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Gravitational instability of scalar fields and formation of primordial black holes

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Summary. The gravitational instability of a spatially uniform state of a relativistic scalar field on a time-independent background is considered.

The instability is demonstrated to be similar to the well-known Jeans instability of dust-like non-relativistic matter. The consideration is generalized to incorporate 'hydrodynamics', i.e. self-interaction of the field. It is demonstrated that 'hydrodynamic' effects may drastically alter the character of the instability; in particular, the dependence of the instability growth rate on perturbation wavenumber may become non-monotone. The non-linear stage of instability development is analysed half-quantitatively. It is demonstrated that, the 'hydrodynamics' being stabilizing, the instability growth may result in establishing stable spatially periodic structures in the one-dimensional situation. The results obtained are applied to estimating the probability of primordial black hole formation from perturbations growing due to gravitational instability in a universe dominated by a scalar axion field.

1 Introduction

Modern inflationary universe scenarios (Guth 1981; Linde 1982; Albrecht & Steinhardt 1982) and investigation of cosmological consequences of the invisible axion models (Kim 1979; Dine, Fishler & Srednicki 1981; Preskill, Wise & Wilczek 1983; Abbot & Sikivie 1983; Dine & Fishler 1983; Ipser & Sikivie 1983; Stecker & Shafi 1983; and Turner, Wilczek & Zee 1983) broach the question concerning the role of a classical scalar field in cosmological evolution (Turner 1983) and, particularly, its influence on development of cosmological inhomogeneities. This question gains especial relevance in relation to the study of astrophysical effects of supersymmetric grand unification models where the plausible solution of the relic gravitino problem is related to the prolonged stage of the classical scalar field oscillation dominance (Weinberg 1982; Nanopoulos, Olive & Srednicki 1983). For a scalar field φ with the potential energy $V(\varphi)=(1/2)m^2\varphi^2+O(\varphi^4)$ the stage with the Hubble constant $H \ll m$ (in $\hbar=c=1$ units) corresponds to the dust-like expansion stage (Turner 1983) at which the gravitational instability (Lifshits 1946) and formation of the structure of inhomogeneities are possible. As was demonstrated in (Khlopov & Polnarev 1980, 1984), the development of the inhomogeneity structure on the dust-like stage of expansion may be followed by generation of black holes (Zeldovich & Novikov 1966; Hawking 1971). Their formation probability depends on the initial amplitude of perturbations [in the case of a complex scalar field, the role of initial perturbations which start up the growth of the gravitational instability may be played by, e.g. deviations from spatial uniformity related to the presence of strings (Zeldovich 1980)]. That is why constraints on the number of these primordial black holes (PBH) make it possible to obtain restrictions for the chaotic inflation model parameters. To obtain these constraints detailed analysis of the development of classical scalar field instability subject to the form of the potential $V(\varphi)$ is required. The simplest case $V_0=(1/2)m^2\varphi^2$ corresponds to a wave equation linear in φ. Deviation of V from V_0 changes the behaviour of shortwave perturbations.

This analysis is accomplished in the present paper. In Section 2 we consider the gravitational instability of a free real or complex scalar field, i.e. that with the quadratic potential $V_0(\varphi)$, and we demonstrate its instability to be completely analogous (in the linear approximation) to the Jeans–Lifshits gravitational instability of dust-like matter. Section 3 deals with the cases when the potential contains a quartic term representing 'hydrodynamics', i.e. the direct self-interaction of the scalar field. Of especial interest is $V(\varphi)=(1/2)m^2|\varphi|^2-(1/4)\lambda^2|\varphi|^4$ with the 'wrong' sign of the coupling constant λ^2. This potential arises in the Coleman–Weinberg model for not too large fields (Coleman & Weinberg 1973). In Section 3 we show that the quartic term generates 'hydrodynamic' instability of a spatially uniform state of the field. Section 4 is devoted to the qualitative analysis of the non-linear stage of the inhomogeneity evolution. This analysis is based on analogy with similar problems in non-linear wave theory. At last, in Section 5, following the lines of Khlopov & Polnarev (1980, 1984), we discuss formation of PBH from small initial inhomogeneities of the scalar field, depending on the form of $V(\varphi)$.

2 Gravitational instability of free scalar field

The gravitational instability of a spatially uniform state of dust-like matter described by classical non-relativistic equations has been first investigated in the well-known paper by Jeans (1929). In this section we shall consider the analogous problem for a massive free scalar field, its self-interaction being mediated by a weak gravitational field. A time-independent background is assumed. In the Newtonian approximation the equation of motion for the field φ is

$$\varphi_{tt}-\Delta\varphi+m^2(1+2\phi)\varphi=0, \tag{1}$$

ϕ being the gravitational potential, Δ standing for the Laplacian (we assume $c=\hbar=1$), m being the mass of classical particles described by the φ field. Note that in the Newtonian approximation the expression $1-2\phi$ is equal to the component g_{00} of the metric tensor, and the applicability condition for this approximation is

$$\phi\ll 1. \tag{2}$$

However, in the same approximation other diagonal components of the metric tensor also differ from the Galilean values (Landau & Lifshits 1973):

$$g_{\alpha\alpha}=-1-2\phi, \qquad \alpha=1,2,3. \tag{3}$$

The corrections to (1) related to (3) may be disregarded if the φ field spatial gradients, which multiply these corrections, are themselves small as compared with the time derivative φ_t (Landau & Lifshits 1973). This assumption (to be quantitatively formulated below) along with the inequality (2) constitute the applicability conditions for the equation of motion (1). This assumption is equivalent to taking particles at rest initially. As to the equation for the gravitational potential ϕ, in the Newtonian approximation it reduces to the Poisson equation, the source term in it being the total φ field energy density (i.e. the T_{00} component of the energy–momentum tensor). If we are interested in the gravitational instability of the initially spatially uniform state of the complex field

$$\varphi=a_0\exp(imt) \tag{4}$$

or real field

$$\varphi=a_0\cos(mt) \tag{5}$$

we should substract the source corresponding to the uniform fields (4) or (5) from the total source. In the complex field case

$$T_{00}=(m^2/2)|\varphi|^2+(1/2)|\nabla\varphi|^2+(1/2)|\varphi_t|^2, \tag{6}$$

and in the real field case

$$T_{00}=(m^2/2)\varphi^2+1/2(\nabla\varphi)^2+(1/2)\varphi_t^2, \tag{7}$$

∇ standing for the gradient. It should be emphasized that the above procedure is sensible if the gravitational radius of the mass concentrated inside the volume of the size $\sim m^{-1}$ is much smaller than this size, i.e. if

$$Ga_0^2\ll 1. \tag{8}$$

The complex field can be conveniently represented in the form

$$\varphi=a(t,\mathbf{r})\exp[imt+i\Psi(t,\mathbf{r})], \tag{9}$$

a and Ψ being real functions (amplitude and phase). In the case of real field the analogous representation is useful:

$$\varphi=a(t,\mathbf{r})\cos[mt+\Psi(t,\mathbf{r})], \tag{10}$$

a and Ψ being supposed to vary slowly in comparison with $\cos(mt)$.

Substituting (9) into (6) and subtracting the value of T_{00} for the uniform fields (4) and (5) yields the Poisson equation

$$\Delta\phi=4\pi G[m^2(a^2-a_0^2)+ma^2\Psi_t+(1/2)a^2\Psi_t^2+(1/2)(\nabla a)^2+(1/2)a^2(\nabla\Psi)^2], \tag{11}$$

where G is the gravitation constant [in the case of real field the right-hand side of (11) should be multiplied by 1/2]. The equation (1) after substituting (9) or (10) takes the following form:

$$a_{tt}-2ma\Psi_t-a(\Psi_t)^2-\Delta a+2m^2\phi a+a(\nabla\Psi)^2=0,$$

$$a\Psi_{tt}+2ma_t-a\Delta\Psi-2\nabla a\cdot\nabla\Psi=0. \tag{12}$$

To study the stability of the solutions (4) and (5) we shall take a perturbed solution in the form

$$a=a_0+a_1\exp(\Omega t+i\mathbf{kr}),$$

$$\Psi=\Psi_1\exp(\Omega t+i\mathbf{kr}), \tag{13}$$

$$\phi=\phi_1\exp(\Omega t+i\mathbf{kr}),$$

Ω being the instability growth rate, $\mathbf{k}$ being the wave vector and a_1, Ψ_1, ϕ_1 being the small amplitudes of the perturbation. Inserting (13) into (11)–(12) after simple calculations yields the dispersion equation relating Ω to k:

$$\Omega_\pm^2=\pm 2m\sqrt{m^2+k^2+4\pi Gm^2a_0^2}-(k^2+2m^2) \tag{14}$$

(in the case of real field the term with G in (14) should be multiplied by 1/2). As we see from (14), the instability exists in the wavenumber range

$$0 \leqslant k^2 < k_J^2 \equiv 4\sqrt{\pi G} m^2 a_0. \tag{15}$$

The wavenumber k_J is commonly called the Jeans wavenumber. The instability growth rate monotonically falls when k^2 increases from 0 to k_J^2.

As was pointed out above, the underlying equation (1) is applicable provided the φ field gradients are small in comparison with φ_t, i.e. the wavenumbers $k \leqslant k_J$, at which the instability occurs, should be small in comparison with the frequencies $\sim m$. As is seen from (15), this condition coincides with the condition (8) of applicability of our approach. Under this condition, the expression (14) can be expanded at small k:

$$\Omega_+^2 = 4\pi G m^2 a_0^2 - 2\pi G a_0^2 k^2 - k^4/4m^2 \equiv (\Omega^2)_0 + (\Omega^2)_1 k^2 + (\Omega^2)_2 k^4 \tag{16}$$

[in the case of real field $(\Omega^2)_0$ is two times as small].

Note that the energy density of the fields (4) and (5) is, respectively, $m^2 a_0^2$ and $(1/2) m^2 a_0^2$. Thus it follows from (16) that in the both cases $\Omega_+(k^2=0) = 4\pi G T_{00}$ which coincides with the well-known Jeans' relation (Jeans 1929).

The gravitational instability growth rate expansion at small wavenumbers analogous to (16) was considered in Zeldovich (1982), where a diffusion equation was demonstrated to apply to the study of longwave perturbations. The sound velocity c in the long-wave range is determined by the coefficient before the $\sim k^2$ term in the expansion (16):

$$c^2 = 2\pi G a_0^2, \tag{17}$$

i.e. according to (8), this velocity is small in comparison with the light velocity (equal to unity in our notation).

The eigenmode of the growing perturbation can also be found. Taking account of (8), the amplitudes defined in (13) are related as follows:

$$a_1 \approx -\frac{\Omega_+^2 + k^2}{2m\Omega_+} a_0 \Psi_1;$$
$$\phi_1 \approx \frac{4\pi G a_0^2}{\Omega_+} m \Psi_1. \tag{18}$$

As to the neutrally stable ('sound') perturbations at $k^2 > k_J^2$, their frequency $\omega = \sqrt{-\Omega_\mp^2}$ takes the form, at $k^2 \to \infty$,

$$\omega \approx k(1 \pm m/k). \tag{19}$$

As is seen from (18), the group and phase velocities of 'sound' waves at $k^2 \to \infty$ coincide with the light velocity (as already stated, the sound velocity for long waves is much smaller than that of light).

In the case of the complex field (that may be viewed as a system of two real fields) only one branch of the expression (14) results in the instability. This agrees with the result of Grishchuk & Zeldovich (1981) according to which there is only one unstable branch of the dispersion dependence in a many-component system. In the real field case the branch of (13) corresponding to the lower sign implies large values $-\Omega_-^2 \approx 4m^2$. Actually it is a superfluous root of the dispersion equation since proceeding to the variables a and Ψ and employing the equations (10) and (11) for the stability investigation are legitimate only if $|\Omega| \ll m$.

3 Hydrodynamic instability

In this section we shall study instability of a uniform state of the complex scalar field with the potential

$$V(\varphi)=(1/2)m^2|\varphi|^2-(\lambda^2/4)|\varphi|^4. \tag{20}$$

We shall consider the cases of both negative and positive coupling constant λ^2. The former case corresponds to usual self-interaction ('hydrodynamics') of a scalar field while the latter one corresponds to the range of not too large fields in the Coleman–Weinberg model (Coleman & Weinberg 1973), so both cases are interesting from the viewpoint of cosmological applications.

The equation of motion for the φ field takes the following form with regard to (20) [*cf.* (1)]:

$$\varphi_{tt}-\Delta\varphi+m^2(1+2\phi)\varphi-\lambda^2|\varphi|^2\varphi=0. \tag{21}$$

The solution to (21) which describes the spatially uniform state of the scalar field has the obvious form [*cf.* (4)]:

$$\varphi=a_0\exp(i\omega t), \tag{22}$$

where

$$\omega^2=m^2-\lambda^2 a_0^2. \tag{23}$$

As one sees from (23), the amplitude of the uniform state obeys the inequality:

$$\lambda^2 a_0^2<m^2. \tag{24}$$

A general solution should be looked for in the form [*cf.* (9)]

$$\varphi=a(t,\mathbf{r})\exp[i\omega t+i\Psi(t,\mathbf{r})]. \tag{25}$$

Substituting (25) into (21) yields the equations of motion for the amplitude and phase [*cf.* (12)]:

$$\begin{aligned} &a_{tt}-2\omega a\Psi_t-a\Psi_t^2-\Delta a+a(\nabla\Psi)^2+2m^2\phi a-\lambda^2 a(a^2-a_0^2)=0,\\ &a\Psi_{tt}+2\omega a_t+2a_t\Psi_t-a\Delta\Psi-2\nabla a\cdot\nabla\Psi=0. \end{aligned} \tag{26}$$

At last, the Poisson equation takes the following form instead of (11):

$$\begin{aligned} \Delta\phi=4\pi G[&(1/2)m^2a^2+(1/2)\omega^2a^2-(1/4)\lambda^2a^4+\omega a^2\Psi_t+(1/2)a^2\Psi_t^2\\ &+(1/2)(\nabla a)^2+(1/2)a^2(\nabla\Psi)^2-A], \end{aligned} \tag{27}$$

$$A=(1/2)m^2a_0^2-(1/4)\lambda^2a_0^4+(1/2)\omega^2a_0^2=m^2a_0^2-(3/4)\lambda^2a_0^4.$$

The perturbed solution is again searched for in the form (13). The dispersion equation resulting from inserting (13) into (26)–(27) can be conveniently written as follows:

$$(\Omega^2+k^2-2\lambda^2a_0^2)(\Omega^2+k^2)+4\omega^2\Omega^2-16\pi Gm^2a_0^2(m^2-\lambda^2a_0^2)=0. \tag{28}$$

As it is seen from (28), the condition (24) provides for the existence of just one unstable branch of the dispersion dependence $\Omega(k^2)$ as well as in the case of the purely gravitational instability considered in the preceding section. The perturbation eigenmode resulting in instability is (13) with [*cf.* (18)]

$$a_1=-\frac{\Omega^2+k^2}{2\omega\Omega}a_0\Psi_1,$$

$$\phi_1=\frac{4\pi Ga_0^2(m^2-\lambda^2a_0^2)}{\omega\Omega}\Psi_1.$$

The value of $\Omega_0 \equiv \Omega(k=0)$ is determined by the equation

$$(\Omega_0^2)^2 + 2(2m^2 - 3\lambda^2 a_0^2)\,\Omega_0^2 - 16\pi G a_0^2 m^2 (m^2 - \lambda^2 a_0^2) = 0. \tag{29}$$

The subsequent analysis of the equations (28) and (29) simplifies if one takes into account the condition (8), provided $|\lambda^2|$ being not too small:

$$|\lambda^2| \gg G a_0^2.$$

First of all, if $\lambda^2 < 0$, i.e. we deal with usual self-interaction, it immediately follows from (28) and (29) that this self-interaction partially damps the gravitational instability (Fig. 1). Indeed, it follows from (29) that at $k=0$

$$\Omega_+^2 \approx \frac{8\pi G a_0^2 m^2 (m^2 - \lambda^2 a_0^2)}{2m^2 - 3\lambda^2 a_0^2} = 4\pi G a_0^2 m^2 \left[1 + \frac{\lambda^2 a_0^2}{2(2m^2 - 3\lambda^2 a_0^2)}\right] \tag{30}$$

which in the case $\lambda^2 < 0$ is always smaller than the analogous value (16) at $k=0$. The difference between (16) and (30) vanishes in the limit $a_0^2 \to 0$. The similarity of the gravitational instabilities of a free scalar field and dust-like matter was pointed out in Turner (1983). As to the Jeans wavenumber, it can be readily found from (28):

$$k_J^2 \approx \frac{8\pi G m^2 (m^2 + |\lambda|^2 a_0^2)}{|\lambda|^2}. \tag{31}$$

Comparing (31) with (15), we see that the ratio of the two values is

$$\sqrt{\pi G a_0^2} \cdot \frac{m^2 + |\lambda|^2 a_0^2}{|\lambda|^2 a_0^2} \ll 1,$$

i.e. under the stabilizing effect of the self-interaction, the instability range strongly contracts on the wavenumber scale (Fig. 1).

In the opposite case $\lambda^2 > 0$ the situation is altogether different. In this case the quartic term in (19) results in additional (besides gravitational) instability which can be naturally called hydrodynamic.

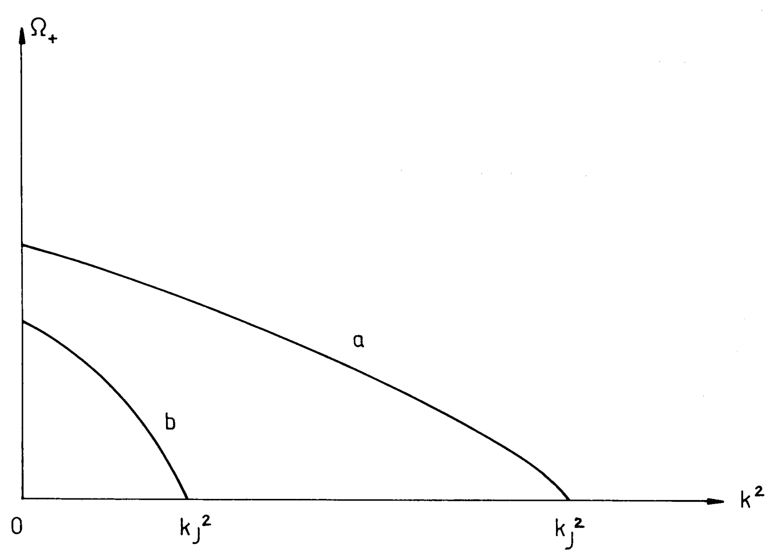

Figure 1. The gravitational growth rate Ω_+ versus the squared perturbation wavenumber k^2. (a) The instability of the free field; (b) the instability of the field with the usual self-interaction.

For further analysis it is convenient to introduce the dimensionless parameter $\xi=\lambda^2 a_0^2/m^2$. We should separately consider the three cases. First, if

$$Ga_0^2 \ll \xi < 2/3 \tag{32}$$

the value of Ω_+ at $k=0$, as before, is determined by formula (30). However this time ($\lambda^2>0$) the value (30) is larger than the analogous value (16) at $k=0$. A still more drastic difference is that the dependence $\Omega_+(k^2)$ is not monotonic now (Fig. 2). The value k_m^2 of k^2 corresponding to the maximum of Ω_+ can be readily found from (28):

$$k_m^2 \approx \frac{\lambda^2 a_0^2(4m^2-5\lambda^2 a_0^2)}{4(m^2-\lambda^2 a_0^2)}. \tag{33}$$

The maximum value of Ω_+^2 is

$$\Omega_{max}^2=\Omega^2(k_m^2)=\frac{\lambda^4 a_0^4}{4(m^2-\lambda^2 a_0^2)}. \tag{34}$$

Obviously, (34) is much larger than (16) at $k=0$. At last, the Jeans wavenumber

$$k_J^2 \approx 2\lambda^2 a_0^2 \tag{35}$$

is also large as compared with the value (15) that corresponds to the purely gravitational instability. Since the expressions (33)–(35) do not contain the gravitational parameter Ga_0^2 we infer that in the present case the instability is actually purely hydrodynamic (corresponding to negative mean pressure), except for the very long wave range $k^2 \ll G\lambda^{-2}m^4$ where, as it is seen from (30), the instability is gravitational as before. By completely neglecting gravitation, i.e. setting $G=0$ in (28), we would obtain the dependence $\Omega_+(k^2)$ depicted in Fig. 2 by the dashed line: the purely hydrodynamic instability growth rate vanishes at $k=0$ if $\xi<2/3$.

In the next case,

$$2/3<\xi<4/5, \tag{36}$$

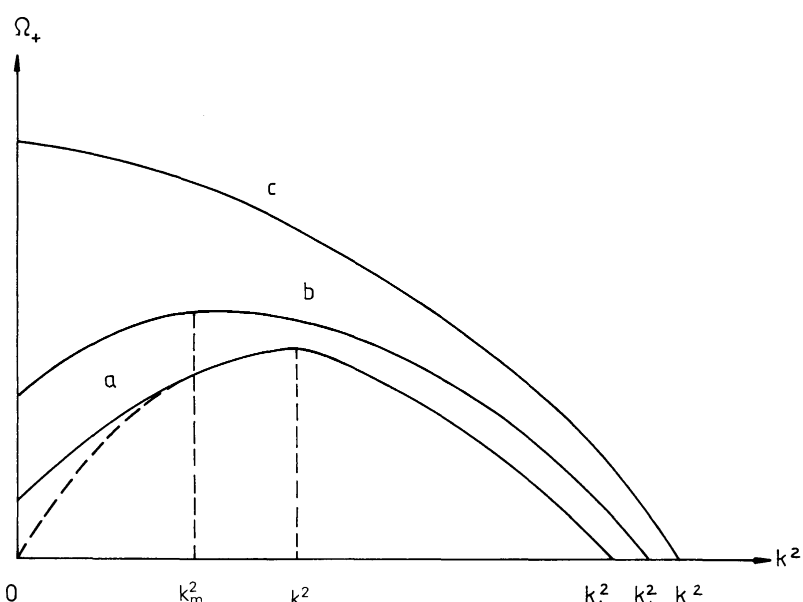

Figure 2. The hydrodynamic instability growth rate Ω_+ versus k^2. (a) The case $\xi\equiv\lambda^2 a_0^2/m^2<2/3$ (the dashed line corresponds to $G=0$); (b) the case $2/3<\xi<4/5$; (c) the case $4/5<\xi<1$.

the instability is hydrodynamic everywhere including the long-wave range. In particular, at $k^2 \ll m^2$

$$\Omega_+^2 \approx 2(3\lambda^2 a_0^2 - 2m^2) + \frac{4m^2 - 5\lambda^2 a_0^2}{2m^2 - 3\lambda^2 a_0^2} k^2 \equiv (\Omega^2)_0 + (\Omega^2)_1 k^2. \quad (37)$$

As is seen from (37), in the present case $(\Omega^2)_0$ is much larger than the 'gravitation-induced' value (30). In other respects this case does not differ from the preceding one, and the formulae (33)–(35) remain valid.

In the last case, $4/5 < \xi < 1$, the dependence $\Omega_+(k^2)$ again becomes monotonic (Fig. 2) which is also seen from the fact that $(\Omega^2)_1$ in (37) becomes negative in the range (36); at the same time the formulae (35), (37) remain relevant.

The dependence of k_m^2 on ξ in all the range $Ga_0^2 \ll \xi < 1$ is depicted in Fig. 3. The small value of ξ_0, at which the graph $k_m^2(\xi)$ 'takes off' from the abscissa axis, can be found from (29):

$$\xi_0 \approx 4\pi G a_0^2 \ll 1.$$

The maximum value of k_m^2

$$(k_m^2)_{max} \approx (1/2)(3 - \sqrt{5}) m^2 \quad (38)$$

is attained at the value $\xi = 1 - 1/\sqrt{5}$ [note that $1 - 1/\sqrt{5} < 2/3$, i.e. the maximum of k_m^2 lies in the range (32)].

To conclude this section we should note that there is one more physically interesting possibility, when $\lambda^2 < 0$ and $m^2 < 0$ in (20) (it corresponds to the standard Higgs' spontaneous symmetry breaking model). In this situation the spatially uniform state of the complex scalar field (this is the state with the excited Goldstone field) is described by the following solution instead of (22)–(23):

$$\varphi = a_0 \exp(i\omega t), \quad (39)$$

$$\omega^2 = |\lambda^2| a_0^2 - |m^2|, \quad (40)$$

i.e. according to (40), the existence condition $\omega^2 > 0$ is

$$a_0^2 > m^2/\lambda^2 \quad (41)$$

[*cf.* (24)]. The dispersion equation takes the form:

$$(\Omega^2 + k^2 + 2|\lambda^2| a_0^2)(\Omega^2 + k^2) + 4(|\lambda|^2 a_0^2 - |m|^2)\Omega^2 + 16\pi G |m^2| a_0^2 (|\lambda^2| a_0^2 - |m|^2) = 0 \quad (42)$$

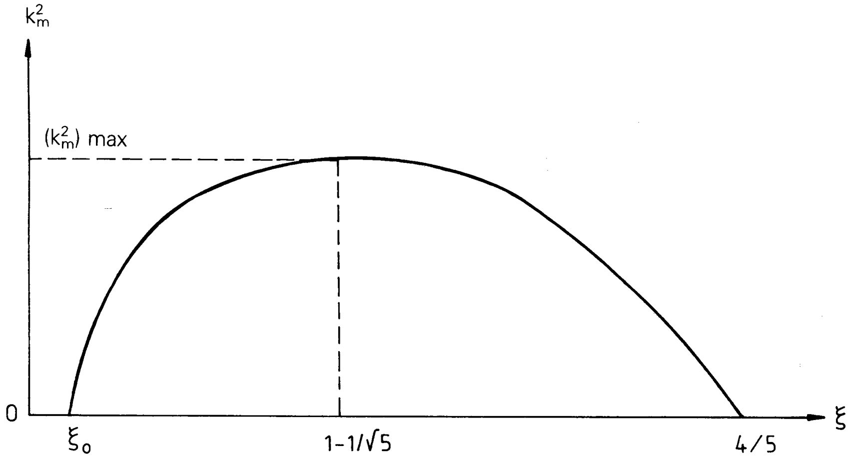

Figure 3. The squared wavenumber k_m^2 corresponding to the maximum of Ω_+ versus ξ.

It is easy to verify that, with account of (8) and (41), the equation (42) for Ω^2 has only negative real roots, i.e. the solution (39)–(40) is neutrally stable against both gravitational and hydrodynamic perturbations.

4 The qualitative analysis of the non-linear stage in the one-dimensional case

Above we concentrated on the linear stability investigation for the spatially uniform state of the scalar field. In the two- or three-dimensional case the further non-linear evolution of perturbations apparently results in collapse. Spherically symmetric collapse implies formation of PBH which will be discussed in the next section. However, in the one-dimensional case collapsing does not take place. This case is interesting, e.g. in relation to the known 'pancake' theory (Shandarin, Doroshkevich & Zeldovich 1983) which is actually one-dimensional.

We shall be interested in one-dimensional stationary solutions to the full non-linear system (26), (27), i.e. solutions with

$$a_t=0, \qquad \phi_t=0, \qquad \Psi_{tt}=0. \tag{43}$$

Then the second equation (26) integrates to yield

$$\Psi_x=c/a^2, \tag{44}$$

c being an arbitrary constant. As is seen from (44), when $c\neq 0$ the φ field acquires some mean wavenumber $k=\langle c/a^2\rangle$, the angular brackets standing for spatial averaging. We may eliminate k by transforming to a moving frame of reference (then the phase $\omega t+kx$ of the φ field under the action of Lorenz transformation turns into $\omega' t'$, the primes concerning the quantities in the moving reference frame).

Therefore we shall suppose $c=0$, i.e. $\Psi_x=0$, reducing (26) and (27) to

$$a_{xx}=-2\omega\chi a-\chi^2 a+2m^2\phi a-\lambda^2 a(a^2-a_0^2), \tag{45}$$

$$\phi_{xx}=4\pi G[(1/2)m^2a^2+(1/2)\omega^2a^2-(1/4)\lambda^2a^4+\omega\chi a^2+(1/2)a^2\chi^2+(1/2)(\nabla a)^2-A], \tag{46}$$

where the constant $\chi\equiv\Psi_t$. We shall look for a spatially periodic solution to (45)–(46) in the form

$$a=a_0+a_1\sin(\varkappa x)+a_{20}a_1^2+a_{21}a_1^2\cos(2\varkappa x)+O(a_1^3), \tag{47}$$

$$\phi=\phi_1 a_1\sin(\varkappa x)+\phi_2 a_1^2\cos(2\varkappa x)+O(a_1^3), \tag{48}$$

$$\chi=\chi_2 a_1^2+O(a_1^3), \tag{49}$$

$$\varkappa^2=(\varkappa^2)_0+(\varkappa^2)_2 a_1^2+O(a_1^3), \tag{50}$$

where a_1 is a small arbitrary amplitude. The expression (48) implies the spatially averaged value of ϕ to be zero (otherwise it can be eliminated by means of a trivial transformation). Substituting (47)–(50) into (45)–(46) in the first approximation with respect to a_1 determines $(\varkappa^2)_0$ and ϕ_1:

$$(\varkappa^2)_0=k_J^2, \tag{51}$$

$$\phi_1=(2\lambda^2a_0^2-\varkappa^2)/(2m^2a_0). \tag{52}$$

In the next approximation we find the coefficients before the terms in (47)–(50) of a_1^2. For the sake of simplicity we write these coefficients for the purely gravitational case $\lambda^2=0$:

$$a_{20}=-(4a_0)^{-1}, \qquad a_{21}=-3/(20a_0), \qquad \phi_2=\sqrt{\pi G}/(5a_0), \qquad \chi_2=-m\sqrt{\pi G}/a_0, \tag{53}$$

$$(\varkappa^2)_2=-(3/2)\sqrt{\pi G}/a_0. \tag{54}$$

The expressions (47)–(50) are valid provided the second harmonic amplitudes are small in comparison with the first harmonic amplitudes, i.e. as it follows from (53) and (54),

$$3a_1/20a_0 \ll 1, \qquad a_1/10a_0 \ll 1, \qquad 3a_1^2/8a_0^2 \ll 1. \tag{55}$$

As we see, the approximation (47)–(54) reasonably describes the solution not only at $a_1/a_0 \ll 1$ but, also, when $a_1 \lesssim a_0$.

Thus, the stationary equations (45)–(46) possess the family of periodic solutions with period $2\pi/\varkappa$ dependent on their amplitude a_1, the period increasing, according to (54), with the growth of a_1^2. The limit solution from this family corresponding to infinite period is the solution with asymptotic behaviour at $|x| \to \infty$:

$$a - a_0, \phi \sim \exp(-q|x|), \tag{56}$$

where q can be readily found after substituting (56) into (45)–(46), provided $|\lambda^2| \gg Ga_0^2$:

$$q^2 \approx 8\pi Ga_0^2(m^2 - \lambda^2 a_0^2)/\lambda^2 a_0^2) \ll k_J^2 \tag{57}$$

(in the case $\lambda^2 = 0$ it follows from (46) that $q = k_J$).

Of principle interest is the question of stability of the stationary solutions. It is easy to realize that in the case $\lambda^2 \geq 0$ the effective non-linearity in the equations of motion is destabilizing: it results in 'self-pinching' of matter (e.g. in the case $\lambda^2 = 0$ this is self-attracting interaction via gravitational field; the presence of the hydrodynamic non-linearity with $\lambda^2 > 0$ implies additional hydrodynamic 'self-pinching'). In the one-dimensional case the equilibrating effect of the pressure prevents collapse under the action of this 'self-pinching'. However this character of the self-interaction makes all the above stationary periodic solutions unstable. As to the soliton, its 'tail' which asymptotically coincides with the spatially uniform solution (22)–(23) is surely unstable, while its 'kern' may be stable. Therefore, proceeding from the analogy with similar problems in the non-linear wave theory (Whitham 1974; Karpman 1968) (the problem about the evolution of an initial state for the non-linear Schroedinger equation with attraction seems quite analogous to ours), it is natural to expect the eventual state of our system in the one-dimensional case to be 'turbulent'. It may be interpreted as stochastic 'gas' of the solitons (56)–(57), the intersolitonic distance being $\sim q^{-1}$.

The situation is altogether different the case $\lambda^2 < 0$. If

$$|\lambda^2| a_0^2 \gtrsim Ga_0^2 m^2 \tag{58}$$

the stabilizing (repulsive) hydrodynamic nonlinearity dominates over the gravitational self-attraction. Of course, the stationary solutions (47)–(54) with small amplitude a_1 [see (55)] are themselves unstable because they are close to the unstable trivial solution (22)–(23). As to the solutions with a_1 not small, they can be rigorously investigated only numerically. However, we may adduce some simple qualitative arguments enabling us to make general conclusions about the stability.

As seen from (14) or (28), there are four branches of the dependence $\Omega(k)$, the two neutrally stable ($\Omega = \pm i\sqrt{-\Omega_-^2}$) resulting in fast oscillations with the frequencies $\sim m$, while the two others ($\Omega = -\sqrt{\Omega_+^2}$ stable and $\Omega = +\sqrt{\Omega_+^2}$ unstable) implying slow motions with the frequencies $\sim \sqrt{Ga_0^2 m}$. Averaging with respect to fast oscillations yields the effective slow motion equation of the form:

$$u_{tt} = \Omega_+^2(-\Delta) - Q(u), \tag{59}$$

$\Omega_+^2(-\Delta)$ being the linear operator obtained by inserting the operator $-\Delta$ instead of k^2 into the function $\Omega_+^2(k^2)$, and $Q(u)$ being some non-linear and, generally speaking, non-local functional. Substituting into the equations of motions (26) the solution to the Poisson equation (27) written by means of the Green's function, it is easy to verify that $Q(u)$ is a uniform cubic functional.

The equation (59) has a rather scaring form when written explicitly. However, one can get an idea of the properties of its solutions by studying the more simple model equation

$$u_{tt}=(\Omega^2)_0 u-(\Omega^2)_1\Delta u+(\Omega^2)_2\Delta^2 u-\alpha u^3, \qquad \alpha>0, \qquad (\Omega^2)_1>0, \qquad (\Omega^2)_2>0, \tag{60}$$

[the relation of model equations to genuine ones is discussed in Zeldovich & Malomed (1982), Malomed & Zhabotinsky (1983) and Malomed (1984a, b)]. Equation (60) can be written in the dimensionless form

$$u_{tt}=u+\beta\Delta u-\Delta^2 u-(4/3)\,u^3, \tag{61}$$

where

$$\beta=\frac{|(\Omega^2)_1|}{[(\Omega^2)_0|(\Omega^2)_2|]^{1/2}}. \tag{62}$$

Equations analogous to (61) often occur in the theory of non-linear waves in active-dissipative media (Zeldovich & Malomed 1982; Malomed & Zhabotinsky 1983; Malomed 1983, 1984a, b; Malomed & Staroselsky 1983). The difference is that in these equations the second time derivative is replaced by the first one. Therefore the stationary solutions for the equations of the both types coincide, while the instability growth rate Ω in the case of a conservative equation [e.g. (61)] is related to that Γ in the case of a dissipative equation as follows:

$$\Omega^2=\Gamma. \tag{63}$$

Therefore stability in the dissipative case is the necessary but not sufficient condition for the stability of a solution to a conservative problem.

Using this analogy we immediately infer that in the case $\beta\gg 1$, which is actually equivalent to the possibility of neglecting the term $-\Delta^2 u$ in the rhs of (61), all the stationary periodic solutions are unstable since, as is well known (Zeldovich & Malomed 1982; Malomed 1984a), those of the equation

$$u_t=\Delta u-(4/3)u^3$$

are unstable. Proceeding to the values $\beta\lesssim 1$ we shall scrutinize the case $\beta\ll 1$ because the case $\beta\sim 1$ involves calculations which are too tedious. Pursuant to (62), $\beta\ll 1$ is equivalent to

$$\lambda^4\ll Gm^4/a_0^2 \tag{64}$$

[(64) is compatible with (58) as $Ga_0^2\ll 1$]. Then we may omit the second term in the rhs of (61) arriving at the equation

$$u_{tt}=u-\Delta^2 u-(4/3)\,u^3. \tag{65}$$

We shall search for approximate solutions to (65) in the quasi-harmonic form (Malomed & Zhabotinsky 1983; Malomed & Staroselsky 1983)

$$u=a_1(t,x)\sin[\delta(t,x)]+a_3(t,x)\sin[3\delta(t,x)]+\ldots, \tag{66}$$

implying a_1 and a_3 to be slowly varying functions of x in comparison with the phase $\delta(t, x)$. Inserting (66) into (65) yields the family of stationary solutions:

$$\delta=kx, a_1^2=1-k^4 \qquad (k^4<1); \tag{67}$$

$$a_3/a_1=(1-k^4)/(243\,k^4-3). \tag{68}$$

Proceeding to the stability investigation we shall look for the perturbed solution in the form [*cf.* (13)]:

$$a_1=\sqrt{1-k^4}+a_1^{(1)}\exp(\Omega t+ipx),$$
$$\delta=kx+\delta^{(1)}\exp(\Omega t+ipx), \tag{69}$$

$a_1^{(1)}$ and $\delta^{(1)}$ being the small perturbation amplitudes. Substituting (69) into (65) yields after standard calculations [following the lines of Zeldovich & Malomed (1982), Malomed & Zhabotinsky (1983); Malomed (1983, 1984a, b) and Malomed & Staroselsky (1983)] the dispersion equation

$$\Gamma^2+2[6k^2p^2+p^4+(1-k^2)]\Gamma+p^4(6k^2+p^2)^2+2p^2(1-k^2)(6k^2+p^2)-16k^2p^2(k^2+p^2)=0, \tag{70}$$

where Γ is determined in (63). At $p=0$ (70) possesses the evident root $\Gamma(0)=0$ owing to the translational invariance of the underlying equation (65) (Zeldovich & Malomed 1982). The branch of roots $\Gamma(p^2)$ originating from $\Gamma(0)=0$ is critical for the stability since it may go to either the unstable range $\Gamma>0$ or the stable one $\Gamma\leqslant 0$ (Zeldovich & Malomed 1982; Malomed & Zhabotinsky 1983; Malomed 1984a). Thus the condition for stability against the perturbations with small p^2 takes the form

$$\left.\frac{\partial\Gamma}{\partial p^2}\right|_{p^2=0}\leqslant 0, \tag{71}$$

or, as is seen from (70),

$$k^4\leqslant 3/7 \tag{72}$$

[*cf.* the results of Malomed & Staroselsky (1983) for an analogous problem]. The condition (72) is sensible in the range of k where it is compatible with the quasi-harmonicity condition

$$|a_3/a_1|\ll 1 \tag{73}$$

that was implied when constructing and investigating the solution (67)–(68). According to (68), at the edge of the range (72)

$$|a_3/a_1|\approx 0.0056, \tag{74}$$

i.e. (72) is indeed compatible with (73). Of course, (73) fails at sufficiently small k. To provide for the complete stability of the solutions (67)–(68) one should also take into account the perturbations with $p\sim k$ [see (69)]. It is known (Zeldovich & Malomed 1982) that in the case of an equation with cubic non-linearity the most dangerous perturbations of this sort have the wave number $2k$. The most convenient way for their investigation is employing the linearized version of (65):

$$(u_1)_{tt}=u_1-(u_1)_{xxxx}-4u^2u_1, \tag{75}$$

u_1 being a small perturbation. Taking it as

$$u_1=\exp(\sqrt{1-\gamma t})\,w(x) \tag{76}$$

brings (75) to the form of the eigenvalue problem:

$$\gamma w=+w_{xxxx}+4u^2w. \tag{77}$$

Since the problem (77) is self-conjugate, all the eigenvalues are real, and the smallest one, $\gamma_{\min}$, which is the most dangerous from the stability viewpoint, is determined by the variational principle:

$$\gamma_{\min}=\min\left\{\left[\int(w_{xx}^2+4u^2w^2)\,dx\right]\Big/\int w^2x\right\},\tag{78}$$

where the minimum is realized with respect to varying the probe function w. The good way for approximate calculating $\gamma_{\min}$ is to substitute into (78) the probe function of the form

$$w=w_0+\cos(2kx+\varepsilon),\tag{79}$$

w_0 and ε being independent variation parameters. The minimum of (78) on the function (79) is attained at

$$w_0=4k^4/(1-k^4)+\sqrt{16k^4/(1-k^4)+1/2},$$

$$\cos\varepsilon=8k^4w_0/[(1-k^4)(w_0^2-1/2)].\tag{80}$$

The value of γ corresponding to (79)–(80) is

$$\gamma_{\min}=2(1-k^4)-(1/2)(1-k^4)^2/[4k^4+\sqrt{16k^8+(1/2)(1-k^4)^2}].\tag{81}$$

As is seen from (76), the condition for stability against the perturbations of the form (76), (79) is, instead of (71),

$$\gamma_{\min}-1>0.\tag{82}$$

In particular, for $k^4=3/7$ [see (72)] substituting (81) into (82) yields

$$\gamma_{\min}-1\approx 0.0959.\tag{83}$$

Thus we may assert that the solutions (67)–(68) satisfying (72)–(73) are stable against all the perturbations. It is important to note that the corrections to (83) related to inharmonicity may be $\lesssim|a_3/a_1|$. Confronting (83) with (74) we infer that the 'stability margin' (83) is sufficient.

So, our conclusion is that, under the conditions (58) and (64), the gravitational instability results in establishing stable one-dimensinal structures.

5 PBH formation on the stage of scalar field domination

At the times $t\gg m^{-1}$ the potential $V(\varphi)=(1/2)m^2\varphi^2\pm 0(\varphi^4)$ results in establishing the dust-like stage of cosmological expansion with equation of state $p=0$. According to Khlopov & Polnarev (1980, 1984), in this the non-linear evolution of inhomogeneities may imply formation of primordial black holes (PBH). To estimate the minimum probability of PBH formation the results of Khlopov & Polnarev (1984), applicable for both non-relativistic particle domination and scalar field domination, may be used. The following conditions are to be satisfied in order that density perturbation may convert into PBH in the course of gravitation–'hydrodynamic' instability development (Khlopov & Polnarev):

(i) spherical symmetry of perturbations;
(ii) its homogeneity.

If a considered configuration retains the isotropy of contraction at the non-linear stage of its evolution up to gravitational self-closing, and contraction homogeneity conditions are satisfied, the configuration inevitably forms PBH. The degree of possible deviations of an initial configuration from spherical symmetry and homogeneity is determined by the amplitude of the initial fluctuation $\delta=\delta\varrho/\varrho$. Quantitatively, the spherical symmetry of contraction is conserved until self-closing of the configuration with the Schwarzschild radius

$$r_g=2GM/c^2$$

and with the radius R_1 corresponding to the non-linear stage beginning moment, provided the deviations from the spherical symmetry being smaller than $x=r_g/R_1$. The same magnitude restricts possible deviations of initial density distribution from the homogeneous one. For the 'flat spectrum' of scalar field initial fluctuations the probability of realization of such spherically symmetric homogeneous configurations is given by Khlopov & Polnarev

$$W_{\rm PBH} \sim \delta^{13/2} \times \begin{cases} (M/M_0)^{10/3}, & M \leqslant M_0 \\ 1, & M_{\max} \geqslant M \geqslant M_0, \end{cases} \tag{84}$$

M_0 being the mass comprised by the cosmological horizon at the dust-like stage beginning moment, and $M_{\max}=m_{\rm pl}\times(\tau/t_{\rm pl})\delta^{-3/2}$, where τ is the moment when the scalar field domination ends due to conversion of the field into relativistic particles. These results were obtained in Khlopov & Polnarev (1984) with no regard to 'hydrodynamic' instability effects which, as it was demonstrated in Section 3, are essential in the case of the potential (20) with $\lambda^2>0$. However, the most considerable effect of this instability, i.e. essential increase of the Jeans wavenumber k, actually does not affect the predicted spectrum of PBH, since this effect results in expansion of the range of small configuration masses $M<M_0$ where the PBH formation probability is damped as $(M/M_0)^{10/3}$. The hydrodynamic instability may increase the PBH formation probability if the scalar field initial amplitude proves to be close to the maximum of the potential $V(\varphi)$: $\varphi_0 \sim \varphi_{\max} \sim m/\lambda$, i.e. if the field from the very beginning is near the hydrodynamic instability threshold (the vacuum stability boundary). In this case the perturbation evolution non-linear stage is stipulated by loss of the hydrodynamic stability, the quantity δ in (89) being realized as $\delta\varphi/(\varphi_{\max}-\varphi_0)$ (it is implied that $\varphi_{\max}-\varphi_0<\varphi_0$).

Thus the hydrodynamic instability development should result in PBH formation with the minimum probability determined by the field inhomogeneity initial amplitude.

For the invisible axion models the mass M_0 is $10^{15} M_\odot$. The existing restriction on the relic radiation temperature fluctuations $\delta T/T<10^{-4}$ yields the maximum value $\delta=\delta\varrho/\varrho \sim 3\times10^{-4}$ which corresponds to the PBH formation probability $\sim 13\times10^{-24}$ with the mass within the interval 10^{15}–$10^{17} M_\odot$. Thus, even if the modern Universe were dominated by invisible axions, their presence would weakly affect the black hole formation. Taking into account indefiniteness in the early Universe conditions, we may expect essentially larger short-scale density perturbation amplitudes. In this case observational astrophysical restrictions for the PBH spectrum would enable the restriction of the possible scalar field dominance stages in the early Universe. In the course of this consideration one should bear in mind that the PBH formation probability increases provided the initial amplitude a_0 of field oscillations being close to the instability threshold.

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