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Gravitational Field as a Generalized Gauge Field

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It is shown that a symmetric tensor field of the second rank $A_{\mu\nu}(x)$ should be introduced in order to retain the invariance of the action-integral under a generalized translation $x^\mu \to x^\mu + \xi^\mu(x)$, provided that the original action-integral is invariant under inhomogeneous Lorentz transformations. It is further proved that the generalized gauge field $A_{\mu\nu}$ should appear in the Lagrangian in exactly the same fashion as the metric tensor $g_{\mu\nu}$ does in Einstein's theory of gravitation.

Some general feature is also discussed with respect to a law of conservation of some physical quantity which becomes no longer valid when the interaction with the generalized gauge field takes place, provided that the associated group is non-Abelian.

§ 1. Introduction

A gravitational field was first interpreted as a kind of generalized gauge fields by one of the present authors[1] by introducing a system of tetrads $h_a{}^\mu(x)$ and extending the Lorentz transformation of the tetrads at each world point to a larger group depending upon six arbitrary functions of x instead of six parameters. Besides this article, some authors[2),3)] tried to introduce a gravitational field by extending the translation group to a general transformation of coordinates

$$x^\mu \to x^\mu + \xi^\mu(x),$$

but their arguments seem rather unsatisfactory and complicated.

Many groups of transformations depending on parameters have been found in connection with the different kinds of conservation laws. Among these groups it is well known that the group of phase-transformations of complex fields was extended to the gauge transformation depending on an arbitrary scalar function $\lambda(x)$ connected with the existence of an electromagnetic field. The invariance under rotations in the iso-spin space was extended to the invariance under a generalized rotation group by an adjoined introduction of the Yang-Mills field. The most well-known group, namely the translation group, has been conjectured to be related with the gravitational field because the gravitational field is, following Einstein's equation, produced by the energy-momentum tensor of material fields, the conservation of which holds owing to the invariance of the material system under a translation of coordinates. In spite of such a conjecture, however, there has not been any convincing article which shows the gravitational field being derivable from the postulate that the action integral of a matrial system is invariant under a group of generalized translations depending upon four arbitrary functions of x, The aim of the present paper is to show that a tensor field of *the second rank* should be introduced in order to retain the invariance of the action-integral and that this tensor field should appear in the original Lagrangian of the material field in exactly the same way as the metric tensor $g_{\mu\nu}$ does in Einstein's theory of gravitation. This conclusion is derived from the assumption that the original Lagrangian is invariant under inhomogeneous Lorentz transformations but the invariance under rotations of tetrads has not been assumed.

In addition to the derivation of the gravitational field, some general feature is discussed with respect to the laws of conservation. It is shown that a physical quantity owned by some field, say $\phi_A(x)$, which is conserved owing to the invariance of the action-integral of ϕ_A under some parameter-group of transformations, becomes unable to satisfy the law of conservation when the original ϕ-field begins to interact with a generalized gauge field associated with the group mentioned above, provided that this group of transformations is non-Abelian. The conservation is recovered only when the quantity carried by the generalized gauge field is taken into account together with that possessed by the field ϕ_A.

The present procedure of introducing the interaction of a gravitational field with a material system might include its application in a derivation of an S-matrix for a material system interacting with a gravitational field if the Lorentz-invariant S-matrix is known for this matrial system without the gravitational interaction.

§ 2. Fundamental postulate

Consider a field $\phi_A(x)$ $(A=1, 2\cdots N)$ with a Lagrangian density

$$L(\phi_A, \phi_{A,\mu}), \qquad \phi_{A,\mu}=\partial\phi_A/\partial x^\mu.$$

Let us assume that the action-integral

$$I=\int L d^4x$$

is invariant under the following groups of transformations:

i) translation

$$x^\mu \to x^{\mu\prime}=x^\mu+a^\mu, \qquad (a^\mu=\text{constant parameter})$$

$$\delta\phi_A=\phi_A{}'(x')-\phi_A(x)=0\,,$$

ii) Lorentz transformation

$$x^\mu \to x^{\mu\prime}=x^\mu+\delta x^\mu=x^\mu+\epsilon^\mu{}_\nu\cdot x^\nu,$$

$$\delta\phi_A=\phi_B\cdot C^{B\mu}_{A\nu}\cdot\frac{\partial\delta x^\nu}{\partial x^\mu}=\phi_B\cdot C^{B\mu}_{A\nu}\cdot\epsilon^\nu{}_\mu\,,$$

where $\epsilon^\mu{}_\nu$ is an infinitesimal parameter and is restricted by the condition

$$\epsilon_{\mu\nu}=\eta_{\mu\rho}\cdot\epsilon^{\rho}{}_{\nu}=-\epsilon_{\nu\mu}\,,$$

$$\eta_{\mu\nu}=\eta^{\mu\nu}=\begin{cases}1 & \mu=\nu=1,2,3\,,\\ 0 & \mu\neq\nu\,,\\ -1 & \mu=\nu=0\,.\end{cases}$$

Here it has been assumed that the field ϕ_A is a kind of tensor and the transformation-coefficient $C^{B\mu}_{A\nu}$ is an appropriate sum of products of Kronecker's δ.

Our fundamental postulate is that the action integral should be invariant under the generalized translations which is a generalization of (i) and (ii) depending upon four infinitesimal arbitrary functions of x, in place of four parameters a^μ, i.e.

$$x^\mu\rightarrow x^{\mu\prime}=x^\mu+\xi^\mu(x)\,. \qquad (2\cdot1)$$

In order to realize this postulate, the original arguments of the Lagrangian, for example $\partial_\lambda\phi_A$, should be replaced with an appropriately defined "covariant derivative" $\nabla_\lambda\phi_A$ by introducing a new field $A_K(x)$.

Let a Lagrangian

$$L_1(\phi_A,\,\phi_{A,\lambda},\,A_K,\,A_{K,\lambda})$$

be a substitute for the original one. Since the parameter a^μ in (i) behaves as a vector under Lorentz transformations, it is plausible to assume that the new field A_K is a covariant tensor of the r-th rank, following the conclusion of the theory of generalized gauge fields.[4] Thus the field $A_{\mu_1\cdots\mu_r}(x)$ has to be transformed under the transformation (2·1) in the following way:

$$\delta A_{\mu_1\cdots\mu_r}(x)=A_{\nu_1\cdots\nu_r}\cdot D\binom{\nu_1\cdots\nu_r}{\mu_1\cdots\mu_r}{}^{\nu}_{\mu}\cdot\xi^\mu{}_{,\nu}\,, \qquad (2\cdot2)$$

where the transformation coefficient D is

$$D\binom{\nu_1\cdots\nu_r}{\mu_1\cdots\mu_r}{}^{\nu}_{\mu}=-\sum_{i=1}^{r}\delta^{\nu_i}_{\mu}\cdot\delta^{\nu}_{\mu_i}\{\delta^{\nu_1}_{\mu_1}\cdots\delta^{\nu_{i-1}}_{\mu_{i-1}}\cdot\delta^{\nu_{i+1}}_{\mu_{i+1}}\cdots\delta^{\nu_r}_{\mu_r}\}\,. \qquad (2\cdot3)$$

The expression of $\delta\phi_A$ for the Lorentz transformation has the form

$$\delta\phi_A=\phi_B\cdot C^{B\,\nu}_{A\mu}\cdot\xi^\mu{}_{,\nu}$$

for the variation of ϕ_A under the transformation (2·1). This expression of $\delta\phi_A$ gives rise to not only terms being proportional to $\partial\xi/\partial x$ but also terms having $\partial^2\xi/\partial x\cdot\partial x$ in the variation of the action-integral. In order to cancel these terms, it is necessary (a) to let the Lagrangian L_1 depend upon $\partial A_K/\partial x$ in addition to A_K if A_K is a tensor as we have assumed, or (b) to change the definition (2·2) of δA_K in such a way that a term having $\partial^2\xi/\partial x\cdot\partial x$ appears in the definition of δA_K, provided that $\partial A_K/\partial x$ should not appear in L_1. The approach (b) means an abandonment of the tensor character of the A-field. In the present paper, however, we assume the A-field to be a tensor, and consequently L_1 depends on both A_K and $\partial A_K/\partial x$.

The postulate that the new action-integral

$$I_1 = \int L_1 d^4x$$

should be invariant under (2·1) leads to the following various identities (see Appendix A):

$$\partial_\nu \{ [L_1]^A \cdot C^{B\,\nu}_{A\rho} \cdot \phi_B + [L_1]^{a_1 \cdots a_r} \cdot D\binom{b_1 \cdots b_r}{a_1 \cdots a_r}^{\nu}_{\rho} \cdot A_{b_1 \cdots b_r} \} \\ + [L_1]^A \cdot \phi_{A,\rho} + [L_1]^{a_1 \cdots a_r} \cdot A_{a_1 \cdots a_r, \rho} \equiv 0\,, \tag{2·4}$$

$$\partial_\lambda \{ [L_1]^A \cdot C^{B\,\lambda}_{A\rho} \cdot \phi_B + [L_1]^{a_1 \cdots a_r} \cdot D\binom{b_1 \cdots b_r}{a_1 \cdots a_r}^{\lambda}_{\rho} \cdot A_{b_1 \cdots b_r} - T^{\lambda}_{(1)\rho} \} \equiv 0\,, \tag{2·5}$$

$$[L_1]^A \cdot C^{B\mu}_{A\rho} \cdot \phi_B + [L_1]^{a_1 \cdots a_r} \cdot D\binom{b_1 \cdots b_r}{a_1 \cdots a_r}^{\mu}_{\rho} \cdot A_{b_1 \cdots b_r} \\ + \partial_\lambda \left\{ \frac{\partial L_1}{\partial \phi_{A,\lambda}} \cdot C^{B\mu}_{A\rho} \cdot \phi_B + \frac{\partial L_1}{\partial A_{a_1 \cdots a_r, \lambda}} \cdot D\binom{b_1 \cdots b_r}{a_1 \cdots a_r}^{\mu}_{\rho} \cdot A_{b_1 \cdots b_r} \right\} \equiv T^{\mu}_{(1)\rho}\,, \tag{2·6}$$

$$\left\{ \frac{\partial L_1}{\partial \phi_{A,\nu}} \cdot C^{B\,\mu}_{A\rho} \cdot \phi_B + \frac{\partial L_1}{\partial A_{a_1 \cdots a_r, \nu}} \cdot D\binom{b_1 \cdots b_r}{a_1 \cdots a_r}^{\mu}_{\rho} \cdot A_{b_1 \cdots b_r} \right\} \\ + \{\mu \text{ and } \nu \text{ interchanged}\} \equiv 0\,, \tag{2·7}$$

where the following abbreviations have been used:

$$[L_1]^A = \frac{\partial L_1}{\partial \phi_A} - \partial_\lambda \left(\frac{\partial L_1}{\partial \phi_{A,\lambda}} \right),$$

$$[L_1]^{a_1 \cdots a_r} = \frac{\partial L_1}{\partial A_{a_1 \cdots a_r}} - \partial_\lambda \left(\frac{\partial L_1}{\partial A_{a_1 \cdots a_r, \lambda}} \right),$$

$$T^{\mu}_{(1)\rho} = \frac{\partial L_1}{\partial \phi_{A,\mu}} \phi_{A,\rho} + \frac{\partial L_1}{\partial A_{a_1 \cdots a_r, \mu}} \cdot A_{a_1 \cdots a_r, \rho} - \delta^{\mu}_{\rho} L_1\,.$$

§ 3. Determination of the type of gauge field $A_{(a_1 \cdots a_r)}$

The identity (2·7) shows that $\phi_{A,\nu}$ and $A_{a_1 \cdots a_r, \lambda}$ should be included in L_1 only through a particular linear combination $\nabla_\lambda \phi_A$ of the following type:

$$\nabla_\lambda \phi_A = \partial_\lambda \phi_A + \phi_B \cdot M^{Ba_1 \cdots a_r \nu}_{A\lambda}(x) \cdot A_{a_1 \cdots a_r, \nu}\,, \tag{3·1}$$

where the coefficient M is to be determined later and probably depends on x. $\nabla_\lambda \phi_A$ is called in what follows a "covariant derivative" of ϕ_A.

The Lagrangian L_1 can be rewritten in terms of $\nabla_\lambda \phi_A$ as

$$L_1(\phi_A, \phi_{A,\lambda}, A, \partial A/\partial x) = L_2(\phi_A, \nabla_\lambda \phi_A, A)\,. \tag{3·2}$$

Making use of (3·1) and rewriting (2·7) in terms of L_2, we have

$$\frac{\partial L_2}{\partial \nabla_\lambda \phi_A} \cdot \phi_B \cdot [\{ \delta^{\nu}_{\lambda} \cdot C^{B\mu}_{A\rho} + M^{Ba_1 \cdots a_r \nu}_{A\lambda} \cdot D\binom{b_1 \cdots b_r}{a_1 \cdots a_r}^{\mu}_{\rho} \cdot A_{b_1 \cdots b_r} \} \\ + \{\nu \text{ and } \mu \text{ interchanged}\}] \equiv 0\,.$$

This expression suggests that the coefficient M should be a linear combination of $C^{B\mu}_{A\rho}$:

$$M^{Ba_1\cdots a_r\nu}_{A\lambda}(x) = C^{B\beta}_{A\alpha}\cdot Y^{a_1\cdots a_r\nu\alpha}_{\lambda\beta}(x). \tag{3·3}$$

By substituting (3·3) for M's in the above identity, (2·7) gives

$$-2\delta^\alpha_\rho\cdot\delta^{(\nu\mu)}_{(\lambda\beta)} \equiv -\delta^\alpha_\rho(\delta^\nu_\lambda\delta^\mu_\beta+\delta^\nu_\beta\delta^\mu_\lambda)$$
$$\equiv A_{b_1\cdots b_r}(x)\cdot[\{Y^{a_1\cdots a_r\nu\alpha}_{\lambda\beta}\cdot D(^{b_1\cdots b_r}_{a_1\cdots a_r})^\mu_\rho\} + \{\mu \text{ and } \nu \text{ interchanged}\}]. \tag{3·4}$$

Since the left-hand side of (3·4) is symmetric with respect to the subscripts λ and β, the same should also be true for the right-hand side. Thus we find*)

$$Y^{a_1\cdots a_r\nu\alpha}_{\lambda\beta} = Y^{a_1\cdots a_r\nu\alpha}_{\beta\lambda} = Y^{a_1\cdots a_r\nu\alpha}_{(\beta\lambda)}.$$

In what follows, for the sake of simplicity, let the discussion be restricted to the case that the field A is an irreducible tensor of the r-th rank with a full symmetry:

$$A_{(a_1\cdots a_r)}(x).$$

The double contraction of (3·4) by putting $\nu=\lambda$ and $\mu=\beta$ and making use of the definition (2·3) gives a relation

$$20\delta^\alpha_\rho = 2r\cdot Y^{(a_1\cdots a_{r-1}\cdot\mu)\nu\alpha}_{(\nu\mu)}(x)\cdot A_{(a_1\cdots a_{r-1}\rho)}(x).$$

If we define $A^{(a_1\cdots a_r)}(x)$ by

$$A^{(a_1\cdots a_r)}(x) \propto Y^{(a_1\cdots a_{r-1}\mu)\nu a_r}_{(\mu\nu)}, \tag{3·5}$$

the above relation is written as

$$A^{(a_1\cdots a_{r-1}\alpha)}\cdot A_{(a_1\cdots a_{r-1}\rho)} \propto \delta^\alpha_\rho. \tag{3·6}$$

This result shows that r should be >1 otherwise (3·6) leads to a contradiction.

The relation (3·5) allows us to represent Y in terms of $A^{(a_1\cdots)}$ as

$$Y^{(a_1\cdots a_r)\nu\alpha}_{(\lambda\beta)} = A^{(b_1\cdots b_r)}(x)\cdot Z^{(a_1\cdots a_r)\nu\alpha}_{(b_1\cdots b_r)(\lambda\beta)}, \tag{3·7}$$

where the coefficient Z on the right-hand side is an appropriate sum of products of Kronecker's δ.

Substituting (3·7) for Y in (3·4) and making use of the definition (2·3) of D, we have an important relation

$$2\delta^\alpha_\rho\delta^{(\mu\nu)}_{(\lambda\beta)} \equiv r\cdot A^{(b_1\cdots b_r)}\cdot A_{(a_1\cdots a_{r-1}\rho)}\{Z^{(a_1\cdots a_{r-1}\mu)\nu\alpha}_{(b_1\cdots b_r)(\lambda\beta)} + Z^{(a_1\cdots a_{r-1}\nu)\mu\alpha}_{(b_1\cdots b_r)(\lambda\beta)}\}. \tag{3·8}$$

From our assumption that $A_{(a_1\cdots a_r)}$ and $A^{(b_1\cdots b_r)}$ are both fully symmetric with respect to their suffices, it is plausible to assume that the undetermined coefficient Z is also symmetric with respect to both superscripts $(a_1\cdots a_r)$ and subscripts $(b_1\cdots b_r)$, in addition to the symmetric pair of subscripts $(\lambda\beta)$. Since we have no information about the symmetry of Z with respect to the extra superscripts

*) $(\beta\lambda)$ means that Y is symmetric with respect to suffices inside the parentheses.

ν and α, Z having the above mentioned symmetry with respect to the suffices can be expressed in terms of the products of Kronecker's δ in the following way:

$$Z^{(a_1\cdots a_r)\nu\alpha}_{(b_1\cdots b_r)(\lambda\beta)}=a\cdot\delta^{(a_1\cdots a_r)}_{(b_1\cdots b_r)}\delta^{(\nu\alpha)}_{(\lambda\beta)}+b\sum_{i=1}^{r}[\delta^{(a_1\cdots a_r)}_{(b_1\cdots\lambda\cdots b_r)}\cdot\delta^{\nu}_{b_i}\delta^{\alpha}_{\beta}+\delta^{(a_1\cdots a_r)}_{(b_1\cdots\beta\cdots b_r)}\delta^{\nu}_{b_i}\delta^{\alpha}_{\lambda}]$$

$$+c\sum_{i=1}^{r}[\delta^{(a_1\cdots a_r)}_{(b_1\cdots\lambda\cdots b_r)}\cdot\delta^{\alpha}_{b_i}\delta^{\nu}_{\beta}+\delta^{(a_1\cdots a_r)}_{(b_1\cdots\beta\cdots b_r)}\delta^{\alpha}_{b_i}\delta^{\nu}_{\mu}]+d\sum_{i,j(i\neq j)}^{r}\delta^{(a_1\cdots a_r)}_{(b_1\cdots\lambda\cdots\beta\cdots b_r)}\delta^{(\nu\alpha)}_{(b_ib_j)}\,, \tag{3·9}$$

where a, b, c and d are undetermined constants. $\delta^{(\cdots)}_{(\cdots)}$ in (3·9) means

$$\delta^{(a_1\cdots a_r)}_{(b_1\cdots b_r)}=\frac{1}{r!}\sum_{(\mathrm{perm})}\delta^{a_1}_{b_i}\delta^{a_2}_{b_j}\cdots\delta^{a_r}_{b_k}\,,$$

where the summation should be taken over all the permutations of $(b_1\cdots b_r)$.

Inserting the expression (3·9) into (3·8), and taking contractions of (3·8) with respect to many different pairs of suffices, we arrive at the result

$$r=2\,,\qquad a=b=0\,,\qquad c=-d=\tfrac{1}{4} \tag{3·10}$$

with the normalization

$$A^{(\mu\rho)}(x)\cdot A_{(\rho\nu)}(x)=\delta^{\mu}_{\nu}. \tag{3·11}$$

The details of the derivation of this result are given in Appendix B. (3·10) determines Z, Y and $\nabla_{\lambda}\phi_A$ as follows:

$$Z^{(a_1a_2)\nu\alpha}_{(b_1b_2)(\lambda\beta)}=\tfrac{1}{4}[\delta^{(a_1a_2)}_{(b_1\lambda)}\delta^{\alpha}_{b_2}\delta^{\nu}_{\beta}+\delta^{(a_1a_2)}_{(b_2\lambda)}\delta^{\alpha}_{b_1}\delta^{\nu}_{\beta}$$

$$+\delta^{(a_1a_2)}_{(b_1\beta)}\delta^{\alpha}_{b_2}\delta^{\nu}_{\lambda}+\delta^{(a_1a_2)}_{(b_2\beta)}\delta^{\alpha}_{b_1}\delta^{\nu}_{\lambda}-2\delta^{(a_1a_2)}_{(\lambda\beta)}\delta^{(\nu\alpha)}_{(b_1b_2)}], \tag{3·12}$$

$$Y^{(a_1a_2)\nu\alpha}_{(\lambda\beta)}(x)\cdot A_{(a_1a_2),\nu}=\tfrac{1}{2}A^{\alpha a}\{A_{a\lambda,\beta}+A_{\beta a,\lambda}-A_{\lambda\beta,a}\}=\varDelta^{\alpha}_{\beta\lambda}(x), \tag{3·13}$$

$$\nabla_{\lambda}\phi_A=\partial_{\lambda}\phi_A+\phi_B\cdot C^{B\beta}_{A\alpha}\cdot\varDelta^{\alpha}_{\beta\lambda}\,. \tag{3·14}$$

$\varDelta^{\alpha}_{\beta\lambda}$ in (3·13) is nothing but the Christoffel's $\varGamma^{\alpha}_{\beta\lambda}$ provided that our $A_{\mu\nu}$ is identified with the metric tensor $g_{\mu\nu}$.

§ 4. Derivation of Lagrangian

Before beginning a discussion about (2·6), let us consider a little extension of the Lorentz-invariance of the original Lagrangian L.

The Lorentz-invariance of the original Lagrangian can be made manifest by writing explicitly the metric tensor $\eta_{\mu\nu}$. It is easily seen, however, that this manifestly invariant expression of the action-integral can also be invariant under an affine transformation*)

$$x^{\mu}\to x^{\mu\prime}=a^{\mu}{}_{\nu}\cdot x^{\nu}+a^{\mu},\qquad \det(a^{\mu}{}_{\nu})\neq 0\,, \tag{4·1}$$

*) In the case of an affine transformation, since the coefficient $a^{\mu}{}_{\nu}$ has no such restriction as $\eta_{\mu\nu}a^{\mu}{}_{\alpha}\cdot a^{\nu}{}_{\beta}=\eta_{\alpha\beta}$, there cannot exist such a covariant affine tensor as $\eta_{\mu\nu}$ whose components are kept unchanged under the transformation (4·1).

if the metric tensor $\eta_{\mu\nu}$ in L is replaced with a constant covariant affine tensor $\zeta_{\mu\nu}$ and at the same time, if L is replaced with

$$\boldsymbol{L}(\phi_A, \phi_{A,\lambda}, \zeta_{\mu\nu}) = \sqrt{-\zeta}\cdot L(\phi_A, \phi_{A,\lambda}, \zeta_{\mu\nu}),$$
$$\zeta = \det(\zeta_{\mu\nu}). \tag{4·2}$$

The invariance of $I=\int \boldsymbol{L} d^4x$ under an infinitesimal affine transformation

$$x^\mu \to x^{\mu\prime} = x^\mu + \Delta^\mu{}_\nu \cdot x^\nu + a^\mu$$

gives the following identities:

$$[\boldsymbol{L}]^A \cdot \phi_{A,\rho} = -\partial_\lambda \boldsymbol{T}^\lambda{}_\rho \tag{4·3}$$

and

$$[\boldsymbol{L}]^A \cdot \phi_B \cdot C^{B\mu}_{A\nu} - [\boldsymbol{L}]^A \cdot \phi_{A,\nu} \cdot x^\mu$$
$$-2\frac{\partial \boldsymbol{L}}{\partial \zeta_{\mu\rho}} \cdot \zeta_{\nu\rho} + \partial_\lambda \left[\frac{\partial \boldsymbol{L}}{\partial \phi_{A,\lambda}} \cdot \phi_B \cdot C^{B\mu}_{A\nu} - \boldsymbol{T}^\lambda{}_\nu \cdot x^\mu \right] \equiv 0\,, \tag{4·4}$$

where

$$\boldsymbol{T}^\lambda{}_\nu = \frac{\partial \boldsymbol{L}}{\partial \phi_{A,\lambda}} \cdot \phi_{A,\nu} - \delta^\lambda_\nu \cdot \boldsymbol{L}\,.$$

Making use of (4·3), we can rewrite (4·4) as

$$\frac{\partial \boldsymbol{L}}{\partial \phi_A} \cdot C^{B\mu}_{A\nu} \cdot \phi_B + \frac{\partial \boldsymbol{L}}{\partial \phi_{A,\lambda}} \phi_{B,\lambda} \cdot C^{B\mu}_{A\nu} - 2\frac{\partial \boldsymbol{L}}{\partial \zeta_{\mu\rho}} \cdot \zeta_{\nu\rho} \equiv \boldsymbol{T}^\mu{}_\nu\,. \tag{4·5}$$

Now let us return to a discussion about (2·6) which is rewritten in terms of L_2 defined by (3·2) as follows:

$$\frac{\partial L_2}{\partial \phi_A} \cdot \phi_B \cdot C^{B\mu}_{A\rho} - 2\frac{\partial L_2}{\partial A_{\mu\nu}} A_{\rho\nu} + \frac{\partial L_2}{\partial \nabla_\lambda \phi_A} \cdot C^{B\mu}_{A\rho} \cdot \nabla_\lambda \phi_B$$
$$-\left(\frac{\partial L_2}{\partial \nabla_\mu \phi_A} \cdot \nabla_\rho \phi_A - \delta^\mu_\rho L_2 \right) \equiv \frac{\partial L_2}{\partial (\nabla_\lambda \phi_A)} \cdot \{\cdots\}, \tag{4·6}$$

where the following relations have been employed:

$$\frac{\partial L_1}{\partial \phi_A} = \frac{\partial L_2}{\partial \phi_A} + \frac{\partial L_2}{\partial \nabla_\lambda \phi_A} \cdot C^{A\alpha}_{B\beta} \cdot \Delta^\beta_{\alpha\lambda}\,,$$

$$\frac{\partial L_1}{\partial A_{ab}} = \frac{\partial L_2}{\partial A_{ab}} - \frac{\partial L_2}{\partial \nabla_\lambda \phi_A} \cdot \phi_B \cdot C^{B\alpha}_{A\beta} \cdot A^{ma} A^{nb} \cdot Z^{(rs)\nu\beta}_{(mn)(\lambda\alpha)} \cdot A_{rs,\nu}\,.$$

The expression $\{\cdots\}$ on the right-hand side of (4·6) vanishes owing to the commutation relation

$$[\boldsymbol{C}^\alpha_\beta, \boldsymbol{C}^\mu_\rho] = \delta^\mu_\beta \cdot \boldsymbol{C}^\alpha_\rho - \delta^\alpha_\rho \cdot \boldsymbol{C}^\mu_\beta, \tag{4·7}$$

where $\boldsymbol{C}^\alpha_\beta$ is a $N\times N$ matrix whose (A, B) element is $(A|\boldsymbol{C}^\alpha_\beta|B) \equiv C^{A\alpha}_{B\beta}$. The commutation relation (4·7) holds owing to the fact that the matrix $\boldsymbol{C}^\alpha_\beta$ is a representation of the N-th order of generators of the group of affine transformations and, in fact, we can easily derive the relation (4·7) by considering a simple example.

Comparing (4·5) with (4·6) and remembering that the right-hand side of (4·6) identically vanishes, we are led to the conclusion that L_2 should have the same functional form as $\boldsymbol{L}$, namely

$$L_2 \equiv \sqrt{|A|} \cdot L(\phi_A, \nabla_\lambda \phi_A, A_{\mu\nu})$$
$$= \boldsymbol{L}(\phi_A, \nabla_\lambda \phi_A, A_{\mu\nu}),$$

where

$$A = \det(A_{\mu\nu}).$$

The discussion so far developed neither compelles us to interpret $A_{\mu\nu}$ as the gravitational field nor gives any information on the signature of $A_{\mu\nu}$, but in order to let the field equation of ϕ_A be hyperbolic, $A_{\mu\nu}$ should have the signature $-$, $+$, $+$, $+$. In place of our $A_{\mu\nu}$, one can consider a particular tensor field $B_{\mu\nu}$ which can be derived by introducing a system of curvilinear coordinates u^μ into the Minkowskian space, that is,

$$ds^2 = \eta_{\alpha\beta} \cdot dx^\alpha \cdot dx^\beta = B_{\mu\nu}(u) \cdot du^\mu \cdot du^\nu,$$
$$B_{\mu\nu}(u) = \frac{\partial x^\alpha}{\partial u^\mu} \cdot \frac{\partial x^\beta}{\partial u^\nu} \cdot \eta_{\alpha\beta}.$$

The question whether our $A_{\mu\nu}$ is identical with the fictitious gravitational field $B_{\mu\nu}$ or is an entity being completely different from $B_{\mu\nu}$, giving a non-vanishing "curvature tensor",*) is to be answered by the field equation of $A_{\mu\nu}$. Thus if Einstein's equation is taken as a field equation, our $A_{\mu\nu}$ describes a permanent gravitational field produced by the material field ϕ_A.

§ 5. Law of conservation derived from identities (2·4) and (2·5)

The equation of the field $\phi_A(x)$

$$[L_1]^A = 0 \tag{5·1}$$

gives rise to an interesting relation when it is inserted into the identities (2·4) and (2·5).

By recalling the definition (2·3), the identity (2·5) becomes

$$\frac{\partial T^\lambda_{(1)\rho}}{\partial x^\lambda} = \frac{\partial}{\partial x^\lambda} \boldsymbol{S}^\lambda{}_\rho, \tag{5·2}$$

while (2·4) reads

*) The term "curvature tensor" means the Riemann-Christoffel's curvature tensor when $A_{\mu\nu}$ is substituted for the metric tensor $g_{\mu\nu}$ in the ordinary definition of the curvature tensor.

$$\frac{\partial \boldsymbol{S}^{\nu}{}_{\rho}}{\partial x^{\nu}}=\frac{1}{2}\boldsymbol{S}^{ab}\cdot A_{ab,\rho}\,, \tag{5·3}$$

where the following notations have been employed:

$$\boldsymbol{S}^{\mu\nu}=-2[L_1]^{\mu\nu}=-2\left\{\frac{\partial L_1}{\partial A_{\mu\nu}}-\partial_\lambda\left(\frac{\partial L_1}{\partial A_{\mu\nu,\lambda}}\right)\right\},$$

$$\boldsymbol{S}^{\mu}{}_{\nu}=\boldsymbol{S}^{\mu\rho}\cdot A_{\rho\nu}\,. \tag{5·4}$$

The relationship between $\boldsymbol{S}^{\mu}{}_{\nu}$ and $T^{\mu}_{(1)\nu}$ is given by (2·6) with the aid of (5·1):

$$\boldsymbol{S}^{\mu}{}_{\rho}=T^{\mu}_{(1)\rho}-\partial_\lambda F^{[\lambda\mu]}_{\rho}, \tag{5·5}$$

where

$$F^{[\lambda\mu]}_{\rho}\equiv-F^{[\mu\lambda]}_{\rho}=\frac{\partial L_1}{\partial\phi_{A,\lambda}}\cdot C^{B\mu}_{A\rho}\cdot\phi_B-2\frac{\partial L_1}{\partial A_{\mu a,\lambda}}A_{\rho a}\,.$$

The antisymmetry of F with respect to the superscripts $[\lambda\mu]$ is due to the identity (2·7).

The relations (5·5) and (5·2) and the fact that $\boldsymbol{S}^{\mu\nu}$ is a symmetric tensor density, as easily seen from the definition (5·4), show that $\boldsymbol{S}^{\mu\nu}$ should be regard ed as a energy-momentum tensor density of the field ϕ_A interacting with the field $A_{\mu\nu}$.

Taking into account the transformation property of $\boldsymbol{S}^{\mu\nu}$ under (2·1), we can rewrite (5·3) in a covariant form

$$\nabla_\mu S^{\nu}{}_{\rho}=0\,, \tag{5·3$'$}$$

where

$$S^{\nu}{}_{\rho}=\boldsymbol{S}^{\nu}{}_{\rho}/\sqrt{|A|}\,,$$

and the covariant derivative of $S^{\nu}{}_{\rho}$ is

$$\nabla_\lambda S^{\nu}{}_{\rho}=\partial_\lambda S^{\nu}{}_{\rho}+\Delta^{\nu}_{\lambda\sigma}\cdot S^{\sigma}{}_{\rho}-\Delta^{\sigma}_{\lambda\rho}\cdot S^{\nu}{}_{\sigma}\,.$$

The existence of the non-vanishing right-hand side of (5·3) means that the energy-momentum of ϕ_A is not conserved owing to the interaction of ϕ_A with A. It is well known that (5·3) can be transformed into the expression

$$\partial_\nu\{\boldsymbol{S}^{\nu}{}_{\rho}+\boldsymbol{t}^{\nu}{}_{\rho}\}=0 \tag{5·6}$$

by the aid of the field equation of A, where $\boldsymbol{t}^{\nu}{}_{\rho}$ represents a pseudo energy-momentum tensor density of the field A.

This result that the energy-momentum of ϕ_A can no longer be conserved when the interaction of ϕ_A with A takes place, is a consequence of the general theory of the non-Abelian gauge fields on which a brief explanation will be given in the next section.

§ 6. General remarks on the law of conservation*)

Consider a field $\phi_A(x)$, $(A=1, 2, \cdots N)$, the field equation of which is derived from the action-integral

$$I=\int L_{(\phi)}(\phi_A, \phi_{A,\mu})\, d^4x\,.$$

Let us assume that I is invariant under a group of transformations depending on parameters ϵ_a, $(a=1, 2\cdots r)$:**)

$$\delta\phi_A(x)=\phi_A{}'(x)-\phi_A(x)=\phi_B\cdot \overset{a}{C}{}^B_A\cdot\epsilon_a\,.$$

For simplicity, the transformation of coordinates is excluded from our discussion.

The invariance of I leads to a set of identities

$$[L_{(\phi)}]^A\cdot\phi_B\cdot\overset{a}{C}{}^B_A+\partial_\lambda\left(\frac{\partial L_{(\phi)}}{\partial\phi_{A,\lambda}}\cdot\phi_B\cdot\overset{a}{C}{}^B_A\right)\equiv 0\,. \tag{6·1}$$

The postulate that I should be invariant even under an extended group which depends upon arbitrary function $\lambda_a(x)$'s instead of ϵ_a's, necessitates an introduction of a generalized gauge field $A_{a\mu}(x)$ with a transformation property

$$\delta A_{a\mu}=A_{b\mu}\cdot M^b{}_a{}^c\cdot\lambda_c(x)+\lambda_{a,\mu}\,. \tag{6·2}$$

This postulate of invariance gives rise to the following identities, if one follows a similar line of argument to that given in Appendix A:

$$[L_1]^A\cdot\overset{a}{C}{}^B_A\cdot\phi_B+\frac{\partial L_1}{\partial A_{b\mu}}\cdot M^c{}_b{}^a\cdot A_{c\mu}-\partial_\mu\left(\frac{\partial L_1}{\partial A_{a\mu}}\right)\equiv 0\,, \tag{6·3}$$

$$\partial_\lambda\left[\frac{\partial L_1}{\partial\phi_{A,\lambda}}\cdot\overset{a}{C}{}^B_A\cdot\phi_B+\frac{\partial L_1}{\partial A_{a\lambda}}\right]\equiv 0\,, \tag{6·4}$$

$$\frac{\partial L_1}{\partial\phi_{A,\lambda}}\cdot\overset{a}{C}{}^B_A\cdot\phi_B+\frac{\partial L_1}{\partial A_{a\lambda}}\equiv 0\,, \tag{6·5}$$

where the new Lagrangian L_1 is

$$L_1(\phi_A, \phi_{A,\lambda}, A_{a\mu})\,.$$

The identity (6·5) implies that L_1 should depend upon $\phi_{A,\lambda}$ and $A_{a\lambda}$ only through an "invariant derivative" $\nabla_\lambda\phi_A$ defined by

$$\nabla_\lambda\phi_A=\partial_\lambda\phi_A-A_{a\lambda}\cdot\overset{a}{C}{}^B_A\cdot\phi_B\,. \tag{6·6}$$

Thus L_1 can be written as

$$L_1(\phi_A, \phi_{A,\lambda}, A_{a\lambda})=L_2(\phi_A, \nabla_\lambda\phi_A)\,. \tag{6·7}$$

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*) The first half of the content of this section is a review of the paper I.I.I., but our definition of $j^{(a)}$ is different from that given by (1.27) on page 1601 of I.I.I.

**) For brevity, coordinate transformations are not considered in this section.

The identity (6·3) is transformed into the following expression with the aid of (6·6) and (6·7):

$$\frac{\partial L_2}{\partial \phi_A}\cdot \overset{a}{C}{}^B_A\cdot \phi_B + \frac{\partial L_2}{\partial \nabla_\lambda \phi_A}\cdot \overset{a}{C}{}^B_A\cdot \nabla_\lambda \phi_B \equiv 0\,, \tag{6·3}'$$

where Lie's commutation relations*)

$$[\overset{a}{\boldsymbol{C}}, \overset{b}{\boldsymbol{C}}] = f^a{}_c{}^b\cdot \overset{c}{\boldsymbol{C}}$$

have been used together with the reasonable assumption

$$M^a{}_c{}^b = f^a{}_c{}^b.$$

A comparison of (6·3)′ with (6·1) suggests that L_2 should be chosen to have the same functional form as $L_{(\phi)}$, namely,

$$L_1(\phi_A, \phi_{A,\lambda}, A_{a\lambda}) = L_{(\phi)}(\phi_A, \nabla_\lambda \phi_A)\,.$$

Let us define the "(a)-current" $j^{(a)\lambda}_{(\phi)}$ by

$$j^{(a)\lambda}_{(\phi)} = \frac{\partial L_{(\phi)}}{\partial \nabla_\lambda \phi_A}\cdot \overset{a}{C}{}^B_A\cdot \phi_B \equiv -\frac{\partial L_1}{\partial A_{a\lambda}}\,. \tag{6·8}$$

It is easily seen from the definition (6·8) that $j_{(\phi)}$ has a transformation property

$$\delta j^{(a)\lambda}_{(\phi)} = j^{(b)\lambda}_{(\phi)}\cdot f^c{}_b{}^a\cdot \lambda_c(x)\,,$$

which leads to the definition of the "invariant derivative" of $j_{(\phi)}$:

$$\nabla_\lambda j^{(a)\mu}_{(\phi)} = \partial_\lambda j^{(a)\mu}_{(\phi)} - A_{c\lambda}\cdot f^c{}_b{}^a\cdot j^{(b)\mu}_{(\phi)}. \tag{6·9}$$

Making use of (6·9) and assuming the field equation of ϕ_A

$$[L_1]^A = 0\,,$$

we can derive the equations of continuity for $j_{(\phi)}$ from the identity (6·3):

$$\nabla_\mu j^{(a)\mu}_{(\phi)} = 0\,. \tag{6·10}$$

(6·10) shows that the "(a)-charge" of the ϕ_A-field defined by

$$Q^{(a)} = \int_{-\infty}^{\infty} j^{(a)0}_{(\phi)} d^3x$$

is no longer conserved except for the Abelian case $f^a{}_c{}^b \equiv 0$.

Let us assume that a Lagrangian density $L_{(A)}(A_{a\lambda}, A_{a\lambda,\mu})$ is chosen in such a way as that the action-integral $I_A = \int L_{(A)} d^4x$ is invariant under the transformation (6·2). Then in a completely similar way we can derive many identities, among which the following identity corresponds to (6·4):

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*) The (B, A) element of the $N\times N$ matrix $\overset{a}{\boldsymbol{C}}$ is defined by

$$(B|\overset{a}{\boldsymbol{C}}|A) = \overset{a}{C}{}^B_A.$$

$$\partial_\lambda\left[\frac{\partial L_{(\phi)}}{\partial \nabla_\lambda \phi_A}\cdot \overset{a}{C}{}^B_A\cdot\phi_B+\frac{\partial L_{(A)}}{\partial A_{b\mu,\lambda}}\cdot f^c{}_b{}^a\cdot A_{c\mu}+[L_{\rm tot}]^{a\lambda}\right]\equiv 0\,, \tag{6·11}$$

where

$$L_{\rm tot}=L_{(\phi)}(\phi_A,\nabla_\lambda\phi_A)+L_{(A)}(A_{a\mu},A_{a\mu,\lambda}).$$

If the field equation of $A_{a\mu}$

$$[L_{\rm tot}]^{a\mu}=\frac{\partial L_{(A)}}{\partial A_{a\mu}}-\partial_\lambda\left(\frac{\partial L_{(A)}}{\partial A_{a\mu,\lambda}}\right)-j^{(a)\mu}_{(\phi)}=0 \tag{6·12}$$

is employed, (6·11) reads

$$\partial_\lambda[j^{(a)\lambda}_{(\phi)}+j^{(a)\lambda}_{(A)}]=0\,, \tag{6·13}$$

where the second term defined by

$$j^{(a)\lambda}_{(A)}=\frac{\partial L_{(A)}}{\partial A_{b\mu,\lambda}}\cdot f^c{}_b{}^a\cdot A_{c\mu} \tag{6·14}$$

is interpreted as the "(a)-charge" carried by the A-field. The relation (5·6) in § 5 is a particular example of (6·13). In the former case, the energy and momentum stand for the "(a)-charge" of the present section.

The field equation (6·12) shows that the A-field emerges from a current density of the "(a)-charge". This interpretation (6·12) together with the fact that the A-field possesses the "(a)-charge" as is shown by (6·13), leads to the conclusion that the A-field can be produced by itself. This is the reason why a generalized gauge field associated with a non-Abelian group should obey a set of non-linear field equations.

Appendix

A. *Derivation of identities (2·4)～(2·7)*

Consider a variation of

$$I_1=\int_\Omega L_1(\phi_A,\phi_{A,\lambda},A_{(a_1\cdots)},A_{(a_1\cdots),\lambda})\,d^4x$$

due to the variations of ϕ_A and $A_{(a_1\cdots)}$

$$\begin{aligned}\phi_A'(x')-\phi_A(x)&=\delta\phi_A=\phi_B\cdot C^{B\nu}_{A\mu}\cdot\xi^\mu{}_{,\nu}\,,\\ A'_{(a_1\cdots)}(x')-A_{(a_1\cdots)}(x)&=\delta A_{(a_1\cdots)}=A_{(b_1\cdots)}\cdot D\left(\begin{smallmatrix}b_1\cdots\\ a_1\cdots\end{smallmatrix}\right)^\nu_\mu\cdot\xi^\mu{}_{,\nu}\end{aligned} \tag{A·1}$$

which are caused by a transformation of coordinates

$$x^\mu\to x^{\mu\prime}=x^\mu+\delta x^\mu=x^\mu+\xi^\mu(x).$$

δI_1 of the first order with respect to ξ^μ is given by

$$\delta I_1=\int_\Omega L_1\{\phi_A+\delta\phi_A,\phi_{A,\lambda}+\delta\phi_{A,\lambda},\cdots\}\frac{\partial(x^{0\prime},\cdots x^{3\prime})}{\partial(x^0,\cdots x^3)}\cdot d^4x-I_1$$

$$= \int_\Omega \Big\{ \frac{\partial L_1}{\partial \phi_A}\delta\phi_A + \frac{\partial L_1}{\partial \phi_{A,\lambda}}\delta\phi_{A,\lambda} + \frac{\partial L_1}{\partial A_{(a_1\cdots)}}\cdot\delta A_{(a_1\cdots)} + \frac{\partial L_1}{\partial A_{(a_1\cdots),\lambda}}\delta A_{(a_1\cdots),\lambda} + \lambda_1\cdot\xi^\mu_{,\mu}\Big\} d^4x\,.$$

Let us introduce another kind of variation defined by

$$\bar\delta\phi_A(x) \equiv \phi_A'(x) - \phi_A(x) \equiv \delta\phi_A(x) - \phi_{A,\lambda}\cdot\delta x^\lambda, \tag{A·2}$$

which has a convenient property

$$\frac{\partial}{\partial x^\lambda}\bar\delta\phi_A = \bar\delta\Big(\frac{\partial\phi_A}{\partial x^\lambda}\Big).$$

In terms of this variation, δI_1 is written as

$$\delta I_1 = \int_\Omega \Big[[L_1]^A\cdot\bar\delta\phi_A + [L_1]^{(a_1\cdots a_r)}\cdot\bar\delta A_{(a_1\cdots a_r)} + \partial_\lambda\Big\{\frac{\partial L_1}{\partial \phi_{A,\lambda}}\delta\phi_A + \frac{\partial L_1}{\delta A_{(a_1\cdots),\lambda}}\delta A_{(a_1\cdots)} - T^\lambda{}_\rho\cdot\delta x^\rho\Big\}\Big] d^4x\,. \tag{A·3}$$

Substituting the definitions (A·1) for $\delta\phi_A$ and $\delta A_{(a_1\cdots)}$ in the above expression and putting $\delta I_1 \equiv 0$, we are led to the identities. Especially when $\xi^\mu(x) = a^\mu + \Delta^\mu{}_\nu\cdot x^\nu$ and $A_{(a_1\cdots)}(x)$ is replaced with a constant affine tensor $\zeta_{\mu\nu}$ (and consequently $A_{(a_1\cdots),\lambda}=0$), the coefficient of each parameter a^μ or $\Delta^\mu{}_\nu$ in δI_1, should identically vanish because the domain of integration can be arbitrarily chosen. The relations thus obtained are the identities (4·3) and (4·4) where $\boldsymbol{L}$ takes the place of the present L_1.

On the contrary if $\xi^\mu(x)$ is an arbitrary function of x, the first and the second terms of the right-hand side in (A·3) can be transformed by a partial-integration to the following form:

$$\begin{aligned}\delta I_1 = &- \int_\Omega \xi^\rho\cdot[\partial_\nu\{[L_1]^A\cdot C^{B\nu}_{A\rho}\cdot\phi_B \\ &+ [L_1]^{(a_1\cdots a_r)}\cdot D(^{b_1\cdots b_r}_{a_1\cdots a_r})^\nu_\rho\cdot A_{(b_1\cdots b_r)}\} + [L_1]^A\cdot\phi_{A,\rho} + [L_1]^{(a_1\cdots a_r)}\cdot A_{(a_1\cdots a_r),\rho}]\,d^4x \\ &+ \int_\Omega \partial_\lambda\Big[[L_1]^A\cdot C^{B\lambda}_{A\rho}\cdot\phi_B\cdot\xi^\rho + [L_1]^{(a_1\cdots a_r)}\cdot D(^{b_1\cdots b_r}_{a_1\cdots a_r})^\lambda_\rho A_{(b_1\cdots)}\cdot\xi^\rho \\ &+ \frac{\partial L_1}{\partial\varphi_{A,\lambda}}C^{B\mu}_{A\rho}\cdot\varphi_B\cdot\xi^\rho{}_{,\mu} + \frac{\partial L_1}{\partial A_{(a_1\cdots a_r),\lambda}}\cdot D(^{b_1\cdots b_r}_{a_1\cdots a_r})^\mu_\rho\cdot A_{(b_1\cdots b_r)}\cdot\xi^\rho{}_{,\mu} - T^\lambda{}_\rho\cdot\xi^\rho\Big]d^4x \equiv 0\,.\end{aligned} \tag{A·4}$$

If ξ^ρ and $\xi^\rho{}_{,\mu}$ are chosen to vanish on the boundary surface of Ω, the second integral in (A·4) vanishes and we have

$$\delta I_1 = -\int_\Omega \xi^\rho[\cdots]_\rho\cdot d^4x \equiv 0\,, \tag{A·5}$$

where ξ^ρ can take any value inside Ω. Thus we have the identity

$$[\cdots]_\rho = [\partial_\nu \{[L_1]^A \cdot C^{B\nu}_{A\rho} \cdot \phi_B + [L_1]^{(a_1\cdots)} \cdot D\binom{b_1\cdots}{a_1\cdots}^\nu_\rho \cdot A_{(b_1\cdots)}\} + [L_1]^A \cdot \phi_{A,\rho} + [L_1]^{(a_1\cdots)} \cdot A_{(a_1\cdots),\rho}] \equiv 0 \qquad (\mathrm{A}\cdot 5)'$$

which is nothing but the identity (2·4). Inserting (A·5)′ into (A·4), we have

$$\delta I_1 = \int_\Omega \partial_\lambda [\cdots]^\lambda \cdot d^4x \equiv 0$$

for arbitrary ξ^μ's. By putting equal to zero the coefficients of ξ^ρ, and its derivatives in the above identity, (2·5) ~ (2·7) can be derived.

B. *Determination of* $A_{(a_1\cdots a_r)}$

Let us begin with the substitution of (3·9) for Z's in the identity (3·8). The latter is written as

$$\begin{aligned} 2\delta^\alpha_\rho \delta^{(\mu\nu)}_{(\lambda\beta)} \equiv\; & r\cdot a[A^\mu_\rho \cdot \delta^{(\nu\alpha)}_{(\lambda\beta)} + A^\nu_\rho \cdot \delta^{(\mu\alpha)}_{(\lambda\beta)}] \\ & + r\cdot b[2A^\nu_\rho \cdot \delta^{(\alpha\mu)}_{(\lambda\beta)} + 2A^\mu_\rho \cdot \delta^{(\alpha\nu)}_{(\lambda\beta)} + 2(r-1)\{\delta^\alpha_\beta A^{\mu\nu}_{\lambda\rho} + \delta^\alpha_\lambda A^{\mu\nu}_{\beta\rho}\}] \\ & + r\cdot c[4A^\alpha_\rho \delta^{(\mu\nu)}_{(\lambda\beta)} + (r-1)\{\delta^\nu_\beta A^{\mu\alpha}_{\lambda\rho} + \delta^\nu_\lambda A^{\mu\alpha}_{\beta\rho} + \delta^\mu_\beta A^{\nu\alpha}_{\lambda\rho} + \delta^\mu_\lambda A^{\nu\alpha}_{\beta\rho}\}] \\ & + r(r-1)\cdot d\cdot [2(r-2)\cdot A^{\mu\nu\alpha}_{\lambda\beta\rho} + \delta^\mu_\beta A^{\nu\alpha}_{\lambda\rho} + \delta^\mu_\lambda A^{\nu\alpha}_{\beta\rho} + \delta^\nu_\beta A^{\mu\alpha}_{\lambda\rho} + \delta^\nu_\lambda A^{\mu\alpha}_{\beta\rho}], \end{aligned} \qquad (\mathrm{B}\cdot 1)$$

where the following abbreviations have been employed:

$$A^\alpha_\rho \equiv A^{(\alpha a_2\cdots a_r)} A_{(\rho a_2\cdots a_r)},$$

$$A^{\alpha\nu}_{\beta\rho} \equiv A^{(\alpha\nu a_3\cdots a_r)} A_{(\beta\rho a_3\cdots a_r)},$$

etc.

The double contraction of (B·1) by putting $\mu=\lambda$ and $\nu=\beta$ leads to

$$A^\alpha_\rho \{5ra + 2(2r+3)\cdot r\cdot b + 10\cdot r(r+3)\cdot c + 2r(r-1)(r+3)\cdot d\} \equiv 20\delta^\alpha_\rho, \qquad (\mathrm{B}\cdot 2)$$

which allows to put

$$A^\alpha_\rho \equiv A^{(\alpha a_2\cdots a_r)} \cdot A_{(\rho a_2\cdots a_r)} = \delta^\alpha_\rho \qquad (\mathrm{B}\cdot 3)$$

and consequently

$$A^{(a_1\cdots a_r)} \cdot A_{(a_1\cdots a_r)} = 4\,. \qquad (\mathrm{B}\cdot 3)'$$

Inserting (B·3) into (B·2), we have

$$5ra + 2(2r+3)rb + 10r(r+3)c + 2r(r-1)(r+3)d = 20\,. \qquad (\mathrm{B}\cdot 2)'$$

In a similar way, the contraction of (B·1) by putting $\alpha=\rho$ and $\mu=\lambda$ leads to

$$6ra + 2r(2r+3)b + 10r(r+3)\cdot c + 2r(r-1)(r+3)d = 20\,, \qquad (\mathrm{B}\cdot 4)$$

where the normalization (B·3) and (B·3)′ have been employed. The third type of contraction $\alpha=\lambda$ and $\mu=\rho$ gives a relation

$$15ra + 10r(2r+3)b + 2r(7r+3)c + 2r(2r+3)(r-1)d = 10\,. \qquad (\mathrm{B}\cdot 5)$$

A subtraction of (B·2)′ from (B·4) gives

$$ra=0$$

or

$$a=0\,. \tag{B·6}$$

If (B·1) is contracted by putting $\mu=\lambda$, we obtain a relation

$$\{2(r-1)\cdot r\cdot b+6(r-1)\cdot r\cdot c+2r(r-1)(r+1)d\}A^{\alpha\nu}_{\beta\rho}$$
$$=\{5-rb-r(r+a)c-r(r-1)\cdot d\}\delta^{\alpha}_{\rho}\delta^{\nu}_{\beta}-2r(r+2)b\cdot\delta^{\alpha}_{\beta}\delta^{\nu}_{\rho}. \tag{B·7}$$

Since the left-hand side of (B·7) is symmetric with respect to the pairs of suffices (α,ν) and (β,ρ) the same should be true for the right-hand side. Thus we have

$$5-rb-r(r+9)c-r(r-1)\cdot d=-2r(r+2)\cdot b\,, \tag{B·8}$$

and consequently (B·7) becomes

$$\{2r(r-1)b+6(r-1)\cdot r\cdot c+2r(r-1)(r+1)\cdot d\}A^{\alpha\nu}_{\beta\rho}=-4r(r+2)\cdot b\cdot\delta^{(\alpha\nu)}_{(\beta\rho)}\,. \tag{B·7)′$$

Similarly, contractions $\mu=\rho$ and $\alpha=\lambda$ lead to

$$\{cr+r(r-1)^2d\}A^{\alpha\nu}_{\lambda\beta}=\{1-(2r+3)rb-r(r+1)c-r(r-1)d\}\delta^{(\alpha\nu)}_{(\lambda\beta)} \tag{B·9}$$

and

$$\{5r(r-1)b+r(r-1)c+r(r-1)^2d\}A^{\alpha\nu}_{\lambda\beta}=\{1-5rb-r(r+1)c-r(r-1)d\}\delta^{(\alpha\nu)}_{(\lambda\beta)} \tag{B·10}$$

respectively.

In (B·7)′, (B·9) and (B·10), if the coefficients of $A^{::}_{::}$ do not vanish, these relations show that $A^{::}_{::}$ should be proportional to $\delta^{(::)}_{(::)}$. Recalling the normalization (B·3), we have to put

$$A^{\alpha\nu}_{\beta\rho}=\tfrac{2}{5}\delta^{(\alpha\nu)}_{(\beta\rho)}\,. \tag{B·11}$$

Thus (B·9) and (B·10) give

$$5(2r+3)rb+(5r+7)r\cdot c+(2r+3)(r-1)\cdot r\cdot d=5 \tag{B·9)′$$

and

$$5(2r+3)rb+(7r+3)rc+(2r+3)(r-1)rd=5 \tag{B·10)′$$

respectively.

A subtraction of (B·9)′ from (B·10)′ gives

$$(2r-4)\cdot r\cdot c=0\,. \tag{B·12}$$

If we choose $r=2$, then (B·11) becomes unsolvable relations

$$A^{\alpha\nu}(x)\cdot A_{\beta\rho}(x)=\tfrac{2}{5}\delta^{(\alpha\nu)}_{(\beta\rho)}\,.$$

On the contrary if $c=0$ is chosen, we have three relations:

$$(\mathrm{B}\cdot 8)\to(2r+3)\cdot r\cdot b=r(r-1)d-5\,,$$

$$(\mathrm{B}\cdot 9)',\ (\mathrm{B}\cdot 10)'\to 5(2r+3)\cdot r\cdot b+(2r+3)(r-1)\cdot r\cdot d=5\,,$$

$$(\mathrm{B}\cdot 7)'\to 3(2r+3)\cdot b+(r-1)(r+1)d=0\,.$$

These three equations are unsolvable with respect to b and d. Therefore (B·12) cannot hold.

Thus we arrive at a conclusion that all the coefficients of both sides of relations (B·7)′, (B·9) and (B·10) should vanish. Vanishing of the right-hand side of (B·7)′ gives

$$b=0 \tag{B·13}$$

while the left-hand side of (B·7)′ gives a relation when it vanishes:

$$3c+(r+1)d=0\,. \tag{B·14}$$

Similarly, (B·9) gives a couple of relations

$$c+(r-1)^2d=0\,, \tag{B·15}$$

$$1=r(r+1)c+r(r-1)d\,. \tag{B·16}$$

Finally from (B·10) we obtain relations

$$c+(r-1)d=0\,, \tag{B·17}$$

$$1=r(r+1)c+r(r-1)d\,. \tag{B·18}$$

A comparison of (B·15) with (B·17) gives

$$(r-2)\cdot d=0\,.$$

Since the case $d=0$ leads to a contradiction as easily seen, we obtain

$$r=2\,.$$

It is easily verified that the solution of our problem is

$$a=b=0\,,\qquad c=-d=\tfrac{1}{4}\,.$$

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