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Graphs with prescribed Levi form trace

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Abstract We prove existence and uniqueness of a viscosity solution of the Dirichlet problem related to the prescribed Levi mean curvature equation, under suitable assumptions on the boundary data and on the Levi curvature of the domain. We also show that such a solution is Lipschitz continuous by proving that it is the uniform limit of a sequence of classical solutions of elliptic problems and by building Lipschitz continuous barriers.

Keywords Levi mean curvature · Quasilinear degenerate elliptic PDE's · Viscosity solutions · Comparison principle · Global Lipschitz estimates

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1 Introduction

If M is a hypersurface in $\mathbb{R}^{n+1}$, and if Π is its second fundamental form, then the eigenvalues of Π are the principal curvatures of M and the trace of Π is called the mean curvature of M. For a real hypersurface $M \subset \mathbb{C}^{n+1}$, let J be the canonical complex structure and let H denote the J-invariant n-dimensional complex subspace of the complexified tangent space to M. The restriction of the second fundamental form of M on H is a Hermitian form Λ, which is called the Levi form. More

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precisely, if M is a real manifold of class C^2 which is locally defined by ρ, then the Levi form $\Lambda(\rho)$ is the restriction to the complex tangent space H of the Hermitian form associated with the complex Hessian matrix $Hess_{\mathbb{C}}\rho = \left(\frac{\partial^2\rho}{\partial z_\ell \partial \bar{z}_p}\right)_{\ell,p=1}^{n+1}$ of ρ. The Levi form Λ itself depends on the defining function for the domain, while the normalized Levi form $\tilde{\Lambda}(\rho) = \frac{\Lambda(\rho)}{|\partial\rho|}$ is independent of the defining function ρ and depends only on the domain (a proof of this assertion can be found in [8, Proposition A.1]). Bedford and Gaveau were the first to remark this fact and in [1] they used the normalized Levi form to bound the domain over which M can be defined as the graph of a function of class C^2. The eigenvalues of $\tilde{\Lambda}$ correspond to mean curvatures in certain complex directions and, more generally, symmetric functions in the eigenvalues of $\tilde{\Lambda}$ are complex curvatures of M. The sum of the eigenvalues of $\tilde{\Lambda}$, corresponding to the complex version of the mean curvature of M, is the scalar function $K_M(\cdot)$ defined by

$$K_M(z) = -\frac{1}{n|\partial\rho(z)|^3} \sum_{1\le i_1<i_2\le n+1} \det \begin{pmatrix} 0 & \partial_{\overline{i_1}}\rho(z) & \partial_{\overline{i_2}}\rho(z) \\ \partial_{i_1}\rho(z) & \partial_{i_1\overline{i_1}}\rho(z) & \partial_{i_1\overline{i_2}}\rho(z) \\ \partial_{i_2}\rho(z) & \partial_{i_2\overline{i_1}}\rho(z) & \partial_{i_2\overline{i_2}}\rho(z) \end{pmatrix}.$$

We will call $K_M(z)$ *the mean Levi curvature* of M at a point $z \in M$. Here $\partial_j, \partial_{\bar{j}}, \partial_{\ell\bar{j}}$ denote respectively the derivatives $\frac{\partial}{\partial z_j}, \frac{\partial}{\partial \bar{z}_j}, \frac{\partial^2}{\partial z_\ell \partial \bar{z}_j}$ and $\partial\rho = (\partial_1\rho, \ldots, \partial_{n+1}\rho)$.

Example 1.1 (Levi mean curvature of a ball) If M is the ball of radius r with center at zero, then by choosing as defining function $\rho = |z_1|^2 + \cdots + |z_{n+1}|^2 - r^2$, we have

$$K_M(z) = \frac{1}{n}|r|^{-3} \sum_{1\le i_1<i_2\le n+1} (|z_{i_1}|^2 + |z_{i_2}|^2) \equiv r^{-1}.$$

Example 1.2 (Levi mean curvature of a cylinder) Let $B(0,r) \subset \mathbb{C}^n \times \mathbb{R}$ be a ball of radius r. We consider the following cylinder

$$B(0,r) \times i\mathbb{R} = \{(z,w) \in \mathbb{C}^n \times \mathbb{C} : |z|^2 + \left(\frac{w+\bar{w}}{2}\right)^2 - r^2 < 0\}.$$

It is easy to check that

$$\frac{2n-1}{2nr} \le K_{\partial B(0,r)\times i\mathbb{R}}(z,w) = \frac{2nr^2 - |z|^2}{2nr^3} \le \frac{1}{r}$$

for every $(z,w) \in \partial B(0,r) \times i\mathbb{R}$.

In [8, Proposition 2.1 and formula (17) p.316] it was proved that if $\partial_{n+1}\rho(z) \neq 0$ then we can write the Levi mean curvature of M at $z \in M$ as follows:

$$K_M(z) = \frac{1}{n}|\partial\rho|^{-1} \text{trace}\left(\left(I_n - \frac{\alpha^T \cdot \bar{\alpha}}{1+|\alpha|^2}\right) A(\rho)\right) \tag{1.1}$$

where

$$\alpha = (\alpha_1, \ldots, \alpha_n), \quad \alpha_j(z) = \frac{\partial_j\rho(z)}{\partial_{n+1}\rho(z)}, \tag{1.2}$$

and $A(\rho)$ is the $n\times n$ Hermitian matrix with coefficients

$$A_{j,\bar{\ell}}(\rho)=\partial_{j,\bar{\ell}}\rho-\bar{\alpha}_\ell\partial_{j,\overline{n+1}}\rho-\alpha_j\partial_{n+1,\bar{\ell}}\rho+\alpha_j\bar{\alpha}_\ell\partial_{n+1,\overline{n+1}}\rho. \tag{1.3}$$

If M is the graph of a function $u:\Omega\to\mathbb{R}$, $\Omega\subset\mathbb{C}^n\times\mathbb{R}\simeq\mathbb{R}^{2n+1}$, for every $x\in\Omega$ we recognize that

$$K_M(x,u)=\text{trace}\left(\sigma(Du)\sigma^T(Du)D^2u\right) \tag{1.4}$$

where Du and D^2u are respectively the Euclidean gradient and the Hessian of u in $\mathbb{R}^{2n+1}$. Here σ is a $2n+1\times 2n$ real values matrix, whose coefficients are C^1 functions of Du. This is a crucial point in the study of viscosity solutions in view of the results in [7] and we will prove (1.4) in the Section 2. In the sequel we will call the right hand side of (1.4) *Levi mean curvature* operator, and we will denote it by L

$$Lu=\text{trace}\left(\sigma(Du)\sigma^T(Du)D^2u\right).$$

Even if the Levi mean curvature has some geometric properties similar to the Euclidean mean curvature we must stress that the *Levi mean curvature* operator is never strictly elliptic. In this paper we consider the Dirichlet problem of finding a non parametric hypersurface with prescribed mean Levi curvature k on a domain $\Omega\subset\mathbb{C}^n\times\mathbb{R}\subset\mathbb{C}^n\times\mathbb{C}$ where $\Omega\times i\mathbb{R}$ is strongly pseudoconvex. The problem can be formulated as follows. Given $\varphi\in C(\partial\Omega)$ and $k\geq 0$ continuous, find $u\in C(\overline{\Omega})$ such that

$$u|_{\partial\Omega}=\varphi \quad \text{and} \quad Lu=k(\cdot,u) \text{ on } \Omega. \tag{1.5}$$

The Dirichlet problem for the Levi equation for $n=1$ was first considered by A. Debiard and Gaveau [5], who gave an estimate for the modulus of continuity of the solution and by Z. Slodkowski and G. Tomassini in [9].

The existence of classical solution of (1.5) for $n>1$ is an interesting open problem, while for $n=1$ it has been solved in [2]. The main aim of this paper is to show the existence and the uniqueness of a Lipschitz continuous viscosity solution of (1.5). To this purpose we use the main tools of the theory of viscosity solutions. We recall that the theory of viscosity solutions provides a convenient partial differential equations framework for dealing with the lack of the existence of classical solutions. For a complete survey of the results obtained within the theory of viscosity solutions for the second-order case we refer to the "Users'guide" of Crandall, Ishii and Lions [3].

One of the main tools to prove the existence and the uniqueness of a continuous solution to (1.5) is to provide a comparison principle between semicontinuous sub and supersolutions to (1.5). Indeed the existence follows easily through the Perron's method by Ishii [3].

Hereafter we suppose that $\Omega\subset\mathbb{R}^{2n+1}$ is a bounded domain with boundary of class C^2. We list below some basic assumptions we use throughout the paper.

We assume that $k\colon\overline{\Omega}\times\mathbb{R}\to[0,+\infty)$ is a continuous bounded function satisfying

(H1) for all $R>0$, there exists $\ell_R>0$, such that, for every $x\in\overline{\Omega}$, and $-R\leq v\leq u\leq R$

$$\ell_R(u-v)\leq k(\cdot,u)-k(\cdot,v), \tag{1.6}$$

(H2) for all $R>0$, for all $(x,y)\in\overline{\Omega}$ and $|u|\le R$, there exists a modulus of continuity ω_R such that $\omega_R(s)\to 0$ as $s\to 0^+$ and

$$|k(x,u)-k(y,u)|\le \omega_R(|x-y|)\,.$$

Conditions **(H1)** and **(H2)** will be used in Section 3 to prove a comparison principle between viscosity semicontinuous sub- and supersolution to the problem (1.5). In Section 4 to solve the Dirichlet problem by using the Perron's method we will use the following additional assumptions on k and Ω:

(H3) $\partial\Omega$ is of class C^2, $\Omega\times i\mathbb{R}$ is strongly pseudoconvex [1] and, for all $x_0\in\partial\Omega$, $\sup_{\overline{\Omega}\times\mathbb{R}} k < K_{\partial\Omega\times i\mathbb{R}}(x_0)$.

Condition **(H3)** will allow to build local barriers to the problem (1.5).

We prove the following theorems.

Theorem 1.1 [The strictly monotone case] *Assume (H1)–(H3). Then for any $\varphi\in C(\partial\Omega)$ there exists a unique continuous viscosity solution u of (1.5).*

The proof of Theorems 1.1 follows classical arguments from the theory of viscosity solutions (see e.g [3]). To get the Lipschitz continuity of the solution we use the elliptic regularization technique. For every $\varepsilon>0$ we define the $(2n+1)\times(2n+1)$ matrix

$$\sigma_\varepsilon(p)=\begin{pmatrix}\sigma(p) & \begin{pmatrix}0\\ \varepsilon\end{pmatrix}\end{pmatrix}$$

where 0 is null vector in $\mathbb{R}^{2n}$, and we consider the quasilinear elliptic equation

$$L_\varepsilon u = k(x,u),\quad L_\varepsilon := \operatorname{trace}\left(\sigma_\varepsilon(Du)\sigma_\varepsilon^T(Du)D^2u\right).$$

By [6, Theorem 15.10] if $\partial\Omega\in C^{2,\gamma}$ and $\phi\in C^{2,\gamma}(\bar\Omega)$ then for every $\varepsilon>0$ there exists a unique solution $u\in C^{2,\gamma}(\bar\Omega)$ of the Dirichlet problem

$$\begin{cases} L_\varepsilon u = k(x,u), & \text{in } \Omega\\ u=\varphi, & \text{on } \partial\Omega.\end{cases}\tag{1.7}$$

The main strategy then is to prove a maximum principle for $|Du|^2$ and boundary estimate for the gradient of u independent of ε.

If $k\in C^1(\overline{\Omega}\times\mathbb{R})$ then it satisfies

$$\frac{D_xk\cdot p + D_uk|p|^2}{(1+|p|^2)^{1/2}}+\frac{2n}{2n+1}k^2\ge 0\tag{1.8}$$

for every $(x,u)\in\Omega\times\mathbb{R}$ and for all $|p|\ge L$. Then by [6, Theorem 15.1] the maximum principle for $|Du|^2$ holds and we have

$$\sup_{\bar\Omega}|Du|\le\max\{L,\sup_{\partial\Omega}|Du|\}.\tag{1.9}$$

By using local barriers, we then get a priori estimates of the Lipschitz constant and of the L_∞-norm of the solution of the approximating problem. The result is

[1] An open set $D=\{\rho<0\}\subset\mathbb{C}^{n+1}$ is strongly pseudoconvex if the Levi form is positive definite at every point of its boundary.

Theorem 1.2 [The Lipschitz continuous case] *Assume* ***(H1)*** *–* ***(H3)*** *and* $k \in C^1(\overline{\Omega} \times \mathbb{R})$. *Then for every* $\varphi \in C^{1,1}(\partial\Omega)$ *there exists a unique Lipschitz continuous viscosity solution* u *of (1.5).*

We should stress that for $n = 1$ this argument was used by Slodkowsky and Tomassini to get existence of Lipschitz continuous viscosity solutions in [9] and by Slodkowsky and Tomassini and Da Lio and the second author to prove existence and uniqueness of viscosity solutions of the Levi Monge Ampère equation in [10], [4] respectively.

In order to include the case when the prescribed function k is constant, **(H1)** may be relaxed to

(H4) $k(x,u) = k(u)$ for all $(x,u) \in \Omega$ and for all $R > 0$, and $-R \le v \le u \le R$

$$0 \le k(u) - k(v). \tag{1.10}$$

Theorem 1.3 [The x-independent case] *Assume* ***(H2)*** *–* ***(H4)***. *Then, for every* $\varphi \in C(\partial\Omega)$, *there exists a unique continuous viscosity solution* u *of (1.5). Moreover, if* $\varphi \in C^{1,1}(\partial\Omega)$, *then the viscosity solution is Lipschitz continuous.*

Our paper is organized as follows. In Section 2 we prove (1.4). In Section 3 we give a precise viscosity formulation of the Dirichlet problem (1.5) and we prove comparison principles between viscosity semicontinuous sub- and supersolutions to the problem (1.5) assuming either conditions **(H1)** and **(H2)**, or **(H2)** and **(H4)**. In Section 4 under the hypothesis **(H3)** we get the existence of a continuous solution to (1.5) for all continuous boundary data, via the comparison results and the Perron's method.

We then show the Lipschitz continuity of the viscosity solution to (1.5) for all $C^{1,1}$boundary data, by using an approximation argument and some a priori estimates.

2 The trace of the Levi form for graphs

In this section we prove (1.4). If $D = \{z \in \mathbb{C}^{n+1} : \rho(z) < 0\}$ is an open set with C^2 boundary and $\rho_{n+1}(z) \neq 0$ for every $z \in \partial D$, by (1.1) we have

$$K_{\partial D} := \frac{1}{n}\frac{1}{|\partial\rho|} tr(H(\rho)A(\rho)), \quad with\, H(\rho) = I_n - \frac{\alpha^T \cdot \bar{\alpha}}{1+|\alpha|^2}.$$

Lemma 2.1 *The Hermitian matrix* $H(\rho)$ *is positive definite.*

Proof For every $\zeta \in \mathbb{C}^n, \zeta \neq 0$ we have

$$\langle H(\rho)\zeta,\zeta\rangle = |\zeta|^2 - \frac{\langle \bar{\alpha}\zeta, \bar{\alpha}\zeta\rangle}{1+|\alpha|^2} \ge |\zeta|^2 - \frac{|\alpha|^2|\zeta|^2}{1+|\alpha|^2} = \frac{|\zeta|^2}{1+|\alpha|^2} > 0.$$

□

Let us remark that the Hermitian matrix $H(\rho)$ is the square of a Hermitian matrix. Indeed we have the following

Lemma 2.2 *Let us set*

$$H_\gamma(\rho) = I_n - \gamma\frac{\alpha^T\cdot\bar{\alpha}}{1+|\alpha|^2}, \quad \gamma\in\mathbb{R}.$$

For

$$\gamma = \frac{\sqrt{1+|\alpha|^2}}{1+\sqrt{1+|\alpha|^2}} \tag{2.1}$$

we have $H_\gamma(\rho)\cdot H_\gamma(\rho) = H(\rho)$.

Proof The proof is computational and we leave it to the reader. □

We now identify $\mathbb{C}^{n+1}$ with $\mathbb{R}^N\times\mathbb{R}$, where $N = 2n+1$, as $z\approx(x,y,t,s)$, $(z_1,\dots,z_{n+1})\approx(x_1,\dots,x_n,y_1,\dots,y_n,t,s)$. Let $\rho(z) = u(x,y,t)-s$, with $u\in C^2(\Omega)$, $\Omega\subseteq\mathbb{R}^N$ an open set and take $D = Epi(u) = \{(x,y,t,s)\in\mathbb{R}^N\times\mathbb{R} : u(x,y,t)<s\}$, $\partial D = Graph(u) = \{(x,y,t,s)\in\mathbb{R}^N\times\mathbb{R} : u(x,y,t) = s\}$. We have $\partial\rho = \frac{1}{2}(u_{x_1} - iu_{y_1},\dots,u_{x_n}-iu_{y_n},u_t+i)$, and $|\partial\rho| = \frac{1}{2}(|Du|^2+1)^{\frac{1}{2}}$ where $Du = grad(u)$.

In order to explicitly write the equation (1.1) for graphs we set $J = \begin{pmatrix} I_n \\ iI_n \\ -\overline{\alpha}\end{pmatrix}$ and write the matrix A in (1.3) for graphs as: $A(\rho) = \frac{1}{4}\overline{J}^T D^2u\, J$. The coefficients α in (1.2) for graphs are $\alpha = -a+ib$ where

$$\begin{aligned} a &= (a_1,\dots,a_n), \quad a_j = \frac{-u_{x_j}u_t + u_{y_j}}{(u_t^2+1)} \\ b &= (b_1,\dots,b_n), \quad b_j = \frac{-u_{y_j}u_t - u_{x_j}}{(u_t^2+1)}. \end{aligned} \tag{2.2}$$

Moreover, $1+|\alpha|^2 = 1+|a|^2+|b|^2 = \frac{|Du|^2+1}{u_t^2+1}$. We introduce two real $n\times n$ matrices as $P = Re(\alpha^T\overline{\alpha})$, $\quad Q = Im(\alpha^T\overline{\alpha})$. Since $\alpha^T\overline{\alpha} = (-a+ib)^T(-a-ib) = a^Ta + b^Tb + i(a^Tb - b^Ta) = P+iQ$ we can recognize that

$$P = a^Ta + b^Tb = P^T, \quad Q = a^Tb - b^Ta = -Q^T. \tag{2.3}$$

We can now write the prescribed curvature equation for graphs.

Proposition 2.1 *For every* $(x,y,t)\in\Omega$ *we have*

$$K_M((x,y,t),u) = \text{trace}\left(\sigma(Du)\sigma^T(Du)D^2u\right)$$

where

$$\sigma = \frac{1}{(2n)^{1/2}(1+|Du|^2)^{1/4}}\begin{pmatrix} I_n - \frac{\gamma P}{1+|a|^2+|b|^2} & \frac{-\gamma Q}{1+|a|^2+|b|^2} \\ \frac{\gamma Q}{1+|a|^2+|b|^2} & I_n - \frac{\gamma P}{1+|a|^2+|b|^2} \\ \frac{a}{\sqrt{1+|a|^2+|b|^2}} & \frac{b}{\sqrt{1+|a|^2+|b|^2}} \end{pmatrix} \tag{2.4}$$

with γ, *and* a, b *and* P, Q *defined as in* (2.1),(2.2), (2.3) *respectively. Moreover, the coefficients of* σ *are Lipschitz continuous functions of* Du.

Corollary 2.1.1 *The matrix* $\sigma\sigma^T$ *is nonnegative definite with minimum eigenvalue identically zero. The* $\text{rank}(\sigma\sigma^T) = 2n$ *at every point and*

$$\text{trace}\,\sigma\sigma^T = \frac{1+u_t^2+2n(1+|Du|^2)}{2n(1+|Du|^2)^{3/2}} \geq \frac{1}{(1+|Du|^2)^{1/2}}. \tag{2.5}$$

3 Comparison principles

In the sequel we shall denote by x a point in $\mathbb{R}^N$, with $N = 2n+1$. We first give a precise formulation of the Dirichlet problem (1.5) in a viscosity sense. To this purpose we consider the operator $F : \Omega \times \mathbb{R} \times \mathbb{R}^N \times \mathscr{S}^N \to \mathbb{R}$ with $N = 2n+1$, defined by

$$F(x,u,p,X) := k(x,u) - \text{trace}\,(\sigma(p)\sigma^T(p)X). \tag{3.1}$$

Definition 3.1 We say that $u \in USC(\overline{\Omega})$ (resp. $v \in LSC(\overline{\Omega})$) is a viscosity subsolution (resp. supersolution) of (3.1) if for all $\phi \in C^2(\overline{\Omega})$ the following holds: at each local maximum x_0 (resp. local minimum) point of $u - \phi$ $(v - \phi)$

$$F(x_0,u(x_0),D\phi(x_0),D^2\phi(x_0)) \geq 0$$
$$(resp.\, F(x_0,v(x_0),D\phi(x_0),D^2\phi(x_0)) \leq 0).$$

Definition 3.2 A function $u \in USC(\overline{\Omega})$ (resp. $v \in LSC(\overline{\Omega})$) is said to be a viscosity subsolution (resp. supersolution) of Dirichlet problem

$$\begin{cases} F(x,u,Du,D^2u) = 0 \text{ in } \Omega, \\ u(x) = \varphi(x), \qquad \text{on } \partial\Omega, \end{cases} \tag{DP}$$

where $\varphi \in C(\partial\Omega)$ iff u is a viscosity subsolution (resp. v is a supersolution) of (3.1) such that $u = \varphi$ (resp. $v = \varphi$) on $\partial\Omega$.

In the sequel when we talk about sub- and supersolutions of (DP), we will always mean in a viscosity sense.

We explicitly remark that $u \in C^2(\Omega) \cap C(\bar{\Omega})$ is a viscosity solution of (DP) iff u is a classical solution of (DP).

In this section we provide two comparison principles between viscosity semicontinuous subsolutions and supersolutions of (DP).

As a by-product of the these comparison results and the Perron 's method we get the existence of a unique viscosity solution of (DP).

The first comparison result of this section is the following theorem, which holds under the assumption that the function k is strictly increasing with respect to u. The proof of this result is standard and we refer the reader to [3].

Theorem 3.1 *Assume* **(H1)–(H3)**. *Let* $u \in USC(\overline{\Omega})$, $v \in LSC(\overline{\Omega})$ *be respectively a bounded viscosity subsolution and supersolution of* (DP). *Then* $u \leq v$ *in* $\overline{\Omega}$.

Next we shall prove a comparison result by assuming the weaker condition **(H4)**. When there is not a strict monotonicity with respect to u, one of the classical approaches from the theory of viscosity solutions, is to try to find a strict subsolution or supersolution either of the original equation or of a suitable approximation of it. To this purpose we need the following two Lemmas.

Lemma 3.1 *There is a function $\psi \in C^2(\overline{\Omega})$ such that*

$$\inf_{p\in\mathbb{R}^{2n+1}} (1+|p|^2)^{1/2}\text{trace}\left(\sigma(p)\sigma^T(p)D^2\psi\right) = \nu > 0.$$

Proof Let us take $\psi(x) = g\left(\frac{\|x\|^2}{2}\right)$, with $g \in C^2(\mathbb{R})$ and $g', g'' > 0$. We have $D\psi(x) = g'x$, $D^2\psi(x) = g''x^T \cdot x + g'Id$ and by (2.5)

$$(1+|p|^2)^{1/2}\text{trace}\left(\sigma(p)\sigma^T(p)D^2\psi\right) \geq g'. \qquad \square$$

Lemma 3.2 *If $u \in USC(\overline{\Omega})$ is a bounded viscosity subsolution of $F = 0$, then $u_m = u + \frac{1}{m}\psi$, with ψ as in the previous lemma, is a strictly viscosity subsolution of*

$$\begin{aligned}&f(Du_m - D\psi/m)\left(-\text{Tr}\left((\sigma\sigma^T)(Du_m - D\psi/m)D^2u_m\right)\right)\\&f(Du_m - D\psi/m)k(x, u_m - \psi/m) = -\frac{\nu}{m},\end{aligned}$$

where $f(p) = (1+|p|^2)^{1/2}$.

Now we shall prove a comparison principle, by assuming that **(H4)** holds, i.e. $k\colon \mathbb{R} \to [0, +\infty)$ is a continuous function which does not depend on x.

Theorem 3.2 *Assume* **(H2)–(H4)**. *Let $u \in USC(\overline{\Omega})$, $v \in LSC(\overline{\Omega})$ be respectively a bounded viscosity sub- and supersolution of (DP) and assume that u (or v) is Lipschitz continuous. Then $u \leq v$ in $\overline{\Omega}$.*

Proof We consider $u_m = u + \frac{1}{m}\psi$ with ψ as in Lemma 3.1. We may suppose without restriction that $x \neq 0$ in $\overline{\Omega}$, otherwise in the definition of ψ we replace x with $x - x_0$ with a suitable x_0. Moreover we choose g in such a way that $\|\psi\|_\infty < +\infty$. Our aim is to show that $\sup_\Omega(u_m - v) \leq \frac{1}{m}\|\psi\|_\infty$. Suppose by contradiction that for all m large enough we have $M_m = \max_{\overline{\Omega}}(u_m - v) > \frac{1}{m}\|\psi\|_\infty$. Since by **(H3)** we have $u(x) \leq \varphi(x) \leq v(x)$ for all $x \in \partial\Omega$, such a maximum is achieved at an interior point $\tilde{x}$ (depending on m). For all $\varepsilon > 0$ let us consider the auxiliary function

$$\Phi_\varepsilon(x,z) = u_m(x) - v(z) - \frac{|x-z|^2}{\varepsilon^2}.$$

Let $(x_\varepsilon, z_\varepsilon)$ be a maximum of Φ_ε in $\overline{\Omega} \times \overline{\Omega}$. By standard arguments we get, up to subsequences, $x_\varepsilon, z_\varepsilon \to \tilde{x} \in \overline{\Omega}$, and

$$\begin{aligned}&\frac{|x_\varepsilon - z_\varepsilon|^2}{\varepsilon^2} = o_\varepsilon(1) \ \text{ as } \varepsilon \to 0,\\&u_m(x_\varepsilon) - v(z_\varepsilon) \to u_m(\tilde{x}) - v(\tilde{x}) = M_m,\\&u_m(x_\varepsilon) \to u_m(\tilde{x}),\ v(z_\varepsilon) \to v(\tilde{x}).\end{aligned}$$

Since $\tilde{x}$ is necessarily in Ω, for ε small enough we have $x_\varepsilon,\ z_\varepsilon \in \Omega$ as well. Hence the equation holds for both u_m and v respectively in x_ε and z_ε.

There exist $X, Y \in \mathscr{S}^N$ such that, if $p_\varepsilon := 2\frac{(x_\varepsilon - z_\varepsilon)}{\varepsilon^2}$, we have [2]

$$(p_\varepsilon, X) \in \mathscr{J}^{2,+} u_m(x_\varepsilon), \ \ (p_\varepsilon, Y) \in \mathscr{J}^{2,-} v(z_\varepsilon),$$

$$-\frac{8}{\varepsilon^2} Id \le \begin{pmatrix} X & 0 \\ 0 & -Y \end{pmatrix} \le \frac{3}{\varepsilon^2} \begin{pmatrix} I & -I \\ -I & I \end{pmatrix} \tag{3.2}$$

and by Lemma 3.2

$$\begin{array}{l} f(p_\varepsilon - D\psi/m) F(x_\varepsilon, u_m, p_\varepsilon - D\psi/m, X) \\ + f(p_\varepsilon - D\psi/m)\left(k(u_m - \psi/m) - k(u_m)\right) < -\frac{\nu}{m} \\ F(z_\varepsilon, v, p_\varepsilon, Y) \ge 0. \end{array} \tag{3.3}$$

Set $\Sigma_1 = \sigma(p_\varepsilon - \frac{1}{m} D\psi)$ and $\Sigma_2 = \sigma(p_\varepsilon)$, where σ is the $N \times 2n$ matrix defined in (2.4). Multiply both sides of the inequality (3.2) by the matrix $\begin{pmatrix} \Sigma_1 & \Sigma_2 \end{pmatrix}$ on the left, and by the transpose of its conjugate on the right, to get

$$\Sigma_1 X \overline{\Sigma}_1^T - \Sigma_2 Y \overline{\Sigma}_2^T \le \frac{3}{\varepsilon^2} (\Sigma_1 - \Sigma_2)(\overline{\Sigma_1 - \Sigma_2})^T = \frac{3}{\varepsilon^2} \eta \otimes \bar{\eta}^T \tag{3.4}$$

with $|\eta| \le \frac{C}{m} |D\psi|$. By subtracting the two inequalities in (3.3) and by using **(H2)**, **(H4)** and (3.4), we finally obtain

$$\frac{\nu}{m} = \frac{g'}{m} \le \frac{C|D\psi|^2}{\varepsilon^2 m^2} \le C \frac{(g' \|x\|)^2}{\varepsilon^2 m^2}, \tag{3.5}$$

where C is a positive constant independent of m and ε. Now we take $g(s) = \exp(\beta s - \lambda)$ with $\beta > 0$ and λ to be determined as follows. We have $g' = \beta g$, and $g'' = \beta^2 g$. Since $x \ne 0$, if we choose $m = \varepsilon^{-4}$ then for $\beta = \varepsilon^{-1}$ and λ such that $g \le 1$, we get a contradiction in (3.5). □

Corollary 3.2.1 *Assume* **(H1)–(H3)** *or* **(H2)–(H4)**. *Let $u \in USC(\overline{\Omega})$ and $v \in LSC(\overline{\Omega})$ be respectively a bounded viscosity sub- and supersolution of* (1.7) *and assume that u (or v) is Lipschitz continuous. Then $u \le v$ in $\overline{\Omega}$.*

Remark 3.1 One can prove a variant of Theorems 3.1 and 3.2 and Corollary 3.2.1 in which the condition $u \le v$ on $\partial\Omega$ is dropped and the conclusion is changed to $u - v \le \sup_{\partial\Omega}(u - v)^+$, (see e.g. User's guide [3]).

[2] We recall that $\mathscr{J}^{2,+} u(x_0)$ is the set of $(p, X) \in \mathbb{R}^n \times \mathscr{S}(N)$ such that $u(x) \le u(x_0) + \langle p, (x - x_0)\rangle + \frac{1}{2}\langle X(x - x_0), (x - x_0)\rangle + o(|x - x_0|^2)$ as $x \to x_0$.
The set $\mathscr{J}^{2,-} u(x_0)$ is analogously defined.

4 Lipschitz estimates and proofs of Theorems 1.1, 1.2, 1.3.

We show the existence of a Lipschitz continuous viscosity solution of (DP) under suitable assumptions on k and geometric conditions on the domain. To this purpose we follow two different approaches. More precisely in the case when k depends on x and u, we use an approximation argument and the Bernstein method, while in the particular case that k is independent of the variable x we adapt the method of translation (see e.g. [7]).

We introduce the following notation : for $\gamma > 0$ we set $\Omega_\gamma := \{x \in \overline{\Omega} \, : d(x) < \gamma\}$. We recall that if $\partial\Omega$ is of class C^2 then for $\gamma > 0$ small the distance function $d \in C^2(\Omega_\gamma)$. We have the following lemma.

Lemma 4.1 *Assume* **(H3)***, $\varphi \in C^2(\partial\Omega)$. Then there are $\lambda' > 0$, and $0 < \gamma' \leq \gamma$ such that for all $\lambda \geq \lambda'$ the functions $\underline{u}(x) = \varphi(x) - \lambda d(x)$, and $\overline{u}(x) = \varphi(x) + \lambda d(x)$ are respectively classical subsolution and supersolution of (DP) in $\Omega_{\gamma'}$ and $\underline{u}(x) = \overline{u}(x) = \varphi(x)$ in $\partial\Omega$. Moreover, if $\varphi \in C^{1,1}(\partial\Omega)$ the functions*

$$\underline{v} = \begin{cases} \underline{u}(x), & x \in \Omega_{\gamma'} \\ c|x|^2 - M_1, & x \in \Omega \setminus \Omega_{\gamma'} \end{cases}, \quad \overline{v} = \begin{cases} \overline{u}(x), & x \in \Omega_{\gamma'} \\ M_2, & x \in \Omega \setminus \Omega_{\gamma'} \end{cases} \tag{4.1}$$

are respectively viscosity sub- and supersolution of (DP) in Ω for $c > \dfrac{\sup_{\overline{\Omega}\times\mathbb{R}} k}{4n}$, $M_1 \geq \sup_{\Omega_{\gamma'}}(c|x|^2 - \underline{u})$, $M_2 \geq \sup_{\Omega_{\gamma'}} \overline{u}$. In particular $\underline{v}$ and $\overline{v}$ are Lipschitz continuous in $\Omega_{\gamma'}$.

Proof Let φ be the smooth extension of φ to $\overline{\Omega}$.

Subsolution case : For $x \in \Omega_\gamma$ we have $D\underline{u}(x) = D\varphi(x) - \lambda Dd(x)$, $D^2\underline{u}(x) = D^2\varphi(x) - \lambda D^2 d(x)$ and by **(H3)**

$$\lim_{\lambda\to\infty} F(x, \varphi(x) - \lambda d(x), D\varphi(x) - \lambda Dd(x), D^2\varphi(x) - \lambda D^2 d(x)) \leq -K_{\partial\Omega\times i\mathbb{R}}(x) + \sup_{\Omega\times\mathbb{R}} k < 0.$$

Then, there are $\gamma' \leq \gamma$ and $\lambda' > 0$ depending on $\|D^2\phi\|_\infty$ such that for all $\lambda \geq \lambda'$, $\underline{u}(x)$ is a classical subsolution of (DP) in $\Omega_{\gamma'}$.

Supersolution case : We have

$$\lim_{\lambda\to\infty} F(x, \varphi(x) + \lambda d(x), D\varphi(x) + \lambda Dd(x), D^2\varphi(x) + \lambda D^2 d(x)) \geq K_{\partial\Omega\times i\mathbb{R}}(x) > 0.$$

□

Proof (of Theorem 1.1) The Dirichlet problem (1.5) is equivalent to (DP). By Theorem 3.1, the Perron method (see [3]) and Lemma 4.1 for every $\varphi \in C^{1,1}(\partial\Omega)$ there exists a unique continuous viscosity solution u of (DP). If $\varphi \in C(\partial\Omega)$ we consider a sequence of C^2 functions $\{\varphi_\varepsilon\}_{\varepsilon>0}$ converging uniformly to φ on $\partial\Omega$. Let u_ε be the solution to (DP) with boundary data φ_ε. By Remark 3.1 we have $\sup_\Omega |u_\varepsilon - u_{\varepsilon'}| = \sup_{\partial\Omega} |u_\varepsilon - u_{\varepsilon'}| = \sup_{\partial\Omega} |\varphi_\varepsilon - \varphi_{\varepsilon'}|$. Hence u_ε converges uniformly to the unique solution of (DP). □

Proof (of Theorem 1.2) Let u be a solution of (1.7) and let $c > \frac{\sup_{\overline{\Omega}\times\mathbb{R}} k}{4n}$ and $v(x) = c|x|^2$. We have $-L_\varepsilon(v) + k(x,v) \le -2c(2n) + k(x,v) \le 0$ and $-L_\varepsilon(-v) + k(x,-v) \ge 0$. By the comparison principle (see Remark 3.1) we have $\sup_\Omega(v - u) \le \sup_{\partial\Omega}(v-u)^+$, $\inf_\Omega(u-v) \ge \inf_{\partial\Omega}(u-v)^-$ and $\sup_\Omega(u+v) \le \sup_{\partial\Omega}(u+v)^+$. Hence, there is a positive constant C independent of ε such that

$$\sup_\Omega |u| \le \sup_{\partial\Omega} |u| + C. \tag{4.2}$$

Next, we fix γ' such that $\underline{u} \le u \le \overline{u}$ in $\Omega_{\gamma'}$, with $\underline{u}, \overline{u}$ as in Lemma 4.1. On $\partial\Omega$ we have

$$\frac{\partial \underline{u}}{\partial \nu} \le \frac{\partial u}{\partial \nu} \le \frac{\partial \overline{u}}{\partial \nu} \tag{4.3}$$

with ν the interior normal to $\partial\Omega$. Since $k \in C^1(\Omega \times \mathbb{R})$ is strictly monotone with respect to u then inequality (1.8) is satisfied and by (1.9) and (4.3) and the stability of viscosity solutions with respect to uniform convergence we can conclude. $\square$

Next we prove the existence of a unique Lipschitz continuous viscosity solution to (DP) under the assumption that k does not depend on x.

Proof (of Theorem 1.3) By Theorem 3.2, the Perron method (in the form of [7, p.32]) and Lemma 4.1 for $\varphi \in C(\partial\Omega)$ there exists a unique continuous viscosity solution u of (DP). If $\varphi \in C^{1,1}(\partial\Omega)$, let u be a solution of (1.7). By (4.2) we can fix γ' such that $\underline{u} \le u \le \overline{u}$ in $\Omega_{\gamma'}$, with $\underline{u}, \overline{u}$ as in Lemma 4.1. To show that the Lipschitz constant of u is independent of ε we adapt the method of translations (see [7]). Given $h \in \mathbb{R}^N$, the function $u(\cdot + h)$ is a viscosity solution of the same equation as that for u but set in $\Omega - h$, since the equation does not depend on x. Corollary 3.2.1 and Remark 3.1 yield

$$\begin{aligned}\sup_{\Omega_{\gamma'}\cap(\Omega_{\gamma'}-h)} |u - u(\cdot+h)| &\le \sup_{\partial(\Omega_{\gamma'}\cap(\Omega_{\gamma'}-h))} |u - u(\cdot+h)| \\ &\le \sup_{\partial(\Omega_{\gamma'}\cap(\Omega_{\gamma'}-h))} \max\{|\overline{u} - \underline{u}(\cdot+h)|, |\underline{u} - \overline{u}(\cdot+h)|\} \le C|h|.\end{aligned}$$

Thus the Lipschitz constant of u in $\Omega_{\gamma'}$ is independent of ε. Next we show that this implies that the Lipschitz constant of u in Ω is independent of ε. Indeed by Corollary 3.2.1 and Remark 3.1 we have

$$\sup_{\Omega\cap(\Omega-h)} |u - u(\cdot+h)| \le \sup_{\partial(\Omega\cap(\Omega-h))} |u - u(\cdot+h)|. \tag{4.4}$$

For $|h| \le \gamma'$, $\sup_{\partial(\Omega\cap(\Omega-h))} |u - u(\cdot+h)| \le C|h|$ by the above estimates and by the stability of viscosity solutions with respect to uniform convergence we can conclude. $\square$

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