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Gravitational radiation and the stability of rotating stars

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Summary. It has been noted that all rotating stars are unstable to the radiation of gravitational waves by non-radial stellar modes. However small the rate of stellar rotation, for each set of modes (for example the p-modes) there is always a critical azimuthal wavenumber m_{crit}, such that modes with $m > m_{\mathrm{crit}}$ are unstable. Using a scalar theory of gravitation we calculate the growth rate of the instability explicitly in the slow-motion regime. We find that the instability grows on astronomically interesting timescales only for neutron stars with rotational periods $\lesssim$ a few milliseconds. We indicate why general relativity is likely to yield the same conclusion. We speculate briefly on a situation in which an accreting neutron star can radiate a large fraction of the accretion luminosity as gravitational waves.

1 Introduction

Some attention has focused recently on an instability found to exist in the General Theory of Relativity which shows in particular that all rotating stars are unstable (Friedman & Schutz 1975; Bardeen *et al.* 1977; Friedman 1977). The nature of the instability is relatively easy to understand. Consider a particular stellar mode of frequency ω and with an azimuthal wavenumber m. If the star rotates slowly with angular velocity Ω the mode is split into two modes, with frequencies differing by $\sim 2m\Omega$, one of which travels forwards (in the sense of rotation) relative to the star, and the other backwards. If $m\Omega \ll \omega$ the modes rotate in the same sense relative to the star as they do in an inertial frame. The effect of gravitational radiation on these modes is to radiate positive (negative) angular momentum from the forward (backward) travelling mode and so to damp them both. If, however, we increase Ω sufficiently (or, equivalently, if we consider modes of sufficiently high m) such that $m\Omega \gtrsim \omega$ we may find that the mode moving backwards relative to the star is now moving forwards relative to an inertial frame. The effect of gravitational radiation is now to remove *positive* angular momentum from the mode. Since the mode moves backwards relative to the star its response to this (in linear theory) is to grow.

Such instabilities are not limited to the effects of gravitational radiation – for example the sound waves produced by a water wave which moves backwards relative to the stream, yet forwards relative to the air have a similar effect. They occur, therefore, in studies of the Kelvin–Helmholtz instability (see, e.g. Gill 1965).

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We have noted that for a given star with $\Omega \neq 0$, modes with sufficiently high m are susceptible to the instability, and that therefore all rotating stars must display this instability (Bardeen *et al.* 1977). This does not mean, contrary to the assertion by Thorne (1977), that the modes with the largest m are the most unstable. Estimates of the growth timescale τ_g for the modes have been given in the literature (Bardeen *et al.* 1977; Friedman & Schutz 1977) and may be written as

$$\tau_g \sim \tau_d \left(\frac{c}{v}\right)^{2m+1} \tag{1.1}$$

where τ_d is the dynamical timescale of the star and v is, for example, its surface rotational velocity ($v = a\Omega$ where a is the stellar radius).

Computation of the growth timescale using general relativity is difficult, not just because of algebraic complexity but also because of the intrinsic non-linearity of that theory. For this reason we have considered the same problem using scalar gravity. Since the problem is now linear we can write down a complete solution for the slow motion regime (Section 2). We indicate in the Appendix why we are confident that the solution of the general relativistic problem will lead to a similar functional form of solution, albeit with different numerical coefficients. We find that (1.1) is a reasonable approximation when $am\Omega \ll c$. We consider the implications of our findings in Section 3 and show that only for neutron stars spinning with periods $P \lesssim$ a few milliseconds is the instability of physical importance. Accreting neutron stars can be spun up by accreted material. We speculate that spin up is curtailed by the growth of this instability. We consider a particular example in which a substantial fraction of the accretion luminosity is radiated away as gravitational waves.

2 Scalar gravity

The equations for the fluid are the standard inviscid Navier–Stokes equations

$$\frac{D\mathbf{v}}{Dt} = -\frac{1}{\rho}\nabla p - \nabla\psi \tag{2.1}$$

$$\frac{D\rho}{Dt} = -\operatorname{div}(\rho v) \tag{2.2}$$

$$\frac{Dp}{Dt} = \frac{\gamma p}{\rho}\frac{D\rho}{Dt}. \tag{2.3}$$

Here $\mathbf{v}$ is the fluid velocity, p the pressure, ρ the density, γ a constant specific heat ratio and ψ the scalar gravitational potential. We take ψ to satisfy the scalar wave equation

$$\nabla^2\psi - \frac{1}{c^2}\frac{\partial^2\psi}{\partial t^2} = 4\pi G\rho. \tag{2.4}$$

We perturb the equations about a stationary configuration which rotates with constant angular velocity Ω. For simplicity, we take the equilibrium configuration to have coincident pressure and density surfaces. We assume that the perturbed quantities, which we denote by dashes, vary as $\exp\{i\sigma t + im\phi\}$. We use the Lagrangian displacement $\boldsymbol{\xi}$, which is related to the Eulerian velocity variation $\mathbf{v}'$ by $\mathbf{v}' = i(\sigma + m\Omega)\boldsymbol{\xi}$. Equations (2.1)–(2.3) then give

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$$\rho' = -\operatorname{div}(\rho\boldsymbol{\xi}) \tag{2.5}$$

$$p' = \frac{\gamma p}{\rho}\rho' - \boldsymbol{\xi}\cdot\left(\nabla p - \frac{\gamma p}{\rho}\nabla\rho\right) \tag{2.6}$$

$$-(\sigma + m\Omega)^2\boldsymbol{\xi} + 2\mathrm{i}(\sigma + m\Omega)\,\boldsymbol{\Omega}\times\boldsymbol{\xi} = -\frac{1}{\rho}\nabla p' + \frac{\rho'}{\rho^2}\nabla p - \nabla\psi'. \tag{2.7}$$

Equation (2.4) becomes

$$\nabla^2\psi' + \frac{\sigma^2}{c^2}\psi' = 4\pi G\rho'. \tag{2.8}$$

Since we are looking for a growing mode we shall assume σ to have a small negative imaginary part. We then seek solutions of the perturbed wave equation which vanish at infinity. We multiply (2.7) by $\rho\boldsymbol{\xi}^*$ (* denotes complex conjugate) integrate over all space and do several integrations by parts, making use of (2.5) and (2.6). We then multiply (2.8) by $-\psi'^*/4\pi G$, integrate over all space, and add the resulting equation to that obtained from (2.5)–(2.7). This yields an equation of the form

$$-\sigma^2 A + \sigma B = C \tag{2.9}$$

where each of A, B and C are real. Note that convergence of the integrals is assured only when σ has a negative imaginary part.

We write

$$A = A_0 + W$$

where

$$A_0 = \int \rho\,|\boldsymbol{\xi}|^2\,d\tau$$

and

$$W = \int \frac{|\psi'|^2\,d\tau}{4\pi G c^2}.$$

We find

$$B = 2\int \rho\boldsymbol{\xi}^*\cdot[\mathrm{i}\boldsymbol{\Omega}\times\boldsymbol{\xi} - m\Omega\boldsymbol{\xi}]\,d\tau$$

and

$$C = m^2\Omega^2\int\rho\,|\boldsymbol{\xi}|^2\,d\tau - \int\gamma p\,|\operatorname{div}\boldsymbol{\xi}|^2\,d\tau - \int(\boldsymbol{\xi}\cdot\nabla p)\operatorname{div}\boldsymbol{\xi}^*\,d\tau - \int(\boldsymbol{\xi}^*\cdot\nabla p)\operatorname{div}\boldsymbol{\xi}\,d\tau$$
$$-\int(\boldsymbol{\xi}^*\cdot\nabla p)(\boldsymbol{\xi}\cdot\nabla\rho)\frac{d\tau}{\rho} - 2\mathrm{i}m\Omega\int(\boldsymbol{\Omega}\times\boldsymbol{\xi})\cdot\boldsymbol{\xi}^*\rho\,d\tau$$
$$+\int\psi'\operatorname{div}(\rho\boldsymbol{\xi}^*)\,d\tau + \int\psi'^*\operatorname{div}(\rho\boldsymbol{\xi})\,d\tau - \int\frac{|\nabla\psi'|^2}{4\pi G}\,d\tau.$$

We also write

$$C = C_0 - W_1 + W_2$$

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where

$$W_1 = \int \frac{|\nabla \psi'|^2 d\tau}{4\pi G}$$

$$W_2 = \int \frac{|\nabla \Psi|^2 d\tau}{4\pi G}$$

and by definition Ψ satisfies Poisson's equation

$$\nabla^2 \Psi = 4\pi G \rho'.$$

We then rewrite (2.9) as

$$-\sigma^2 A_0 + \sigma B = C_0 + \sigma^2 W - W_1 + W_2. \tag{2.10}$$

We now write $\sigma = \sigma_0 - i\gamma$ and assume $\gamma \ll \sigma_0$. In the non-radiating case ($c \to \infty$) the real part of (2.10) becomes

$$-\sigma_0^2 A_0 + \sigma_0 B = C_0.$$

We have therefore,

$$\sigma_0 = \frac{B}{2A_0} \pm \sqrt{\frac{-C_0}{A_0} + \frac{B^2}{4A_0^2}}. \tag{2.11}$$

Corrections to this from the W-terms are of order c^{-1} and we may neglect them. Furthermore, to the same order of accuracy, A_0, B and C_0, can be evaluated with the eigenfunctions for $c^{-1} = 0$. The imaginary part may be written

$$\frac{-W\sigma_0}{\pm A_0\sqrt{-C_0/A_0 + B^2/4A_0^2}} = 1. \tag{2.12}$$

At first sight, this equation appears to be of little use since it does not appear to contain γ. However since we expect $\psi' \propto \exp(-\gamma r/c)$ for large r, we note that the integral $W \propto |\gamma|^{-1}$. Thus for small γ we may write

$$|\gamma| = \frac{-\sigma_0}{\pm A_0\sqrt{-C_0/A_0 + B^2/4A_0^2}} \lim_{\gamma\to 0} (|\gamma| W). \tag{2.13}$$

We note at this juncture that for self-consistency the right-hand side of (2.13) must be positive. For this we require that the sign of σ_0 is not the same as that of the square root (for the appropriate choice of its sign). This is possible only if $C_0 > 0$, which is just the secular stability criterion evaluated in the inertial frame. We also remark that (2.12) is just a statement of the physical fact that the wave energy permeating all space is minus the mode energy (which is negative), as indeed it must be for such a mode which has grown at the expense of radiation.

We now evaluate γ for a rotating star. The solution of (2.8) which satisfies the appropriate boundary conditions at $r = 0$ and $r = \infty$ is

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$$\psi' = \sum_{l,m} 4\pi G\left(\frac{2l+1}{4\pi}\right)\frac{(l-m)!}{(l+m)!}\frac{i\pi}{2}P_l^m(\mu)\exp(im\phi)\times\iint\Big\{F_l(r)\int_0^\infty F_l(r')\rho'(\mathbf{r}')r'^2dr'$$

$$-i(-1)^{l+1}\left[G_l(r)\int_0^{r'}F_l(r')\rho'(\mathbf{r}')r'^2dr' + F_l(r)\int_r^\infty G_l(r')\rho'(\mathbf{r}')r'^2dr'\right]\Big\}$$

$$\times P_l^m(\mu')\exp(-im\phi')\,d\mu'\,d\phi' \tag{2.14}$$

where

$\mu = \cos\theta$

$F_l(r) = r^{-1/2}J_{l+1/2}(\sigma r/c)$

and

$G_l(r) = r^{-1/2}J_{-(l+1/2)}(\sigma r/c)$.

Here $P_l^m(\mu)$ is the Legendre polynomial and $J_l(z)$ is the Bessel function. At large distances ($r \gg cl/\sigma$) this becomes,

$$\psi' \sim \sum_{l,m} 4\pi G\left(\frac{2l+1}{4\pi}\right)\frac{(l-m)!}{(l+m)!}\,i\left(\frac{\pi c}{2\sigma}\right)^{1/2}P_l^m(\mu)\exp(im\phi)\times\iint\left\{\int_0^\infty F_l(r')\rho'(\mathbf{r}')r'^2dr'\right\}$$

$$\times P_l^m(\mu')\exp(-im\phi')\,d\mu'\,d\phi'\times r^{-1}\exp\left[-i\left(\frac{\sigma r}{c}-\tfrac{1}{2}(l+1)\pi\right)\right]. \tag{2.15}$$

Thus we find that

$$\lim_{\gamma\to 0}\sigma_0|\gamma|W = \pi^2 G\sum_{l,m}\left(\frac{2l+1}{4\pi}\right)\frac{(l-m)!}{(l+m)!}\times\left|\iiint\rho'(\mathbf{r}')J_{l+1/2}\left(\frac{\sigma_0 r'}{c}\right)r'^{3/2}P_l^m(\mu')\right.$$

$$\left.\times\exp(-im\phi')\,dr'\,d\mu'\,d\phi'\right|^2. \tag{2.16}$$

This, together with (2.13), gives us a general formula for γ. As an example let us consider the oscillations of a homogeneous, incompressible sphere of density ρ_0 and radius a. In this case we have

$\boldsymbol{\xi} = \nabla[r^l P_l^m(\mu)\exp(im\phi)]$

and

$\rho'(\mathbf{r}') = \rho_0\delta(r'-a)\,\xi_r(r',\mu',\phi')$.

Evaluating (2.16) we find

$$\gamma = \frac{\pi^2 G\rho_0 l\,[J_{l+1/2}(\sigma_0 a/c)]^2}{\sqrt{-C_0/A_0 + B^2/4A_0^2}}. \tag{2.17}$$

Confidence in (2.17) is obtained by noting that the case of an incompressible rotating cylinder of constant density, the dispersion relation can be obtained explicitly. For small γ this gives

$$\gamma = \frac{\pi^2 G\rho_0 m\,[J_m(\sigma_0 a/c)]^2}{\sqrt{-C_0/A_0 + B^2/4A_0^2}} \tag{2.18}$$

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which is closely analogous to (2.17). We note that for large m,

$$|\sigma_0| \sim |m|\Omega \quad \text{and} \quad \sqrt{-C_0/A_0 + B^2/4A_0^2} \sim \sigma_0 + |m|\Omega.$$

3 Application

The growth time of the instability τ_g is a rapidly decreasing function of v/c where $v = a\Omega$ is the rotational velocity of the stellar surface. For this reason, the objects for which the instability is most likely to be important are neutron stars (the only objects more compact than these are black holes, which are known to be stable (see, e.g. Thorne 1977). We take the f-modes of a Newtonian homogeneous star as being representative, and we assume that the eigenfrequencies, σ, are well approximated by

$$(\sigma + m\Omega)^2 = \frac{2GMl(l-1)}{a^3(2l+1)}. \tag{3.1}$$

This implies a shift of $m\Omega$ on the purely spherical modes. For instability we require that

$$m^2\Omega^2 \geqslant \frac{2GMl(l-1)}{a^3(2l+1)}. \tag{3.2}$$

Since $l > |m|$ and since τ_g is a rapidly increasing function of l, we consider the most rapidly growing modes, that is, those for which $l = m$. Equation (3.2) may then be written

$$l \gtrsim \left(\frac{GM}{ac^2}\right)\cdot\left(\frac{c}{v}\right)^2. \tag{3.3}$$

For neutron stars we take $GM/ac^2 = 0.2$. Equation (2.8) then gives the growth time approximately as

$$\tau_g = \tau_d\,[J_{l+1/2}(lv/c)]^{-2} \tag{3.4}$$

where l satisfies (3.3) and for neutron stars we take $\tau_d = 10^{-4}$ s. Note that this formula corresponds to (1.1) when $am\Omega \ll c$. The growth time is shown as a function of c/v in Table 1. The rapidity of the increase of τ_g as a function of c/v (or rotation period) is well illustrated. The most rapidly spinning neutron star known (the Crab pulsar with rotation period of 33 ms) has $c/v \sim 160$ and a growth timescale to the instability which is truly astronomical!

Table 1.

c/v	m	τ_g (s)
4	3	4.0
5	5	2.7×10^3
6	7	1.2×10^7
7	10	1.8×10^{12}
8	13	1.3×10^{18}
160	5×10^3	$\exp(4.8 \times 10^4)$

The growth rate of the instability τ_g is tabulated as a function of c/v where $v = a\Omega$ is the rotational velocity of the stellar surface. The critical azimuthal wavenumber, m, above which all modes are unstable is also shown. For a given c/v, τ_g is a rapidly increasing function of m so that the fastest growing mode is the one for the value of m shown in the table.

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For the modes considered we see that for neutron stars spinning with periods $P \gtrsim 1.5$ ms, the instability is of no physical importance. A similar calculation can be made for the p-modes, and although the growth times will differ from those for the f-modes the conclusion that for neutron stars spinning with periods $\gtrsim$ a few milliseconds the instability is of no importance is still valid. Moreover since, when the gravitational radiation is calculated properly according to the general theory of relativity, the functional form for τ_g is unlikely to be altered, the same conclusion probably holds in general.

Although the instability is of no importance for any *known* object in the Universe, we speculate briefly on a situation in which the instability may play a role. Consider a non-magnetic neutron star which is accreting matter from an accretion disc on a timescale $\tau_a = M/\dot{M}$. The neutron star is spun up until it reaches a rotation period for which the timescale on which it accretes angular momentum $\sim \tau_a(a^3\Omega^2/GM)^{1/2}$ is equal to the timescale on which it can radiate it away $\sim \tau_g$. For an accretion luminosity of $L_a \sim 10^{38}$ erg/s, we find $\tau_a \sim 10^8$ yr and, from Table 1, the equilibrium rotation period is $\sim 1\frac{1}{2}$ ms. The fastest growing mode has $m \approx 10$. In this situation the star radiates gravitational radiation with a luminosity $L_G \sim (a^3\Omega^2/GM)^{1/2} L_a \sim 3 \times 10^{37}$ erg/s at a frequency of $\sim 6 \times 10^3$ Hz.

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Appendix: generalization to other wave fields

We have calculated the growth rate for the case of scalar gravity. The procedure can be generalized to other wave fields including general relativity. The algebra is tedious and we just present the results, which will be seen to be physically reasonable.

It turns out that the expression for the growth rate can be generalized to the form

$$\gamma = -\lim_{\gamma \to 0} [\gamma U] / \{2A \pm \sigma_0 \sqrt{-C/A + B^2/4A^2}\} \tag{A.1}$$

where U is the total energy in waves. In the case of general relativity we write the metric tensor

$$g_{\alpha\beta} = \eta_{\alpha\beta} + h_{\alpha\beta}$$

where

$$h_{\alpha\beta} = h^{(0)}_{\alpha\beta} + h'_{\alpha\beta}.$$

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We also choose the gauge so that

$$\partial/\partial x^{\alpha}(\eta^{\alpha\beta}\tilde{h}_{\rho\beta})=0$$

where

$$\tilde{h}_{\rho\kappa}=h'_{\rho\kappa}-\tfrac{1}{2}\eta_{\rho\kappa}\eta^{\alpha\beta}h'_{\alpha\beta},$$

(see, e.g. Weinberg 1972).

Then the gravitational wave energy can be written (Misner, Thorne & Wheeler 1973)

$$U=\frac{4\sigma_0^2c^2}{64\pi G}\int\sum_{p,q}[|\tilde{h}_{pq}|^2\eta^{pp}\eta^{qq}-\tfrac{1}{2}|\tilde{h}_{pq}\eta^{pq}|^2]\,d\tau$$

and thus

$$\gamma=\lim_{\gamma\to 0}\frac{\gamma\int\sum_{p,q}[|\tilde{h}_{pq}|^2\eta^{pp}\eta^{qq}-\tfrac{1}{2}|\tilde{h}_{pq}\eta^{pq}|^2]\,d\tau\cdot\sigma_0c^2}{\pm 32\pi GA\sqrt{-C/A+B^2/4A^2}}.$$

The $\tilde{h}_{pq}$ satisfy the equations

$$\nabla^2\tilde{h}_{\rho\kappa}+\sigma^2\tilde{h}_{\rho\kappa}=\frac{16\pi G}{c^2}S'_{\rho\kappa}.$$

The difficulty of non-linearity now manifests itself by the fact that the $S'_{\rho\kappa}$ contains the $\tilde{h}_{\rho\kappa}$. However, the lowest order contribution to each $S_{\rho\kappa}$ value inside the light cylinder can be evaluated as: $(i,j=1\text{–}3)$

$$S_{00}=\rho';\quad S_{0i}=(\rho v_i/c)'$$

$$S_{ij}=\left[\frac{\rho v_iv_j}{c^2}\right]'+\frac{p'}{c^2}\delta_{ij}-\frac{1}{4\pi Gc^2}\left[\frac{\partial\phi}{\partial x^i}\frac{\partial\phi}{\partial x^j}+2\phi\frac{\partial^2\phi}{\partial x^i\partial x^j}\right]'+\frac{\delta_{ij}}{2\pi Gc^2}[\phi\nabla^2\phi+\tfrac{3}{4}(\nabla\phi)^2]'$$

where ϕ is the standard Newtonian potential which satisfies Poisson's equation.

Each of the $\tilde{h}_{\rho\kappa}$ contributes an integral to the growth rate similar to the scalar gravity case. In fact if we ignore all $\tilde{h}_{\rho\kappa}$ but $\tilde{h}'_{00}$ we get precisely the scalar gravity result. The other $\tilde{h}_{\rho\kappa}$ contribute terms of comparable magnitude. This can be seen both by examining the gauge condition and by inspecting the resulting integrals directly. We thus expect the only difference between this case and the scalar gravity case to be reflected in a numerical coefficient.

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