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Gravitational polarization and the phenomenology of MOND

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Abstract
The modified Newtonian dynamics (MOND) has been proposed as an alternative to the dark matter paradigm; the philosophy behind is that there is no dark matter and we witness a violation of the Newtonian law of dynamics. In this paper, we interpret the phenomenology sustaining MOND differently, as resulting from an effect of 'gravitational polarization', of some cosmic fluid made of dipole moments, aligned in the gravitational field, and representing a new form of dark matter. We invoke an internal force, of non-gravitational origin, in order to hold together the microscopic constituents of the dipole. The dipolar particles are weakly influenced by the distribution of ordinary matter; they are accelerated not by the gravitational field, but by its gradient or tidal gravitational field.

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1. Introduction

The observed discrepancy between the dynamical mass and the luminous mass of bounded astrophysical systems is generally attributed to the existence of an invisible form of matter, coined as the missing mass or dark matter. The nature of the dark matter particles is unknown but extensions of the standard model of particle physics provide a number of candidates [1]. The dark matter triggers the formation of large-scale structures by gravitational collapse and predicts the scale dependence of density fluctuations. Simulations suggest some universal dark matter density profile around ordinary matter distributions [2]. An important characteristic of dark matter, required by the necessity of clustering matter on small scales, is that it should be cold or non-relativistic at the epoch of the galaxy formation. However, the dark matter hypothesis has some difficulties [3] in naturally explaining the flat rotation curves of galaxies, one of the most persuasive evidences for the existence of dark matter, and the Tully–Fisher empirical relation [4] between the observed luminosity and the asymptotic rotation velocity of spiral galaxies.

On the other hand, the modified Newtonian dynamics (MOND) has been proposed by Milgrom [5–7] as an alternative to the dark matter paradigm. It imputes the mass discrepancy not to the presence of some additional non-baryonic matter, but to the failure of the Newtonian law of gravity in an appropriate regime—a drastic change of paradigm. MOND involves a single parameter a_0 as a constant acceleration scale, which delineates the specific MOND regime, corresponding to accelerations much smaller than a_0, from the Newtonian regime, for which the accelerations are much larger. Several relativistic extensions of MOND, assuming the existence of extra fields associated with gravity, besides the spin-2 metric field of general relativity, have been proposed [13, 14]. Such extensions have culminated in the scalar–vector–tensor theory of Bekenstein and Sanders [15–17].

MOND has been very successful at fitting the flat rotation curves of galaxies, and at naturally recovering the Tully–Fisher relation (see [8–12] for reviews). Intriguingly, the numerical value of a_0 that fits the data is close to the Hubble scale, $a_0 \approx cH_0$. MOND may have also some observational problems. There are some counter-examples of galaxies where MOND does not seem to account for the observed kinematics [18], and, most importantly, the mass discrepancy at the scale of clusters of galaxies is not entirely explained by MOND [19, 20].

In this paper and the next one [21] (hereafter paper II), we take the view that MOND does not represent a violation of the fundamental law of gravity, but, rather, provides us with an important hint on the (probably unorthodox) nature of the elusive dark matter. More precisely, we interpret the phenomenology behind MOND as resulting from an effect of gravitational polarization, of some cosmic fluid made of dipole moments, and representing a new form of dark matter. The dipole moments get aligned in the gravitational field produced by ordinary masses, thereby enhancing the magnitude of the field and yielding MOND. Such an effect is the gravitational analogue of the usual electrostatic effect of polarization of a dielectric medium in an applied electric field [22].

In the present paper we imagine, as a model for the dipole, a doublet of particles, one having a positive gravitational mass, the other having a negative and opposite gravitational mass and both particles being endowed with positive inertial masses. The gravitational behaviour involving masses of this type is governed by a negative Coulomb law—like masses attract and unlike masses repel [23]. As a result the dipole moment cannot be stable. Even if we neglect the repulsive gravitational force between the two particles, they will accelerate apart from each other in an exterior gravitational field produced by ordinary matter. We shall therefore invoke an internal force, of non-gravitational origin, between the two particles constituting the dipole, to bound them in a gravitational field. The MOND acceleration scale a_0 will appear to be related to the properties of this internal 'microscopic' force at short distances. We find that the motion of the dipolar particles violates the equivalence principle, and is driven by the *tidal* gravitational field of ordinary matter, rather than the gravitational field itself. In this sense the dark matter is only weakly influenced by the distribution of ordinary matter.

Summarizing, in our approach the dark matter is described by a 'digravitational' medium, which is subject to polarization in a gravitational field, and is otherwise essentially static (an 'ether'). An alternative interpretation of this dark matter is by the gravitational analogue of a plasma in electromagnetism, i.e. composed of positive and negative gravitational masses, and oscillating at the natural plasma frequency. In a gravitational field the mean position of the masses is displaced from equilibrium and the plasma acquires a dipolar polarization. The observational predictions of the present (non-relativistic) model are the same as for MOND.

In paper II we shall propose a relativistic model of dipolar particles, based on an action principle in general relativity. This model will be consistent with the equivalence principle, and as a result the dynamics of dipolar particles, even in the non-relativistic limit, will be different from that of the present quasi-Newtonian model. The present paper and paper II provide two distinct models, both of them suggest a close connection between the phenomenology of MOND and some form of gravitationally polarized dipolar dark matter.

Section 2 investigates the formal analogy between the MOND equation and the electrostatics of nonlinear media. Sections 3 and 4 introduce a microscopic quasi-Newtonian description of the gravitational dipole moment. In section 5, we derive the expression of the non-gravitational internal force in the MOND regime. In section 6, we present an alternative though equivalent formulation of the dipolar medium in terms of a polarized gravitational plasma.

2. Analogy with electrostatics

The MOND equation, in the variant derivable from a non-relativistic Lagrangian [13], takes the form of the modified Poisson equation

$$\partial_i \left[\mu \left(\frac{g}{a_0} \right) g^i \right] = -4\pi G \rho, \tag{1}$$

where ρ denotes the density of ordinary matter, $g^i = \partial_i U$ is the gravitational field in the non-relativistic limit (so that $a^i = g^i$ is the acceleration of ordinary matter) and U is the gravitational potential[1]. In equation (1) the Milgrom function μ depends on the ratio g/a_0, where $g = |g^i|$ is the norm of the gravitational field and a_0 is the constant acceleration scale. The MOND regime corresponds to the limit of weak gravity, much below the scale a_0, i.e. $g \ll a_0$; in this limiting regime we have $\mu(g/a_0) \approx g/a_0$ [5–7]. When $g \gg a_0$, the function $\mu(g/a_0)$ asymptotes to one, and we recover the usual Newtonian law. Sometimes we shall consider the formal 'Newtonian' limit $g \to \infty$; however we always assume that the gravitational field is non-relativistic. Various forms of the function μ have been proposed (cf [24]), but most of them appear to be rather ad hoc.

Taking the MOND equation (1) at face, we note a striking analogy with the usual equation of electrostatics describing the electric field inside a dielectric medium, namely $\partial_i D^i = \rho_\mathrm{e}$, where the electric induction D^i is proportional to the electric field E^i (at least for not too large electric fields): $D^i = \mu_\mathrm{e}\varepsilon_0 E^i$. Here $\mu_\mathrm{e} = 1 + \chi_\mathrm{e}$ is the dielectric coefficient and χ_e denotes the electric susceptibility of the dielectric medium, which depends on the detailed microscopic properties of the medium (see, e.g. [22]). In nonlinear media the susceptibility is a function of the norm of the electric field, $\chi_\mathrm{e}(E)$ with $E = |E^i|$. The electric polarization is proportional to the electric field, and is given by $\Pi^i_\mathrm{e} = \chi_\mathrm{e}\varepsilon_0 E^i$. The density of electric charge due to the polarization is $\rho_\mathrm{e}^{\mathrm{pol}} = -\partial_i \Pi^i_\mathrm{e}$. Generically we have $\chi_\mathrm{e} > 0$, which corresponds to the *screening* of electric charges by the polarization charges, and reduction of the electric field inside the dielectric.

In keeping with this analogy, let us interpret the MOND function μ entering equation (1) as a 'digravitational' coefficient, and write

$$\mu = 1 + \chi, \tag{2}$$

[1] Spatial indices i, j are raised and lowered using the Euclidean metric δ_{ij}; the summation convention is used throughout.

where χ would be a coefficient of '*gravitational susceptibility*', parametrizing the relation between some '*gravitational polarization*', say Π^i, and the gravitational field:

$$\Pi^i = -\frac{\chi}{4\pi G} g^i. \tag{3}$$

Since we have seen that the MOND function (2) depends on the magnitude of the gravitational field, $\mu(g/a_0)$, the same is true of the so-defined gravitational susceptibility, $\chi(g/a_0)$, in a close analogy with the electrostatics of nonlinear media. Hence we expect that χ should characterize the response of some nonlinear digravitational medium to an applied gravitational field. The mass density associated with the polarization would then be given by the same formula as in electrostatics,

$$\rho_{\mathrm{pol}} = -\partial_i \Pi^i. \tag{4}$$

With these notations, equation (1) can be rewritten as

$$\Delta U = -4\pi G(\rho + \rho_{\mathrm{pol}}). \tag{5}$$

In such a rewriting of MOND, we see that the Newtonian law of gravity is not violated, but, rather, we are postulating the existence of a new form of matter, to be called dark matter, which contributes in the normal way to the right-hand side (RHS) of the Poisson equation (5). The dark matter consists of polarization masses with the volume density ρ_{pol} given by (4).

3. Sign of the susceptibility coefficient

For the moment we have restated the MOND equation in the form (4)–(5) and proposed a formal interpretation. To check this interpretation, let us view the digravitational medium as consisting of individual dipole moments π^i with the number density n, so that the polarization vector reads

$$\Pi^i = n\pi^i. \tag{6}$$

We suppose that the dipoles are made of a doublet of sub-particles, one with a positive mass $+m$ and one with a negative mass $-m$, the masses which we are referring here are the *gravitational masses* of these particles, i.e. the gravitational analogue of the electric charges, $m_{\mathrm{g}} = \pm m$. Clearly the exotic nature of this dark matter shows up here, when we suggest the notion of negative gravitational masses. If the two masses $\pm m$ are separated by the spatial vector d^i, pointing in the direction of the positive mass, the dipole moment is

$$\pi^i = md^i. \tag{7}$$

Let us further suppose that the two sub-particles are endowed with *inertial masses* which are *positive*, and given by $m_{\mathrm{i}} = m$. The dipole moment thus consists of an ordinary particle, say $(m_{\mathrm{i}}, m_{\mathrm{g}}) = (m, m)$, associated with an exotic particle, $(m_{\mathrm{i}}, m_{\mathrm{g}}) = (m, -m)$.

The ordinary particle will always be attracted by an external mass distribution made of ordinary matter; however, the other particle $(m_{\mathrm{i}}, m_{\mathrm{g}}) = (m, -m)$ will always be *repelled* by the same external mass. In addition, the two sub-particles will repel each other. We see therefore that the gravitational dipole is unstable, and we shall invoke a non-gravitational force to sustain it. We also expect that the external gravitational field will exert a torque on the dipole moment in such a way that its orientation has the positive mass $+m$ oriented in the direction of the external mass, and the negative mass $-m$ oriented in the opposite direction. We thus find that π^i and Π^i should both point towards the external mass, i.e. be oriented in the same direction as the external gravitational field g^i. From equation (3) we therefore conclude that the susceptibility coefficient χ, in the gravitational case, must be *negative*:

$$\chi < 0. \tag{8}$$

This corresponds to an '*anti-screening*' of the ordinary mass by the polarization masses, and *enhancement* of the gravitational field in the presence of the digravitational medium. This simply results from the fact that, in contrast to electrostatics, alike gravitational charges or masses always attract. Result (8) is nicely compatible with the prediction of MOND; indeed we have $0 \leqslant \mu < 1$ in a straightforward interpolation between the MOND and Newtonian regimes, hence $-1 \leqslant \chi < 0$. The stronger gravitational field predicted by MOND may thus be naturally interpreted by a process of anti-screening by polarization masses.

4. Microscopic model for the dipole

To give more substance to the model, suppose that some interaction F^i between the two constituents of the dipole is at work, and let the dipole moment be embedded into the gravitational field $g^i = \partial_i U$. The equations of motion of the sub-particles $(m_\mathrm{i}, m_\mathrm{g}) = (m, m)$ and $(m, -m)$, having positions x^i_+ and x^i_- respectively, are

$$m \frac{\mathrm{d}^2 x^i_+}{\mathrm{d}t^2} = m g^i(x_+) - F^i(x_+ - x_-), \tag{9}$$

$$m \frac{\mathrm{d}^2 x^i_-}{\mathrm{d}t^2} = -m g^i(x_-) + F^i(x_+ - x_-). \tag{10}$$

The internal force F^i is proportional to the relative separation vector $d^i = x^i_+ - x^i_-$, namely

$$F^i = F \frac{d^i}{d}. \tag{11}$$

The norm of F^i is a function of the separation distance, $F = F(d)$ where $d = \left|x^i_+ - x^i_-\right|$, and is expected to also depend on the magnitude of the gravitational field, $g = |g^i|$. The force F^i is assumed to be *attractive*, $F > 0$ (hence it is non-gravitational). This force is indispensable if we are looking for configurations in which the constituents of the dipole remain at constant distance from each other. For simplicity, we suppose that the gravitational force between the two sub-particles, i.e. $F_\mathrm{g} = -Gm^2/d^2$, which is repulsive, is negligible or included in the definition of F.

Next we introduce the centre of *inertial* masses, $x^i = \left(x^i_+ + x^i_-\right)/2$, and transform the system of equations (9)–(10) into an *equation of motion* for the 'dipolar particle',

$$2m \frac{\mathrm{d}^2 x^i}{\mathrm{d}t^2} = \pi^j \partial_{ij} U + \mathcal{O}(d^2), \tag{12}$$

and an *evolution equation* for the dipole moment,

$$\frac{\mathrm{d}^2 \pi^i}{\mathrm{d}t^2} = 2m g^i - 2F^i + \mathcal{O}(d^2). \tag{13}$$

In both equations (12) and (13), we neglect terms of the order of the square of the separation distance d, assuming that $d \ll |x^i|$. On the RHS of (12)–(13), g^i and $\partial_{ij} U$ are evaluated at the position of the centre-of-mass x^i. The torque produced by the forces acting on the two sub-particles on the RHS of (9)–(10) is given by

$$C^i = \varepsilon^{ijk} \pi^j g^k + \mathcal{O}(d^2). \tag{14}$$

This torque will tend to align the dipole moment with (and in the same direction of) the gravitational field.

The prominent feature of the equation of motion (12) is the violation of the equivalence principle by the dipolar particle, as a result of the fact that the particle's inertial mass is $2m$

while its gravitational mass is zero. Note that equation (12) has a structure different from the equation we obtain in the non-relativistic limit of the relativistic model of paper II—indeed, contrarily to the present quasi-Newtonian model, the relativistic model is consistent with the equivalence principle (see paper II for discussion). While the inertial and passive/active gravitational masses of the dipolar particle are given here by $M_{\mathrm{i}} = 2m$ and $M_{\mathrm{p}} = M_{\mathrm{a}} = 0$, the relativistic model of paper II will have $M_{\mathrm{i}} = M_{\mathrm{p}} = 2m$ and $M_{\mathrm{a}} = \mathcal{O}(c^{-2})$ in the non-relativistic limit $c \to \infty$. This represents a fundamental difference between the two models. In addition, of course, since the present model is Newtonian it does not *a priori* allow (in contrast to paper II) us to answer questions related to cosmology or the motion of relativistic particles. However, despite these differences, we shall recover in paper II the main characteristics of the mechanism of gravitational polarization.

From equation (12) the dipolar particle is expected to accelerate slowly in a given gravitational field, as compared to an ordinary particle. More precisely, we find that the particle is not directly subject to the gravitational field, but, rather, to its gradient, namely the *tidal gravitational field* $\partial_{ij}U$. In the potential $U \sim 1/R$ (respectively the MOND analogue $U \sim \ln R$), the acceleration is typically of the order of $1/R^3$ (respectively $1/R^2$).[2] The observational consequence is that the dark matter consisting of a fluid of dipole moments is necessarily cold, and even 'colder' than ordinary non-relativistic matter. This property may be consistent with the observation of galactic structures. Thus, the dipolar dark matter appears as a medium whose dynamics are weakly influenced by the distribution of ordinary galaxies.

On the other hand, the evolution equation (13) shows that a situation of equilibrium, where the distance d between the pair of particles remains constant, is possible. The equilibrium is realized when the internal force F^i exactly compensates for the gravitational force[3],

$$F^i = mg^i + \mathcal{O}(d^2). \tag{15}$$

Note that here as everywhere else, g^i represents the *total* gravitational field, the sum of the contributions due to the ordinary masses and the polarization masses. Because F^i is proportional to d^i, equation (11), we see that when the equilibrium holds, the dipole moment is in the direction of the gravitational field, $\pi^i = md^i \propto g^i$. The polarization vector $\Pi^i = n\pi^i$ is aligned with the gravitational field and the medium is polarized. Hence, the equilibrium condition (15) provides a mechanism for verifying the crucial equation (3).

Let us now get information on the susceptibility coefficient χ as a function of $g = |g^i|$. From equation (3) and the relation $\Pi = nmd$ [see (6)–(7)], we get

$$\chi = -4\pi G m n \frac{d}{g}. \tag{16}$$

The condition (15) implies that d is necessarily a function of g, say $d = d(g)$, obtained by inversion of the relation $F(d) = mg$ or, rather, since as we have seen the force should also depend on g, of the relation $F(d, g) = mg$. Thus, χ is a certain function of g, depending on the properties of the internal force F^i, and we are able, in principle, to relate the MOND function $\mu = 1 + \chi$ to the internal structure of the dipolar particles. We obtain

$$\mu\left(\frac{g}{a_0}\right) = 1 - 4\pi G m n \frac{d(g)}{g}. \tag{17}$$

Note that such a function is expected to be a complicated function of g, because it is made of the inverse of $F(d, g) = mg$, and especially because it depends on the spatial distribution of the

[2] This type of coupling to the tidal gravitational field is well known; for instance it corresponds to the non-relativistic limit of the coupling between the spin and the Riemann curvature tensor, for particles with spin moving on an arbitrary background [25, 26].

[3] In section 6 we shall be more precise about what is meant by equilibrium condition; see equation (32).

dipole moments, characterized by their number density n. The distribution of n is determined by the gravitational field *via* the equation of motion (12), together with the Eulerian continuity equation $\partial_t n + \partial_i(nv^i) = 0$, where $v^i = \mathrm{d}x^i/\mathrm{d}t$. However, as we have seen, the motion of the dipolar particle is sensitive in the first approximation only to the tidal gravitational field. Thus, a reasonable approximation is probably to consider that the velocity field v^i remains small; hence the number density n is nearly constant and uniform. In the following we shall neglect the tidal gravitational fields, and shall treat n as a constant.

5. Internal force law

In the limit $g \to 0$, we enter the deep MOND regime—a nonlinear regime characterized by $\mu = g/a_0 + \mathcal{O}(g^2)$.[4] Comparing with equation (17), we deduce that the dipole separation d should behave in terms of the gravitational field in the MOND regime like

$$d = \frac{g}{4\pi Gmn}\left[1 - \frac{g}{a_0} + \mathcal{O}(g^2)\right]. \tag{18}$$

Thus, in the first approximation, d is found to be proportional to the gravitational field (recall that n is assumed to be constant); this means that using $F = mg$ the force must be dominantly proportional to d. More precisely, we find that the force law $F(d)$ that is necessary to account for the MOND phenomenology is

$$F(d) = 4\pi Gm^2nd\left[1 + \frac{4\pi Gmn}{a_0}d + \mathcal{O}(d^2)\right]. \tag{19}$$

Interestingly, this force becomes weaker when the particles constituting the dipole moment get closer. As we see from (19), the MOND acceleration scale a_0 happens to parametrize, in this model, the expansion of the internal force at short distances, i.e. $d \to 0$ (in the regime where $g \to 0$).

Equation (19) represents the value of the force at equilibrium, i.e. when (15) is satisfied, and it depends on the number n of particles. However, the internal force F^i itself—not necessarily at equilibrium—is defined by the equations of motion (9)–(10) for a single dipole moment without reference to n, and in this sense is *intrinsic* to the dipole moment. Nevertheless, it seems unusual that the equilibrium force (19) should depend on the surrounding density n of the medium. In section 6, we shall show that this force is actually the one corresponding to a harmonic oscillator describing the oscillations of a 'gravitational plasma' at its natural plasma frequency. The dependence on n of equation (19) will then appear to be that involved into the usual expression [22] of the plasma frequency, as given by equation (29).

In addition, we want to recover the usual Newtonian gravity when $g \gg a_0$, and we see from equation (17) that it suffices that $\mathrm{d}(g)/g \to 0$ when $g \to \infty$. There is a large number of possibilities; many force laws $F(d)$ do it in practice. For instance we can adopt $d \propto g^{1-\epsilon}$ with any $\epsilon > 0$, which corresponds to the power force law $F(d) \propto d^\alpha$ with $\alpha = 1/(1-\epsilon)$, and we see that any powers α, except those with $0 \leqslant \alpha \leqslant 1$, are possible. In the discussion below we shall choose $\epsilon > 1$ in order to ensure that $d \to 0$ in the Newtonian regime. The particular case $\epsilon = 3/2$ gives back the $1/d^2$ type force law. In this case we can adopt for the susceptibility function

$$\chi = -\left(\frac{a_0}{g}\right)^{3/2}\left[1 + \mathcal{O}\left(\frac{1}{g}\right)\right]. \tag{20}$$

[4] For simplicity we assume a power-law expansion when $g \to 0$.

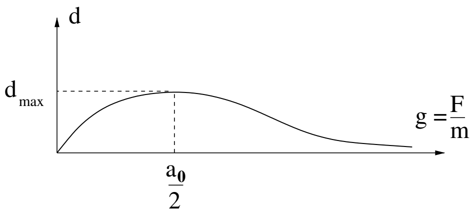

Figure 1. The dipolar separation distance d as a function of g. The equilibrium condition $F = mg$ is satisfied. Such a graph is valid when the number density n of dipole moments is constant. More generally, for non-constant n, the physically meaningful analogue of d would be the polarization $\Pi = nmd$ (cf figure 4 in paper II).

This corresponds to the dipolar separation

$$d = \frac{a_0}{4\pi Gmn}\left(\frac{a_0}{g}\right)^{1/2}\left[1 + \mathcal{O}\left(\frac{1}{g}\right)\right], \tag{21}$$

and internal force

$$F = ma_0\left(\frac{a_0}{4\pi Gmnd}\right)^2[1 + \mathcal{O}(d^2)]. \tag{22}$$

Note that because the expression (22) is positive, it represents a Coulombian force, i.e. attractive between unlike masses.

Since d tends to zero in both the Newtonian and MOND regimes, we see that the function $d \to F(d)$ is actually two-valued. We have already noted that F depends not only on d but also on g; the expression of the force (19) is valid in the MOND regime where $g \to 0$, while equation (22) holds in the Newtonian regime where $g \to \infty$. An alternative (but rather ad hoc) choice, encompassing both types of behaviour, is provided by the susceptibility function $\chi = -\mathrm{e}^{-g/a_0}$. In this case we have

$$d = \frac{g}{4\pi Gmn}\mathrm{e}^{-g/a_0}, \tag{23}$$

and the force law $F(d)$ at equilibrium is obtained by substituting $g = F/m$ on the RHS and looking for the two-valued inverse.

The generic form of the distance function $d(g)$ is illustrated in figure 1. From the figure we comment on the physical picture one might have in mind. In the absence of gravitational fields, i.e. in the absence of ordinary matter, the dipole moments do not exist (at least classically) since their separation d is zero. Indeed, when $g = 0$ we have $d = 0$ by equation (18). The dipolar ether does not produce any noticeable effect. Suppose that some external mass, made of ordinary matter, is steadily approached. The dipole moments start feeling a weak gravitational field and they become active. According to equation (18) and figure 1, the dipoles open up and get aligned in the gravitational field in order to maintain the equilibrium (15). The medium is polarized, we are in the MOND regime, and the gravitational field is dominated by the contribution of the polarization masses. Further approaching the ordinary mass, the dipolar separation d eventually reaches a maximal value. From (18) we find, approximately,

$$d_{\max} \approx \frac{a_0}{16\pi Gmn}, \tag{24}$$

which is reached for $g \approx a_0/2$. If at that point we continue to increase the gravitational field (putting nearer the external mass), d will begin to decrease and the dipole moments will close up. Finally, for strong gravitational fields, the dipole moments become inactive again (indeed $d = 0$ when $g \to \infty$). The gravitational field is dominated by the contribution of the ordinary matter, and we recover the Newtonian regime.

6. Gravitational plasma

The medium of dipolar dark matter described above can also be interpreted, in an alternative formulation, as a polarizable 'gravitational plasma' consisting of the two species of particles $(m_\mathrm{i}, m_\mathrm{g}) = (m, \pm m)$. Such formulation will essentially be equivalent to the previous one, but is simpler and more appealing physically.

Suppose that the two types of particles $(m, \pm m)$ are in equal numbers so that the plasma is globally neutral. There is an equilibrium point where the plasma is *locally* neutral, so the number *densities* of the two particle species are equal, at each point, to some common constant and uniform value n. As it stands the equilibrium is unstable because the gravitational force between unlike masses is repulsive. To ensure a stable equilibrium we must postulate like in equations (9)–(10) some restoring non-gravitational force F^i, acting between the masses $m_\mathrm{g} = \pm m$, and superseding the gravitational force. We introduce some associated internal field f^i such that

$$F^i = -m f^i, \tag{25}$$

and assert that this field obeys a Gauss law in the non-relativistic limit,

$$\partial_i f^i = -\frac{4\pi G m}{\chi}(n_+ - n_-). \tag{26}$$

The number densities of the particles species $(m, \pm m)$ are denoted by $n_\pm$. Here we have introduced the susceptibility coefficient χ to represent a dimensionless coupling constant characterizing the internal interaction. Note that since χ is negative, equation (8), the force law (26) is attractive between unlike particles and repulsive between like ones. Furthermore, supposing that the plasma is bathed by an external gravitational field g^i, constant and uniform over some region of consideration[5], we expect that the coupling constant should reflect the presence of this gravitational field, and we assume a dependence on its norm g, namely $\chi = \chi(g)$.

Let x^i_+ and x^i_- (in short $x^i_\pm$) be the displacement vectors of the masses from the equilibrium position characterized by the density n. The particles are accelerated by the internal field f^i and by the applied external gravitational field g^i. The equations of motion have already been given in equations (9)–(10) and now read

$$m\frac{\mathrm{d}^2 x^i_\pm}{\mathrm{d}t^2} = \pm m(f^i + g^i). \tag{27}$$

Consider a small departure from equilibrium, corresponding to small displacement vectors $x^i_\pm$. The density perturbations are given by $n_\pm = n\big(1 - \partial_i x^i_\pm\big)$ to first order in the displacements $x^i_\pm$. Using equation (26), we readily integrate for f^i and inject the solution into (27). In this way we find that $x^i_+ + x^i_- = 0$ (the centre of inertial masses is at rest—neglecting tidal fields), together with the following harmonic oscillator for $\pi^i = m d^i$ where $d^i = x^i_+ - x^i_-$,

$$\frac{\mathrm{d}^2\pi^i}{\mathrm{d}t^2} + \omega^2\pi^i = 2mg^i. \tag{28}$$

Actually this computation is the classic derivation of the plasma frequency [22] which is found for the case at hand to be

$$\omega = \sqrt{-\frac{8\pi G m n}{\chi}}. \tag{29}$$

[5] We adopt the frame associated with the plasma's equilibrium configuration. The gravitational field g^i is defined in that frame.

The frequency depends on the density n of the plasma at equilibrium and on the strength of the internal interaction which is encoded into the coupling constant χ. The solution we get for the internal field is

$$f^i = -\frac{\omega^2}{2m}\pi^i \quad \Longrightarrow \quad F^i = \frac{\omega^2}{2}\pi^i, \tag{30}$$

so we can check that equation (28) is equivalent to our previous equation (13) (recall that here g^i is considered to be constant and uniform). The particles oscillate around some non-zero mean position that is determined by the ambiant gravitational field as

$$\langle\pi^i\rangle = \frac{2m}{\omega^2}g^i. \tag{31}$$

The mean value of the force is

$$\langle F^i\rangle = mg^i, \tag{32}$$

and coincides with the equilibrium condition postulated in equation (15).

This result (31) is classical but can easily be recovered from standard quantization of the harmonic oscillator (28). The spectrum of 'plasmons' is discrete ($k \in \mathbb{N}$) with energy levels shifted by the gravitational field [27],

$$E_k = \left(k + \frac{3}{2}\right)\hbar\omega - \frac{mg^2}{2\omega^2}. \tag{33}$$

The eigenstates are $|\psi_{p_1p_2p_3}\rangle = |\phi_{p_1}\rangle|\phi_{p_2}\rangle|\phi_{p_3}\rangle$ where $k = p_1 + p_2 + p_3$; the eigenvalues have degeneracy $(k+1)(k+2)/2$. The one-dimensional eigenstate function $\phi_{p_i}(\pi^i) = \langle\pi^i|\phi_{p_i}\rangle$ is of the form $\varphi_{p_i}\left(\pi^i - \frac{2m}{\omega^2}g^i\right)$ where (H_p being the Hermite polynomial)

$$\varphi_p(\rho) = \frac{1}{2^{\frac{p}{2}}\sqrt{p!}}\left(\frac{\omega}{\pi\hbar m}\right)^{1/4} H_\mathrm{p}\left(\sqrt{\frac{\omega}{\hbar m}}\rho\right)\mathrm{e}^{-\frac{\omega}{2\hbar m}\rho^2}. \tag{34}$$

The expectation value $\langle\pi^i\rangle = \langle\psi_{p_1p_2p_3}|\pi^i|\psi_{p_1p_2p_3}\rangle$ does not vanish due to the presence of the gravitational field; it can be computed using $\int_{-\infty}^{+\infty}\mathrm{d}\pi\,\pi\left|\varphi_p\left(\pi - \frac{2m}{\omega^2}g\right)\right|^2 = \frac{2m}{\omega^2}g$ and the result agrees with (31).

We analyse the effect of the mean dipole moment vector (31) on the equation for the gravitational field g^i (now supposed to be generated by external sources). More precisely we argue that the mean value is really a quantum expectation value as we have just proved, so that by adopting a 'semi-classical' approach this expectation value $\langle\pi^i\rangle$ should have a direct effect as the source for the *classical* gravitational field g^i. The polarization readily follows from the expression of the plasma frequency (29) as

$$\Pi^i = n\langle\pi^i\rangle = -\frac{\chi}{4\pi G}g^i, \tag{35}$$

so that we recover equation (3) exactly. Note that the constant density n of the plasma at equilibrium is equal to the density of dipole moments. The polarization is automatically proportional to the gravitational field. This is an attractive feature of the present formulation based on a gravitational plasma; there is no need to invoke a mechanism to align Π^i with g^i. Following the same reasoning as in section 2, the polarization (35) gives rise to the density of polarization masses (4), which is added to the RHS of the Poisson law, equation (5). The MOND equation (1) readily follows.

Let us comment on the field equation (26) for the supposedly fundamental internal interaction f^i. In the absence of gravity we have $\chi = -1$; hence equation (26) becomes that of a *negative* Poisson equation corresponding to a negative Newtonian force (with 'anti-gravitational' constant $-G$). On the other hand, in strong gravity ($g/a_0 \to \infty$) we have $\chi \to 0$ and we see that the strength of the internal field f^i becomes infinite. Therefore the plasma gets locked in its undisturbed equilibrium state for which $\pi^i = 0$ (strictly speaking this is true if $g\chi \to 0$ when $g \to \infty$). In this limit the dipolar medium is inactive; there is no induced polarization and the Newtonian law holds. Note that there is clearly some amount of fine tuning in the present model (and the one of paper II). Namely the function $\chi(g)$ is not determined from first principles within the model but is tuned to agree with the phenomenology of MOND. In particular $\chi(0) = -1$ is not 'explained' in the model but comes from astronomical observations. Similarly $\chi(\infty) = 0$ is imposed in order to recover the Newtonian regime where it is observed to be valid.

To conclude, the phenomenology of MOND suggests the existence of a polarization mechanism at work at the scale of galactic structures, and which could be viewed as the gravitational analogue of the electric polarization of a dielectric material. The dark matter would consist of the polarization masses associated with some gravitational dipole moments aligned with the gravitational field of ordinary masses. We find that the properties of the dipolar dark matter are governed by the internal non-gravitational force linking together the constituents of the dipolar medium. The formulation of this medium in terms of an oscillating gravitational plasma polarized in the gravitational field is particularly attractive. In paper II we shall show how the notion of dipolar particle can be made compatible with the framework of general relativity.

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