GRAVITATIONAL RADIATION DAMPING*

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An analysis is presented of the emission of gravitational waves by a star in nonradial pulsation. This analysis, rigorous to first order in the pulsation amplitude and to all orders in G and c, exhibits damping of the pulsation as the waves are emitted (gravitational radiation reaction).

There is a long-standing uncertainty and controversy in general relativity over radiation damping: A number of people have analyzed the emission of gravitational waves by binary star systems and have reached varying conclusions about the amount of energy radiated and the effect of the radiation on the emitting bodies. Some analyses predict damping of the orbital motion[1]; others predict "antidamping"[2]; and still others predict that the orbital motion is unaffected.[3,4] Even those analyses which agree qualitatively usually disagree quantitatively. Why is there such great discrepancy? Because of the following: (i) The analyses make use of approximation methods (usually involving δ-function idealizations of the stars) which are of uncertain validity; (ii) the calculations are enormously complicated and are full of potential sources of error; (iii) the final mathematical answers emerge in such complicated form that their physical meaning is difficult to discern.

The purpose of this Letter is to present an analysis of gravitational radiation damping which is much simpler and more reliable than previous analyses. The new analysis differs from previous work in these major respects: (i) Instead of studying binary star systems where radiation damping is extremely small, it treats a neutron star in nonradial pulsation where radiation damping is extremely large; (ii) instead of idealizing the radiating star as a δ-function singularity and working solely with the vacuum field equations, it employs a realistic stellar model and solves simultaneously for the fluid motions, the near-zone oscillations in the gravitational field, and the far-zone gravitational waves; (iii) instead of doing a perturbation expansion about flat, empty space-time, it expands about the strongly curved space-time of the fully relativistic, unperturbed stellar model; (iv) the approximation methods on which it relies are not complex and uncertain, rather they are the well-understood methods by which one studies the quantum mechanical leakage of a particle out of a potential well.

The main result of the analysis is a solution for the nonradial oscillations of a star, valid to first order in the amplitude, in which gravitational waves flow out radially, in which the fluid motions exhibit gradual damping ($\delta\vec{r} \propto \cos(\sigma t)e^{-t/\tau}$), and in which the power carried by the waves and the rate of decrease of pulsation energy, calculated independently, are found to be equal.

The equations of motion for a star in nonradial pulsation have been worked out recently by Thorne and Campolattaro (paper I)[5] and extended by Price and Thorne (paper II).[6] For pulsations of spherical harmonic indices l and m and parity $\pi = (-1)^l$, the gravitational field takes the form

$$ds^2 = e^{\nu}(1 + H_0 Y_m{}^l)dt^2 + 2H_1 Y_m{}^l dtdr - e^{\lambda}(1 - H_2 Y_m{}^l)dr^2 - r^2(1 - KY_m{}^l)(d\theta^2 + \sin^2\theta d\varphi^2). \quad (1)$$

This line element corresponds to a particular, uniquely defined choice of coordinates. Here ν and λ are the functions of r which characterize the unperturbed star; $Y_m{}^l(\theta, \varphi)$ is the ordinary scalar spherical harmonic; and H_0, H_1, H_2, K are functions of r and t which describe the perturbation in the gravitational field. The stellar pulsation is described not only by the gravitational amplitudes H_0, H_1, H_2, and K, but also by amplitudes W and V for the radial and tangential displacements of the fluid

$$\delta r = \frac{e^{-\lambda/2}}{r^2} W Y_m{}^l; \quad \delta\theta = -\frac{V}{r^2}\partial_\theta Y_m{}^l;$$
$$\delta\varphi = -\frac{V}{r^2}\frac{\partial_\varphi Y_m{}^l}{\sin^2\theta}. \quad (2)$$

Equations (1) and (2) and the analysis which follows are accurate to first order in the amplitudes H_0, H_1, H_2, K, W, V.

Of the six perturbation functions only K, W, V represent true dynamical degrees of freedom. H_0, H_1, H_2 are fixed in terms of K, W, V by certain initial-value equations. The stellar pulsation and the emission of gravitational waves are governed by hyperbolic differential equations (the perturbed, dynamical Einstein field equations) of the form

$$(\partial^2/\partial t^2)\{K, W, V\} = \mathcal{L}\{K, W, V\}, \tag{3}$$

plus certain boundary conditions. Here $\mathcal{L}$ is a particular third-order, linear differential operator in r [cf. paper I, Eq. (14); also paper II, Eq. (6a)].

By a combination of analytical and numerical techniques, the equations of motion (3) have been solved for several realistic models of neutron stars. The method of analysis is similar to, but differs slightly from, the method proposed in paper I: One first splits the pulsations into complex normal modes

$$\{K, W, V\} = \{K_\omega(r), W_\omega(r), V_\omega(r)\}e^{i\omega t}. \tag{4}$$

The complex eigenfunctions satisfy the eigenvalue equation

$$\mathcal{L}\{K_\omega, W_\omega, V_\omega\} = -\omega^2\{K_\omega, W_\omega, V_\omega\}. \tag{5}$$

Far from the star (radiation zone) the eigensolution represents a mixture of ingoing and outgoing gravitational waves

$$K_\omega = C^{(\mathrm{O})}\exp\{-i\omega[r+2M\ln(r-2M)]\} + C^{(\mathrm{I})}\exp\{+i\omega[r+2M\ln(r-2M)]\}, \tag{6}$$

which takes the alternative form, in a better behaved coordinate system ("radiation gauge"; cf. paper II)

$$g_{\mu\nu} = (g_{\mu\nu})_{\text{unperturbed}} + h_{\omega\mu\nu}e^{i\omega t},$$

$$\frac{h_{\omega\theta\theta}}{r^2} = -\frac{h_{\omega\varphi\varphi}}{r^2\sin^2\theta} = \frac{i}{\omega r}\left[\frac{2l+1}{4\pi}\frac{(l-m)!}{(l+m)!}\right]^{1/2}\left[\frac{l(l+1)}{2}P_l^m + \frac{\partial^2 P_l^m}{\partial\theta^2}\right]e^{im\varphi}K_\omega(r),$$

$$\frac{h_{\omega\theta\varphi}}{r^2\sin\theta} = -\frac{m}{\omega r}\left[\frac{2l+1}{4\pi}\frac{(l-m)!}{(l+m)!}\right]^{1/2}\left[\frac{\partial P_l^m/\partial\theta}{\sin\theta} - \frac{\cos\theta}{\sin^2\theta}P_l^m\right]e^{im\varphi}K_\omega(r),$$

$$\text{(all other physical components of } h_{\omega\mu\nu}) = O(1/r^2) \text{ or smaller.} \tag{7}$$

Here M is the star's mass in "geometrized units" ($c = G = 1$). This approximates at large r the canonical form for a weak gravitational wave in flat space.[7]

We concentrate attention on the "poles of the S matrix"–i.e., on those complex frequencies $\omega = \sigma + i/\tau$ at which the ingoing amplitude $C^{(\mathrm{I})}$ vanishes. For realistic stellar models these poles lie very close to the real frequency axis (see end of this paper for a concrete example). Consequently, on the real frequency axis near a particular pole one has

$$C^{(\mathrm{I})} = \alpha e^{i\beta}(\omega-\sigma-i/\tau),$$
$$C^{(\mathrm{O})} = C^{(\mathrm{I})*} = \alpha e^{-i\beta}(\omega-\sigma+i/\tau). \tag{8}$$

(The latter equality results from the fact that real ω implies real K_ω.) Here α and β are real constants which characterize the residue at the pole of the S matrix. By combining Eqs. (6) and (8), one obtains

$$K_\omega = 2\alpha[(\omega-\sigma)^2 + 1/\tau^2]^{1/2} \times \mathrm{Re}\exp\{i\omega[r+2M\ln(r-2M)] + i(\beta-\tfrac{1}{2}\pi) + i\tan^{-1}[(\omega-\sigma)\tau]\}, \tag{9}$$

which is valid far from the star ($r \gg M$ and $r \gg R$) in physical space and near the pole ($|\omega-\sigma| \ll \sigma$) on the real frequency axis of frequency space. Equation (9) assumes that near the star's center the fluid amplitude has been normalized to some fixed value, independent of ω.

The eigenfunctions (7), (9) represent analytically the amplitude of a standing gravitational wave in a spherical cavity of infinite radius with the star at its center, for a real frequency ω near the star's resonant frequency σ. One calculates the eigenfunctions for the corresponding fluid motion and for the near-zone gravitational field by integrating numerically the eigenequations (5).

One's greatest interest is not in these idealized standing-wave modes, but in realistic pulsations with purely outgoing waves. As in quantum theory, so also here, one builds the realistic solutions by a suitable superposition of standing-wave modes with real frequencies near the pole of the S matrix $|\omega-\sigma| \ll \sigma$. The required superposition can be constructed analytically inside the star and in the radiation zone, but must be constructed numerically in the near-field zone. The superposition, suitably constructed (see Appendix C of paper II), yields a solution of Einstein's equations which represents the following:

At time $t=0$ the star is set pulsating by some unspecified agent. The pulsation is confined initially to the star's interior and to the near zone; in the radiation zone the gravitational field is that of Schwarzschild augmented, perhaps, by a small amount of angular momentum due to rotational motions of the fluid. After the time $t=0$ gravitational waves flow out from the near zone into the radiation zone, and the star's pulsations begin to damp. Behind the wave front, after the decay of a few transients, the stellar pulsations and gravitational waves are described by the real part of the complex normal mode at the pole of the S matrix [real part of Eqs. (4), (6), and (7) with $\omega=\sigma+i/\tau$ and $C^{(\mathrm{I})}=0$]. Consequently, the angular frequency of pulsation of the star as measured by a distant observer is σ, the real part of the frequency at the pole of the S matrix, and the damping rate is $1/\tau$, the complex part of that frequency.

The location of the poles of the S matrix have been calculated numerically for the quadrupole pulsation of several realistic neutron-star models. For all models examined the poles lie in the upper half of the complex frequency plane ($\tau > 0$), corresponding to damping rather than antidamping. The pulsation periods $T=2\pi/\sigma$ are typically between 10^{-4} and 10^{-3} sec; and the damping times τ are typically between 0.3 and 3 sec. The energies radiated for initial amplitudes $\langle|\delta\vec{r}|/r\rangle$ of 0.1 are roughly 10^{51} erg (~0.1% of the rest mass of the star). These results are in qualitative agreement with calculations based on linearized general relativity[8]—which calculations we had no a priori right to trust.

The method used to locate the poles of the S matrix and obtain the above results involved a calculation of the standing-wave eigenfunctions [numerical integration of Eq. (5) for real ω], followed by a comparison of Eq. (9) with the resultant gravitational wave amplitude K_ω. As predicted by Eq. (9), K_ω showed a number of resonances corresponding to poles of the S matrix. The resonances are exhibited most clearly in terms of the ratio

$$E_M/E_W = \mathrm{const}/[(\omega-\sigma)^2+(1/\tau)^2],$$

where E_W is the energy in one wavelength of the standing gravitational waves far from the star, as diagnosed from Isaacson's[9] stress-energy tensor, and E_M is the pulsation energy of the star. (See Fig. 1 for an example.) The resonant frequency was σ, and the resonance half-width was $1/|\tau|$. The sign of τ and also checks on the values of σ and $|\tau|$ were obtained by examining the resonant change with ω of the phase of K_ω [Eq. (9)].

A check on the accuracy of the numerical integrations and on the entire formulation of the problem was provided by energy conservation —an aspect of general relativity which played no other role in the analysis: For a realistic pulsation formed by superposing standing-wave modes with $\omega \approx \sigma$, the energy carried off by the gravitational waves was compared with the energy of pulsation of the fluid. It was found that energy is

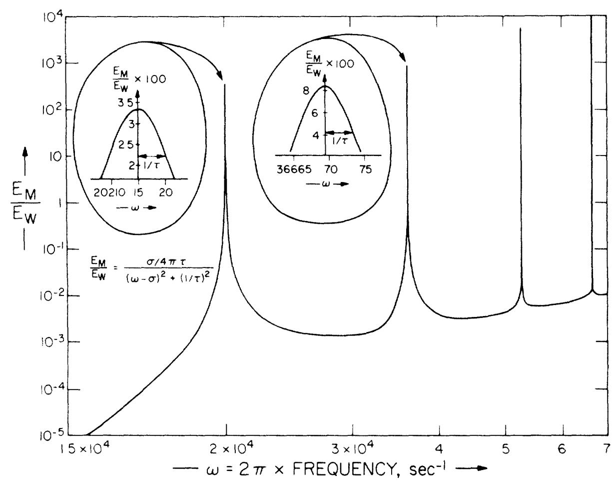

FIG. 1. Resonances in the quadrupole ($l=2$) standing-wave normal modes for a H-W-W neutron star [for a discussion of the H-W-W neutron-star models, see B. K. Harrison, K. S. Thorne, M. Wakano, and J. A. Wheeler, Gravitational Theory and Gravitational Collapse (University of Chicago Press, Chicago, Ill., 1965)] of central density 6×10^{15} g/cm^3. Corresponding to each value of the angular frequency ω there are four independent standing-wave modes ($m=-2, -1, 0, +1, +2$). Plotted as a function of ω is the ratio of the total pulsation energy E_M of the matter in the star to the energy E_W in one wavelength of the standing gravitational waves. This ratio is the same for all four modes (degeneracy in m). The energy ratio shows sharp resonances at frequencies near the (complex) poles of the S matrix [cf. Eq. (10)]. This figure is based on numerical computations of paper III (Ref. 11).

conserved at all times in this realistic problem if and only if the constant appearing in formula (10) for the standing-wave modes is equal to $\sigma/4\pi\tau$. Happily, the numerical calculations confirm this equality (cf. Fig. 1).

The detailed results of the numerical work and a discussion of its astrophysical implications will be presented elsewhere (paper III[10]).

Much of the analytic work reported here was carried out in collaboration with Campolattaro (paper I) and Price (paper II). The analysis was also influenced strongly by discussions with S. Chandrasekhar, C. W. Misner, and J. A. Wheeler. B. A. Zimmerman provided valuable assistance with the numerical calculations. Part of the work was performed while I was participating in the International Research Group in Relativistic Astrophysics at the Institut d'Astrophysique in Paris, France (summer 1967). I thank Professor Evry Schatzman for his kind hospitality there.

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PROPERTIES OF THE P AND P' REGGE TRAJECTORIES FROM LOW-ENERGY πN AND KN SCATTERING AMPLITUDES*

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We show that the resonance-dominance approximation for the low-energy part of finite-energy sum rules for the $C=+1, I=0$ t-channel πN and KN elastic amplitudes reproduces correctly properties of the P' trajectory and residue functions. The Pomeranchukon is fully accounted for by the low-energy background. The Gell-Mann ghost-eliminating mechanism is favored for the P' trajectory.

Finite-energy sum rules[1] (FESR) enable one to relate the phenomenological Regge description of high-energy scattering amplitudes to the properties of low-energy resonances or background amplitudes. The resonance-dominance approximation for the low-energy region has been successful in computing various properties of trajectories other than the Pomeranchukon[2]; while in the case of $C=+1, I=0$ t-channel amplitudes it is difficult to separate the contributions of the P and P' trajectories in a straightforward manner. It was recently proposed[3] that this difficulty can be removed if we assume that the Pomeranchukon is mostly "built" (in the FESR sense) from the nonresonating "background" part of the low-energy amplitude while all other "ordinary" trajectories, including P', are mainly generated by the low-energy resonances and can be appropriately analyzed in terms of the resonance-dominance approximation.[4] This hypothesis has al-