can now represent the removal of an edge from a single cycle algebraically, by replacing the current value of w_i with 0, and geometrically, by assigning e a weight of 0 in cycle C_i. In the example below, we only show weights on the cycles containing a given edge. Note that the weights not shown are all 0.

Example 4.1. Let G be the graph in Fig. 3 whose edges e_1, e_2, e_3 are weighted by $L = \{a, b, c\}$. Let G' be the graph obtained by removing e_1 from the cycle C_1. In G the weight vectors of e_1, e_2, e_3 are $(a, a, 0)$, $(0, b, b)$, and $(c, 0, c)$. In G' the weight vectors of e_1, e_2, e_3, respectively, are $(0, a, 0)$, $(0, b, b)$, and $(c, 0, c)$.

Definition 4.2. If e is not contained in any cycle of $\mathcal{B}$, we call e a **free** edge. If e is contained in only one C in $\mathcal{B}$, we call e an **exterior** edge. If e is contained in two or more cycles, we call e an **interior** edge.

Our goal is to remove all the interior edges from a cycle C without changing the isomorphism class of $Syz^r(G)$. If we can do this, we say that the cycle C splits off from G.

Lemma 4.3. *Fix (L, r) and let G_1 and G_2 have the same weighted cycles and the same number of edges. Then*

$$Syz^r(G_1) \cong Syz^r(G_2),$$

as graded S-modules.

Proof. The matrix M is determined by the cycles of G and the number of edges of G, so any two graphs weighted by (L, r) with the same weighted set of cycles will have the same matrix. Thus the syzygy modules are isomorphic. $\square$

Lemma 4.4. *If $G = A \sqcup B$, the disjoint union of graphs, then*

$$Syz^r(G) = Syz^r(A) \oplus Syz^r(B),$$

as graded S-modules.

Proof. If A and B are disjoint, then they do not have any edges in common. Since the columns of M are indexed by the edges of G, this means M will be a block matrix with two components, one for A and one for B. If either A or B is acyclic, we can add a row of zeros to M (without changing the kernel) and this will be the corresponding block. Thus the kernel of M is the direct sum of the kernels of each block. The result now follows. □

Definition 4.5. If e_1 is an interior edge of C such that $\ell_1^{r+1} \in \text{span}\{\ell_e^{r+1} : e \in C, e$ is exterior$\}$, we say that e_1 is ***r*-removable** from C. If G' is obtained from G by a sequence of r-removals of interior edges from cycles, and G'' has the same matrix as G', we say that G **decomposes** into G'' with respect to (L, r). If a cycle C has no interior edges in G', we say that C **splits off** from G with respect to (L, r).

Notice that when C splits off, it does so with the r-removable edges deleted and their endpoints identified.

Example 4.6. Consider the graphs G and G' in Fig. 3. If we can remove the edge with linear form c from C_1 in G', then we can split C_1 off from G'. The cycle that splits off will have three edges, those that were originally exterior.

Theorem 4.7. *If G decomposes into G'' with respect to (L, r), then*

$$Syz^r(G) \cong Syz^r(G''),$$

as graded S-modules.

Proof. Recall that $Syz^r(G)$ is the kernel of a matrix M with rows indexed by the cycles in a given basis for $\mathcal{C}$ and columns indexed by the edges of G. The (i, j)th entry of M is 0 if edge e_j is not contained in cycle C_i and $\pm\ell_j^{r+1}$ depending on the orientation of edge j in cycle C_i. If e is an exterior edge, ℓ_e only appears in one row of M, so we can use its column to perform column operations on M to create a new matrix M' that differs only in that row. This will not change the image of M, i.e., the module generated by the columns of M.

Let e_j be an r-removable edge of C_i where $e_1, \ldots, e_{j-1}$ are the exterior edges of C_i. Then there are constants $b_1, \ldots, b_{j-1}$ such that

$$\ell_j^{r+1} = \sum_{p=1}^{j-1} b_p \ell_p^{r+1}$$

and by column operations we can replace the (i, j)th entry of M with 0. Thus edge e_j is no longer represented in row i, making it no longer a part of the cycle C_i, and this new matrix M' corresponds to the graph G' obtained by removing e_j from C_j, giving e_j a weight of 0 in C_i.

To see that $\ker(M) \cong \ker(M')$, consider the map $\varphi\colon \ker(M) \to \ker(M')$ given by

$$(\alpha_1, \dots, \alpha_{j-1}, \alpha_j, \dots, \alpha_m) \mapsto (\alpha_1 + b_1\alpha_j, \dots, \alpha_{j-1} + b_{j-1}\alpha_j, \alpha_j, \dots, \alpha_m).$$

Since the b_k's are constants, this is easily seen to be an isomorphism. Notice that changing $\alpha_1, \dots, \alpha_{j-1}$ only affects row C_i, since the corresponding edges are exterior. Finally, since φ preserves polynomial degree, this is a graded isomorphism. □

Corollary 4.8. *If C splits off from G as C' relative to (L, r), then*

$$Syz^r(G) \cong Syz^r(G - C') \oplus Syz^r(C'),$$

as graded S-modules.

Proof. If C_i splits off G as C_i', then C_i' has no interior edges in G'. Let $G'' = (G - C_i') \sqcup C_i'$, where in the second summand we identify edges so that C_i' is a cycle, and let e be an edge of G'. If e is an edge of C_i', then its weight vector is 0 except in the ith component. Thus the corresponding edge e in G'' will have the same weight vector. If e is not in C_i', then its weight vectors in G' and $G - C_i'$ will be the same, thus it will be the same in G''. By Lemma 4.3, we have that $Syz^r(G') \cong Syz^r(G'')$ and the rest of the corollary follows from Lemma 4.4 and Theorem 4.7. □

Let $Hilb(A)$ denote the Hilbert series of the graded module A.

Corollary 4.9. *If G' is a decomposition of G with respect to (L, r) with disjoint cycles $C_1, \dots, C_k$ and p free edges, then:*

1. *$Syz^r(G) \cong Syz^r(C_1) \oplus \cdots \oplus Syz^r(C_k) \oplus S^p$, as graded S-modules.*
2. $\mathrm{hd}(Syz^r(G)) = \mathrm{Max}_{i=1}^k(rank\, C_i - 2)$.
3. *$Syz^r(G)$ is free if and only only if each C_i has rank 2.*
4. $Hilb(Syz^r(G)) = Hilb(Syz^r(C_1)) + \cdots + Hilb(Syz^r(C_k)) + p/(1-t)^d$.

Proof. By repeated applications of Corollary 4.8, we get that $Syz^r(G) \cong Syz^r(C_1) \oplus \cdots \oplus Syz^r(C_k) \oplus Syz^r(T^p)$ where T^p is the acyclic part of G'. Since T^p has no cycles, its matrix M is a row of p zeros, so $ker(M) = S^p$. This proves part 1, and parts 2 and 3 follow from Proposition 2.6. To prove part 4, we first note that the Hilbert series of S^p is $p/(1-t)^d$. Since the isomorphism in part 1 is graded and the Hilbert series is additive, part 4 now follows. □

When G decomposes into cycles, we can easily compute the Hilbert series of $Syz^r(G)$, if we know the result in the one cycle case. We now describe conditions on G and L for such a decomposition to occur. We start with the following theorem.

Theorem 4.10. *Suppose that G contains no interior edges, let $C_1, \dots, C_k$ be a basis for the cycle space of G consisting of cycles of minimal rank, and let p be the number* *of free edges of G. Then for any r:*

1. $Syz^r(G) \cong Syz^r(C_1) \oplus \cdots \oplus Syz^r(C_k) \oplus S^p$, *as graded S-modules.*
2. $\mathrm{hd}(Syz^r(G)) = \mathrm{Max}_{i=1}^k(rank\, C_i - 2)$.
3. $Syz^r(G)$ *is free if and only only if each C_i has rank* 2.
4. $Hilb(Syz^r(G)) = Hilb(Syz^r(C_1)) + \cdots + Hilb(Syz^r(C_k)) + p/(1-t)^d$.

Proof. If G contains no interior edges, then each cycle splits off from G as itself, and we have satisfied the hypotheses of Corollary 4.9. □

In the case where G contains interior edges, there are two ways to try to remove them with respect to (L, r). First, if an interior edge e of C contains the same linear form as an exterior edge of C, then e is r-removable for every r. We formalize this below.

Definition 4.11. Let e_1 be an interior edge of G. If there exists an exterior edge e_2 in some cycle C containing e_1 such that ℓ_1 is a constant multiple of ℓ_2, then we call e_1 a **removable** edge of C.

Lemma 4.12. *If e_1 is a removable edge of C, then it is an r-removable edge of C for all $r \geq 0$.*

Proof. If e_1 is removable from C, then ℓ_1 is a constant multiple of ℓ_2 which means that ℓ_1^{r+1} is a constant multiple of ℓ_2^{r+1} for all $r \geq 0$, so $\ell_1^{r+1} \in \mathrm{span}\{\ell_e^{r+1} : e \in C, e$ is exterior$\}$, and hence e_1 is r-removable from C for all r. □

Example 4.13. Consider the graph G in Fig. 4. Note that each interior edge is removable, since its linear form also lies on an exterior edge. By Lemma 4.12, each interior edge is r-removable for all r. Hence G decomposes into G' and finally into disjoint cycles as in G''.

In the example above, $Syz^r(G)$ will be free for all r because G decomposes into disjoint cycles of rank 2. What happens if we take the same graph but change the linear forms? In the next section we will see that this is akin to altering the embedding of a polyhedral complex in $\mathbf{R}^d$. In particular, the graph in Fig. 4 is the dual graph of a 3×3 grid of parallelograms in the plane.

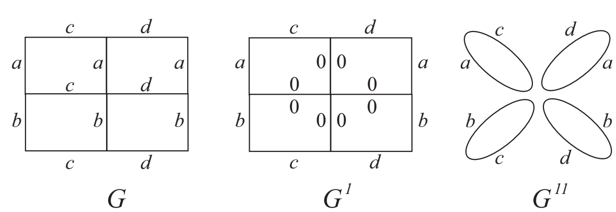

Fig. 4. Decomposing G into cycles.

The second way to remove interior edges is as follows. When the linear forms on a cycle C have distinct spans, we can remove an interior edge if there are enough exterior edges. If an interior edge lies on only two cycles C_1 and C_2, then removing it from C_1 makes it an exterior edge in C_2. Thus, we may now be able to remove another interior edge of C_2 that was not possible previously.

Lemma 4.14. *Let C be a cycle of G of rank* 2*. Suppose e is an interior edge of C, and the exterior edges of C contain $r+2$ linear forms with distinct spans. Then e is s-removable for all $s \leq r$.*

Proof. By a change of coordinates, we may view all linear forms in a cycle of rank 2 in terms of two variables. The number of monomials of degree s in two variables is $s+1$, so any $s+2$ polynomials of degree s will be linearly dependent. On the other hand, since $s+2 \leq r+2$, there exist $s+1$ distinct polynomials $\{\ell_1^{s+1}, \ldots, \ell_{s+1}^{s+1}\}$ of degree s. By Lemma 5.1 of [13] these must be linearly independent. Thus $\ell_e^{r+1}, \ell_1^{s+1}, \ldots, \ell_{s+1}^{s+1}$ are dependent, and ℓ_e must be a linear combination of the others. □

In fact, the proof of Lemma 4.14 shows that any r-removable edge of a cycle of rank 2 is also s-removable for all $s \leq r$. The following results are generalizations of Theorem 5.3(a) in [13] and Theorem 5.2 in [12], respectively.

Theorem 4.15. *If every interior edge is a removable edge of some cycle C, then G decomposes into cycles and the results of Theorem* 4.10 *hold for all r.*

Proof. If every interior edge is removable, then we can decompose G into G' with no interior edges by a finite sequence of edge removals. This procedure will work for any r. We now apply Theorem 4.10. □

Theorem 4.16. *Let G be a planar graph such that every cycle has rank* 2*. Suppose there exists an ordering $e_1 < \cdots < e_m$ of edges of G such that for each interior edge e_i, either e_i is a removable edge of some cycle C or for some C, $C \cap \{e_1, \ldots e_{i-1}\}$ contains $r+2$ distinct linear forms. Then G decomposes into cycles with respect to (L, r) and the results of Theorem* 4.10 *hold for all $s \leq r$.*

Proof. Let $s \leq r$. Since G is a planar graph, an edge lies on at most two cycles. Let e_{i_1} be the smallest interior edge (with respect to the given ordering). By the hypotheses together with Lemma 4.14, e_{i_1} is either removable or s-removable from some cycle C, which means it is s-removable. Now, e_{i_1} lies in only one cycle C', which means it is now an exterior edge of C'. We consider the next smallest e_{i_2}, and by the same reasoning we can remove e_j from one of its cycles, making it an exterior edge of its other cycle. Note that at each stage $C \cap \{e_1, \ldots, e_{i-1}\}$ will consist of exterior edges of the cycle C. When we are done, we will have decomposed G into G' with no interior edges with respect to (L, s). We now apply Corollary 4.9. □

If we can recursively split off all of the cycles of G, Schenck and Stillman assert in [12] that this is the best possible result using only local data.

5. Applications to Modules of Splines

Let Δ be a hereditary polyhedral subdivision of a region in $\mathbf{R}^{d-1}$, and let $\hat{\Delta}$ in $\mathbf{R}^d$ be the join of Δ with a vertex in $\mathbf{R}^d$ outside of the affine span of Δ. A classical problem in approximation theory is to find dimensions of and bases for the vector subspaces $C^r_m(\Delta)$ of $C^r(\Delta)$ consisting of splines of total polynomial degree at most m.

One of the primary goals in the study of $C^r(\Delta)$ has been to determine whether it is a free module, and to compute its Hilbert series when it is a graded module. We saw in Section 3 that it is sufficient to study the same questions for the module $B^r(\hat{\Delta}) = Syz^r(G_{\hat{\Delta}})$. Although we are interested in $C^r(\Delta)$ we restrict our work here to $C^r(\hat{\Delta})$. The benefit of studying $C^r(\hat{\Delta})$ as opposed to $C^r(\Delta)$ is that $C^r(\hat{\Delta})$ is a graded module, which means we have many computational and theoretical tools at our disposal. Moreover, we can get the dimensions of $C^r_m(\Delta)$ from the Hilbert series of $C^r(\hat{\Delta})$, and when $C^r(\hat{\Delta})$ is free, a module basis for $C^r(\hat{\Delta})$ gives rise to vector space bases of $C^r_m(\Delta)$ for each m. (See [5], [6], and [10] for details.)

We now interpret the results from the previous section in terms of spline modules. We focus on the case where Δ is a subdivision of a region in the plane. In this case G_Δ and $G_{\hat{\Delta}}$ will be planar graphs, and hence an edge lies in at most two cycles. Let $G = G_{\hat{\Delta}}$, where $\Delta \subset \mathbf{R}^2$, let $\{C_1, \ldots, C_k\}$ be a basis for the cycle space of G, and let p denote the number of free edges of G. We assume that Δ is bounded and may contain one-dimensional holes. We define the **outside boundary** of Δ to be the usual boundary if we fill in the holes.

Theorem 5.1. *Let Δ be a hereditary polygonal subdivision of a region in $\mathbf{R}^2$. If every non-boundary edge of Δ meets the outside boundary of Δ, then for all $r \geq 0$:*

1. $Syz^r(G) \cong Syz^r(C_1) \oplus \cdots \oplus Syz^r(C_k) \oplus S^p$, *as graded S-modules.*
2. $Syz^r(G)$ *is free if and only if G has a basis of rank 2 cycles.*
3. $Hilb(Syz^r(G)) = Hilb(Syz^r(C_1)) + \cdots + Hilb(Syz^r(C_k)) + p/(1-t)^3$.

Proof. Any edge of Δ that meets the outside boundary of Δ corresponds to be an exterior edge of G. Thus, we can apply Theorem 4.10. Part 2 follows because the syzygy modules of a cycle is free if and only if the cycle has rank 2. □

If Δ is simplicial and satisfies the hypotheses of Theorem 5.1, then $Syz^r(G)$ *is free if and only if Δ is a topological disk*. This is because for a topological disk, G always contains a basis of cycles around vertices, and such cycles will always have rank 2. In fact, Schenck and Stillman prove, for any simplicial Δ, that $C^r(\hat{\Delta})$ free implies Δ is topologically trivial [13]. The following example shows that this result is false for polyhedral complexes, i.e., it is possible for $Syz^r(G)$ to be free without Δ being acyclic.

Example 5.2. Let Δ be the polygonal complex in Fig. 5. Its dual graph is given by a single cycle with linear forms (a, b, c). Although Δ has non-trivial homology, if the lines given by a, b, and c meet at a point, then the cycle (a, b, c) will have rank 2. Hence $C^r(\hat{\Delta})$ will be free for all r. If we embed this complex generically, the cycle will have rank 3, and $C^r(\hat{\Delta})$ will not be free for any r.

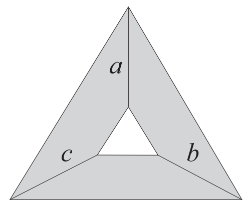

Fig. 5. A polygonal complex.

Based on this example and Theorem 5.1, we make the following conjecture:

Conjecture 1. *Let Δ be a polyhedral subdivision of a region in* $\mathbf{R}^d$. *If* $C^r(\hat{\Delta})$ *is free, then G contains a basis of cycles of rank* 2.

Using the terminology in [13], we define a non-boundary edge of Δ to be a **pseudoboundary edge** if its affine span is the same as that of an edge meeting the outside boundary, and both edges share a cycle in G. The following theorem is a extension of Theorem 5.3(b) in [12] to polygonal complexes.

Theorem 5.3. *Let Δ be a hereditary polygonal subdivision of a region in* $\mathbf{R}^2$. *If all non-boundary edges in Δ are pseudoboundary edges, then G decomposes into cycles and the results of Theorem* 5.1 *hold for all r.*

Proof. A pseudoboundary edge of Δ corresponds to a removable edge of G. Thus, we can apply Theorem 4.10. □

Example 5.4. Let Δ be a 3×3 grid of parallelograms. Then G is the graph seen in Fig. 4, where a, b, and c are distinct linear forms. Because Δ is symmetric, all edges will be pseudoboundary edges, hence Theorem 5.3 applies.

The example above will work for any $m \times n$ grid of parallelograms. However, if we embed Δ generically, then the interior edges will no longer be pseudoboundary edges. We can still try to remove edges from cycles using the following theorem, which is essentially Theorem 5.2 of [12] extended to polyhedral subdivisions of a disk.

Theorem 5.5. *Let Δ be a polygonal complex in the plane such that G has a basis of cycles of rank* 2. *Suppose there exists an ordering $e_1 < \cdots < e_m$ of edges of Δ such that for each non-boundary edge e_i, either e_i is a pseudoboundary edge of some cycle C or for some C, $C \cap \{e_1, \ldots e_{i-1}\}$ contains $r + 2$ distinct linear forms. Then G decomposes into cycles with respect to (L, r) and the results of Theorem* 5.1 *hold for all $s \leq r$.*

Proof. Since every edge is r-removable, Theorem 4.16 applies. □

Using local cohomology and other homological techniques, Schenck and Stillman get several other results that we have not been able to reproduce with our methods. The

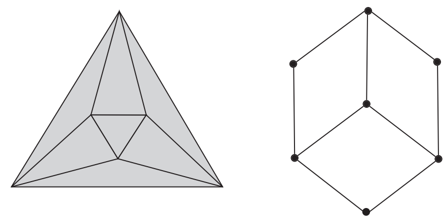

Fig. 6. Δ and its dual graph G.

following non-freeness theorem (Theorem 5.3(b) in [13]) is perhaps most relevant to our work here.

Theorem 5.6 [13]. *If Δ is a triangulation of a disk in the plane and if Δ has at least one edge which is not a pseudoboundary edge, then there exists s such that for $r \geq s$, $C^r(\hat{\Delta})$ is not free.*

Thus, for most generically embedded complexes, we cannot split G into cycles. However, for any particular r, we can subdivide Δ in such a way that we create enough new exterior edges to remove all of the interior edges of a given cycle. The following example illustrates this procedure.

Example 5.7. Let Δ be the planar triangulation of an octahedron given in Fig. 6 together with its dual graph G. Using Theorem 5.3 of [13], we see that $C^r(\hat{\Delta})$ cannot be free when $r = 3$. Let Δ' be the subdivision Δ created by adding $r = 3$ new edges to the lower triangle as in Fig. 7. Then in G' we will now have a cycle with $r + 2 = 5$ distinct exterior edges. Thus we can split off this cycle from G'. We can continue to subdivide Δ in this way until we have split off each cycle. Since each cycle has rank 2, the resulting syzygy module will be free for any $r \leq 3$.

This procedure will work for any polygonal complex Δ in the plane such that G has a basis of cycles of rank 2. This is because in any (finite) planar graph we can always find a cycle with an exterior edge. This edge will correspond to an edge in Δ between adjacent polygons P_1 and P_2, and we can subdivide one or both of these polygons to create a cycle with the required number of exterior edges. When Δ is simplicial, this cycle will necessarily go around a vertex of Δ, so we can easily add new edges through this vertex

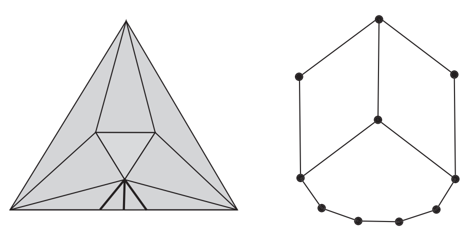

Fig. 7. Δ' and its dual graph G'.

out to the boundary of P_2, as in the previous example. If Δ is not simplicial we could be in the situation of Example 5.2. In this case the affine span of the edge between P_1 and P_2 will necessarily meet the affine spans of all other edges between polygons in the cycle. This is akin to a cycle going around a vertex, even if the vertex is not in Δ. When we add edges to P_1, for example, we just have to make sure that their affine spans also go through this common point. In either case we can recursively split off all of the cycles of G.

When G decomposes into cycles of rank 2, we also get a nice result about the freeness of $C^r(\hat{\Delta})$ that is false in general. See [7] for a counterexample in the general case, and a proof of this result for any planar Δ.

Theorem 5.8. *Let Δ be a hereditary polyhedral subdivision of a region in $\mathbf{R}^{d-1}$. If G decomposes into cycles of rank 2 with respect to (L, r), then $C^r(\hat{\Delta})$ free implies $C^s(\hat{\Delta})$ is free for all $s \leq r$.*

Proof. We defined edge removals in terms of a power of a linear form lying in the span of another set of powers of linear forms. If all cycles have rank 2, then for a given interior edge e, either e is removable or we are in the situation of Lemma 4.14. In either case e is s-removable for all $s \leq r$, and we can also split off cycles with respect to (L, s) for all $s \leq r$. Since all cycles have rank 2, the syzygy module is free by part 3 of Corollary 4.9. ☐

Acknowledgement

The author thanks the reviewers for helpful comments and suggestions.

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Received August 29, 2003, *and in revised form March* 25, 2004. *Online publication August* 30, 2004.