*J. Austral. Math. Soc.* (*Series A*) **44** (1988), 46–63

# GRAPHS WHOSE FULL AUTOMORPHISM GROUP IS A SYMMETRIC GROUP

MARTIN W. LIEBECK

(Received 16 May 1986; revised 5 September 1986)

Communicated by L. Caccetta

**Abstract**

We address the problem of describing all graphs $\Gamma$ such that $\operatorname{Aut}\Gamma$ is a symmetric group, subject to certain restrictions on the sizes of the orbits of $\operatorname{Aut}\Gamma$ on vertices. As a corollary of our general results, we obtain a classification of all graphs $\Gamma$ on $v$ vertices with $\operatorname{Aut}\Gamma \cong S_n$, where

$$v < \min\{5n, \tfrac{1}{2}n(n-1)\}.$$

1980 *Mathematics subject classification* (*Amer. Math. Soc.*): 05 C 25.

## Introduction

It has been known since Frucht's paper [1] of 1938 that, given any finite group $G$, there is a graph $\Gamma$ such that the automorphism group of $\Gamma$ is isomorphic to $G$. For certain groups $G$, such as $S_n$, this result is obvious, and it is more interesting to investigate the more general problem of describing all graphs $\Gamma$ such that $\operatorname{Aut}\Gamma \cong G$. In this paper we address this problem for the symmetric groups $S_n$. This was considered for graphs with less than $2n$ vertices in [2] and [3]. Here we investigate the graphs $\Gamma$ such that $\operatorname{Aut}\Gamma \cong S_n$, under the following far less restrictive hypothesis:

(*) all orbits of $\operatorname{Aut}\Gamma$ on the set $V\Gamma$ of vertices of $\Gamma$ have size less than $\frac{1}{2}n(n-1)$.

It is an elementary consequence of (*) (see Proposition 1.2 below) that for $n > 6$, all the orbits of $\operatorname{Aut}\Gamma$ on $V\Gamma$ have size 1 or $n$. Let $t$ be the number of orbits of size $n$. In Theorem 1.4 we show that $\Gamma$ must satisfy various strong necessary conditions; and we conjecture (1.5) that these conditions on an arbitrary graph $\Delta$ are also sufficient to imply that $\operatorname{Aut}\Delta \cong S_n$. We prove Conjecture 1.5 for $1 \leqslant t \leqslant 4$ (see Theorem 2.7). In particular this gives a classification of all graphs $\Gamma$ with $\operatorname{Aut}\Gamma \cong S_n$ $(n > 6)$ and $|V\Gamma| < \min\{5n, \frac{1}{2}n(n-1)\}$. This substantially improves the results of [2] and [3], and also solves various problems raised in [3, Section 4]. We include general descriptions of these graphs in an Appendix.

Finally, we remark that the methods of this paper will extend to the analysis of graphs with automorphism group $S_n$ under weaker hypotheses than (*) (see Remark 3 after Theorem 2.7).

NOTATION. If $G$ is a permutation group on a set $\Omega$ and $\Delta \subseteq \Omega$ then $G_{\{\Delta\}}$ denotes the setwise stabilizer of $\Delta$ in $G$; and if $\Psi$ is a fixed set of $G$ then $G^{\Psi}$ denotes the action of $G$ on $\Psi$. Also $\operatorname{Alt}(\Omega)$ and $\operatorname{Sym}(\Omega)$ denote, respectively, the alternating and symmetric groups on $\Omega$.

## 1. A general result and a conjecture

We begin with an elementary proposition.

PROPOSITION 1.1. *Let $n > 6$ and let $H$ be a proper subgroup of $S_n$ with $|S_n : H| < \frac{1}{2}n(n-1)$. Then $H$ is $A_n$, $S_{n-1}$ or $A_{n-1}$.*

PROOF. If $H$ is transitive and imprimitive on the $n$ points with blocks of size $a$ and $ab = n$ $(a \neq 1, b \neq 1)$, then $|H| \leqslant (a!)^b b!$, so

$$\tfrac{1}{2}n(n-1) > |S_n : H| \geqslant n!/\big((a!)^b b!\big)$$

which forces $n \leqslant 6$, a contradiction. If $H$ is primitive on the $n$ points and $H \neq A_n$ then a result of Bochert (Theorem 14.2 of [4]) gives $\frac{1}{2}n(n-1) > |S_n : H| \geqslant [\frac{1}{2}(n+1)]!$, forcing $n \leqslant 6$ or $n = 8$. The latter is impossible (an easy check) so this case cannot occur. Finally if $H$ is intransitive with an orbit of size $r$ then $\frac{1}{2}n(n-1) > |S_n : H| \geqslant \binom{n}{r}$, so that $r$ is 1 or $n-1$ and $H$ is $S_{n-1}$ or $A_{n-1}$.

From Proposition 1.1 we see that if $\Gamma$ is a graph with $\operatorname{Aut}\Gamma \cong S_n$ $(n > 6)$ and all orbits of $\operatorname{Aut}\Gamma$ on $V\Gamma$ have size less than $\frac{1}{2}n(n-1)$ then these orbit sizes all lie in $\{1, 2, n, 2n\}$. We shall easily show below (Proposition 1.2) that the orbit sizes 2 and $2n$ cannot occur, so for the remainder of this section we concentrate on the set $\mathscr{E}_n$ of graphs defined as follows:

DEFINITION. Let $\Gamma$ be a graph and let $n \geqslant 2$. Then $\Gamma \in \mathscr{E}_n$ if and only if $\operatorname{Aut}\Gamma$ has a subgroup $G$ isomorphic to $S_n$ such that all orbits of $G$ on $V\Gamma$ have size 1 or $n$.

Let $\Gamma \in \mathscr{E}_n$. Then we may take $\Gamma$ to be a graph on $tn + r$ vertices $\{\alpha_{11}, \ldots, \alpha_{1n}, \ldots, \alpha_{t1}, \ldots, \alpha_{tn}, \phi_1, \ldots, \phi_r\}$ such that $\operatorname{Aut}\Gamma$ has a subgroup $G$ isomorphic to $S_n$ with $r$ fixed points $\phi_1, \ldots, \phi_r$ and $t$ orbits $\Delta_1, \ldots, \Delta_t$ of size $n$, where $\Delta_i = \{\alpha_{i1}, \ldots, \alpha_{in}\}$ $(i = 1, \ldots, t)$. It is clear that each subgraph $\Delta_i$ is either the complete graph $K_n$ or the null graph $V_n$ and that for any $i$, $j$, $\phi_j$ is joined to all or no vertices in $\Delta_i$. For any $i$, $j$, $k$ define

$$\Gamma_j(\alpha_{ik}) = \{\alpha_{jl} \in \Delta_j \mid \alpha_{jl} \text{ is joined to } \alpha_{ik} \text{ in } \Gamma\}.$$

Then $\Gamma_j(\alpha_{ik})$ is a union of orbits of the stabiliser $G_{\alpha_{ik}}$ on $\Delta_j$. Now if $n \neq 6$ then $S_n$ has just one conjugacy class of subgroups of index $n$, so we may assume in this case that $G_{\alpha_{ik}} = G_{\alpha_{jk}}$ for all $i$, $j$, $k$; and $S_6$ has two conjugacy classes of subgroups of index 6, one class containing the stabilizer of one of the 6 points and the other containing a subgroup $S_5$ transitive on the 6 points. Hence for any $n \geqslant 2$ and any $i$, $j$, $k$ we may assume that $G_{\alpha_{ik}}$ is either transitive on $\Delta_j$ or has orbits $\{\alpha_{jk}\}$ and $\Delta_j \setminus \{\alpha_{jk}\}$ on $\Delta_j$. Consequently $\Gamma_j(\alpha_{ik})$ is one of the sets $\varnothing$, $\Delta_j$, $\{\alpha_{jk}\}$ and $\Delta_j \setminus \{\alpha_{jk}\}$; and for any $k$, $l$, if $\Gamma_j(\alpha_{ik})$ is $\varnothing(\Delta_j, \{\alpha_{jk}\}, \Delta_j \setminus \{\alpha_{jk}\})$ then $\Gamma_j(\alpha_{il})$ is $\varnothing(\Delta_j, \{\alpha_{jl}\}, \Delta_j \setminus \{\alpha_{jl}\}$ respectively).

Note that the above analysis goes through if we replace $S_n$ by the alternating group $A_n$. Using this analysis we now prove the result promised above.

PROPOSITION 1.2. *Let $\Gamma$ be a graph with* $\operatorname{Aut}\Gamma \cong S_n$ $(n > 6)$ *and suppose that all orbits of* $\operatorname{Aut}\Gamma$ *on* $V\Gamma$ *have size less than* $\frac{1}{2}n(n-1)$. *Then all these orbits have size* 1 *or* $n$.

PROOF. Write $G = \operatorname{Aut}\Gamma$. By Proposition 1.1 all $G$-orbits on $V\Gamma$ have size 1, 2, $n$ or $2n$. Let $H < G$ with $H \cong A_n$. Then all $H$-orbits on $V\Gamma$ have size 1 or $n$; let those of size $n$ be $\Delta_1, \ldots, \Delta_t$ and let fix $H = \{\phi_1, \ldots, \phi_r\}$. By the above analysis, each $\Delta_i$ is $K_n$ or $V_n$, each $\phi_j$ is joined to all or no vertices of $\Delta_i$ and, writing $\Delta_i = \{\alpha_{i1}, \ldots, \alpha_{in}\}$, we may choose notation so that $\Gamma_j(\alpha_{ik})$ is one of $\varnothing$, $\Delta_j$, $\{\alpha_{jk}\}$ and $\Delta_j \setminus \{\alpha_{jk}\}$ and if $\Gamma_j(\alpha_{ik})$ is $\varnothing$ $(\Delta_j$, $\{\alpha_{jk}\}$, $\Delta_j \setminus \{\alpha_{jk}\})$ then $\Gamma_j(\alpha_{il})$ is $\varnothing(\Delta_j$, $\{\alpha_{jl}\}$, $\Delta_j \setminus \{\alpha_{jl}\}$ respectively) (any $i$, $j$, $k$, $l$). It is clear from this that $\operatorname{Aut}\Gamma$ contains a subgroup $K$ such that $K \cong S_n$, $K$ has orbits $\Delta_1, \ldots, \Delta_t$ and fixes each $\phi_j$. Hence $K = G$ and the orbits of $G$ all have size 1 or $n$.

We now resume our analysis of a graph $\Gamma \in \mathscr{E}_n$ on vertex set $\Delta_1 \cup \cdots \cup \Delta_t \cup \{\phi_1, \ldots, \phi_r\}$ as described above. From $\Gamma$ we define a coloured graph $\Gamma^*$ with vertex set $\{\delta_1, \ldots, \delta_t, \phi_1, \ldots, \phi_r\}$ having 3 vertex-colours (white, black and red) and 5 edge-colours $(0, 1, n-1, n$ and black) as follows:

(i) $\delta_i$ is coloured white if $\Delta_i$ is $V_n$, black if $\Delta_i$ is $K_n$ $(i = 1, \ldots, t)$;

(ii) $\phi_j$ is coloured red $(j = 1, \ldots, r)$;

(iii) $\phi_j$ is joined to $\delta_i$ by a black edge if $\phi_j$ is joined in $\Gamma$ to all vertices of $\Delta_i$, and by no edge at all if not;

(iv) $\phi_i$ is joined to $\phi_j$ by a black edge if $\phi_i$ is joined to $\phi_j$ in $\Gamma$, and by no edge if not;

(v) the vertices $\delta_i$, $\delta_j$ are joined by an edge coloured $0, 1, n-1$ or $n$ as follows (if $n = 2$ then the labels 1 and $n-1$ should represent different colours):

$\delta_i$ ×——0——× $\delta_j$ if $\Gamma_j(\alpha_{ik}) = \varnothing$,

$\delta_i$ ×——1——× $\delta_j$ if $\Gamma_j(\alpha_{ik}) = \{\alpha_{jk}\}$,

$\delta_i$ ×——n-1——× $\delta_j$ if $\Gamma_j(\alpha_{ik}) = \Delta_j \setminus \{\alpha_{jk}\}$,

$\delta_i$ ×——n——× $\delta_j$ if $\Gamma_j(\alpha_{ik}) = \Delta_j$,

(here $\times$ represents a black or a white vertex). The automorphism group $\operatorname{Aut}\Gamma^*$ is the group of permutations of $V\Gamma^*$ preserving all vertex- and edge-colours. Clearly $\Gamma$ can be reconstructed from $\Gamma^*$; so $\Gamma \leftrightarrow \Gamma^*$ is a 1-1 correspondence.

Now we define two further graphs from $\Gamma^*$: firstly, $\Gamma_0^*$ is the subgraph of $\Gamma^*$ on $\delta_1, \ldots, \delta_t$ with all edges coloured 0 or $n$ deleted and all edges coloured 1 or $n-1$ replaced by a black edge; secondly, $\Gamma_1^*$ is obtained from $\Gamma^*$ by the following replacements:

(1) replace all vertices $\delta_i$, $\phi_j$ by black vertices $\delta_i$, $\phi_j$;

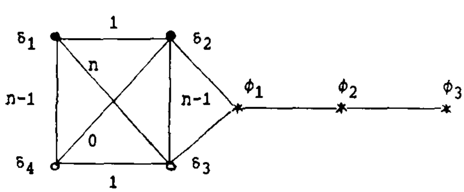
(2) replace any edge coloured 1 or $n$ by a black edge;

(3) delete any edge coloured 0 or $n-1$.

Thus $\Gamma_1^*$ can be regarded as an uncoloured graph.

We aim to obtain necessary and sufficient conditions for $\operatorname{Aut}\Gamma \cong S_n$ purely in terms of the smaller graphs $\Gamma^*$, $\Gamma_0^*$ and $\Gamma_1^*$.

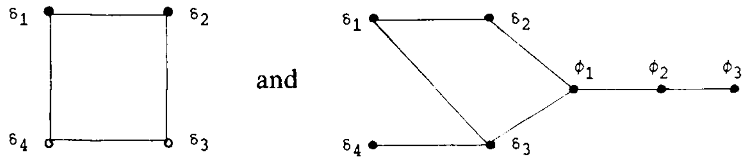
EXAMPLE. If $\Gamma^*$ is



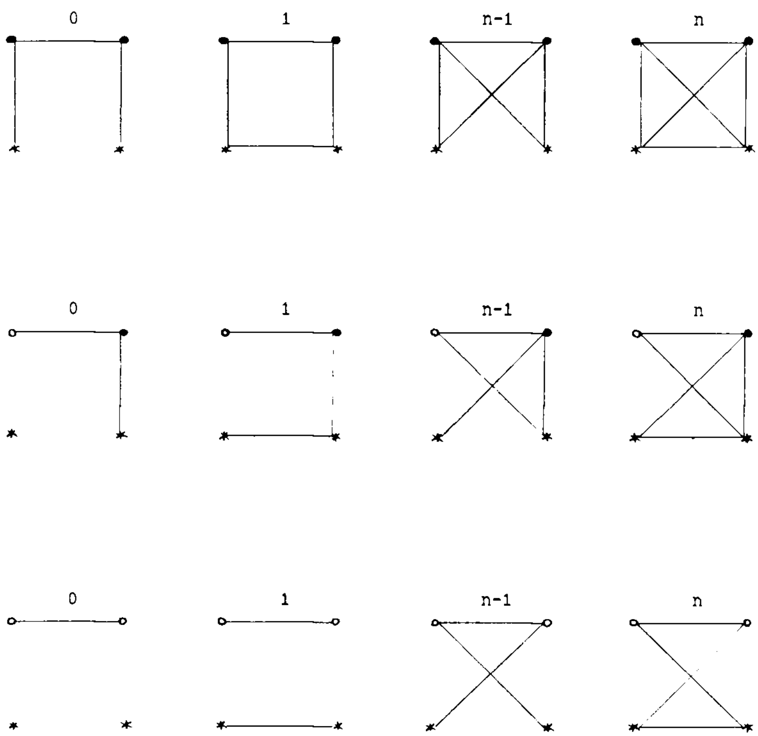

(* represents a red vertex)

https://doi.org/10.1017/S1446788700031360 Published online by Cambridge University Press

then $\Gamma_0^*$, $\Gamma_1^*$ are respectively




Now consider a coloured graph $\Gamma^*$ with $t = r = 2$, that is, with $V\Gamma^* = \{\delta_1, \delta_2, \phi_1, \phi_2\}$. Let $\Gamma$ be the corresponding graph in $\mathscr{E}_n$ with $V\Gamma = \Delta_1 \cup \Delta_2 \cup \{\phi_1, \phi_2\}$. It is easy to see that $\operatorname{Aut}\Gamma$ contains a subgroup isomorphic to $S_{n+1}$ having orbits $\Delta_1 \cup \{\phi_{i_1}\}$ and $\Delta_2 \cup \{\phi_{i_2}\}$ (where $\{i_1, i_2\} = \{1,2\}$) on $V\Gamma$ if and only if $\Gamma^*$ is isomorphic to one of the following 12 graphs:




https://doi.org/10.1017/S1446788700031360 Published online by Cambridge University Press

(where ∘, •, $*$ represent white, black and red vertices respectively). Denote this set of 12 graphs by $\mathscr{C}_n$. Note that the graphs in $\mathscr{C}_n$ correspond to 6 graphs in $\mathscr{E}_n$ and their complements.

LEMMA 1.3. *Let $\Gamma \in \mathscr{E}_n$ have vertex set $\Delta_1 \cup \cdots \cup \Delta_t \cup \{\phi_1, \ldots, \phi_r\}$ as above and let $\Gamma^*$ and $\Gamma_1^*$ be the graphs corresponding to $\Gamma$ as above. Suppose that* $\operatorname{Aut}\Gamma_1^*$ *contains an automorphism $x = (\delta_1\phi_{i_1}) \cdots (\delta_t\phi_{i_t})$ ($i_1, \ldots, i_t$ all distinct) such that*

(i) *$\phi_{i_j}$ is joined to $\delta_j$ in $\Gamma^*$ if and only if $\delta_j$ is black ($j = 1, \ldots, t$), and*

(ii) *for any distinct, $k$, $l$ the subgraph $\{\delta_k, \delta_l, \phi_{i_k}, \phi_{i_l}\}$ of $\Gamma^*$ lies in the set $\mathscr{C}_n$ of 12 graphs defined above.*

*Then $\Gamma \in \mathscr{E}_{n+1}$.*

PROOF. For $j = 1, \ldots, t$ put $\Delta'_j = \Delta_j \cup \{\phi_{i_j}\}$. By (i) each subgraph $\Delta'_j$ is either $K_{n+1}$ or $V_{n+1}$. Write $\phi_{i_j} = \alpha_{j,n+1}$ ($j = 1, \ldots, t$) and for any $i$, $j$, $k$ define

$$\Gamma'_j(\alpha_{ik}) = \left\{\alpha_{jl} \in \Delta'_j \mid \alpha_{jl} \text{ is joined to } \alpha_{ik} \text{ in } \Gamma\right\}.$$

Then by (i) and (ii), $\Gamma'_j(\alpha_{ik})$ is one of the sets $\varnothing$, $\Delta'_j$, $\{\alpha_{jk}\}$ and $\Delta'_j \setminus \{\alpha_{jk}\}$ and for any $k$, $l$, if $\Gamma'_j(\alpha_{ik})$ is $\varnothing$ ($\Delta'_j$, $\{\alpha_{jk}\}$, $\Delta'_j \setminus \{\alpha_{jk}\}$) then $\Gamma'_j(\alpha_{il})$ is $\varnothing$ ($\Delta'_j$, $\{\alpha_{jl}\}$, $\Delta'_j \setminus \{\alpha_{jl}\}$ respectively). Also, since $x \in \operatorname{Aut}\Gamma_1^*$, for any $k \notin \{i_1, \ldots, i_t\}$ and any $j$, $\phi_k$ is joined to all or no vertices of $\Delta'_j$. From these facts we see that $\operatorname{Aut}\Gamma$ contains a subgroup $H \cong S_{n+1}$ having orbits $\Delta'_1, \ldots, \Delta'_t$ and fixing $\phi_k$ for $k \notin \{i_1, \ldots, i_t\}$. Hence $\Gamma \in \mathscr{E}_{n+1}$.

THEOREM 1.4. *Let $\Gamma \in \mathscr{E}_n$ have vertex set $\Delta_1 \cup \cdots \cup \Delta_t \cup \{\phi_1, \ldots, \phi_r\}$ as above and let $\Gamma^*$, $\Gamma_0^*$, $\Gamma_1^*$ be the graphs corresponding to $\Gamma$. Suppose that* $\operatorname{Aut}\Gamma \cong S_n$. *Then*

(a) $\operatorname{Aut}\Gamma^* = 1$;

(b) *$\Gamma_0^*$ is connected (by the black edges);*

(c) $\operatorname{Aut}\Gamma_1^*$ *contains no automorphisms $(\delta_1\phi_{i_1}) \cdots (\delta_t\phi_{i_t})$, with $i_1, \ldots, i_t$ distinct, such that*

(i) *$\phi_{i_j}$ is joined to $\delta_j$ in $\Gamma^*$ if and only if $\delta_j$ is black ($j = 1, \ldots, t$),*

(ii) *for any distinct $k$, $l$ the subgraph $\{\delta_k, \delta_l, \phi_{i_k}, \phi_{i_l}\}$ of $\Gamma^*$ lies in $\mathscr{C}_n$.*

PROOF. (a) Suppose that $h \in \operatorname{Aut}\Gamma^*$ with $h \neq 1$. Define a permutation $g$ on $V\Gamma$ as follows

(1) if $\delta_i h = \delta_j$ put $\alpha_{ik} g = \alpha_{jk}$ ($k = 1, \ldots, n$),

(2) for $i = 1, \ldots, r$ put $\phi_i g = \phi_i h$.

It is easy to check that $g \in \operatorname{Aut}\Gamma$, which contradicts the fact that since $\operatorname{Aut}\Gamma \cong S_n$, $\operatorname{Aut}\Gamma$ has orbits $\Delta_1, \ldots, \Delta_t$ and fixes each $\phi_j$.

https://doi.org/10.1017/S1446788700031360 Published online by Cambridge University Press

(b) Suppose that $\Gamma_0^*$ is disconnected and let $\{\delta_{i_1}, \dots, \delta_{i_u}\}$ $(u < t)$ be a connected component of $\Gamma_0^*$; write $\Delta = \bigcup_{j=1}^{u} \Delta_{i_j}$. Then for any $\beta \in V\Gamma \setminus \Delta$ and any $j \in \{1, \dots, u\}$, $\beta$ is joined to all or no vertices in $\Delta_{i_j}$. Hence $\operatorname{Aut}\Gamma$ contains a subgroup $H \cong S_n$ with orbits $\Delta_{i_1}, \dots, \Delta_{i_u}$ and fixing every vertex in $V\Gamma \setminus \Delta$. Since $u < t$ it is clear that $H \neq \operatorname{Aut}\Gamma$, contradicting the fact that $\operatorname{Aut}\Gamma \cong S_n$. Thus $\Gamma_0^*$ is connected.

(c) This follows directly from Lemma 1.3.

It seems likely that a general converse of Theorem 1.4 holds; since we have only been able to prove this when $t \leqslant 4$, we state the general case as a conjecture.

CONJECTURE 1.5. *Let $n$, $t$ be positive integers with $n > t$. Let $\Gamma^*$ be a graph on vertex set $\{\delta_1, \dots, \delta_t, \phi_1, \dots, \phi_r\}$ with 3 vertex-colours (white and black among the $\delta_i$, red for the $\phi_i$) and 5 edge-colours* $(0, 1, n-1, n$ *for edges between the $\delta_i$, black for any other edges). Let $\Gamma_0^*$ and $\Gamma_1^*$ be the graphs defined from $\Gamma^*$ as above and suppose that these satisfy conditions* (a), (b) *and* (c) *of Theorem* 1.4. *Then if $\Gamma$ is the graph on $tn + r$ vertices corresponding as above to $\Gamma^*$, we have* $\operatorname{Aut}\Gamma \cong S_n$.

In the next sections we prove Conjecture 1.5 for $1 \leqslant t \leqslant 4$ and give some illustrations of its use in describing graphs $\Gamma$ with $\operatorname{Aut}\Gamma \cong S_n$. It should be noted that we have introduced the condition $n > t$ in Conjecture 1.5 solely for convenience in the proofs in §2, and that it seems likely that the conjecture is true for any values of $n$ and $t$ with $n \geqslant 3$.

## 2. Proofs of Conjecture 1.5 for $1 \leqslant t \leqslant 4$

*The case $t = 1$.* We prove Conjecture 1.5 for $t = 1$. Let $\Gamma^*$ be a graph on $\{\delta_1, \phi_1, \dots, \phi_r\}$ coloured as in 1.5. We assume first that $\delta_1$ is black. Writing $H = \operatorname{Aut}\Gamma_1^*$, condition (a) of Theorem 1.4 means that

(1) $H_{\delta_1} = 1$,

condition (b) is vacuously satisfied and condition (c) means that

(2) $H$ contains no automorphism $(\delta_1 \phi_{i_1})$ with $\delta_1$ joined to $\phi_{i_1}$.

Suppose then that (1) and (2) hold and let $\Gamma$ be the corresponding graph on $n + r$ vertices $\Delta_1 \cup \{\phi_1, \dots, \phi_r\}$, where $\Delta_1 = \{\alpha_{11}, \dots, \alpha_{1n}\}$ is $K_n$ since $\delta_1$ is black. Write $G = \operatorname{Aut}\Gamma$. We show that $G \cong S_n$.

Since $H_{\delta_1} = 1$ it is clear that $G_{\{\Delta_1\}}$ fixes $V\Gamma \setminus \Delta_1$ pointwise; thus $G_{\{\Delta_1\}} \cong S_n$. Suppose that there exists $g \in G \setminus G_{\{\Delta_1\}}$. Then $G_{\{\Delta_1 g\}}$ fixes $V\Gamma \setminus \Delta_1 g$ pointwise and $G_{\{\Delta_1 g\}} \cong S_n$, so if $\Delta_1 \cap \Delta_1 g = \varnothing$ then $H_{\delta_1}$ has a subgroup isomorphic to $S_n$, contradicting (1). Hence $\Delta_1 \cap \Delta_1 g \neq \varnothing$. Write $\Sigma = \Delta_1 \cup \Delta_1 g$. It is easy to see that $G_{\{\Sigma\}} = \langle G_{\{\Delta_1\}}, G_{\{\Delta_1 g\}} \rangle \cong \operatorname{Sym}(\Sigma)$ and that $G_{\{\Sigma\}}$ fixes $V\Gamma \setminus \Sigma$ pointwise.

https://doi.org/10.1017/S1446788700031360 Published online by Cambridge University Press

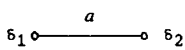
Consequently $\Sigma$ is a complete subgraph of $\Gamma$ and if we choose $\phi_{i_1} \in \Sigma \setminus \Delta_1$ then $(\alpha_{11}\phi_{i_1}) \in \operatorname{Aut}\Gamma$. This forces $(\delta_1\phi_{i_1}) \in \operatorname{Aut}\Gamma_1^*$, contradicting (2), as $\phi_{i_1}$ is joined to $\delta_1$.

The case where $\delta_1$ is white follows from the above argument by considering the complement of the corresponding graph $\Gamma$. Hence Conjecture 1.5 is proved for $t = 1$.

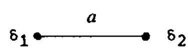
Descriptions of the graphs characterized by this result can be found in the Appendix.

*The cases* $2 \leqslant t \leqslant 4$. We prove Conjecture 1.5 just for $t = 4$, as the cases $t = 2$ and $t = 3$ are similar and easier. In the proof we shall need, for $2 \leqslant u \leqslant 4$ and $n > u$, a description of all coloured graphs $\Gamma^*$ on $\{\delta_1, \ldots, \delta_u\}$ which give rise as in §1 to vertex-transitive graphs $\Gamma$ on $un$ vertices. We call $\Gamma^*$ *vertex-monochrome* if all the vertices $\delta_i$ have the same colour.

$u = 2$. The only graphs $\Gamma^*$ on $\{\delta_1, \delta_2\}$ which give rise to a transitive graph $\Gamma$ are

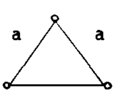


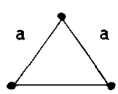

and




where $a$ is 0, 1, $n - 1$ or $n$ (that is, $\Gamma^*$ is any vertex-monochrome graph on $\{\delta_1, \delta_2\}$).

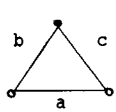
$u = 3$. Suppose that $\Gamma^*$ on $\{\delta_1, \delta_2, \delta_3\}$ gives rise to a transitive graph $\Gamma$. If $\Gamma^*$ is vertex-monochrome then by the regularity of $\Gamma$ it must be one of the following graphs:




$(a \in \{0, 1, n - 1, n\})$.

There are no further such graphs $\Gamma^*$. For suppose that $\Gamma^*$ is not vertex-monochrome. Then we may take $\Gamma^*$ to be




where $a, b, c \in \{0, 1, n - 1, n\}$. Since $\Gamma$ is regular we have

$$a + b = a + c = b + c + n - 1$$

so that $b = c$ and $a = c + n - 1$. Thus $c$ is 0 or 1, which forces $\Delta_3$ to be the unique subgraph $K_n$ of $\Gamma$, contradicting the transitivity of $\Gamma$.

$u = 4$. Suppose that $\Gamma^*$ on $\{\delta_1, \delta_2, \delta_3, \delta_4\}$ gives rise to a transitive graph $\Gamma$. Let $a_{ij} \in \{0, 1, n - 1, n\}$ be the colour of the edge joining $\delta_i$ and $\delta_j$. If $\Gamma^*$ is vertex-monochrome then since $\Gamma$ is regular of valency $b$, say, we have

$$a_{12} + a_{13} + a_{14} = a_{12} + a_{23} + a_{24} = a_{13} + a_{23} + a_{34} = a_{14} + a_{24} + a_{34} = b.$$

https://doi.org/10.1017/S1446788700031360 Published online by Cambridge University Press

This gives $a_{14} = a_{23}$, $a_{13} = a_{24}$, $a_{12} = a_{34}$, so $\operatorname{Aut}\Gamma^*$ contains the subgroup $V_4 = \langle(\delta_1\delta_2)(\delta_3\delta_4), (\delta_1\delta_3)(\delta_2\delta_4)\rangle$. If $\delta_1$ is black and $\delta_2$, $\delta_3$, $\delta_4$ are white then the regularity of $\Gamma$ gives $2(a_{23} - a_{14}) = n - 1$ which is not possible since $n > u = 4$. And if $\delta_1$, $\delta_2$ are black and $\delta_3$, $\delta_4$ are white then it is easy to see that any vertex in $\Delta_1 \cup \Delta_2$ is contained in more subgraphs $K_n$ of $\Gamma$ than any vertex in $\Delta_3 \cup \Delta_4$, contradicting the transitivity of $\Gamma$.

We summarise these results in a lemma:

LEMMA 2.1. *Let $\Gamma^*$ be a coloured graph on $\{\delta_1, \ldots, \delta_u\}$ $(u \leqslant 4)$ which gives rise as in §1 to a transitive graph $\Gamma$ on un vertices $(n > u)$. Then $\Gamma^*$ is vertex-monochrome and $\operatorname{Aut}\Gamma^*$ contains a subgroup $S$, where $S = S_2$ if $u = 2$, $S = S_3$ if $u = 3$ and $S = V_4$ if $u = 4$.*

PROOF OF CONJECTURE 1.5 FOR $t = 4$. Let $\Gamma^*$ be a graph on $\{\delta_1, \delta_2, \delta_3, \delta_4, \phi_1, \ldots, \phi_r\}$ coloured as in 1.5 and suppose that (a), (b) and (c) of Theorem 1.4 hold. Let $n$ be an integer with $n > t = 4$ and let $\Gamma$ be the corresponding graph on $4n + r$ vertices $\Delta_1 \cup \Delta_2 \cup \Delta_3 \cup \Delta_4 \cup \{\phi_1, \ldots, \phi_r\}$ (where $\Delta_i = \{\alpha_{i1}, \ldots, \alpha_{in}\}$ for $i = 1, \ldots, 4$). Write $\Delta = \Delta_1 \cup \Delta_2 \cup \Delta_3 \cup \Delta_4$ and $\Phi = \{\phi_1, \ldots, \phi_r\}$, and let $\Delta^*$ be the subgraph of $\Gamma^*$ on $\{\delta_1, \delta_2, \delta_3, \delta_4\}$. Also write $\phi_i \sim \delta_j$ if $\phi_i$ is joined to $\delta_j$ in $\Gamma^*$. Put $G = \operatorname{Aut}\Gamma$. By the construction of $\Gamma$ from $\Gamma^*$ (explained in §1) it is clear that $G$ has a unique subgroup $H \cong S_n$ having orbits $\Delta_1$, $\Delta_2$, $\Delta_3$, $\Delta_4$, fixing each $\phi_i$, and such that $H_{\alpha_{ij}} = H_{\alpha_{ik}}$ for all $i$, $j$, $k$. We aim to show that $G_{\{\Delta\}} = H$, which we establish in the following two lemmas.

LEMMA 2.2. *If $g \in G_{\{\Delta\}} \setminus H$ then $\Delta_i g \neq \Delta_i$ for some $i \in \{1, 2, 3, 4\}$.*

PROOF. Suppose that $\Delta_i g = \Delta_i$ for all $i$. Then clearly $g^{\Phi}1^{\Delta^*} \in \operatorname{Aut}\Gamma^*$, so $g^{\Phi} = 1$ by (a) of 1.4. Now $H^{\Delta_1} \cong S_n$, so $g^{\Delta_1} = h^{\Delta_1}$ for some $h \in H$. Then $g^{-1}h$ fixes $\Delta_1 \cup \Phi$ pointwise and $g^{-1}h \neq 1$ as $g \notin H$. Hence the sets

$$\Delta' = \bigcup\left\{\Delta_i \mid (g^{-1}h)^{\Delta_i} = 1\right\}, \qquad \Delta'' = \bigcup\left\{\Delta_i \mid (g^{-1}h)^{\Delta_1} \neq 1\right\}$$

are both nonempty. Let $K = \langle(g^{-1}h)^x \mid x \in H\rangle$. Then $K^{\Delta'} = 1$ and for $\Delta_i \subseteq \Delta''$ we have $K^{\Delta_i} \geqslant \operatorname{Alt}(\Delta_i)$ since $K^{\Delta_i} \lhd \operatorname{Sym}(\Delta_i)$. Hence for any $\Delta_i \subseteq \Delta''$ and any $\alpha_{jk} \in \Delta'$, $\alpha_{jk}$ is joined to all or no vertices of $\Delta_i$. Thus in $\Gamma^*$, any edge between a vertex of $\{\delta_i \mid \Delta_i \subseteq \Delta'\}$ and a vertex of $\{\delta_i \mid \Delta_i \subseteq \Delta''\}$ must be coloured 0 or $n$. This forces $\{\delta_i \mid \Delta_i \subseteq \Delta'\}$ to be a union of connected components of the graph $\Gamma_0^*$, contradicting (b) of 1.4.

LEMMA 2.3. *We have* $G_{\{\Delta\}} = H$.

PROOF. Suppose false and pick $g \in G_{\{\Delta\}} \setminus H$. By Lemma 2.2 we have $\Delta_i g \neq \Delta_i$ for some $i$, so if $L = \langle H, g \rangle$ then $L$ has at most 3 orbits on $\Delta$. We prove the lemma by obtaining a contradiction to the fact that $\operatorname{Aut}\Gamma^* = 1$. There are several cases, depending on the number of orbits of $L$ on $\Delta$.

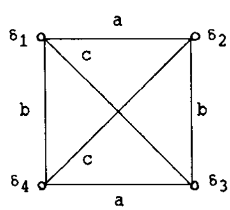
*Case* 1. *$L$ is transitive on* $\Delta$. For any $i, j \in \{1,2,3,4\}$ write $\Delta_i \to \Delta_j$ if there exist $\alpha_{ik} \in \Delta_i$, $\alpha_{jl} \in \Delta_j$ with $\alpha_{ik} g = \alpha_{jl}$. For distinct $i_1, \ldots, i_u \in \{1,2,3,4\}$ $(1 \leqslant u \leqslant 4)$ write $[\Delta_{i_1} \cdots \Delta_{i_u}]$ to mean that $\Delta_{i_1} \to \Delta_{i_2}, \Delta_{i_2} \to \Delta_{i_3}, \ldots, \Delta_{i_u} \to \Delta_{i_1}$.

Now $H$ has orbits $\Delta_1$, $\Delta_2$, $\Delta_3$, $\Delta_4$ on $\Delta$ and $L = \langle H, g \rangle$ is transitive on $\Delta$; it is not hard to see from this that we may assume that one of the following holds

(i) $[\Delta_1 \Delta_2 \Delta_3 \Delta_4]$;
(ii) $[\Delta_1 \Delta_2]$, $[\Delta_1 \Delta_3]$ and $[\Delta_1 \Delta_4]$;
(iii) $[\Delta_1 \Delta_2]$, $[\Delta_1 \Delta_3]$ and $[\Delta_2 \Delta_4]$;
(iv) $[\Delta_1 \Delta_2 \Delta_3]$ and $[\Delta_1 \Delta_4]$;
(v) $[\Delta_1 \Delta_2 \Delta_3]$ and $[\Delta_1 \Delta_2 \Delta_4]$.

Suppose that (i) holds. Then $\alpha_{1i_1} g = \alpha_{2i_2}$, $\alpha_{2j_2} g = \alpha_{3j_3}$, $\alpha_{3k_3} g = \alpha_{4k_4}$, $\alpha_{4l_4} g = \alpha_{1l_1}$ for some $i_1$, $i_2$, etc. Choose $\phi_a \in \Phi$. If $\phi_a$ is joined to $\alpha_{1i_1}$ then $\phi_a g$ is joined to $\alpha_{2i_2}$, hence to every vertex in $\Delta_2$, so $\phi_a g^2$ is joined to $\alpha_{3j_3}$. Thus $\phi_a \sim \delta_1 \Rightarrow \phi_a g^2 \sim \delta_3$. In this way we see that $\phi_a \sim \delta_1 \Rightarrow \phi_a g^2 \sim \delta_3$, $\phi_a \sim \delta_2 \Rightarrow \phi_a g^2 \sim \delta_4$, $\phi_a \sim \delta_3 \Rightarrow \phi_a g^2 \sim \delta_1$ and $\phi_a \sim \delta_4 \Rightarrow \phi_a g^2 \sim \delta_2$. Also by Lemma 2.1 we have $(\delta_1\delta_3)(\delta_2\delta_4) \in \operatorname{Aut}\Delta^*$. It follows that $(g^2)^{\Phi}(\delta_1\delta_3)(\delta_2\delta_4) \in \operatorname{Aut}\Gamma^*$, contradicting the fact that $\operatorname{Aut}\Gamma^* = 1$.

If (ii) holds then for any $a \in \{1, \ldots, r\}$ we have $\phi_a \sim \delta_2 \Leftrightarrow \phi_a g \sim \delta_1 \Leftrightarrow \phi_a \sim \delta_3 \Leftrightarrow \phi_a \sim \delta_4$. Hence any permutation of $\{\delta_2, \delta_3, \delta_4\}$ fixing $\delta_1$ and each $\phi_a$ will be an automorphism of $\Gamma^*$ providing it is an automorphism of the subgraph $\Delta^*$. By Lemma 2.1 we can take $\Delta^*$ to be




where $a, b, c \in \{0, 1, n-1, n\}$. If $a = b$ then $(\delta_2\delta_4) \in \operatorname{Aut}\Gamma^*$, if $b = c$ then $(\delta_3\delta_4) \in \operatorname{Aut}\Gamma^*$ and if $a = c$ then $(\delta_2\delta_3) \in \operatorname{Aut}\Gamma^*$, all of which are contradictions. Hence, $a$, $b$, $c$ are distinct and we may assume that either $a = 0$, $b = 1$, $c \geqslant n-1$ or $a \leqslant 1$, $b = n-1$, $c = n$. Write $m(\alpha_{ij}, \alpha_{kl})$ for the number of mutual adjacencies of $\alpha_{ij}$ and $\alpha_{kl}$ in the subgraph $\Delta$ of $\Gamma$. Then for any $i$, $j$, $k$, $l$

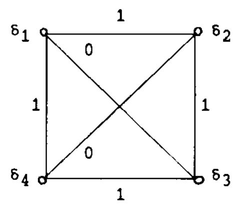
https://doi.org/10.1017/S1446788700031360 Published online by Cambridge University Press

we have $m(\alpha_{1_i}, \alpha_{1_j}) \geqslant n - 2$ and $m(\alpha_{2k}, \alpha_{3l}) \leqslant 2$. However, by assumption (we are in case (ii)) there exist $i$, $j$, $k$, $l$ such that $\alpha_{1_i} g = \alpha_{2k}$, $\alpha_{1j} g = \alpha_{3l}$, which forces $m(\alpha_{1_i}, \alpha_{1_j}) = m(\alpha_{2k}, \alpha_{3l})$; hence $n - 2 \leqslant 2$ or $n \leqslant 4$, contradicting the fact that $n > t = 4$.

In case (iii) we have $\phi_a \sim \delta_1 \Leftrightarrow \phi_a g \sim \delta_2 \Leftrightarrow \phi_a \sim \delta_4$ and $\phi_a \sim \delta_2 \Leftrightarrow \phi_a g \sim \delta_1 \Leftrightarrow \phi_a \sim \delta_3$. Hence $(\delta_1\delta_4)(\delta_2\delta_3) \in \operatorname{Aut}\Gamma^*$ which is a contradiction.

In case (iv) we see similarly that $(\delta_1\delta_2)(\delta_3\delta_4) \in \operatorname{Aut}\Gamma^*$, again a contradiction.

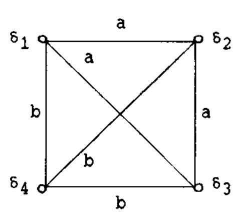
Finally, suppose that (v) holds. Then $\phi_a \sim \delta_3 \Leftrightarrow \phi_a g \sim \delta_1 \Leftrightarrow \phi_a \sim \delta_4$ so $(\delta_3\delta_4) \in \operatorname{Aut}\Gamma^*$ if $b = c$ in the subgraph $\Delta^*$. Thus $b \neq c$. Suppose first that $c \leqslant 1$. Then $m(\alpha_{3_i}, \alpha_{4_j}) \leqslant 2$ for any $i$, $j$, so by application of $g^{-1}$ we see that $m(\alpha_{1k}, \alpha_{1l}) \leqslant 2$ for any distinct $k$, $l$. This forces $a \leqslant 1$ and $b \leqslant 1$. Since $b \neq c$ we may take $b = 1$, $c = 0$; as $\Gamma_0^*$ is connected we have $a = 1$ and $\Delta^*$ is



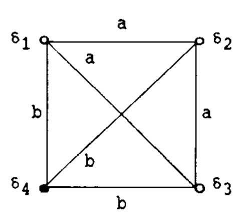

Thus the subgraph $\Delta$ of $\Gamma$ consists of $n$ disjoint squares. Now $\alpha_{1_i} g = \alpha_{2_j}$ for some $i$, $j$. Since $\alpha_{3_i}$ is the unique vertex of $\Delta$ opposite to $\alpha_{1_i}$ in the square containing $\alpha_{1_i}$ and $\alpha_{4_j}$ is similarly opposite to $\alpha_{2_j}$, we must have $\alpha_{3_i} g = \alpha_{4_j}$. In this way we see that $[\Delta_3\Delta_4\Delta_1]$ and $[\Delta_3\Delta_4\Delta_2]$ also hold. The usual argument now shows that $\phi_a \sim \delta_1 \Leftrightarrow \phi_a \sim \delta_2 \Leftrightarrow \phi_a \sim \delta_3 \Leftrightarrow \phi_a \sim \delta_4$ so that $V_4 \leqslant \operatorname{Aut}\Gamma^*$, which is a contradiction. Similar arguments yield a contradiction if $c \geqslant n - 1$.

We have now dealt completely with Case 1.

*Case* 2. *$L$ has orbits $\Delta_1 \cup \Delta_2 \cup \Delta_3$ and $\Delta_4$ on $\Delta$.* In this case we may assume that either (i) $[\Delta_1\Delta_2\Delta_3]$, or (ii) $[\Delta_1\Delta_2]$ and $[\Delta_1\Delta_3]$ holds. Using Lemma 2.2 for $u = 3$ and the fact that each vertex in $\Delta_1 \cup \Delta_2 \cup \Delta_3$ has the same valency, we see that the subgraph $\Delta^*$ can be taken to be one of




and




https://doi.org/10.1017/S1446788700031360 Published online by Cambridge University Press

for some $a, b \in \{0, 1, n-1, n\}$. If (i) holds then $g^{\Phi}(\delta_1\delta_2\delta_3) \in \operatorname{Aut}\Gamma^*$, while in case (ii) we have $\phi_a \sim \delta_2 \Leftrightarrow \phi_a g \sim \delta_1 \Leftrightarrow \phi_a \sim \delta_3$ so that $(\delta_2\delta_3) \in \operatorname{Aut}\Gamma^*$. These contradictions deal with Case 2.

*Case* 3. *L has orbits* $\Delta_1 \cup \Delta_2$ *and* $\Delta_3 \cup \Delta_4$ *or* $\Delta_1 \cup \Delta_2$, $\Delta_3$ *and* $\Delta_4$ *on* $\Delta$. Then either (i) $[\Delta_1\Delta_2]$ and $[\Delta_3\Delta_4]$, or (ii) $[\Delta_1\Delta_2]$, $[\Delta_3]$ and $[\Delta_4]$, holds. In case (i) we have $g^{\Phi}(\delta_1\delta_2)(\delta_3\delta_4) \in \operatorname{Aut}\Gamma^*$ and in case (ii), $g^{\Phi}(\delta_1\delta_2) \in \operatorname{Aut}\Gamma^*$, neither of which can be so.

This completes the proof of Lemma 2.3.

To finish the proof of Conjecture 1.5 for $t = 4$ it remains to show that $G = G_{\{\Delta\}}$. Suppose then that there exists $g \in G \setminus G_{\{\Delta\}}$. Put $M = \langle G_{\{\Delta\}}, G^g_{\{\Delta\}} \rangle = \langle H, H^g \rangle$ and let $\Psi_1, \ldots, \Psi_s$ be the orbits of $M$ on $\Delta \cup \Delta g$. For each $i$ let $X_i = \{ j \mid \Delta_j \subseteq \Psi_i \}$.

LEMMA 2.4. *If* $X_i \neq \varnothing$ *then there is a block system* $\mathscr{B}_i$ *for* $M^{\Psi_i}$ *(possibly with blocks of size* 1), *one of whose blocks* $B_i$ *is contained in* $\{ \alpha_{j1} \mid j \in X_i \}$, *and such that* $M^{\mathscr{B}_i} \geqslant \operatorname{Alt}(\mathscr{B}_i)$.

PROOF. Pick $k \in X_i$, so that $\Delta_k \subseteq \Psi_i$. The lemma is certainly true if $\Delta_k = \Psi_i$ (for then we take $\mathscr{B}_i = \Psi_i$, that is, $\mathscr{B}_i$ to be the set of blocks of size 1); hence we may assume that $\Delta_k \subset \Psi_i$. Let $\mathscr{B}_i$ be a block system for $M^{\Psi_i}$ such that $|\mathscr{B}_i| > 1$ and $\mathscr{B}_i$ contains blocks of maximum possible size. Then $M^{\mathscr{B}_i}$ is primitive. Let $B_i$ be the block of $\mathscr{B}_i$ containing $\alpha_{k1}$. Certainly either $\Delta_k \subseteq B_i$ or $\Delta_k \cap B_i = \{\alpha_{k1}\}$; we show that the latter must hold. Suppose then that $\Delta_k \subseteq B_i$. From the action of $H$ we have

(1) $B_i \cap \Delta$ is a union of $H$-orbits $\Delta_j$.

Next we show that

(2) $B_i \cap \Delta g$ is a union of $H^g$-orbits $\Delta_j g$.

We prove this as follows: if $|\Delta_k \cap \Delta_l g| \leqslant 1$ for all $l$ then since $n > t$ there exists $\alpha_{km} \in \Delta_k \setminus \Delta g$, so that $H^g$ fixes $\alpha_{km}$. Now $\Delta_k \subset \Psi_i$ so we can find $l$ such that $\Delta_k \cap \Delta_l g \neq \varnothing$. Also $\Delta_k \subseteq B_i$, so $B_i \cap \Delta_l g \neq \varnothing$. Since $H^g$ fixes $\alpha_{km}$ this forces $\Delta_l g \subseteq B_i$. The action of $H^g$ now gives (2). If $|\Delta_k \cap \Delta_l g| \geqslant 2$ for some $l$ then $\Delta_l g \subseteq B_i$ again, from which (2) follows as before. Hence (2) is established. Now $M = \langle H, H^g \rangle$ and $\Psi_i$ is a union of sets $\Delta_j$ and $\Delta_j g$ on which $M$ is transitive. It follows from (1) and (2) that $B_i = \Psi_i$, contradicting the fact that $|\mathscr{B}_i| > 1$.

Thus we have shown that $\Delta_k \cap B_i = \{\alpha_{k1}\}$. Since $H_{\alpha_{k1}} = H_{\alpha_{j1}}$ for all $j$, it follows that $B_i \subsetneq \{ \alpha_{j1} \mid j \in X_i \}$. Finally, $M^{\mathscr{B}_i}$ is primitive and contains the subgroup $H^{\mathscr{B}_i} \cong S_n$, so $M^{\mathscr{B}_i}$ contains an element of degree at most 8. From this it follows without much difficulty that $M^{\mathscr{B}_i} \geqslant \operatorname{Alt}(\mathscr{B}_i)$ (see for instance the papers of W. A. Manning referred to at the end of §15 of [4]).

If $X_i = \varnothing$ then $\Psi_i = \Delta_j g$ for some $j$ and $M^{\Psi_i} = \mathrm{Sym}(\Psi_i)$. Put $\mathscr{B}_i = \Psi_i$ in this case (that is, let $\mathscr{B}_i$ be the set of blocks of size 1). Choose notation so that $X_i \neq \varnothing$ for $i = 1, \ldots, s_0$ and $X_i = \varnothing$ for $i = s_0 + 1, \ldots, s$. For $i \in \{1, \ldots, s_0\}$ let $\mathscr{B}_i'$ be the set of blocks of $\mathscr{B}_i$ contained in $\Delta$. Then $|\mathscr{B}_i'| = r_i n$ for some positive integer $r_i$. Write $\mathscr{B} = \bigcup_{i=1}^{s} \mathscr{B}_i$.

LEMMA 2.5. *The following hold*:
(i) $s = s_0$;
(ii) $|\mathscr{B}_j| = |\mathscr{B}_k|$ *for all* $j, k \in \{1, \ldots, s\}$;
(iii) *if* $|\mathscr{B}_1| = b$ *then* $M \cong A_b$ *or* $M \cong S_b$ *and* $M$ *acts similarly on each* $\mathscr{B}_j$ $(j = 1, \ldots, s)$.

PROOF. If $K$ is the kernel of the action of $M$ on $\mathscr{B}$ then $K \leqslant G_{\{\Delta\}}$, so $K = 1$ since $G_{\{\Delta\}} = H$. Hence $M$ acts faithfully on $\mathscr{B}$. Write $N = M'$. Then by Lemma 2.4, $N$ is a subdirect product of $\prod_{i=1}^{s} \mathrm{Alt}(\mathscr{B}_i)$ (that is, $N$ projects surjectively onto each factor). Since each $\mathrm{Alt}(\mathscr{B}_i)$ is simple, $N$ is isomorphic to a direct product of some of the groups $\mathrm{Alt}(\mathscr{B}_i)$ and if we choose $i_0$ such that $|\mathscr{B}_{i_0}| = \max\{|\mathscr{B}_i|\colon i = 1, \ldots, s\}$ then $N$ has a minimal normal subgroup $N_0 \cong \mathrm{Alt}(\mathscr{B}_{i_0})$. Now $g \notin G_{\{\Delta\}}$, so $H^g \neq H$ and so $M \nleqslant G_{\{\Delta\}}$. Consequently $|\mathscr{B}_{i_0}| > n$. Hence if $X_i = \varnothing$ then $|\mathscr{B}_{i_0}| > |\mathscr{B}_i|$. Let

$$J = \left\{ j \mid N_0^{\mathscr{B}_j} = \mathrm{Alt}(\mathscr{B}_j) \right\} \quad \text{and} \quad \mathscr{B}_0 = \bigcup_{j \in J} \mathscr{B}_j.$$

Then $J \subseteq \{1, \ldots, s_0\}$, $|\mathscr{B}_j| = |\mathscr{B}_{i_0}|$ for all $j \in J$ and $N_0$ fixes $\mathscr{B} \setminus \mathscr{B}_0$ pointwise.

Write $H_0 = H'$; then $H_0 \cong A_n$ and $H_0^{\mathscr{B}_0} \leqslant N_0^{\mathscr{B}_0}$. It follows that $N_0$ acts similarly on all $\mathscr{B}_j$ $(j \in J)$ (whether $|\mathscr{B}_{i_0}| = 6$ or not), and hence that $N_0$ contains a nontrivial element $x$ fixing each $\mathscr{B}_j'$ setwise $(J \in J)$. Then $x$ fixes $\mathscr{B} \setminus \mathscr{B}_0$ pointwise, so $x \in G_{\{\Delta\}}$ and so $x \in H$. This forces $J = \{1, \ldots, s_0\}$. If $s > s_0$ then $N$ has a subgroup $L \cong A_n$ fixing $\bigcup_{i=1}^{s_0} \mathscr{B}_i$ pointwise; clearly $L \leqslant G_{\{\Delta\}}$, which is not possible as $G_{\{\Delta\}} = H$. Thus $s = s_0$, $J = \{1, \ldots, s\}$, $N = N_0$ and the lemma follows.

LEMMA 2.6. *We have* $|\mathscr{B}_i'| = n$, *that is*, $r_i = 1$ *for all* $i$.

PROOF. By Lemma 2.5, $M$ has a subgroup $N_1$ fixing eacyh $\mathscr{B}_i'$ setwise and such that $N_1^{\mathscr{B}_i'} \geqslant \mathrm{Alt}(\mathscr{B}_i')$ $(i = 1, \ldots, s)$. Clearly $N_1 \leqslant G_{\{\Delta\}}$. Since $G_{\{\Delta\}} = H \cong S_n$ this forces $|\mathscr{B}_i'| = n$, that is, $r_i = 1$, for all $i$.

We can now complete the proof of Conjecture 1.5 for $t = 4$. First note that from the proof of Lemma 2.5, we have $M \nleqslant G_{\{\Delta\}}$. Hence there exists $k$ such that

$X_k \neq \varnothing$ and $\Psi_k \not\subseteq \Delta$ (equivalently $\mathscr{B}'_k \neq \mathscr{B}_k$). By Lemma 2.6 we have $|\mathscr{B}_k| = n + c$ where $c > 0$ is the number of blocks in $\mathscr{B}_k \setminus \mathscr{B}'_k$. Thus by Lemma 2.5, $X_i \neq \varnothing$, $|\mathscr{B}_i| = n + c$ and $M^{\mathscr{B}_i} \cong S_{n+c}$ $(i = 1, \ldots, s)$. Finally, choose $B'_1 \in \mathscr{B}_1 \setminus \mathscr{B}'_1$. There exists $m \in M$ with $m^{\mathscr{B}_i} = (B_1 B'_1)$. By Lemma 2.5, $M$ acts similarly on all $\mathscr{B}_i$, so $m^{\mathscr{B}_i} = (B_i B'_i)$ for some $B'_i \in \mathscr{B}_i \setminus \mathscr{B}'_i$ $(i = 1, \ldots, s)$. Since the kernel of the action of $M$ on $\mathscr{B}$ is trivial, we have $m^2 = 1$. Hence $m = (\alpha_{11}\phi_{i_1})(\alpha_{21}\phi_{i_2}) \cdots (\alpha_{t1}\phi_{i_t})$ for some $\phi_{i_j} \in \Phi$ $(j = 1, \ldots, t)$. From this it follows that $(\delta_1\phi_{i_1}) \cdots (\delta_t\phi_{i_t}) \in \operatorname{Aut}\Gamma_1^*$ and that for any distinct $k$, $l$ the subgraph $\{\delta_k, \delta_l, \phi_{i_k}, \phi_{i_l}\}$ of $\Gamma^*$ lies in the set $\mathscr{C}_n$ of 12 graphs defined in §1. This contradicts (c) of Theorem 1.4.

This completes the proof of Conjecture 1.5 for $t = 4$.

We summarise the results proved in this section:

THEOREM 2.7. *Let $n$, $t$ be integers with $1 \leqslant t \leqslant 4$ and $n > t$, and let $\Gamma^*$ be a graph on $\{\delta_1, \ldots, \delta_t, \phi_1, \ldots, \phi_r\}$ coloured as described in Conjecture* 1.5. *Suppose that* (a), (b) *and* (c) *of Theorem* 1.4 *are satisfied. Then if $\Gamma$ is the corresponding graph on $tn + r$ vertices, we have* $\operatorname{Aut}\Gamma \cong S_n$.

The results 1.2, 1.4 and 2.7 give a description of all graphs $\Gamma$ on $v$ vertices with $\operatorname{Aut}\Gamma \cong S_n$ $(n > 6)$ and $v < \min\{5n, \frac{1}{2}n(n-1)\}$. This description is illustrated below in the Appendix. For values of $n$ with $n \leqslant 6$ there are some extra possibilities which can easily be determined using the techniques of this paper.

REMARKS. 1. The restriction $n > t$ in Theorem 2.7 is in fact unnecessary—it is not hard to show that the result is true for any $n$, $t$ with $1 \leqslant t \leqslant 4$, $n \geqslant 3$.

2. The obstacle to a general proof of Conjecture 1.5 seems to lie solely in proving Lemma 2.3 in the general case; the subsequent steps of the proof for $t = 4$ do not depend on the value of $t$ and would remain largely unchanged in the general case.

3. The methods of this paper could be used to study graphs with automorphism group $S_n$ having some orbit sizes greater than $\frac{1}{2}n(n-1)$. For example, suppose that we only restrict all orbits to have size less than $n(n-1)(n-2)/6$. Then for $n$ large enough, the proofs of Propositions 1.1 and 1.2 show that all orbits have size 1, $n$ or $\frac{1}{2}n(n-1)$ (with the action of $S_n$ in the latter case being that on the set of pairs of points in an underlying set of size $n$). There are four possible subgraphs on an orbit of size $\frac{1}{2}n(n-1)$: these are the complete graph $K_{\frac{1}{2}n(n-1)}$, the triangular graph $T_n$ and their complements. By introducing a suitable collection of colours to represent these subgraphs and the edges between them, we can proceed in similar fashion to §1.

## Appendix

In this Appendix we give descriptions of some of the graphs characterized by Theorems 1.4 and 2.7. In particular we describe all graphs $\Gamma$ with $\operatorname{Aut}\Gamma \cong S_n$ and $|V\Gamma| \leqslant 3n$ (with $n > 6$). The reader will have no difficulty in extending these descriptions to cover all graphs $\Gamma$ with $\operatorname{Aut}\Gamma \cong S_n$ and

$$|V\Gamma| < \min\{5n, \tfrac{1}{2}n(n-1)\}.$$

It is unfortunately necessary to introduce some fairly complicated notation for these descriptions, so we include a number of small examples for illustration.

Throughout this Appendix, $\Gamma$ denotes a graph on $v$ vertices. For any $n$, let

$$\mathscr{F}_{v,n} = \{\Gamma \mid \operatorname{Aut}\Gamma \cong S_n\}.$$

For any $t$, $r$, $n$ with $n > t$ define

$$\mathscr{G}_{t,r,n} = \{\Gamma \mid v = tn + r, \operatorname{Aut}\Gamma \cong S_n \text{ has } t \text{ orbits of size } n \text{ and } r \text{ fixed points on } V\Gamma\}.$$

Thus by Proposition 1.2, for $n > 6$ and $v < \frac{1}{2}n(n-1)$, we have

$$(1) \qquad \mathscr{F}_{v,n} = \bigcup_{1 \leqslant j \leqslant t} \mathscr{G}_{j,(t-j)n+r,n}$$

where $v = tn + r$ and $0 \leqslant r < n$. Thus to describe $\mathscr{F}_{v,n}$ we must describe the graphs in $\mathscr{G}_{t,r,n}$. This can be done for $t \leqslant 4$ using Theorems 1.4 and 2.7, and we now give such descriptions explicitly, starting with the simplest case $t = 1$. For convenience, if $\mathscr{S}$ is a set of graphs on $v$ vertices, define

$$\mathscr{S}^* = \{\{\Gamma, \overline{\Gamma}\} \mid \Gamma \in \mathscr{S}\},$$

where $\overline{\Gamma}$ denotes the complement of $\Gamma$.

(A) *The case* $t = 1$. We describe $\mathscr{G}_{1,r,n}$. Let $\mathscr{H}_{1,r}$ be the set of (uncoloured) graphs $\Gamma_1^*$ on $1 + r$ vertices $\{\delta_1, \phi_1, \ldots, \phi_r\}$ such that $H = \operatorname{Aut}\Gamma_1^*$ satisfies

(1) $H_{\delta_1} = 1$, and,

(2) $H$ contains no element $(\delta_1 \phi_{i_1})$ with $\delta_1$ joined to $\phi_{i_1}$.

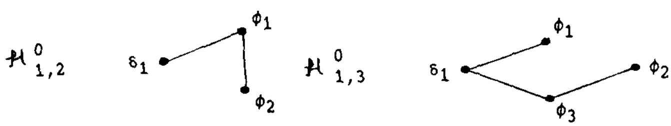
Each graph $\Gamma_1^*$ in $\mathscr{H}_{1,r}$ corresponds as in Section 1 to a graph $\Gamma$ on $n + r$ vertices as follows: $\Gamma$ has vertex set $\Delta_1 \cup \{\phi_1, \ldots, \phi_r\}$, the subgraph $\Delta_1$ is $K_n$, the subgraph $\{\phi_1, \ldots, \phi_r\}$ is as in $\Gamma_1^*$, and $\phi_i$ is joined to all or no vertices in $\Delta_1$ according as $\phi_i$ is or is not joined to $\delta_1$ in $\Gamma_1^*$. By Theorem 2.7 with $t = 1$, we have $\operatorname{Aut}\Gamma \cong S_n$, so that $\Gamma$ is in $\mathscr{G}_{1,r,n}$.

Now the graphs in $\mathscr{G}_{1,r,n}$ are unlabelled, so we choose a subset $\mathscr{H}_{1,r}^0$ of $\mathscr{H}_{1,r}$ containing exactly one member of each orbit of $\operatorname{Sym}\{\phi_1, \ldots, \phi_r\}$ on $\mathscr{H}_{1,r}$. Then $\mathscr{G}_{1,r,n}^*$ is in 1-1 correspondence, as described above, with $\mathscr{H}_{1,r}^0$. We write this as

$$\mathscr{G}_{1,r,n}^* \leftrightarrow \mathscr{H}_{1,r}^0.$$

In particular, for $r < n$ we have by (1), $\mathscr{F}^*_{n+r,n} \leftrightarrow \mathscr{H}^0_{1,r}$. This gives the results of [3].

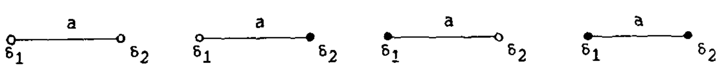
EXAMPLE. We illustrate this with $r = 2$ and $r = 3$. Each of $\mathscr{H}^0_{1,2}$ and $\mathscr{H}^0_{1,3}$ consists of just *one* graph:



Thus $\mathscr{G}_{1,2,n}$ and $\mathscr{G}_{1,3,n}$ consist, respectively, of the graphs


and their complements.

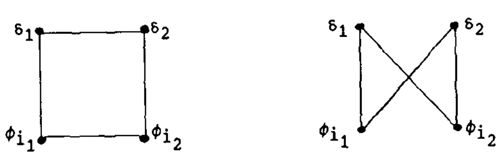
(B) *The case* $t = 2$. We now describe $\mathscr{G}_{2,r,n}$ $(n \geqslant 3)$. If $\Gamma \in \mathscr{G}_{2,r,n}$ and $\Gamma^*$ is the corresponding coloured graph on $\{\delta_1, \delta_2, \phi_1, \ldots, \phi_r\}$ then by (b) of Theorem 1.4 the subgraph of $\Gamma^*$ on $\delta_1$, $\delta_2$ is one of the following:



where $a$ is 1 or $n - 1$. Let $\mathscr{H}_{2,r}$ be the set of (uncoloured) graphs $\Gamma_1^*$ on $\{\delta_1, \delta_2, \phi_1, \ldots, \phi_r\}$ such that $H = \operatorname{Aut}\Gamma_1^*$ satisfies

(1) $H_{\{\delta_1\delta_2\}} = 1$, and

(2) $H$ contains no element $(\delta_1\phi_{i_1})(\delta_2\phi_{i_2})$ such that the subgraph $\{\delta_1, \delta_2, \phi_{i_1}, \phi_{i_2}\}$ is one of

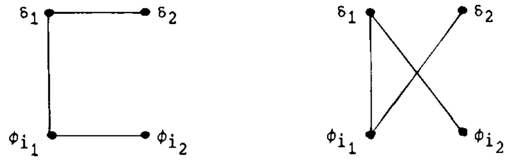


And let $\mathscr{H}'_{2,r}$ be the set of graphs $\Gamma_1^*$ on $\{\delta_1, \delta_2, \phi_1, \ldots, \phi_r\}$ such that

(I) $H_{\delta_1\delta_2} = 1$, and

https://doi.org/10.1017/S1446788700031360 Published online by Cambridge University Press

(II) $H$ contains no element $(\delta_1\phi_{i_1})(\delta_2\phi_{i_2})$ such that the subgraph $\{\delta_1, \delta_2, \phi_{i_1}, \phi_{i_2}\}$ is one of




Choose a subset $\mathscr{H}_{2,r}^0$ of $\mathscr{H}_{2,r}$ containing exactly one member of each orbit of $\mathrm{Sym}\{\phi_1, \ldots, \phi_r\} \times \mathrm{Sym}\{\delta_1, \delta_2\}$ on $\mathscr{H}_{2,r}$; and choose a subset $\mathscr{H}_{2,r}'^0$ containing exactly one member of each orbit of $\mathrm{Sym}\{\phi_1, \ldots, \phi_r\}$ on $\mathscr{H}_{2,r}'$. Then each graph $\Gamma_1^*$ in $\mathscr{H}_{2,r}^0$ corresponds to a unique graph $\Gamma$ in $\mathscr{G}_{2,r,n}$ in which the subgraphs $\Delta_1$ and $\Delta_2$ are both $K_n$ (and $a = 1$ if $\delta_1$ and $\delta_2$ are joined in $\Gamma_1^*$, $a = n - 1$ if not). And each $\Gamma_1^*$ in $\mathscr{H}_{2,r}'^0$ corresponds to a unique graph $\Gamma$ in $\mathscr{G}_{2,r,n}$ in which $\Delta_1$ is $K_n$ and $\Delta_2$ is $V_n$; the complement $\bar{\Gamma}$ then corresponds to the graph $(\Gamma_1^*)^+$, which is the image of the complement $\bar{\Gamma}_1^*$ under the transposition $(\delta_1\delta_2)$. Hence if we write

$$\left(\mathscr{H}_{2,r}'^0\right)^+ = \left\{\left\{\Gamma_1^*, \left(\Gamma_1^*\right)^+\right\} \mid \Gamma_1^* \in \mathscr{H}_{2,r}'^0\right\}$$

then we have

$$\mathscr{G}_{2,r,n}^* \leftrightarrow \mathscr{H}_{2,r}^0 \cup \left(\mathscr{H}_{2,r}'^0\right)^+.$$

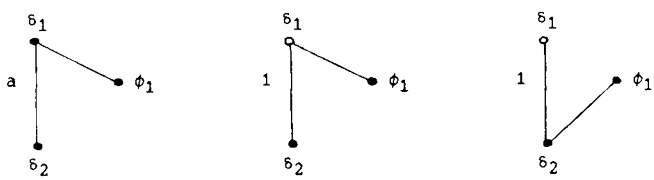
Note that if $v = 2n + r < 3n$ and $n > 6$, then by (1),

$$\mathscr{F}_{v,n} = \mathscr{G}_{1,n+r,n} \cup \mathscr{G}_{2,r,n}$$

so the description of $\mathscr{F}_{v,n}$ is given by (A) and the above.

Example. We illustrate the above by producing the graphs in $\mathscr{G}_{2,0,n}$ and $\mathscr{G}_{2,1,n}$. Those in $\mathscr{G}_{2,0,n}$ correspond to the two coloured graphs


Thus $\mathscr{G}_{2,0,n}$ consists of the *corona* $K_n \circ K_1$ (which is $K_n$ with each vertex joined to just one further vertex) and its complement. This answers a question raised in [3, §4]. The graphs in $\mathscr{G}_{2,1,n}$ are those corresponding to the coloured graphs

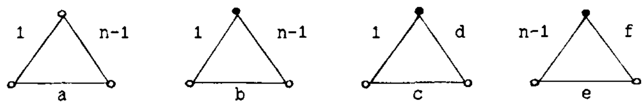



(where $a$ is 1 or $n - 1$), together with their complements.

https://doi.org/10.1017/S1446788700031360 Published online by Cambridge University Press

Descriptions similar to, but rather more complicated than those given in (A) and (B), exist for $t = 3$ and $t = 4$. We leave these to the reader, and offer just one further illustration.

(C) We describe $\mathcal{G}_{3,0,n}$ $(n \geqslant 4)$. By Theorems 1.4 and 2.7, $\mathcal{G}^*_{3,0,n}$ is in 1-1 correspondence with the following set $\mathcal{H}_{3,0}$ of coloured graphs:




(any $b \in \{0, 1, n-1, n\}$, $a, d, f \in \{0, n\}$, $c, e \in \{1, n-1\}$).

Hence for $n > 6$,

$$\mathcal{F}^*_{3n,n} = \mathcal{G}^*_{1,2n,n} \cup \mathcal{G}^*_{2,n,n} \cup \mathcal{G}^*_{3,0,n}$$
$$\leftrightarrow \mathcal{H}^0_{1,2n} \cup \mathcal{H}^0_{2,n} \cup \left(\mathcal{H}'^0_{2,n}\right)^+ \cup \mathcal{H}_{3,0}.$$

## References

[1] R. Frucht, 'Herstellung von Graphen mit vorgegebener abstrakten Gruppe', *Compositio Math.* **6** (1938), 239–250.

[2] A. Gewirtz and L. V. Quintas, 'Connected extremal edge graphs having symmetric automorphism group', *Recent progress in combinatorics*, ed. W. T. Tutte (Academic Press, New York, 1969).

[3] K. Heinrich and W. D. Wallis, 'Graphs with symmetric automorphism group', *J. Graph Theory* **2** (1978), 329–336.

[4] H. Wielandt, *Finite permutation groups* (Academic Press, New York-London, 1964).

Department of Mathematics
Imperial College
London S.W.7
England