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GRAVITATIONAL BOUNCE IN GENERAL RELATIVITY

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Summary

The radial motion of uniform spheres is studied in general relativity, in close analogy with the Newtonian treatment. Attention is concentrated on an equation of the energy conservation type. Conditions are found for an initially inward motion to reverse ('bounce') for various assumed relations between density and central pressure.

1. *Introduction.* In a number of recent papers (**1**)–(**5**) the structure and stability of spherically symmetric systems in general relativity has been discussed. The present paper gives a full analysis (briefly reported earlier (**6**)) of the motions of *uniform* systems of this type when a relation is imposed linking the central pressure with the density.

We first consider the Newtonian case, then use a method discussed by Thompson & Whitrow (**4**) to derive a kind of energy equation, and finally we use this to examine the motion of relativistic models, paying particular attention to the question of whether an initially inward motion later reverses direction ('bounces') or leads to total collapse through the Schwarzschild limit.

2. *Newtonian models.* The physical situation can be clarified by examining the Newtonian case of a spherical self-gravitating mass first discussed by Homer Lane (**7**) and Emden (**8**), applied to the special case of uniform density. Using r as a label moving with the matter and $R(t, r)$ for the distance at time t of the matter so labelled from the centre of our spherically symmetric distribution, we have for the mass m within the sphere labelled r

$$m = m(r) = 4\pi R^3 \rho/3 \tag{1}$$

where $\rho(t)$ is the (uniform) density.

The relation $m = m(r)$ applies since we make the assumption that there is no overtaking of matter by matter in the motion.

By equation (1)

$$R(t, r) = \alpha(t)\, s(r). \tag{1'}$$

Since r is purely a label, we can relabel so that $s(r) = r$ and $m = 4\pi r^3/3$, implying

$$\rho(t) = \alpha^{-3}(t). \tag{2}$$

Taking units so that the constant of gravitation $G = 1$, the equation of motion of a particle in Lagrangian form is

$$\ddot{R} = -\frac{m}{R^2} - \alpha^2(t) p' \tag{3}$$

where $p = p(t, r)$ is the pressure and, as throughout this paper, dots and dashes denote differentiation with respect to t, r. It is immediately seen from equation (3) that

$$-p' = r\left(\frac{\ddot{\alpha}}{\alpha^2}+\frac{4\pi}{3\alpha^4}\right). \tag{4}$$

Thus

$$p(t, r) = \frac{1}{2}(r_s^2-r^2)\left(\frac{\ddot{\alpha}}{\alpha^2}+\frac{4\pi}{3\alpha^4}\right), \tag{5}$$

suffix s denoting surface values, and assuming, as always, that $p_s = 0$. This equation may also be written as

$$\frac{p(t, r)}{\rho(t)} = \frac{1}{2}\frac{R_s^2-R^2}{R_s}\left(\ddot{R}_s+\frac{m_s}{R_s^2}\right), \tag{5'}$$

or, specializing to the centre,

$$\frac{p(t, 0)}{\rho(t)} = \frac{1}{2}R_s\left(\ddot{R}_s+\frac{m_s}{R_s^2}\right). \tag{5''}$$

Note that if the right hand bracket is positive, then the pressure is positive and the pressure gradient is negative throughout the interior. (A negative bracket would imply the surface being accelerated inward faster than by gravity alone, presumably through suction. We do not consider this possibility further.) Also note that $p(t, r)/p(t, 0)$ is time independent.

We now develop the well-known theory in a manner suitable for comparison with the relativistic work later in the paper. Define the surface potential $z = m_s/R_s$. Then

$$\rho = \frac{3m_s}{4\pi R_s^3} = \frac{3}{4\pi m_s^2}z^3. \tag{6}$$

Next we make the physical assumption that there is a functional relation between density and central pressure. This allows us to define a quantity x by using equation (6) in

$$\frac{z}{x}\frac{dx}{dz} = \frac{p_c(\rho)}{\rho}. \qquad (p_c = p(t, 0)) \tag{7}$$

Clearly there will be an arbitrary additive constant in $\log x$. Then equation (5″) becomes

$$\frac{z}{x}\frac{dx}{dz} = \frac{m_s^2}{2z}\ddot{\left(\frac{1}{z}\right)}+\frac{1}{2}z \qquad \left(=\frac{1}{2}R_s\ddot{R}_s+\frac{1}{2}z\right), \tag{5'''}$$

which may be integrated to give the energy type of equation

$$\frac{1}{2}\dot{R}_s^2 = \frac{1}{2}m_s^2\frac{\dot{z}^2}{z^4} = z-2\log x, \tag{8}$$

the constant of integration having been absorbed into the definition of $\log x$, which thus contains both the variable pressure energy of the model and the constant total energy.

We may now use equations (6)–(8) to study the question of the bounce. For any assumed relationship* between p_c and ρ equation (7) yields a family of curves

* Only the relationship between ρ and p_c may be chosen at will. Once this has been done the relation between $p(t, r)$ and ρ follows from the constancy of $p(t, r)/p_c$.

in the (x, z) plane. Whether a model initially moving inward bounces or not depends entirely on whether the (x, z) curve, as defined by the p–ρ relation and the initial conditions, does or does not intersect the curve (to be called S) $z = 2 \log x$. The representative point of the model must, by equation (8), never move below S, and will be on S if the model is instantaneously at rest (see Fig. 1).

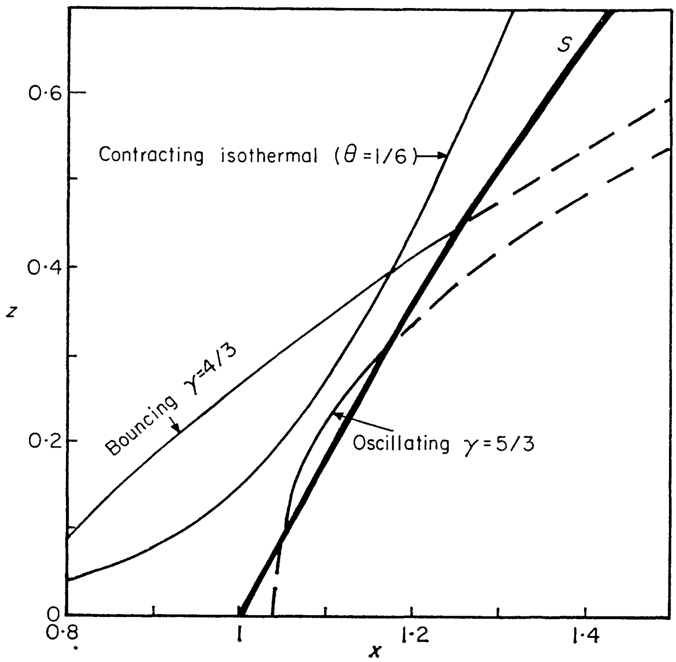

FIG. 1. *The Newtonian case, with some typical representative curves.*

We now investigate a simple case to demonstrate the method. Suppose that the model is ' isothermal ' ($p_c = \theta\rho$), with the ' temperature ' θ a known constant. We also take the initial potential z_0 and the initial velocity v ($= (\dot{R}_s)_0$) to be known. By equation (7)

$$z = Ax^{1/\theta}, \tag{9}$$

A being a constant of integration readily defined by equation (8) since

$$2 \log x_0 = z_0 - \tfrac{1}{2}v^2$$

so that

$$\theta \log (z/z_0) = \log x - \tfrac{1}{2}z_0 + \tfrac{1}{4}v^2. \tag{9'}$$

It is immediately obvious that equation (9′) intersects S at most twice and is never above S between the two intersections, proving that no such model can oscillate. An initially inward moving model will bounce if equation (9′) intersects S for $z > z_0$. This will certainly not be the case if v^2 is too large. In the limit equation (9′) touches S. It is thus readily seen that the conditions for a bounce are

$$\left.\begin{aligned} &z_0 < 2\theta, \\ &\frac{v^2}{2z_0} < 1 + \frac{2\theta}{z_0}\left[\log \frac{2\theta}{z_0} - 1\right] \end{aligned}\right\}. \tag{10}$$

If the temperature is not known, a further initial datum is required. We take this to be the initial acceleration and use the convenient quantity $q = (R_s)_0(\ddot{R}_s)_0$. Note that we may always assume that $q+z_0>0$, for otherwise the inward acceleration of the surface would at least equal that of free fall. By equation (5‴)

$$2\theta = q+z_0.$$

The criteria for a bounce thus become

$$\left.\begin{aligned} &q>0, \\ &\frac{v^2}{2z_0}<\left(1+\frac{q}{z_0}\right)\log\left(1+\frac{q}{z_0}\right)-\frac{q}{z_0} \end{aligned}\right\}. \qquad (10')$$

Thus a bounce can only occur for models with initial outward acceleration and then only if the initial velocity is not too large (see Fig. 1).

In the case that the pressure–density law is of the form

$$p_c = A\rho^\gamma$$

the multiplicative A constant is generally unknown. We thus need z_0, v, q. We have from equations (6), (7) and (5‴) that

$$\frac{z}{x}\frac{dx}{dz} = \frac{q+z_0}{2}\left(\frac{z}{z_0}\right)^{3(\gamma-1)}.$$

Integrating and using equation (8)

$$\log x = \frac{q+z_0}{6(\gamma-1)}\left[\left(\frac{z}{z_0}\right)^{3(\gamma-1)}-1\right]+\frac{1}{2}z_0-\frac{1}{4}v^2. \qquad (11)$$

Consider first the case $\gamma = 4/3$. Then

$$z = z_0+\frac{z_0}{q+z_0}\left[2\log x-z_0+\frac{1}{2}v^2\right]. \qquad (12)$$

It is immediately evident that for $q>0$ this curve must intersect S for some $z>z_0$, while for $q\leq 0$ it cannot do so. Thus an initial outward acceleration is the necessary and sufficient condition for a bounce.

If $\gamma = 5/3$ we have

$$\left(\frac{z}{z_0}\right)^2 = 1+\frac{2}{q+z_0}\left[2\log x-z_0+\frac{1}{2}v^2\right]. \qquad (13)$$

This is bound to intersect S for sufficiently large z so that all such models bounce. If $z_0>q+v^2$ there will also be an intersection for sufficiently small z so that such a model will oscillate. More generally, an examination of the curvature of equation (11) compared with that of S readily shows that such a double intersection with S can occur only if $\gamma>4/3$. Only such models can oscillate and only such models can therefore be in stable equilibrium. Moreover whenever $\gamma>4/3$, there is bound to be an intersection for sufficiently large z, so that these models can never contract to zero radius, which is the fate of all models with $\gamma\leq 4/3$ if initial conditions do not ensure a bounce (see Fig. 1).

All these results are well known and have been long established, but the particular form of presentation here adopted will make easier the comparison with the relativistic results.

3. *The metric.* We now turn to the relativistic case, where the metric of (4) is

$$ds^2 = e^{\nu}dt^2 - e^{\lambda}dr^2 - e^{\mu}d\Omega^2,$$

$$(d\Omega^2 = d\theta^2 + \sin^2\theta\, d\phi^2).$$

It is assumed in (4) that $\lambda = \mu - a(r)$; we follow this assumption as it fits the case of uniform density, but there is no advantage in keeping $a(r)$ general, since a re-labelling giving $a(r) = 2 \log r$ may always be carried out. It is shown in (4) that if the coordinates are co-moving then

$$\exp \nu = \dot{\mu}^2 F(t).$$

We write $\exp \mu = R^2(t, r)$ so that the metric becomes

$$ds^2 = 4F\,\dot{R}^2 R^{-2}\, dt^2 - R^2(r^{-2}\, dr^2 + d\Omega^2), \tag{14}$$

which cannot be admissible at the centre ($R = 0$), unless rR'/R tends to one there. Then, as was shown in (4), the field equations yield a perfect fluid energy tensor with

$$4\pi\rho = \frac{3}{4F} - \frac{r^2}{R^2}\left[2\frac{R''}{R} - \frac{R'^2}{R^2} + 2\frac{R'}{rR} - \frac{1}{r^2}\right] \tag{15}$$

and

$$8\pi p = -\frac{1}{4F}\left[3 - \frac{R\dot{F}}{\dot{R}F}\right] + \frac{r^2}{R^2}\left[2\frac{\dot{R}'}{\dot{R}}\frac{R'}{R} - \frac{R'^2}{R^2} - \frac{1}{r^2}\right] \tag{16}$$

provided (since, in a perfect fluid, $T_1{}^1 = T_2{}^2 = T_3{}^3$) that

$$R'' + \frac{R'}{r} + \frac{R}{r^2} - 2\frac{R'^2}{R^2} = \frac{3}{r^2}S(r), \tag{17}$$

where $S(r)$ is an arbitrary function of r, and

$$S'(r) = -\frac{4\pi}{3}R^3\rho'. \tag{18}$$

The energy tensor leading to these results is diagonal, so that there is no radial energy flow. Hence we are dealing with the adiabatic case as described in (3).

4. *Uniform models.* We now concentrate on the case of a sphere of uniform density surrounded by empty space. Thus

$$\rho = \begin{cases} \rho(t) & (r \le r_s) \\ 0 & (r > r_s) \end{cases}, \text{ so that } \rho' = -\rho(t)\,\delta(r - r_s), \tag{19}$$

and equation (18) is satisfied if $\rho(t)R_s{}^3$ is independent of the time. This is indeed the case since the quantity $m_s = 4\pi\rho\, R_s{}^3/3$ is conserved, provided the surface pressure always vanishes, as we shall assume.* Integration of equation (18) gives merely

$$S(r) = \begin{cases} A & (r \le r_s) \\ A + m_s & (r > r_s) \end{cases} \tag{20}$$

* For a sphere moving in a surrounding medium of non-zero density and pressure the conclusion would not hold.

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where A is an as yet arbitrary constant of integration. If now equation (17) is integrated for $r \leq r_s$ we obtain

$$\frac{r^2 R'^2}{R^2} = \sigma(t)\, R^2 + 1 - \frac{2A}{R}, \tag{21}$$

where σ is a function of integration. The regularity condition at the centre* ($R = 0$) stated immediately after equation (14) ensures that $A = 0$. It follows that at $R = 0$ also $r = 0$.

Integrating equation (21) again we find

$$R = \frac{r\alpha(t)}{1 - r^2\beta(t)} \tag{22}$$

where $\alpha^2 = 4\beta/\sigma$, and β is an arbitrary function of t.

Then, since $3m_s = 4\pi\rho\, R_s{}^3$, by equation (11)

$$\frac{8\pi\rho}{3} = \frac{1}{4F} - \frac{4\beta}{\alpha^2} = \frac{2m_s}{r_s{}^3}\left(\frac{1 - r_s{}^2\beta}{\alpha}\right)^3 \tag{23}$$

which establishes the connection between $F(t)$, $\alpha(t)$ and $\beta(t)$.

Finally

$$\frac{p}{\rho} = -1 - \frac{1}{3}\frac{\dot\rho}{\rho}\frac{R}{\dot R} = -1 + \frac{\dot R_s}{R_s}\frac{R}{\dot R}. \tag{24}$$

It is also interesting to note that the definition of $m(t, r)$ given in equation (11) of (4),† which is there shown to imply

$$m' = 4\pi\rho\, R^2 R',$$

because of equation (22) and equation (23) implies also

$$\frac{m}{m_s} = \left(\frac{r}{r_s}\right)^3 \left(\frac{1 - \beta r_s{}^2}{1 - \beta r^2}\right)^3 = \frac{R^3}{R_s{}^3}. \tag{24'}$$

This shows how the work done by the pressure changes m, and indeed accounts for the occurrence of β in equation (22) as compared with equation (1′).

Thus r_s, m_s, $\alpha(t)$ and $\beta(t)$ may be chosen freely and determine $\rho(t)$ and $p(t, r)$. Note however, that certain inequalities must be satisfied.

In order that R may be bounded, positive, vanishing at the centre, and monotonically increasing‡ with r we must have

$$\alpha > 0, \qquad -1 < r_s{}^2\,\beta(t) < 1. \tag{25}$$

These conditions will also insure that $\rho > 0$, but the signature of the metric will be correct only if $F > 0$. This implies, by equation (23)

$$2\alpha\, r_s{}^3\beta + m_s(1 - r_s{}^2\beta)^3 > 0. \tag{26}$$

* Spheres without centre are examined in (5). We do not consider this possibility here.

† To translate from our symbols to those of (4) replace r^{-1} by $f(r)$, α by $1/B$, β by $-C/B$.

‡ This last condition (yielding $-1 < r_s{}^2\,\beta(t)$) is not required from any strong physical argument and will be assumed only for convenience at present. Conditions in which it may be discarded will be discussed later.

By equation (24) the pressure will automatically vanish at the surface. Furthermore, it can easily be shown that the pressure in the interior will be positive and bounded only if $\dot{\alpha}$, $\dot{\beta}$ are bounded and have the same sign. This will ensure also that $p' < 0$, since

$$-\frac{p'}{p+\rho} = \frac{1}{2}\nu' = \frac{2r\alpha\dot{\beta}}{(1-r^2\beta)[\dot{\alpha}+r^2(\alpha\dot{\beta}-\dot{\alpha}\beta)]} > 0.$$

Next

$$\frac{\dot{R}}{R} = \frac{\dot{\alpha}}{\alpha} + \frac{r^2\dot{\beta}}{1-r^2\beta} \tag{27}$$

so that contraction occurs if $\dot{\alpha} < 0$, $\dot{\beta} < 0$.

The occurrence of $\dot{R}$ in the first term of metric (14) indicates that this metric is primarily only useful when the system is in motion. Indeed, in general the time span measured for a period of motion by a co-moving observer will be finite if the fractional change in R during the period of motion is finite. An exception however, occurs if $F \to \infty$, that is if equation (26) tends to equality. This may lead to asymptotic behaviour, a case apparently of no more importance for relativistic than for Newtonian models.

The question of the pressure gradient deserves discussion. If we consider a general (non-uniform) body then although the pressure gradient must be everywhere negative in the static condition, this need not hold during contraction. If for example we imagine that, at some stage during the contraction of a non-uniform gaseous sphere, a phase change (say dissociation) occurs in a particular layer so that this layer heats up much less than the layer above it, then there may well be a positive pressure gradient for a time. Thus there is nothing *unphysical* about a positive gradient, but, as has been shown above, it cannot occur in uniform relativistic models any more than in uniform Newtonian models.

Finally in this section we link the present model with that of a slowly contracting uniform sphere discussed in (3). Slow contraction occurs in our case if $F \to \infty$. By equation (23) this implies

$$\alpha = \frac{m_s}{2r_s}\frac{(1+b)^3}{b} \qquad \text{where } -r_s^2\beta = b. \tag{28}$$

Applying this to equation (22) for $r = r_s$ we obtain

$$R_s = \frac{m_s(1+b)^2}{2b} \text{ so that } \left(1-\frac{2m_s}{R_s}\right)^{1/2} = \frac{1-b}{1+b}. \tag{28'}$$

Resubstituting equations (28) and (28′) into equation (22) we obtain

$$R = \frac{R_s(r/r_s)}{1-\frac{1}{2}[1-(r/r_s)^2]\{1-[1-2m_s/R_s]^{1/2}\}}$$

which is identical with equation (29) of Ref. (3) when corresponding symbols are used.

5. *The external metric.* By equation (20) we must take $S(r) = m_s$ outside the body. Then from equation (21)

$$\log r = \int_{R_s}^{R} dR[(R^4/4F)+R^2-2m_sR]^{-1/2} + \log\gamma(t) \tag{29}$$

where $\gamma(t)$ is a function of integration, the function $\sigma(t)$ has been identified with $1/4F$ from the condition $\rho = 0$ by equation (23) and the limits of integration have been chosen to make γ well defined. Continuity of r then implies that $\gamma(t) = r_s$. Next equations (17) and (20) guarantee the continuity of R', so that

$$\log(r/r_s) = \int_{R_s}^{R} dR[(R^4/4F)+R^2-2m_sR]^{-1/2} \tag{29'}$$

is the correct continuation of equation (22), though unfortunately we now have to deal with an elliptic integral. Note that as $R\to\infty$, r approaches a finite limit. Of course this metric must be the Schwarzschild metric

$$ds^2 = \left(1-\frac{2m_s}{R}\right)d\tau^2-\left(1-\frac{2m_s}{R}\right)^{-1}dR^2-R^2d\Omega^2 \tag{30}$$

in disguise. Indeed, appropriate substitution shows readily that

$$\tau(t,R) = \int\frac{\dot{R}_s dt}{R_s}\left\{\frac{1}{4F}\left[\frac{R_s^2}{4F}+1-\frac{2m_s}{R_s}\right]\right\}^{-1/2}+R+2m_s\log(R-2m_s)$$
$$+\int_R^\infty\frac{RdR}{R-2m_s}\left\{1-\left[1+\frac{4F}{R^2}\left(1-\frac{2m_s}{R}\right)\right]^{-1/2}\right\}. \tag{31}$$

Note that $R_s = R(t, r_s) = R_s(t)$ since r_s is a constant of the system, so that the first integrand is a function of t only. In the second integral $F(t)$ is to be treated as a constant.* It will be appreciated that, by equation (31), τ is almost a null coordinate for large R. Thus the imaginary matter with which the coordinate system is comoving is for large R moving at very high speed. The particular motion of the coordinates outside the body is dependent on the motion inside the body only through $R_s(t)$ and $F(t)$ and the constants m_s and r_s. The space itself is fully specified by m_s; different r_s, R_s, F only specify different disguises of equation (30).

6. *Flat space*. For $m_s = 0$ the Schwarzschild metric gives flat space, and so must our metric. In this case equation (29′) integrates to give

$$R = \frac{\alpha(t)r}{1-\beta(t)r^2}, \qquad \alpha = 2(4F\beta)^{1/2}. \tag{32}$$

Of course this is the same as the internal metric (22) with $\rho = 0$, $p = 0$. The dependence of R_s and r_s turns out to be tautologous. Thus for arbitrary $\alpha(t)$, $\beta(t)$

$$ds^2 = \frac{1}{4\beta}\left[\dot{\alpha}+\frac{\alpha\dot{\beta}r^2}{1-\beta r^2}\right]^2 dt^2-\frac{\alpha^2}{(1-\beta r^2)^2}(dr^2+r^2d\Omega^2) \tag{33}$$

is flat. The best known example of this class of representation is Milne's universe ($\beta = \frac{1}{4}$, $\alpha = t$)

$$ds^2 = dt^2-\frac{t^2}{(1-\frac{1}{4}r^2)^2}(dr^2+r^2d\Omega^2), \tag{33'}$$

but evidently equation (33) represents a much larger class. Note that for $\beta\geq 0$ the metric (33) represents all space, with r taking $\beta^{-1/2}$ as upper limit, but $\beta<0$ gives metric (33) the wrong signature and so must be rejected. For a

* As in equation (29).

purely internal model with curvature, negative β seems to be admissible, provided $-\beta$ is sufficiently small to imply positive F through equation (23).

7. *The energy equation.* For a comoving observer the metric implies

$$ds^2 = 4F\dot{R}^2\, dt^2/R^2$$

so that, using equation (23),

$$\frac{1}{2}\left(\frac{dR}{ds}\right)^2 = \frac{R^2}{8F} = R^2\left[\frac{2\beta}{\alpha^2}+\frac{m_s(1-\beta r_s{}^2)^3}{\alpha^3 r_s{}^3}\right]$$

$$= \frac{2\beta r^2}{(1-\beta r^2)^2}+\frac{m_s}{R_s{}^3}R^2.$$

Hence, by equation (24′),

$$\frac{1}{2}\left(\frac{dR}{ds}\right)^2-\frac{m}{R} = \frac{2\beta r^2}{(1-\beta r^2)^2}. \tag{34}$$

This is very much of the form of a Newtonian energy equation, provided the Newtonian radius is interpreted as R (i.e. defined by the surface area) and the Newtonian time variable is interpreted as proper time, the right hand side representing the changes in pressure energy. Indeed substituting equation (22) in equation (24) we find

$$\frac{p}{\rho} = \dot{\beta}\,\frac{\alpha(r_s{}^2-r^2)}{(1-r_s{}^2\beta)[\dot{\alpha}+r^2(\alpha\dot{\beta}-\dot{\alpha}\beta)]}, \tag{35}$$

so that in any moving model $\dot{\beta} = 0$ is equivalent to $p = 0$ which implies that in the pressure-free case the right hand side of equation (34) (the 'total mechanical energy') remains constant.

We now define

$$x = 1-\beta r_s{}^2,\quad z = \frac{m_s}{R_s} = \frac{m_s x}{r_s\alpha} \qquad (\text{N.B. } 0<x<2,\ 0<z\leq\tfrac{1}{2}.) \tag{36}$$

Applying equation (35) at the centre

$$\frac{\rho}{p_c} = \frac{x\dot{\alpha}}{\alpha\dot{\beta}r_s{}^2} = -\frac{x\dot{\alpha}}{\alpha\dot{x}} = -\frac{d\log\alpha}{d\log x} = \frac{d\log(z/x)}{d\log x}$$

giving

$$\frac{x}{z}\frac{dz}{dx} = \frac{\rho}{p_c}+1. \tag{37}$$

Using equation (34) at the surface we have

$$\frac{1}{2}\left(\frac{dR_s}{ds}\right)^2 = z-\frac{2(x-1)}{x^2}. \tag{38}$$

Differentiating equation (38) with respect to s, using equation (36) and rearranging we obtain

$$\frac{1}{2}R_s\frac{d^2R_s}{ds^2}+\frac{1}{2}z = \frac{2-x}{x^2}\frac{z}{x}\frac{dx}{dz}. \tag{39}$$

Note that the relativistic equations (37)–(39) are very similar to the Newtonian equations (7), (8) and (5‴). The few differences are readily enumerated:

(a) The Newtonian ρ in equation (7) is replaced by $\rho + p_c$ in the relativistic equation (37). In the Newtonian limit p_c is negligible compared with ρ, but the difference becomes appreciable in highly relativistic cases. If for example $p_c = \rho/3$ we have $z \sim x^3$ by equation (7) but $z \sim x^4$ by equation (37).

(b) The Newtonian time derivative on the left hand side of equation (8) and (5‴) is interpreted as a proper time derivative in equations (38) and (39).

(c) The log x of equation (8) is replaced by $(x-1)/x^2$ in equation (38). Note that the two functions have the same value and gradient for small velocities and potentials (x close to unity).

(d) A factor $(2-x)/x^2$ distinguishes equation (39) from equation (5‴), but again this factor equals unity for $x = 1$.

Thus the methods used in the Newtonian analysis of Section 2 can be used for a relativistic discussion with minor modifications. Differences (b) and (d) relate only to the interpretation of the initial conditions, but (a) tends to make representative curves climb faster, while the curve (to be called Q)

$$z = 2\frac{x-1}{x^2} \tag{40}$$

always lies below curve S

$$z = 2 \log x$$

except at $x = 1$ where S and Q touch.

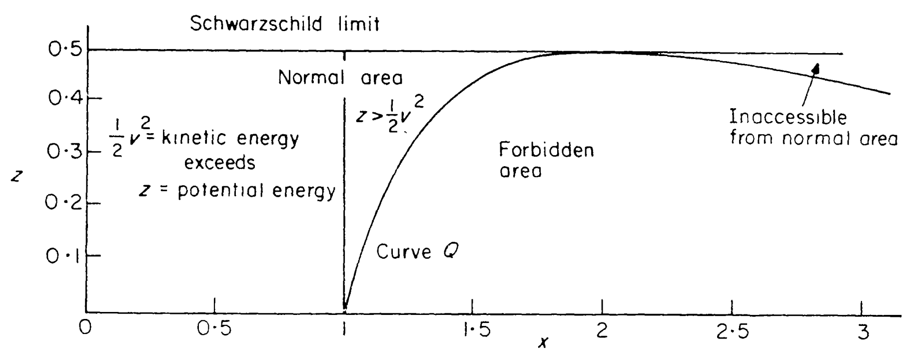

FIG. 2. *The x, z diagram.*

Hence relativistic conditions for a bounce will be more severe than Newtonian ones. If no bounce occurs, the passage of a model through the Schwarzschild limit ($z = \frac{1}{2}$) is now a significant instant in the contraction to a point ($z \to \infty$) since it takes infinite time in the view of an outside observer. Note that since Q never exceeds $\frac{1}{2}$, no bounce can occur after passage through the Schwarzschild limit. Indeed Q attains $z = \frac{1}{2}$ only at $x = 2$. Moreover by equation (37) all models with non-negative densities and pressures have representative curves with positive gradient. Thus the area $x > 2$, $z < \frac{1}{2}$ is inaccessible from $x < 2$. Hence no model can evolve to a state with z below the Schwarzschild limit but with $x > 2$ unless this is initially the case. But $x > 2$ is equivalent to $\beta r_s^2 < -1$, that is a model in which R does not monotonically increase with r. Such models therefore form a completely separate class which we will not consider further. Our area of interest is therefore confined to the strip $0 \leq z \leq \frac{1}{2}$, $0 < x \leq 2$, $z \geq 2(x-1)/x^2$ (see Fig. 2).

Equation (37) allows us to restrict the condition for a bounce further. With any finite pressure it implies that z/x is a non-decreasing function of x. Thus no representative curve can meet Q unless z/x is non-decreasing along Q, which holds only for $x<3/2$ and hence $z<4/9$. Once a model attains the potential 4/9 (long known to be a critical value) no bounce can possibly occur.

More stringent restrictions on p/ρ lead to more severe limits. In general if $p \leq \theta\rho$, no bounce can occur once the model has passed

$$z = \frac{2\theta(2\theta+1)}{(3\theta+1)^2} \tag{41}$$

which for $\theta = \frac{1}{3}$, 1 gives $z = \frac{5}{18}$ (= 0·27), $\frac{3}{8}$ (= 0·375) respectively. The corresponding x values are 6/5 and 4/3.

8. *The isothermal case.* We now develop the case $p_c = \theta\rho$ in strict analogy to the Newtonian case, using again the notation $v = (dR_s/ds)_0$, $q = (R_s d^2R_s/ds^2)_0$ though of course R_s and ds now have somewhat different meanings.

From equation (37), $z \sim x^k$ where $k = (\theta+1)/\theta$. Thus x_0 is determined by equation (38):

$$2\frac{x_0-1}{x_0^2} = z_0 - \frac{1}{2}v^2. \tag{42}$$

To avoid the awkwardness of quadratic equations we do not explicitly evaluate x_0 in terms of z_0 and v. Next

$$z/z_0 = (x/x_0)^k \tag{43}$$

is the representative curve. If θ (and hence k) is known, the model will bounce if the initial point lies below the critical curve of type (43) which touches Q. This is given by

$$z = 2\frac{k+1}{(k+2)^2}\left[x\frac{k+1}{k+2}\right]^k = \frac{2\theta(2\theta+1)}{(3\theta+1)^2}\left[x\frac{2\theta+1}{3\theta+1}\right]^{(\theta+1)/\theta}. \tag{44}$$

Equations (41) and (44) may be put in the form of the conditions for a bounce:

$$\left.\begin{aligned} z_0 &< 2\theta\frac{1+2\theta}{(1+3\theta)^2}, \\ \log x_0 &> \frac{\theta}{1+\theta}\left[\log\frac{z_0}{2\theta}(1+3\theta)+\frac{1+2\theta}{\theta}\log\left(1+\frac{\theta}{1+2\theta}\right)\right]. \end{aligned}\right\} \tag{45}$$

These may be compared with the Newtonian equation (10) (see Fig. 3)

$$\left.\begin{aligned} z_0 &< 2\theta, \\ \log x_0 &> \theta\left[\log\frac{z_0}{2\theta}+1\right], \end{aligned}\right\} \tag{45'}$$

where the second of these equations has been written in terms of x_0 rather than v to facilitate comparison. It is immediately evident that equation (45) approaches (45′) as $\theta \to 0$. The first of the pairs of equations is clearly a more stringent condition in the relativistic than in the Newtonian case and, with a little trouble, it may be shown that this also holds for the second equation.

In the important case $\theta = \frac{1}{3}$, equation (44) becomes

$$z = wx^4, \qquad w^{-1} = 3(1 \cdot 2)^5, \qquad w = 0 \cdot 13396 \ldots . \tag{44'}$$

If θ is not known, we use, as in the Newtonian case, the quantity q. By equation (39)

$$\frac{1}{k} = \frac{\theta}{1+\theta} = \frac{1}{2}\frac{x_0^2}{2-x_0}(q+z_0) \tag{46}$$

now serves to determine θ.

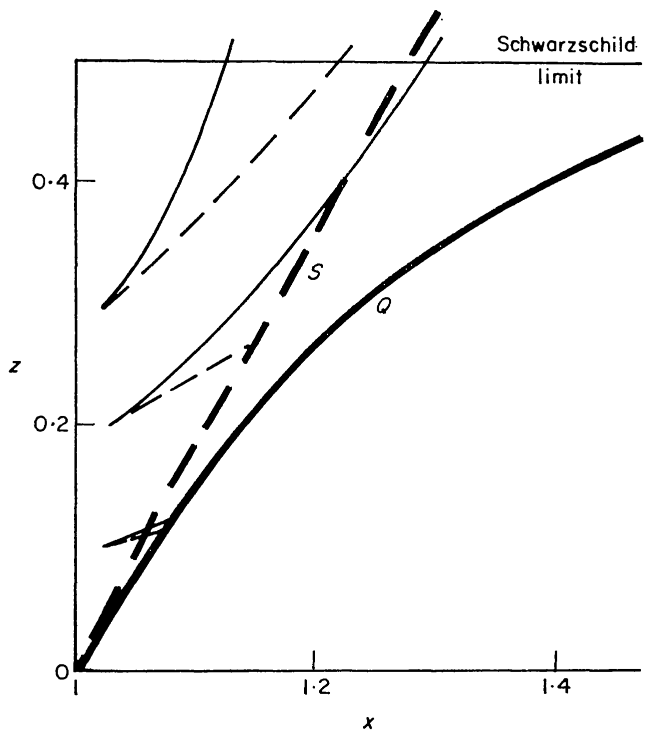

FIG. 3. *Newtonian (– – – –) and relativistic (———) curves for $p_c = \rho_3$ with three sets of identical starting conditions. In the lowest pair both bounce, in the middle one only the Newtonian solution, in the top pair neither bounces.*

It is now convenient to write equation (45) in the form

$$\left.\begin{aligned} z_0 &< 2\theta\,\frac{1+2\theta}{(1+3\theta)^2} = Z\ (\text{say}),\\ z_0 &< Z\left(x_0\,\frac{1+2\theta}{1+3\theta}\right)^{(1+\theta)/\theta}. \end{aligned}\right\} \tag{45''}$$

If $x_0(1+2\theta)/(1+3\theta)<1$, the second condition is the more demanding and otherwise the first condition cannot be met because of the shape of Q. Thus, given x_0 and z_0, θ has to be large enough which by equation (46) implies that q has to be large enough.

The results of the computations are displayed in Fig. 5 in which values of η are displayed as function of z_0 and $y = v^2/2z_0$ (Fig. 4), while ηz_0 is the least value of q leading to a bounce The diagram is restricted to $y \leq 1$ (upper edge) and $\theta \leq \frac{1}{3}$ (right hand curve). On the left hand edge ($z_0 = 0$) the connection between y and η is given by the Newtonian limit (see equation (10′))

$$y = (1+\eta)\log(1+\eta)-\eta$$

leading to $\eta = e-1$ at $y = 1$. Along the bottom edge $\eta = 0$, along the top η

rises slowly from $e-1$ at the left to $1\cdot 5\ (1\cdot 2)^5-1 = 2\cdot 732\ \ldots$ at the right. An even clearer picture is given by plotting values of $\eta y^{-1/2}$ instead of η. This quantity varies along $y = 0$ only between $2^{1/2}$ at the left and $\frac{1}{2}\ 15^{1/2} = 1\cdot 9365\ \ldots$ at the right. With the top edge as before, this quantity varies thus by less than a factor 2 over the whole diagram.

Static models are represented by points on $Q\,(y = 0)$ with their representative curves tangent to $Q\ (\ddot{R} = 0)$. It is evident that they are all unstable.

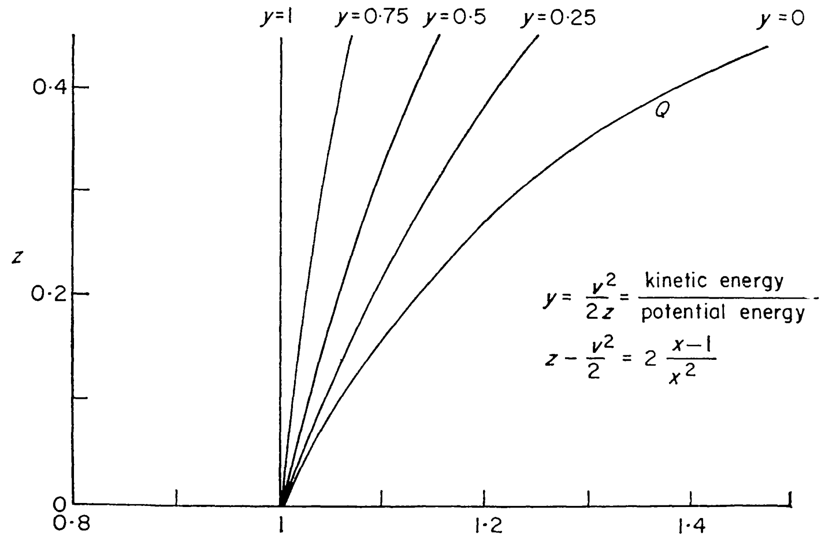

FIG. 4. *x, y, z.*

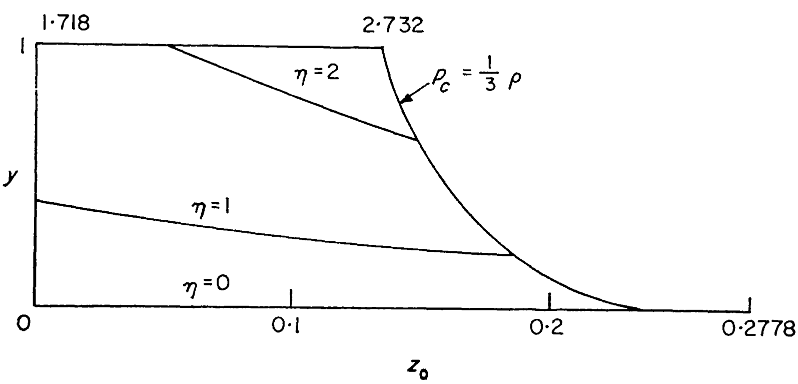

FIG. 5. *Relation between* $\eta = q/z_0$, z_0, y *for least q to give bounce in the isothermal case.*

9. *The case* $\gamma = 4/3$. For general γ

$$p_c \sim \rho^\gamma \sim z^{3\gamma}$$

so that, with some positive constant C,

$$\frac{x}{z}\frac{dz}{dx} = 1 + Cz^{-3(\gamma-1)} \tag{47}$$

giving on integration

$$z^{3(\gamma-1)} = Bx^{3(\gamma-1)} - C \text{ with } p_c/\rho = z^{3(\gamma-1)}/C, \tag{48}$$

where B is a constant of integration.

For $\gamma = 4/3$ these equations taken the simple form

$$z = Bx - C, \qquad p_c/\rho = z/C, \tag{49}$$

so that straight lines result. Again the curvature of Q shows that all equilibrium configurations are unstable. Since now normally neither B nor C are given, we need the relation

$$q + z_0 = 2\,\frac{2-x_0}{x_0^2}\,\frac{z_0}{C+z_0} = 2z_0\,\frac{2-x_0}{Bx_0^3} \tag{39'}$$

together with equation (38) applied at (x_0, z_0) to determine B and C.

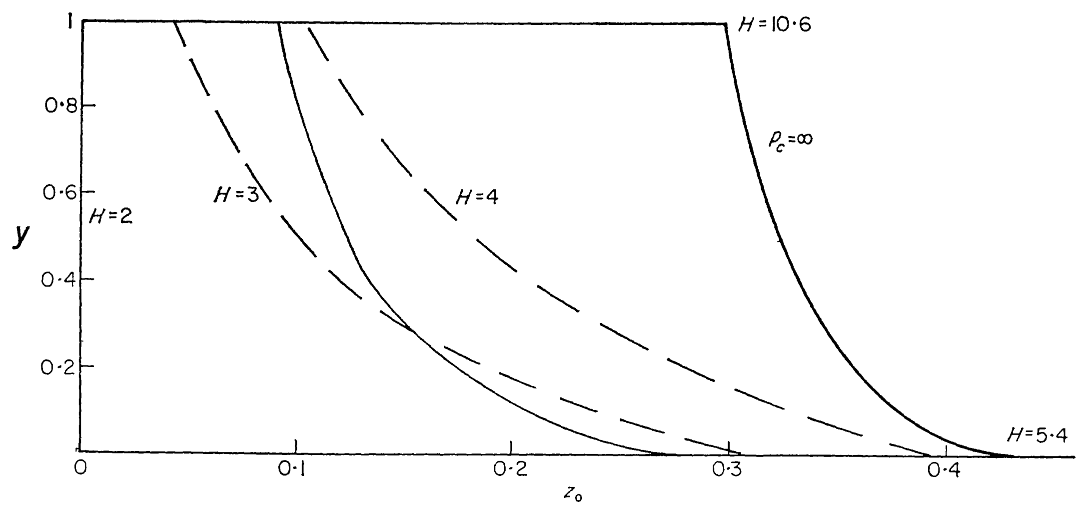

FIG. 6. $\gamma = 4/3$. *Diagram of* $H = qz_0^{-1\cdot 5}\,y^{-0\,5}$ *against* z_0 *and* y. *The right hand edge represents models with infinite central pressure, the middle curve models with maximum* $p_c/\rho = 1/3$. *The dashed lines give* H *values.*

The smallest q and hence the largest B (for a given x_0, z_0) leading to a bounce will give a curve (49) tangential to Q. Supposing this to touch Q at (x_1, z_1), we have

$$B = 2\,\frac{2-x_1}{x_1^3}, \qquad z_1 - z_0 = 2\,\frac{2-x_1}{x_1^3}(x_1 - x_0), \qquad \frac{q}{z_0} = -1 + \frac{2-x_0}{2-x_1}\left(\frac{x_1}{x_0}\right)^3. \tag{50}$$

The results are most conveniently expressed by considering $H = (q/z_0^{3/2}y^{1/2})$ as a function of z_0 and y, and are shown in Fig. 6. Along the left hand edge ($z_0 = 0$) $H = 2$, and increases with increasing z values, more rapidly for large than for small y. The right hand edge of the diagram is given by $C = 0$, i.e. $p_c = \infty$. No bounce can occur from x_0, z_0 if there is no straight line with positive C through it that touches Q. Thus bounces can only occur from initial positions below the highest line with $C = 0$ that touches Q, namely $z = 0\cdot 2963x$. The highest value of H occurs at the top right hand corner and equals $10\cdot 556$. Along the bottom ($y = 0$), representative points lie on Q, and in the limit

$$H = x_1[2(3-x_1)]^{1/2}/(2-x_1).$$

In Newtonian theory *any* outward acceleration will lead to a bounce, i.e. $q > 0$ is the condition there. This agrees with our work in the limit $z \to 0$, as it must.

Next we impose the limitation $p_c \leq \frac{1}{3}\rho$, so that we stick to the validity of equation (49) only for $z \leq C/3$, and then go on to equation (43) with $k = 4$. Thus if we consider a particular initial point x_0, z_0, then the lowest value of B now permitted is the one giving $C = 3z_0$, not, as previously, the one giving $C = 0$, and hence $p_c = \infty$. Along any straight line (49) the ratio p_c/ρ will increase and reach the value $\frac{1}{3}$ at some point (x_2, z_2). If the straight line intersects Q before the point is reached we have a *direct bounce*. If this is not the case we must continue from (x_2, z_2) along a curve (43) with $k = 4$ (to which our line will necessarily be tangential at (x_2, z_2)). If this curve intersects Q, then we have *an indirect bounce*. Otherwise no bounce at all will occur.

The locus of points x_2, z_2 is readily found for any (x_0, z_0) from equation (49) with $z_2 = C/3$, and is

$$\frac{z_2}{z_0} = \frac{x_2}{4x_0 - 3x_2}. \tag{51}$$

If (x_0, z_0) is such that curve (51) intersects Q, then direct bounces can occur but not otherwise. The limit of this zone is given by the locus of points (x_0, z_0) such that their equation (51) touches Q. Expressed in terms of the parameter x_1 (x co-ordinate of point of contact with Q) this boundary is given by

$$x_0 = \frac{3}{4} x_1 \frac{2 - x_1}{3 - 2x_1}, \qquad z_0 = \frac{6}{3 - 2x_1} \left(\frac{x_1 - 1}{x_1}\right)^2. \tag{52}$$

Direct bounces can only occur from points below equation (52) which passes through (1, 0·10993) and ends at (1·2, 0·27778). For such points there will be a range of B values and hence q values giving a direct bounce. This range will always be bordered by a range of indirect bounces for lower B values (i.e. higher q values). The lowest of these indirect bounce curves will be the one where from the start $p_c = \rho/3$ so that $zx^{-4} = z_0 x_0^{-4}$ all the way. Whether there are indirect bounces above (higher B's) the range of direct ones depends on whether the upper intersection of equation (51) and Q occurs above or below the point of Q from which the tangent passes through (x_0, z_0). It can easily be worked out that for points below the tangent to Q at 1·2 there will be no such upper (but certainly lower) indirect bounces whereas for the thin region above this tangent and below curve (52) there are both upper and lower indirect bounces. Finally above curve (52) there are no direct bounces, but there will be indirect ones provided that from the relevant (x_2, z_2) of equation (49) we can reach Q by a curve $z \sim x^4$. This will be the case if (x_0, z_0) is below the limiting such curve, namely equation (44′).

Perhaps it may help to specify the zones numerically for $x_0 = 1$, i.e. $\frac{1}{2}v_0^2 = z_0$. Then (see Fig. 7)

(i) $z_0 < 1/10{\cdot}8 = 0{\cdot}09260$ (i.e. below the tangent from $x_1 = 1{\cdot}2$ on Q). With B too large there is no bounce. A smaller B gives a direct bounce, a still smaller one an indirect one and yet smaller ones are not admitted since they would imply $p_c > \rho/3$ initially.

(ii) $1/10{\cdot}8 < z_0 < 0{\cdot}10993$. As (i), except that a range of B values giving indirect bounces is now interposed between those leading to no bounce and those giving direct ones.

(iii) $0{\cdot}10993 < z_0 < 0{\cdot}13396$. A range of B values leads to indirect bounces. Higher ones lead to no bounce, lower ones are inadmissible.

(iv) $0\cdot13396 < z_0$. No bounces are possible at all with the restriction $p_c \leq \rho/3$.

In the situation previously considered where no such restriction on the central pressure was imposed, bounces could occur for sufficiently low B values for all $z_0 < 0\cdot2963$.

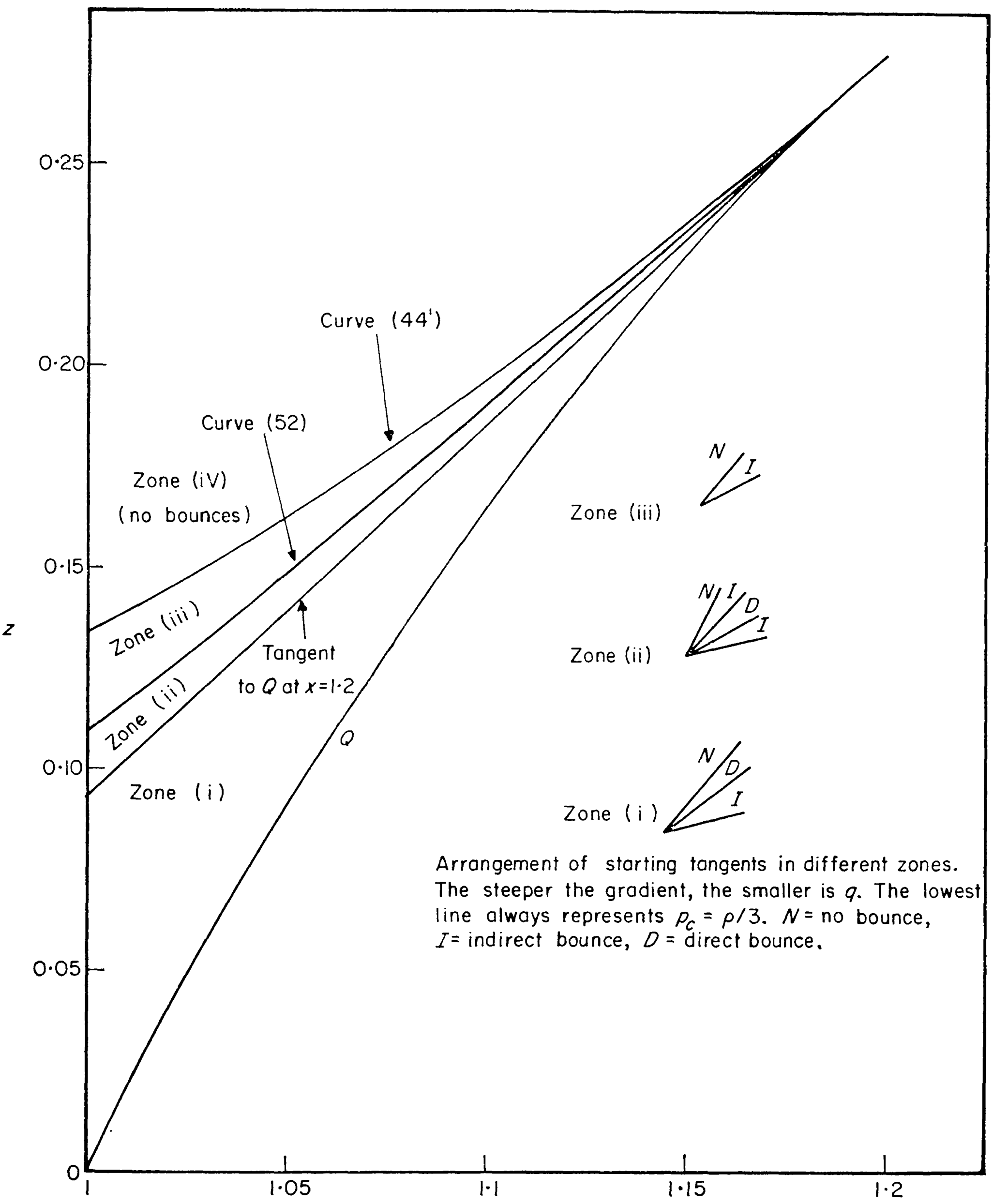

FIG. 7. *Direct and indirect bounces for* $\gamma = 4/3$.

10. *The case* $\gamma = 5/3$. We next consider the case $\gamma = 5/3$. In Newtonian theory this invariably leads to a bounce, but it is well known that this need not be the case in relativity. Equally, equilibrium is always stable in Newtonian theory, but not so in relativity for sufficiently high potentials.

The relation between p_c and ρ integrates to give, by equation (48),

$$z^2 + C = Bx^2 \text{ with } p_c/\rho = z^2/C. \tag{53}$$

To obtain physically meaningful solutions we must therefore have $C > 0$. Hence the representative curves in the (x, z) plane are hyperbolae having the x-axis as their real axis, and thus their curvature is in the same sense as that of curve Q. We therefore see the possibility of stable equilibrium in the shape of a representative curve touching Q from below, so that even after a small perturbation the model will only execute small oscillation about the position of equilibrium. It is readily seen that if curve (53) touches Q at (x_1, z_1) then

$$B = 4\frac{(x_1-1)(2-x_1)}{{x_1}^6}, \qquad C = 4\frac{(x_1-1)(3-2x_1)}{{x_1}^4} \tag{54}$$

and by second differentiation it is immediately established that the curvature of the representative curve exceeds that of Q if and only if

$$x_1 < [15-(33)^{1/2}]/8 = 1\cdot15693 \ldots,$$

i.e.

$$z_1 < [17-(33)^{1/2}]/48 = 0\cdot23449 \ldots . \tag{55}$$

Thus a model of this type is stable if and only if its surface potential does not exceed this limit (see Fig. 8).

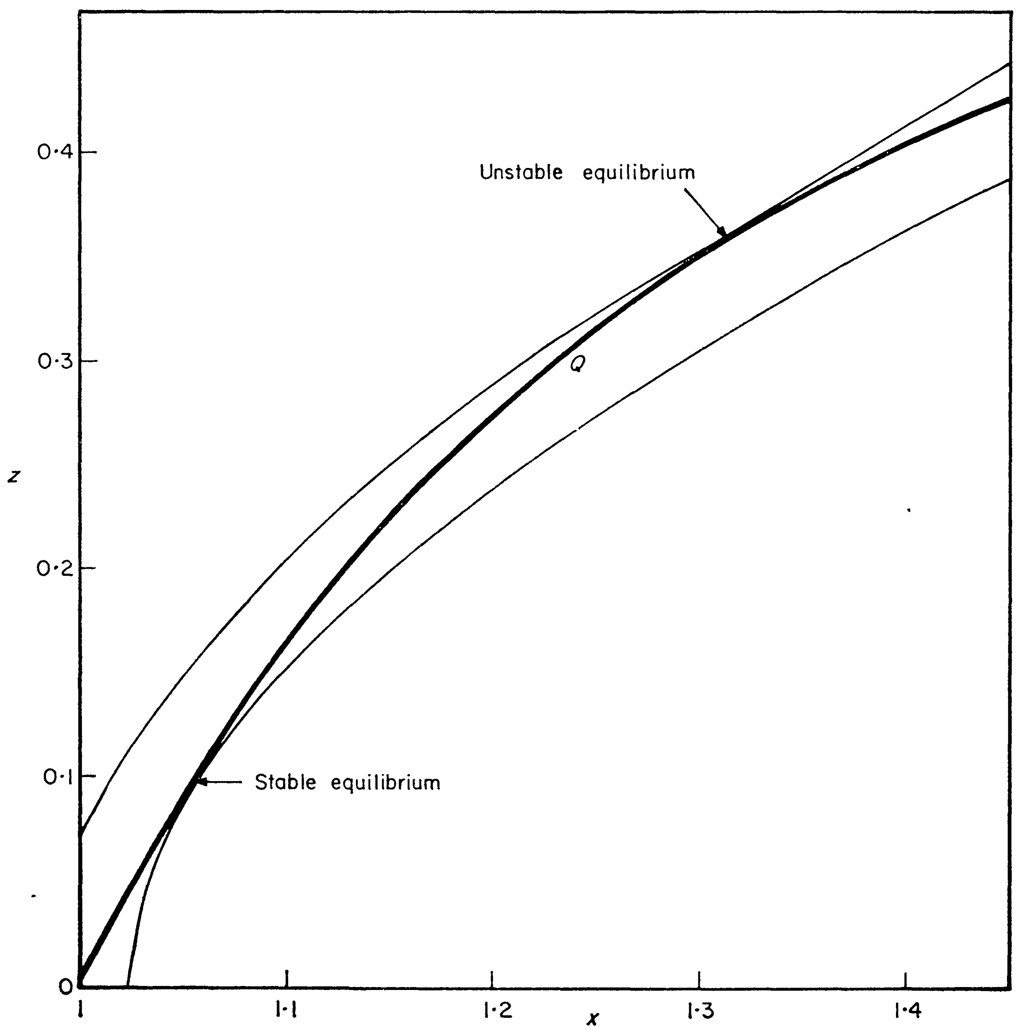

FIG. 8. *Stable and unstable relativistic models with* $\gamma = 5/3$.

Now consider a model initially in a general state of motion. Its mass, radius and surface velocity will define the starting value (x_0, z_0) and through this point there will pass a single-parameter family of hyperbolae (53). We may choose as this parameter the quantity B, which determines the starting gradient and is connected with the initial acceleration parameter q by

$$\frac{1}{2}(q+z_0) = \frac{2-x_0}{Bx_0^4}z_0^2 = \frac{2-x_0}{x_0^2}\frac{z_0^2}{z_0+C}. \tag{56}$$

The resulting calculations are rather tedious and the outcome could be shown on a diagram rather like Fig. 6 displaying the minimum value of q leading to a bounce on a (z_0, y) diagram. The right hand edge is given by $C = 0$ and therefore $p_c = \infty$. On the left hand edge ($z_0 = 0$) $q = 0$. As we move towards the right, at every level q first diminishes through negative values and then rises, becoming again zero on a curve leading from $z_0 = [17-(33)^{1/2}]/48 = 0\cdot 23449 \ldots$ on $y = 0$ to $z_0 = 0\cdot 152 \ldots$ on $y = 1$. Further to the right, q remains zero on $y = 0$, but rises through positive values to the right hand edge, attaining its highest value (46/27) in the top right corner.

Note that equation (56) gives an upper limit of q since by equation (53) C must be positive. On $y = 1$ ($x_0 = 1$) this is $2-z_0$ while on

$$y = 0, \text{ that is } z_0 = 2(x_0-1)/x_0^2$$

this limit is $2(3-2x_0)/x_0^2$, diminishing rather more slowly from 2 at $z_0 = 0$ towards the right. The available interval of q always vanishes on the right hand edge.

In Newtonian theory (12) the only condition for $\gamma = 5/3$ is the general one that the inward acceleration of the surface must not exceed that caused by gravity, that is $q+z_0 \geq 0$. This fits in with our left hand edge and the diminution of q towards the right. Indeed it is readily shown from equations (54) and (56) that the least q for a bounce is for small z_0 given by

$$q = -z_0+nz_0^2+ \ldots,$$

$$n = \frac{2}{B}\left(x_1 = \frac{1}{4}(17^{1/2}+1)\right) = \frac{85\times 17^{1/2}+349}{64} = 10\cdot 929 \ldots .$$

After a bounce a model will expand but its representative hyperbola, after passing the starting point, may have a second intersection with Q, in which case the model oscillates. One such model was found by Bonnor & Faulkner (**2**).

The limiting hyperbola of this type passes through (1, 0), which is on Q, and touches Q for higher z values. The unique such hyperbola has

$$B = C = [85\times 17^{1/2}-349]/8 = 0\cdot 183000 \ldots$$

and touches Q at

$$x_1 = \tfrac{1}{4}[17^{1/2}+1] = 1\cdot 28078 \ldots, \quad z_1 = \tfrac{1}{4}[3\times 17^{1/2}-11] = 0\cdot 34234 \ldots .$$

The region of oscillating models is confined to the segment between this hyperbola and Q. In particular, no model bouncing at $z > 0\cdot 34234 \ldots$ can oscillate.

Next we consider the somewhat more realistic case in which the limit of the law $p_c \sim \rho^{5/3}$ is attained when $p_c = \rho/3$, and for higher densities this last relation is taken to be valid. This will introduce modifications and limitations into the previous work.

First the range of continued validity of the previous conclusions has to be found. Starting from a point (x_0, z_0) consider the ratio $p_c/\rho = z^2/C$ along each hyperbola of the single-parameter family passing through this point. There will be a locus of points where this ratio equals $\frac{1}{3}$, namely

$$\left(\frac{z}{z_0}\right)^2 = \frac{(x/x_0)^2}{4-3(x/x_0)^2}. \tag{57}$$

This locus starts at (x_0, z_0) with gradient $d \log x/d \log x = 4$ and runs to the right with increasing steepness, becoming asymptotic to $x/x_0 = (4/3)^{1/2}$. According to the location of (x_0, z_0), this curve may or may not intersect Q. If it intersects Q, it invariably does so twice. The segment of Q between these two intersections can thus be reached from (x_0, z_0) following the 5/3 power law without the pressure ever attaining a value equal to one third of the density. Again we call this a *direct bounce*. Evidently this is a sub-class of the *general bounces* discussed above, with a larger minimum acceleration, viz. that giving the hyperbola leading to the higher intersection of curve (57) and Q, and a lower maximum acceleration, resulting in the hyperbola leading to the lower intersection of equation (57) and Q. (It will be remembered that in the general case the maximum acceleration corresponded to the straight line $C = 0$ and therefore to infinite central pressure.)

The limiting locus of point (x_0, z_0) from which a direct bounce is just possible is the locus such that equation (57) touches Q. This is given in terms of the co-ordinates (x_3, z_3) of the point of contact by

$$x_0^2 = \frac{3x_3^2}{4}\frac{2-x_3}{3-2x_3}, \qquad z_0^2 = \frac{12(x_3-1)^3}{x_3^4(3-2x_3)}. \tag{58}$$

This locus is the nearly straight curve to be called T reaching Q at $x_3 = 1 \cdot 2$:

x_0	1	1·1	1·2
z_0	0·0868	0·1857	0·2778

Direct bounces can occur only from initial positions below this locus (see Fig. 9).

Consider any starting point between T and Q. The representative point will move along one of the hyperbolae (53). The greater the initial outward acceleration, the greater will be the initial central pressure, the smaller C and B, and the gentler the initial slope of the hyperbola. Consider now curve (57) through our starting point. By hypothesis it will intersect Q in two points, Q_a and Q_b. It is readily seen that always $x_a < 1 \cdot 2$, while x_b may, according to the location of the starting point, exceed 1·2 or not. We shall first assume that $x_b < 1 \cdot 2$.

The gentlest permissible initial slope of hyperbola (53) corresponds to $p_c = \rho/3$. Thus we start on an isothermal solution (43) and are bound to continue along it. With a slightly larger initial slope we will have a hyperbola intersecting equation (57) for $x < x_a$. Thus such a model will contract adiabatically to this point, then isothermally and reverse motion on reaching Q so that we have an indirect bounce. For larger initial slopes, hyperbola (53) will intersect Q between Q_a and Q_b giving a direct bounce, for still larger ones the hyperbola will meet equation (57) between Q_b and the point D where equation (57) intersects the limiting bouncing isothermal model equation (44′), and so we have again an indirect bounce. For yet larger slopes the hyperbola will meet curve (57) above D and so no bounce will occur but the model will contract through the Schwarzschild limit, isothermally if the intersection of the hyperbola and equation (57) occurs below $z = \frac{1}{2}$, otherwise adiabatically.

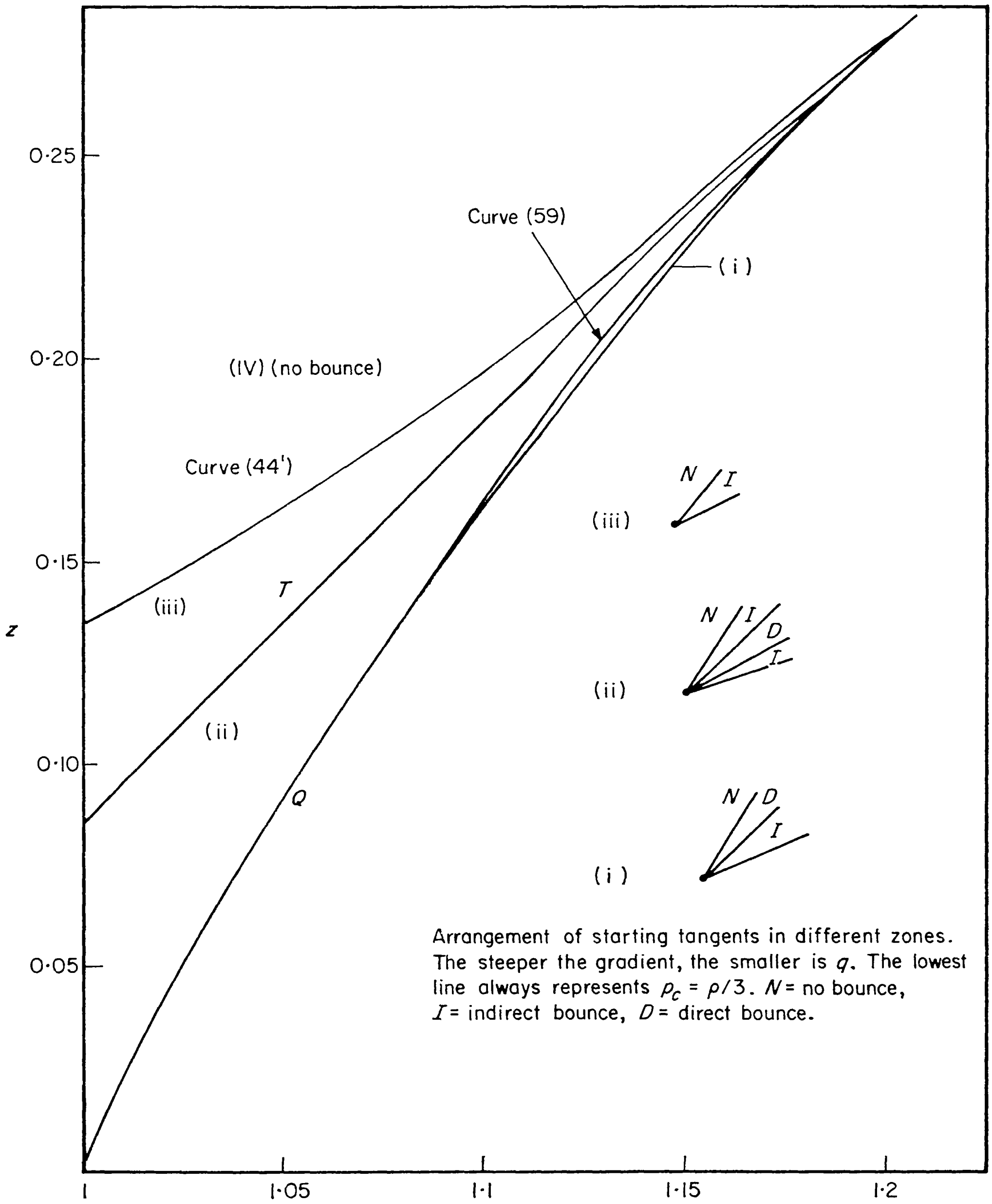

FIG. 9. *Direct and indirect bounces for* $\gamma = 5/3$.

If $Q_b > 1 \cdot 2$, no upper indirect bounce can occur. It is readily seen that $Q_b = 1 \cdot 2$ if, by equation (57)

$$3 \cdot 6\, z_0 = \left[4\left(\frac{x_0}{1 \cdot 2}\right)^2 - 3\right]^{1/2}. \tag{59}$$

The narrow sliver between this curve and Q is the area of initial points of this type. Note that this curve has a lower intersection with Q at $z = 0 \cdot 144 \ldots$.

Between T and equation (44′) indirect bounces will occur with initial accelerations between the maximum ($p_c = \rho/3$) and that leading to the hyperbola meeting equations (64) and (44′) at the same point. For smaller accelerations no bounce occurs.

Finally for initial points above equation (44′) no bounce can occur whatever the initial acceleration.

European Space Research Organisation,
Paris.
1968 *July.*

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Note added in proof. A recent paper by Thompson & Whitrow (1968), *Mon. Not. R. astr. Soc.*, **139,** 499, also uses an equation corresponding to my equation (38) to examine the motion of uniform spheres. The aims and treatments of the papers are very different and their mutual independence is demonstrated by the fact that although Thompson & Whitrow's paper appeared shortly before this one was submitted my note (6) appeared just before their paper was submitted.

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