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Gravitational Instability of the Isothermal Gas Cylinder with an Axial Magnetic Field

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The gravitational instability is investigated for an isothermal gas cylinder with a uniform axial magnetic field. We treat the eigenvalue problem for the linear perturbations and obtain the dispersion relation numerically. It is found that the self-gravitating isothermal cylinder is unstable for axisymmetric perturbations of wavelength $\lambda > \lambda_{cr} = 11.2H$, where H is the radial scale height of gas distribution in the cylinder and that the fastest growing mode appears at the wavelength $\lambda_m \sim 2\lambda_{cr}$. For the cylinder with the radius larger than H, the magnetic field reduces the growth rate but does not change the range of wavelength for unstable ones. The stabilizing effect saturates when the magnetic energy becomes comparable to the thermal energy. On the other hand, when the cylinder has a finite radius smaller than H, the dispersion relation approximates to that of incompressible fluid. The magnetic field is so effective to prevent the instability that both the critical wavelength and the most unstable growing time become longer exponentially as the strength of magnetic field increases.

§ 1. Introduction

Cylindrical structures of astronomical objects are attracting attention in many ways. In observations, the spiral arm of the galaxy has been the outstanding example for a long time and the large-scale structure of universe indicates the filamentary formation of clusters of galaxies. Starlight polarization measurements of our galaxy suggest that the lines of magnetic force are parallel to the filamentary dense region of interstellar medium and several discrete clouds locate with regular separation along it.[1),2)]

There are some theoretical mechanisms on the formation of cylindrical objects. Concerning the star formation process, three-dimensional simulation of the collapse of rotating clouds tells us that the bar-like phase is important to initiate the fragmentation of cloud.[3)] Density enhancements such as galactic density waves or interstellar ionization shock fronts can propagate as plane waves. In general, if such waves collide with each other, they are considered to trigger the cylinder-like dense region at the intersection of waves.[4)] When the ram pressure at such a location is higher than the magnetic pressure, the magnetic field can be parallel to the dense cylinder. If the incompressible gas layer with a plane-parallel magnetic field is formed by a strong shock front for example, it has an instability to break up into filaments along magnetic field lines.[5)]

Are these cylindrical objects gravitationally stable? The gravitational instability of the compressible gas is often discussed according to the Jeans criterion as a first approximation. The self-gravitating system is not stable if its size is greater than the Jeans length:

$$\lambda_J=(\pi C_s^2/G\rho)^{1/2}$$

$$=1.7\times10^{18}\left(\frac{T}{30\mathrm{K}}\right)^{1/2}\left(\frac{\mu}{2.34}\right)^{-1/2}\left(\frac{n}{10^4\mathrm{cm}^{-3}}\right)^{-1/2}\mathrm{cm}\,, \qquad (1\cdot1)$$

where G is the gravitational constant, and C_s, ρ, T, μ and n are the sound velocity, the mass density, the temperature, the mean molecular weight and the number density of gas, respectively. However, is the Jeans criterion valid to apply for the stability of the cylinder? How about the effects of the axial magnetic field?

In the theory of star formation from the interstellar magnetic cloud, which can be approximated as an isothermal ideal gas in the early stage of star formation, the contraction of the cloud proceeds along the lines of magnetic force.[6] As the initial condition of such a contraction, the isothermal gas sphere with the size of $\sim\lambda_J$ threaded by the uniform magnetic field is usually assumed. This means that the cloud has to fragment into the pieces of the length $\sim\lambda_J$ before the contraction starts if it initially extends uniformly along the magnetic field line. But, there is no proof on the possibility of fragmentation from the isothermal magnetic cylinder.

In fact, the isothermal cloud cannot remain stable if the perturbation whose wavelength longer than λ_J is set in. But the Jeans criterion is applicable only to the isotropic homogeneous medium with infinite extent and its dispersion relation tells us that the longer the perturbation wavelength such as $\lambda>\lambda_J$ is, the faster the instability increases. For the fragmentation to occur, it is required that the perturbation with the finite wavelength is the most unstable among various modes of perturbations. Therefore, as a result of the Jeans instability, no fragmentation occurs and the sphere threaded by a uniform magnetic field cannot be obtained. Is there not any change in the cylindrical configuration?

For the incompressible cylinder, the analytical dispersion relation is obtained by Chandrasekhar[7] in the presence of an axial magnetic field and the fragmentation of cylinder is possible. According to this dispersion relation, however, the most unstable wavelength becomes longer exponentially with the strength of magnetic field and much beyond the Jeans length. Therefore, to ascertain the initial condition of star formation from magnetic cloud quantitatively, we have to clarify two points: One is the possibility of fragmentation of isothermal cloud which extends uniformly along the magnetic tube and the other is the effects of magnetic field to its gravitational instability and the size of each fragment.

In this paper, we search the most unstable mode of the self-gravitating isothermal cylinder by treating the eigenvalue problem for linear perturbations and include the effect of magnetic field parallel to the cylinder. In § 2, cylindrical hydrostatic equilibrium solution on which we study the gravitational instability is presented. Linear perturbation equations for the isothermal cylinder with a frozen-in axial magnetic field are given in § 3. In § 4, we have numerical dispersion relations which tell us the nature of gravitational instability of the cylinder. Section 5 is devoted to the comparison of our results for isothermal cylinder with Chandrasekhar's dispersion relation for incompressible one and some astronomical implications are also discussed.

§ 2. Hydrostatic equilibrium

We investigate the gravitational stability of an isothermal ideal gas. As the unperturbed state we assume that the gas extends to infinity along the cylinder axis, which we define as the z-axis of cylindrical coordinates (r, φ, z), and is threaded by a magnetic field $\boldsymbol{B}_0$ parallel to it.

In investigating the effect of magnetic field, we assume the conductivity is large enough to keep the magnetic field frozen into the matter. For simplicity, the unperturbed magnetic field is assumed to be constant as

$$\boldsymbol{B}_0=(0, 0, B_0)\,, \tag{2·1}$$

and it exerts no force on the unperturbed structure. In contrast to the above assumption, Stodolkiewicz,[8] who studied the critical wavelength, assumed magnetic field scales as $B_0 \propto \rho^{1/2}$ so that the unperturbed equilibrium can be supported by both thermal and magnetic pressure. He concluded that the magnetic fields make the system unstable because the critical wavelength becomes short as his parameter $\varkappa^2 = V_a^2/C_s^2$ increases, where V_a is the Alfvén velocity at the center. But his result can be interpreted in another way. The increase of $\varkappa^2$ means the decrease of thermal pressure in the unperturbed state at the same time, so that Jeans-like instability may become manifest for such a cold cylinder. To avoid this confusion, it would be better to take uniform magnetic fields as the unperturbed state.

Therefore, hydrostatic equilibrium should be achieved in the radial direction. The density distribution for such an equilibrium has an analytic expression,[9]

$$\rho_0(r)=\rho_c\left[1+\frac{1}{8}\left(\frac{r}{H}\right)^2\right]^{-2}, \tag{2·2}$$

where scale height

$$H \equiv C_s/(4\pi G\rho_c)^{1/2}\,. \tag{2·3}$$

The total mass per unit length of the cylinder is

$$M=\int_0^\infty 2\pi\rho r dr=\frac{2C_s^2}{G}, \tag{2·4}$$

which is independent of ρ_c. This is an interesting feature of the isothermal cylinder different from that of the plane-parallel layer. That means, for any central density, hydrostatic equilibrium solution does not exist in the cylindrical configuration if the line mass exceeds the above value which is determined only by sound speed, i.e., temperature. Therefore, the isothermal cylinder will collapse to line singularity if the mass is accreted keeping the cylindrical symmety or if the temperature decreases. In this paper, we concentrate on the stability of cylindrical hydrostatic equilibrium. Therefore, we assume the line mass does not exceed the value of Eq. (2·4).

§ 3. The perturbations

We restrict ourselves in the linear analysis of the gravitational instability and investigate the small perturbations in the unperturbed equilibrium state given in § 2.

3.1. *The linear perturbation equations*

Let us denote all the first-order quantities of perturbations by the suffix 1. Then the linearized equations for the perturbed quantities are given as

$$\frac{\partial \boldsymbol{v}_1}{\partial t}=\frac{1}{\rho_0 c}\boldsymbol{j}_1\times\boldsymbol{B}_0-C_s{}^2\nabla\left(\frac{\rho_1}{\rho_0}\right)-\nabla\psi_1\,, \tag{3·1}$$

$$\frac{\partial \rho_1}{\partial t}+\nabla(\rho_0\boldsymbol{v}_1)=0\,, \tag{3·2}$$

$$\nabla^2\psi_1=4\pi G\rho_1\,, \tag{3·3}$$

$$\boldsymbol{j}_1=\frac{c}{4\pi}\nabla\times\boldsymbol{B}_1\,, \tag{3·4}$$

$$\frac{\partial \boldsymbol{B}_1}{\partial t}=\nabla\times(\boldsymbol{v}_1\times\boldsymbol{B}_0)\,, \tag{3·5}$$

where c is the light velocity, $\boldsymbol{v}_1$, $\boldsymbol{j}_1$ are the velocity and the electric current density which appear only in the perturbed state and ψ_1 is the perturbed gravitational potential.

By the normal mode analysis in the cylindrical coordinates, we can write all the perturbed value q_1 in the form,

$$q_1(r,\varphi,z;t)=q_1(r)\exp(ikz+im\varphi-i\omega t)\,. \tag{3·6}$$

Then from Eqs. (3·4) and (3·5) we have for $\boldsymbol{v}_1(r)=(v_r, v_\varphi, v_z)$ and $\boldsymbol{j}_1(r)=(j_r, j_\varphi, j_z)$,

$$j_r=\frac{cB_0}{4\pi\omega}\left[\frac{m}{r}\left(\frac{d}{dr}+\frac{1}{r}\right)v_r+i\left(\frac{m^2}{r^2}+k^2\right)v_\varphi\right], \tag{3·7}$$

$$j_\varphi=\frac{cB_0}{4\pi\omega}\left[i\left(\frac{d^2}{dr^2}+\frac{1}{r}\frac{d}{dr}-\frac{1}{r^2}-k^2\right)v_r-\frac{m}{r}\left(\frac{d}{dr}-\frac{1}{r}\right)v_\varphi\right], \tag{3·8}$$

$$j_z=\frac{cB_0}{4\pi\omega}\left[ik\frac{m}{r}v_r-k\left(\frac{d}{dr}+\frac{1}{r}\right)v_\varphi\right]. \tag{3·9}$$

Using Eqs. (3·1) and (3·7)~(3·9), the perturbation of each velocity component is written as

$$\omega^2 v_r=-\frac{B_0{}^2}{4\pi\rho_0}\left[\left(\frac{d^2}{dr^2}+\frac{1}{r}\frac{d}{dr}-\frac{1}{r^2}-k^2\right)v_r+i\frac{m}{r}\left(\frac{d}{dr}-\frac{1}{r}\right)v_\varphi\right]-i\omega\frac{d}{dr}\chi_1\,, \tag{3·10}$$

$$\omega^2 v_\varphi=-\frac{B_0{}^2}{4\pi\rho_0}\left[i\frac{m}{r}\left(\frac{d}{dr}+\frac{1}{r}\right)v_r-\left(k^2+\frac{m^2}{r^2}\right)v_\varphi\right]+\omega\frac{m}{r}\chi_1\,, \tag{3·11}$$

$$\omega v_z = k\chi_1 , \tag{3·12}$$

where

$$\chi_1 \equiv C_s^2 \frac{\rho_1}{\rho_0} + \psi_1 . \tag{3·13}$$

By use of Eq. (3·6), continuity equation (3·2) and Poisson's equation (3·3) become

$$-i\omega\frac{\rho_1}{\rho_0} + \frac{1}{\rho_0}\left(\frac{d}{dr} + \frac{1}{r}\right)(\rho_0 v_r) + i\frac{m}{r}v_\varphi + ikv_z = 0 \tag{3·14}$$

and

$$\left(\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} - k^2 - \frac{m^2}{r^2}\right)\psi_1 = 4\pi G\rho_1 , \tag{3·15}$$

respectively. We solve Eqs. (3·10)~(3·15) as an eigenvalue problem for ω for a given set of (k,m) with the proper boundary conditions.

3.2. *The boundary conditions for the cylinder with infinite radius*

Isothermal hydrostatic equilibrium solution has an infinite radius for the case of free boundary without external pressure. We give the boundary conditions of regularity at the center and infinite r,

$$v_r,\ v_\varphi,\ \frac{dv_z}{dr},\ \frac{d\chi_1}{dr} \text{ and } \frac{d\psi_1}{dr} \to 0 \qquad \text{for } r \to 0 , \tag{3·16}$$

$$\frac{dv_r}{dr},\ \frac{dv_\varphi}{dr},\ \frac{dv_z}{dr},\ \frac{d\chi_1}{dr} \text{ and } \frac{d\psi_1}{dr} \to 0 \qquad \text{for } r \to \infty . \tag{3·17}$$

The dispersion relations obtained by using these boundary conditions are presented in §§ 4.1 and 4.2.

3.3. *The boundary conditions for the cylinder with finite radius*

For more realistic situation, we include the effect of external pressure and investigate the gravitational instability of the cylinder with a finite radius R. We set up the following boundary condition at the perturbed surface of the cylinder:

$$r = R + \delta r \exp(ikz + im\varphi - i\omega t) . \tag{3·18}$$

First, the r-component of the velocity at the boundary must be compatible with this deformed surface,

$$v_r(R) = -i\omega\delta r . \tag{3·19}$$

As the second boundary condition, we require that the boundary pressure, including the magnetic one, must always balance the external pressure $P_0(R) = C_s^2\rho_0(R)$. To first order in the perturbation, we obtain

$$\left|\frac{dP_0}{dr}\right|_R \delta r + P_1(R) + \frac{B_0}{4\pi}B_{1z}(R) = \frac{B_0}{4\pi}B_{1z}{}^{\mathrm{ex}}(R) . \tag{3·20}$$

The third condition follows from the requirement that the potential and its derivative must be continuous on the deformed surface. This can be written to the first order in the perturbation,

$$\psi_1(R)=\psi_1^{\rm ex}(R) \tag{3·21}$$

and

$$\left|\frac{d\psi_1}{dr}\right|_R+4\pi G\rho_0(R)\delta r=\left|\frac{d\psi_1^{\rm ex}}{dr}\right|_R. \tag{3·22}$$

We assume that the cylinder is surrounded by a hot and tenuous ($\rho_0^{\rm ex}\to 0$) medium, so that the exterior potential $\psi^{\rm ex}$ must satisfy Laplace's equation in cylindrical coordinates for the given perturbation. The suitable solution should be written with the modified Bessel function of order m which is regular at infinity.

$$\psi_1^{\rm ex}(r)=A_1K_m(kr) \qquad \text{for } r>R. \tag{3·23}$$

Eliminating the constant A_1 from Eqs. (3·21) and (3·22), we obtain

$$\left|\frac{d\psi_1}{dr}\right|_R+4\pi G\rho_0(R)\delta r=\frac{kK_m'(x)}{K_m(x)}\psi_1(R), \tag{3·24}$$

where

$$x\equiv kR. \tag{3·25}$$

Moreover, we assume that there is no electric current outside the cylinder. The magnetic field to the first order can be expressed in the form

$$\boldsymbol{B}_1^{\rm ex}=\left(A_2kK_m'(kr),\quad A_2\frac{im}{r}K_m(kr),\quad A_2ikK_m(kr)\right). \tag{3·26}$$

Since the normal component of $\boldsymbol{B}_1$ must be continuous across the boundary, the constant A_2 is determined as

$$A_2\equiv\frac{B_{1r}(R)}{kK_m'(x)}. \tag{3·27}$$

Hence we obtain from Eq. (3·20),

$$\left|C_s^2\frac{d\rho_0}{dr}\right|_R\delta r+C_s^2\rho_1(R)+\frac{B_0}{4\pi}B_{1z}(R)=\frac{B_0}{4\pi}\frac{iK_m(x)}{K_m'(x)}B_{1r}(R). \tag{3·28}$$

We use the above three boundary conditions (3·19), (3·24) and (3·28) to get the dispersion relation of the cylinder with a finite radius and to investigate the effect of external pressure in § 4.3.

§ 4. Dispersion relations

In this section, we give the dispersion relations for the gravitational instability of isothermal gas cylinder and show how the eigenmode of perturbation depends on the strength of magnetic field and the strength of external pressure. The dimensionless perturbation equations and the numerical method that we use to obtain the dispersion relations are compiled in the Appendix.

4.1. *Axisymmetric and non-axisymmetric modes*

In Fig. 1, we show the computed dispersion relation for the axisymmetric mode ($m=0$) of the isothermal gas cylinder without magnetic field.*)

The critical wave number is obtained numerically,

$$k_{cr}=0.561(4\pi G\rho_c)^{1/2}/C_s=0.561/H\,. \tag{4·1}$$

Therefore, the infinite cylinder is unstable for an axisymmetric perturbation of $k<k_{cr}$.

The most unstable wave number is

$$k_m=0.284(4\pi G\rho_c)^{1/2}/C_s\,, \tag{4·2}$$

where the growth rate has the maximum value,

$$|\omega_m|=0.339(4\pi G\rho_c)^{1/2}\,. \tag{4·3}$$

Unlike the Jeans instability, the most unstable perturbation occurs at a finite wavelength and the infinite wavelength perturbation needs infinite growing time, i.e., $\omega\to 0$ as $k\to 0$. This means that the infinite cylinder will break up most probably into the pieces with a length of

$$\lambda_m=2\pi/k_m=22.1\,C_s/(4\pi G\rho_c)^{1/2}\,, \tag{4·4}$$

in the e-folding time scale of $|\omega_m|^{-1}$.

We show also the dispersion relations of the *"kink"* mode ($m=1$) in Fig. 1 in the absence of magnetic field. The infinite cylinder is stable against all the non-axisymmetric mode ($m\geqq 1$) of any wavelength. This is also true in the case of incompressible fluid.[7]

We can understand the above result as follows. In the limit of $\omega\to 0$, using Eqs. (3·12), (3·13) and (3·15), we get

$$\left(\frac{d^2}{dr^2}+\frac{1}{r}\frac{d}{dr}-k^2-\frac{m^2}{r^2}+\frac{\rho_0}{\rho_c H^2}\right)\psi_1=0\,. \tag{4·5}$$

The stability criterion can be represented approximately

$$k^2+\frac{m^2}{\langle r\rangle^2}-\frac{\langle\rho_0\rangle}{\rho_c H^2}>0\,. \tag{4·6}$$

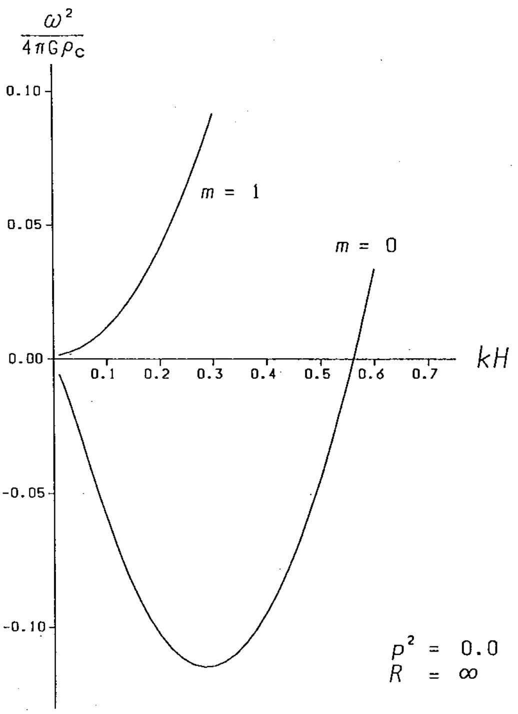

Fig. 1. The dispersion relation of an infinite isothermal cylinder for axisymmetric mode ($m=0$) and non-axisymmetric "*kink*" mode ($m=1$) in the absence of magnetic field ($p^2=0$). The wave number k and the eigenfrequency ω are normalized in the unit $(4\pi G\rho_c)^{1/2}/C_s$ and $(4\pi G\rho_c)^{1/2}$, respectively.

*) In this case, the most unstable mode is also calculated by Hayashi[10] using the energy variational principle and coincides with the lowest line of Fig. 1.

From the definition of critical wave number for $m=0$, $k_{cr}^2 \sim \langle\rho_0\rangle/\rho_c H^2$ and the effective radius $\langle r\rangle$ is the same order as H. Therefore, we rewrite the approximate stability condition Eq. (4·6) to

$$\left(\frac{k}{k_{cr}}\right)^2 + m^2 \frac{\rho_c}{\langle\rho_0\rangle} \gtrsim 1\,, \tag{4·7}$$

so that the cylinder is always stable for $m \geqq 1$. The same argument is possible for the incompressible case replacing the scale height H by the cylinder radius R. It seems to be equivalent to the fact that we cannot give the perturbation in the φ-direction longer than $2\pi H$ or $2\pi R$, which nearly corresponds to the critical wavelength.

Provided that the inner region of the cylinder is empty, that is, in the case of cylindrical shell, the Rayleigh-Taylor instability is dominant and becomes more unstable for non-axisymmetric higher m perturbations as studied by Welter and Schmid-Burgk.[11] Similarly, the interchange instability of high m mode appears when the compressible cylinder is under the influence of an axisymmetric external gravitational field.[12]

4.2. *The effect of the axial magnetic field*

We shall now consider the effect of the uniform magnetic field along the axis of an infinite cylinder on its gravitational instability. As shown in the Appendix, the effect of magnetic field can be represented by one non-dimensional parameter:

$$p^2 \equiv \frac{B_0^2}{4\pi\rho_c C_s^2} = \left(\frac{V_a}{C_s}\right)^2. \tag{4·8}$$

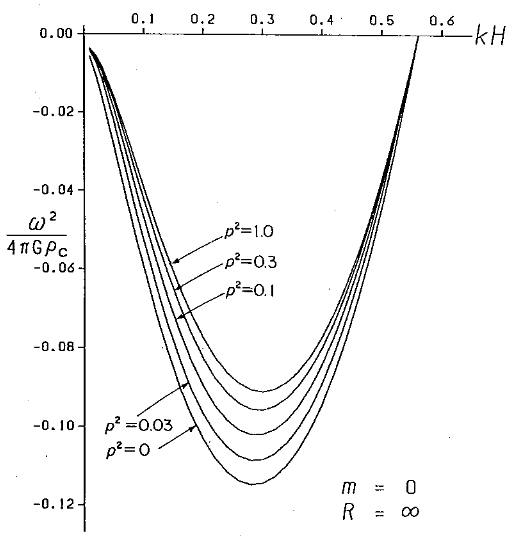

Fig. 2. The dispersion relation for gravitationally unstable mode ($k<k_{cr}$) of an infinite cylinder with the axial magnetic field frozen into the gas. As the magnetic field strength p^2 increases, the growth rate $|\omega|$ of the unstable perturbation decreases but the critical wave number k_{cr} does not change at all. The stabilization effect by the magnetic field saturates so that the lines of $p^2 \gtrsim 1$ are almost overlapped.

The parameter p stands for the ratio of Alfvén velocity to sound velocity. Similar to the cylinder without magnetic field, the cylinder is unstable only for certain axisymmetric perturbations. In Fig. 2 we show the dispersion relation for the gravitationally unstable modes ($k<k_{cr}$) of an infinite cylinder for various values of parameter p^2. The axial magnetic field affects the unstable modes to reduce the growth rate, but it does not change the critical wave number k_{cr}. The latter result can be expected from Eqs. (A·2)~(A·5). In the marginal mode of $\omega\to 0$, p^2 is factored out and disappears from perturbed equations. Therefore k_{cr} is not affected by p^2 unless the boundary condition depends on magnetic field.

This is because the unperturbed state is supported only by the thermal

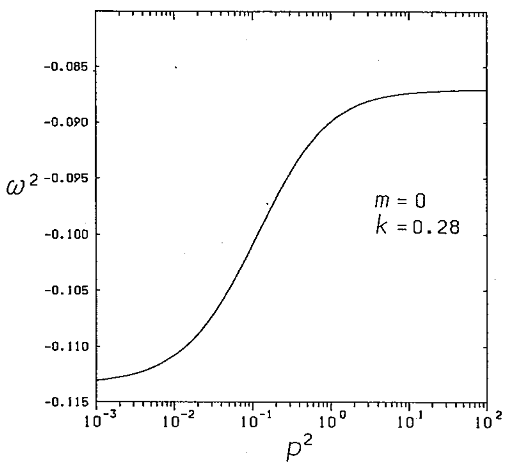

Fig. 3. The saturation of stabilizing effect by the axial magnetic field in the case of the cylinder with infinite radius. The growth rate of the most unstable perturbation $k_m H=0.284$ is suppressed with the increase of magnetic field but it is limited in the range of $p^2 \lesssim 1$.

pressure gradient and the uniform magnetic field has no scale length since the volume integral of the magnetic pressure is infinite, while if the unperturbed magnetic field scales as $B_0 \propto \rho^{1/2}$, the critical wave number depends on the magnetic field due to the change of scale height such as $H \to H(1+V_a^2/C_s^2)^{1/2}$. In the case of the cylinder with a finite radius, even if the unperturbed magnetic field is uniform, the magnetic energy is finite and the characteristic scale length of the magnetic field exists so that there is a possibility for the magnetic field to affect the critical wavelength.

The stabilizing effect of an axial magnetic field on the growth rate of the most unstable mode saturates at

$$|\omega_m|^{(s)}=0.295(4\pi G\rho_c)^{1/2} \qquad (4\cdot9)$$

for $p^2 \gtrsim 1.0$ as shown in Fig. 3. This means that no magnetic field, however strong, can stabilize the cylinder for disturbances of all wavelength.

We understand this result for the isothermal cylinder in the following way. When the magnetic field is present, the deformation which bends the magnetic field line can be stabilized. The magnetic field whose energy is comparable to thermal energy ($p^2 \sim 1$), i.e., half the gravitational energy, is enough to suppress the deformation of the cylinder as if it were bounded by the rigid walls. However, in the unstable density perturbations such that $k < k_{cr}$, the gas is compressed along the axial magnetic field. Therefore, even a magnetic field of infinite strength cannot stop the contraction of gas along the field direction and the stabilization by magnetic fields has to saturate as in Fig. 3. The above nature of the magnetic field is not restricted in the cylindrical geometry. The saturation of the stabilizing effect by the parallel magnetic field is expected also in the disk geometry.[13)]

4.3. *The effect of the external pressure*

We solve the eigenvalue problem of the cylinder with a finite radius to examine the effect of the external pressure. In this case, we use the boundary conditions given in § 3.3.

For cylinders with the radius larger than scale height ($R \gtrsim 10H$), the dispersion relation is almost the same as in Fig. 2, because these models correspond to that of very small external pressure $P_{ex}/P_c \lesssim 10^{-4}$. Figure 4 shows the dispersion relation of the cylinder with radius $R=\sqrt{8}H$, whose density at the boundary is just one fourth of the central density. The critical wave number is slightly larger than that of the cylinder without external pressure. For this model, magnetic field reduces not only the growth rate of unstable perturbations but also the critical wave number due to the

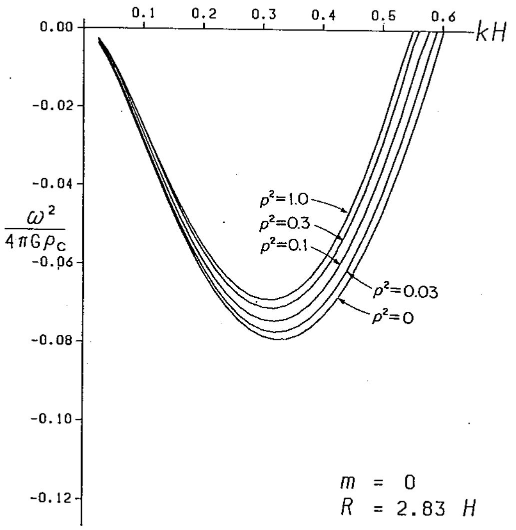

Fig. 4. The dispersion relation for gravitationally unstable mode ($\omega^2<0$) of the cylinder with the radius comparable to its scale height. As the magnetic field strength p^2 increases, the growth rate $|\omega|$ and the critical wave number k_{cr} of the unstable perturbations decrease. The stabilization effect by the magnetic field saturates for $p^2 \gtrsim 1$.

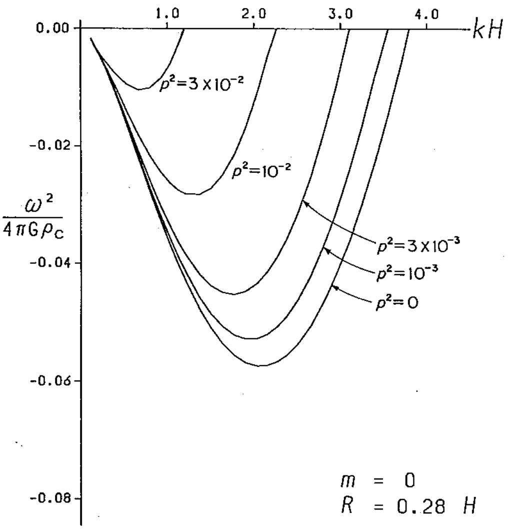

Fig. 5. The dispersion relation for gravitationally unstable mode ($\omega^2<0$) of the cylinder with the radius smaller than its scale height. As the magnetic field p^2 increases, the growth rate $|\omega|$ and the critical wave number k_{cr} of the unstable perturbations decrease. The behavior of this isothermal cylinder is very similar to that of the incompressible cylinder with the axial magnetic field.

surface boundary conditions which depend on p^2. The saturation of stabilization by magnetic field exists also in this case. The dispersion relation does not change for the cylinder with field strength $p^2 \gtrsim 1$.

As the radius becomes much smaller than scale height, the dispersion relation of isothermal cylinder becomes more similar to that of incompressible one. Figure 5 shows the dispersion relation of the cylinder with the radius $R=0.283H$, on which the strong external pressure is exerted. For this cylinder, the density contrast in unperturbed state is very small $\rho(R)/\rho_c=0.980$ and similar to that of incompressible cylinder ($\rho=$const). The dispersion relation also resembles that of incompressible cylinder, which is given by the analytic expression:

$$\frac{\omega^2}{4\pi G\rho}=\frac{xI_1(x)}{I_0(x)}\left[K_0(x)I_0(x)-\frac{1}{2}\right]-\left(\frac{B}{4\pi G^{1/2}R\rho}\right)^2\frac{x}{I_0(x)K_1(x)}, \tag{4·10}$$

where $I_0(x)$ and $I_1(x)$ are the modified Bessel functions which are regular at origin. The critical and the most unstable wavelengths approximate $k_{cr}\to 1.07/R$ and $k_m \to 0.58/R$, respectively. The stabilization by magnetic field is much effective to reduce both the critical wavelength and the growth rate. We find no saturation of stabilizing effect for this cylinder within a resolution of our numerical method.

§ 5. Discussion

In this section, we make a comparison of gravitational instability between isothermal and incompressible cylinder and give some astronomical applications especially on star forming regions.

5.1. *Comparison with the incompressible cylinder*

The isothermal cylinder with infinite radius and the incompressible cylinder are unstable for the perturbations with the wavelength longer than $\lambda_{cr} \sim 2\pi H_{1/2}$ and $2\pi R$, respectively, where $H_{1/2}$ is the half width of the unperturbed state such that $\rho_0(H_{1/2}) = \rho_c/2$. The most unstable wavelength is about twice of the critical one in both cases. Therefore, fragmentation will proceed in the cylindrical configuration irrespective of the equation of state.

The remarkable difference of the gravitational instability of isothermal cylinder and that of incompressible one is the stabilization effect by the axial magnetic field. The result of § 4.2 tells us that no magnetic field, however strong, can stabilize the isothermal cylinder with infinite radius for disturbances of all wavelength. On the contrary, there is no saturation in the case of incompressible fluid.[7] The incompressible cylinder can be stabilized completely by the axial magnetic field B of infinite strength as

$$|\omega_m|/(4\pi G\rho)^{1/2} = 0.21\exp(-B^2/8\pi^2 GR^2\rho^2)\,. \tag{5·1}$$

The above difference can be understood by considering the nature of magnetic fields. The magnetic field stabilizes the gravitational instability by compensating for the gain of gravitational energy with the increase of magnetic energy which follows the field deformation. For the cylinder of an incompressible fluid, only the deformation instability presents so that it could be stabilized effectively by the axial magnetic field.

From the dispersion relations of the isothermal cylinder with a finite radius, we realize that the gravitational instability of the cylindrical isothermal gas has two aspects. One is the deformation instability and the other is the compressible instability. The former corresponds to the *"sausage"* instability of the cylindrical objects and is caused by the global deformation of the surface. The latter is a certain kind of the Jeans instability of compressible uniform medium and does not exist in the case of incompressible fluid. In order to understand the appearance of these two aspects for the finite radius isothermal cylinder, we make use of the tensor virial equation in r-direction,

$$\frac{1}{2}\frac{d^2 I_{rr}}{dt^2} = 2\int_V \left(P + \frac{B_z^2}{8\pi}\right) dV - \int_V \rho r \frac{\partial \psi}{\partial r} dV - \int_S \left(P + \frac{B^2 - 2B_r^2}{8\pi}\right) r dS\,, \tag{5·2}$$

where

$$I_{rr} = \int_V \rho r^2 dV\,.$$

The integral is performed on the unit length of cylinder. In the hydrostatic equilibrium state given by Eqs. (2·1) and (2·4), the three terms of the r.h.s. of Eq. (5·2) should balance,

$$U+W+S=0\,, \tag{5·3}$$

where

$$U=4\pi\int_0^R Pr\,dr\,,$$

$$W=-2\pi\int_0^R \rho\frac{d\psi}{dr}r^2dr$$

and

$$S=-2\pi R^2 P_0(R)\,. \tag{5·4}$$

Figure 6 shows the comparison of the thermal energy term U, the gravitational energy term W and the volume energy term S in the unperturbed states for various cylinder radii R. For the hydrostatic cylinder with a small radius ($R\lesssim 2H$), the volume energy term due to the surface integral of the external pressure is dominant. When the perturbation is given to such a cylinder, the thermal energy term U is almost unchanged due to the conservation of line mass. But the surface deformation can easily decrease the term S and make the cylinder unstable. Hence, the instability

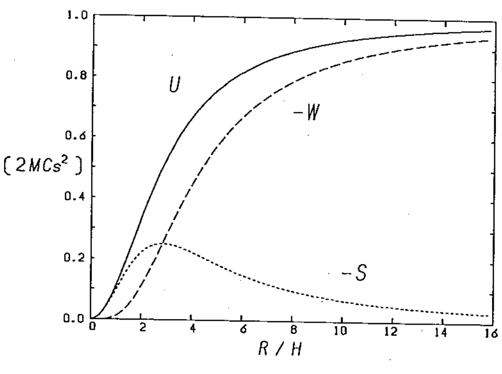

Fig. 6. The comparison of each term (the thermal energy U, the gravitational energy W and the volume energy S) in tensor virial equation as a function of the radius R of cylinder. The unit of the ordinate is twice the thermal energy per unit length of the cylinder with infinite radius. For the hydrostatic cylinder with a small radius ($R\lesssim 2H$), the volume energy term due to the surface integral of the external pressure is dominant. On the other hand, if the radius is much larger than the scale height, the volume energy is negligible and the virial equilibrium is achieved mainly by the balance between U and W.

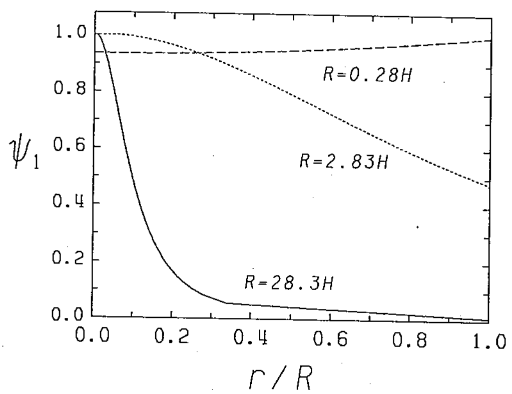

Fig. 7. The eigenfunctions $\psi_1(r)$ for the most unstable axisymmetric mode $m=0$ without magnetic field $p^2=0$. The solid line represents the perturbed gravitational potential of the cylinder with a large radius $R=28.3H$ for the most unstable wavelength $k_m=0.28/H$. The dotted line is for the cylinder with a radius $R=\sqrt{8}H$ and the most unstable wavelength $k_m=0.33/H$. The dashed line is for the cylinder with a small radius $R=0.283H$ and the most unstable wavelength $k_m=2.1/H$. The amplitude is normalized by the maximum value of each eigenfunction.

of small radius isothermal cylinder can be regarded as the deformation instability. While, if the radius is much larger than the scale height, the volume energy is negligible and the virial equilibrium is achieved mainly by the balance between U and W. Therefore, such a cylinder can be unstable by the local density increase in perturbations which enhance self-gravity and decrease the gravitational energy W. This should be called the compressible instability.

The profiles of the eigenfunction also tell us these characteristics. Figure 7 shows the perturbed gravitational potential $\psi_1(r)$ for the most unstable axisymmetric perturbation in the non-magnetic cylinder. For the cylinder with the radius much larger than the scale height, the instability increases in the central region. The increase of central density deepens the gravitational potential and the self-gravity increases the central compression further. On the other hand, if the cylinder radius is smaller than the scale height, the instability appears in the whole region of the cylinder and the amplitude is the largest at the surface boundary. The change in the gravitational force is small, but the deformation of the surface decreases the total energy of the cylinder.

The understanding of these two aspects is consistent with the stabilization effect by the magnetic field. For the cylinder with small radius, the stabilization does not saturate because the deformation instability mainly appears and the magnetic field can prevent it effectively. While in the case of the cylinder with large radius, the compressible instability can occur without bending the field lines so that there arises a limitation of stabilization by the magnetic field.

5.2. *Astrophysical applications*

The dispersion relation tells us that the cylindrical object whose radius is larger than the scale height is unstable for the axisymmetric perturbations with wavelengths larger than λ_{cr} irrespective of the presence of axial magnetic field and the maximum instability occurs at λ_m. The cylindrical cloud will break into the fragments of size λ_m. We obtain the mass of the fragment using Eq. (2·4),

$$M\lambda_m = 12.5(C_s^6/G^3\rho_c)^{1/2}, \tag{5·5}$$

which is nearly equal to the Jeans mass in order of magnitude. Such fragments will appear along the line of magnetic field in the growing time scale of

$$|\omega|^{-1} = 2.95 \sim 3.39\ (4\pi G\rho_c)^{-1/2}, \tag{5·6}$$

which depends on the strength of magnetic field as shown in Fig. 3.

In fact, such a configuration is observed in ρ-Ophiuchi dark cloud region. The region rich in young stellar objects and the region poor of stellar objects locate with the separation along the direction of interstellar magnetic field determined by the measurement of starlight polarization, and the stellar-rich clouds have the mass comparable to the Jeans mass.[14] We may regard these clouds as the fragments caused by the gravitational instability of the cylinder. Moreover each cloud is flattened in the direction of magnetic field,[15),16] which suggests that it is in the process of contraction with magnetic field or in the hydromagnetic equilibrium.

From our numerical results and the above observational support, we can justify the initial condition for the star formation from the magnetic cloud. The fragments with the size $\sim\lambda_f$ can be formed along the field line even if the cloud has infinite and uniform extent in the direction of magnetic fields initially.

What about the cylinder which is confined by the external pressure? According to the dispersion relation of isothermal cylinder in the limit of small radius, a filamentary cloud with the radius $R \ll H$ and uniform density ρ will break up into pieces of length $\lambda_m = 10.83R$ and the fragment mass $M_f = 34.03\rho R^3 \ll \rho H^3$ in the absence of magnetic field. However, one can think that the small radius cylinder, with the axial magnetic field, can accumulate enough mass along the axis to start contraction by self-gravity. In fact, the magnetic field makes the most unstable wavelength longer and hence the fragment mass larger exponentially as

$$\lambda_m = 15.21R\exp(B^2/8\pi^2GR^2\rho^2)\,, \tag{5·7}$$

$$M_f = 47.78\rho R^3\exp(B^2/8\pi^2GR^2\rho^2)\,. \tag{5·8}$$

But at the same time, it reduces the growth rate and the characteristic time scale becomes far beyond the dynamical time scale $1/(4\pi G\rho)^{1/2}$ of our interest. Therefore, in the context of star formation, the gravitational instability of the small radius cylinder has no direct influence because such a cylinder will disrupt into globules with characteristic size smaller than H or remain stable for the time of our interest.

§ 6. Conclusion

The gravitational instability is investigated for the hydrostatic equilibrium of isothermal gas in a cylindrical configuration with a uniform axial magnetic field. From the numerical dispersion relation, we find that the self-gravitating isothermal cylinder is unstable for axisymmetric perturbations of wavelength $\lambda > \lambda_{cr} = 1.77(\pi C_s^2/G\rho_c)^{1/2}$ and that the fastest growing mode with the growth rate $|\omega_m| = 0.34(4\pi G\rho_c)^{1/2}$ appears at the wavelength $\lambda_m = 3.52(\pi C_s^2/G\rho_c)^{1/2}$.

The uniform axial magnetic field has stabilizing effect on the system. For the cylinder with the radius larger than its scale height, it reduces the growth rate but does not affect the range of unstable wavelength. The stabilizing effect saturates when the magnetic energy becomes comparable to the thermal energy. This means any strong axial magnetic field cannot stabilize the infinite cylinder perfectly with respect to all the wavelengths. While the magnetic field of infinite strength can stabilize all the perturbations in the case of incompressible fluid with a finite radius. This can be true even for the isothermal gas when the surface of cylinder is confined by the external pressure and has a finite radius smaller than its scale height. As the radius of the isothermal cylinder becomes smaller, this compressible gas behaves more like incompressible one and the dispersion relation approximates to Chandrasekhar's one.

We can interpret these characteristics as the manifestation of two aspects of gravitational instability of the isothermal gas cylinder, namely, the deformation instability and the compressible instability. The deformation of cylindrical surface brings the deformation instability, while the density perturbation increases in the compressible instability. The axial magnetic field can stabilize chiefly the deformation instability. The magnetic field which has the energy comparable to the thermal energy is enough to suppress the deformation of surface and make it like a rigid wall. However, any strength of magnetic field cannot stabilize the compressible instability because the self-gravitating gas can always accumulate along the field lines when the perturbations such as $\lambda > \lambda_{cr}$ are set in. As the result of this compressible instability, the cylindrical cloud will break into the fragments and each mass, which is determined by the most unstable λ_m, is heavier than the Jeans mass.

As for the pressure-confined isothermal cylinder whose radius is smaller than its scale height, the sausage-type fragments due to the deformation instability are expected. The magnetic field is much effective to prevent this kind of instability. The fragments that will be formed in this way have the smaller mass than the Jeans mass. The star formation from these fragments cannot be expected and only the small cloudlets will be left as the results of this deformation instability.

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Appendix

—— *Dimensionless Perturbation Equations and Numerical Method* ——

In the course of calculation to get the numerical dispersion relation, the variables are normalized as follows:

$$\tilde{\chi}_1=\chi_1/C_s^2\,,\quad \tilde{\psi}_1=\psi_1/C_s^2\,,\quad \tilde{\boldsymbol{v}}=\boldsymbol{v}/C_s\,,$$

$$\tilde{r}=r/H\,,\quad \tilde{\omega}=\omega H/C_s \quad \text{and} \quad \tilde{k}=kH\,. \tag{A·1}$$

We omit the tilde on the new variables hereafter and use the same notations as the old ones for simplicity. Then Eqs. (3·10)~(3·15) can be rewritten in non-dimensional form as

$$\left[\omega^2+\frac{p^2}{f}\left(\frac{d^2}{dr^2}+\frac{1}{r}\frac{d}{dr}-\frac{1}{r^2}-k^2\right)\right]v_r+i\frac{p^2}{f}\frac{m}{r}\left(\frac{d}{dr}-\frac{1}{r}\right)v_\varphi+i\omega\frac{d}{dr}\chi_1=0\,, \tag{A·2}$$

$$i\frac{p^2}{f}\frac{m}{r}\left(\frac{d}{dr}+\frac{1}{r}\right)v_r+\left[\omega^2-\frac{p^2}{f}\left(k^2+\frac{m^2}{r^2}\right)\right]v_\varphi-\omega\frac{m}{r}\chi_1=0\,, \tag{A·3}$$

$$(\omega^2-k^2)\chi_1-\omega^2\psi_1+i\omega\left(\frac{d}{dr}+\frac{1}{r}+g\right)v_r-\omega\frac{m}{r}v_\varphi=0\,, \tag{A·4}$$

$$\left(\frac{d^2}{dr^2}+\frac{1}{r}\frac{d}{dr}-k^2-\frac{m^2}{r^2}+f\right)\psi_1=f\chi_1\,, \tag{A·5}$$

where

$$f(r)\equiv\frac{\rho_0(r)}{\rho_c}=\left(1+\frac{1}{8}r^2\right)^{-2}, \tag{A·6}$$

$$g(r)\equiv\frac{f'(r)}{f(r)} \tag{A·7}$$

and

$$p^2\equiv\frac{B_0{}^2}{4\pi\rho_c C_s{}^2}. \tag{A·8}$$

We solve the eigenvalue problem numerically by using a finite-difference method described by Nakamura.[17] We obtain two second-order differential equations from Eqs. (A·2)~(A·5),

$$\left[(\Omega^2+p^2\omega^2)\frac{d}{dr}\frac{1}{f}\hat{L}_0+\Omega^2\frac{d}{dr}\right]\phi_1-\left[\Omega^2-p^2\left(g\frac{d}{dr}+g'\right)\right]u=0, \tag{A·9}$$

$$\left[\left\{\Omega^2(\omega^2-k^2)-(\Omega^2+p^2\omega^2)\frac{m^2}{r^2}\right\}\frac{1}{f}\hat{L}_0-\Omega^2\left(k^2+\frac{m^2}{r^2}\right)\right]\phi_1$$
$$+\left[\Omega^2\left(\frac{d}{dr}+\frac{1}{r}+g\right)-p^2m^2\frac{g}{r^2}\right]u=0, \tag{A·10}$$

where

$$u\equiv i\omega v_r, \tag{A·11}$$

$$\Omega^2\equiv\omega^2 f-p^2k^2 \tag{A·12}$$

and

$$\hat{L}_0=\frac{d^2}{dr^2}+\frac{1}{r}\frac{d}{dr}-k^2-\frac{m^2}{r^2}. \tag{A·13}$$

The boundary conditions given in § 3.3 for the cylinder with finite radius can also be rewritten in the dimensionless form combined with Eqs. (A·2)~(A·5),

$$\left[(\Omega^2+p^2\omega^2)\frac{d}{dr}\left(\frac{1}{f}\hat{L}_0\right)-\Omega^2\frac{d}{dr}\right]\phi_1-\left[\Omega^2-p^2\left(g\frac{d}{dr}+g'\right)\right]\delta r=0, \tag{A·14}$$

$$\left[\frac{d}{dr}-\frac{kK_m'(x)}{K_m(x)}\right]\phi_1+f\delta r=0 \tag{A·15}$$

and

$$\left[\left\{1-\frac{p^2}{f}\left(\frac{k^2}{\omega^2}-1\right)\right\}\frac{1}{f}\hat{L}_0-\frac{p^2}{f}\frac{k^2}{\omega^2}\right]\phi_1+\left[\left(1+\frac{p^2}{f}\right)g+\frac{p^2}{f}\frac{kK_m(x)}{K_m'(x)}\right]\delta r=0. \tag{A·16}$$

We use the above three boundary conditions (A·14),~(A·16) to get the dispersion relations in § 4.3. Using these boundary conditions, we can write the differential operators of Eqs. (A·9) and (A·10) in a finite difference scheme,

$$\boldsymbol{L}\begin{bmatrix}\bar{\phi}\\ \bar{u}\end{bmatrix}\equiv\begin{bmatrix}\boldsymbol{A}_\phi & \boldsymbol{A}_u\\ \boldsymbol{B}_\phi & \boldsymbol{B}_u\end{bmatrix}\begin{bmatrix}\bar{\phi}\\ \bar{u}\end{bmatrix}=0, \tag{A·17}$$

When we divide the radial coordinate in N grids, $\boldsymbol{A}_u$, $\boldsymbol{B}_u$ and $\boldsymbol{B}_\psi$ are tridiagonal $N\times N$ matrices, $\boldsymbol{A}_\psi$ is a penta-diagonal $N\times N$ matrix and $\bar{\psi}$, $\bar{u}$ are N vectors. Equation (A·14) has a non-trivial solution only when $\det\boldsymbol{L}=0$. Therefore we can determine the eigenvalue ω^2 for given values of k, m and p^2 by searching the zero point of $\det\boldsymbol{L}=0$ iteratively.

Although we can rewrite Eqs. (A·9) and (A·10) into one fourth-order differential equation for ψ_1 as

$$\hat{L}_4\psi_1\equiv\left[\hat{L}_1\alpha\left(\hat{L}_3-\frac{p^2}{\Omega^2}g\hat{L}_2\right)+\hat{L}_2\right]\psi_1=0\,, \tag{A·18}$$

where

$$\hat{L}_1=\frac{d}{dr}+\frac{1}{r}+\varepsilon g\,,$$

$$\hat{L}_2=(\omega^2\varepsilon-\nu^2)\frac{1}{f}\hat{L}_0-\nu^2\,,$$

$$\hat{L}_3=\left(1+\frac{p^2\omega^2}{\Omega^2}\right)\frac{d}{dr}\left(\frac{1}{f}\hat{L}_0\right)+\frac{d}{dr}\,,$$

$$\alpha\equiv\left[1+\frac{p^2}{\Omega^2}\left(\varepsilon g^2+\frac{g}{r}-g'\right)\right]^{-1},$$

$$\varepsilon\equiv(\omega^2 f-p^2\nu^2)/\Omega^2 \tag{A·19}$$

and

$$\nu^2\equiv k^2+\frac{m^2}{r^2}\,,$$

we use Eqs. (A·9) and (A·10) instead of Eq. (A·18) for the accuracy and convenience of numerical calculation. Equation (A·18) is reduced to Welter and Schmid-Burgk's expression[11] in the case $k=p=0$.

For the cylinder with an infinite radius, we have transformed the coordinate $r=[0,\infty]$ to $\xi=[0,1]$ by

$$\xi=(1+r^2/8)^{-1}\,, \tag{A·20}$$

and used the matrix which represents the transformed differential operator in accordance.

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