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Gravitational Radiation from a Kerr Black Hole. I

—— *Formulation and a Method for Numerical Analysis* ——

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A class of new inhomogeneous equations governing gravitational perturbations of the Kerr geometry is presented. It is shown that, contrary to the case of the Teukolsky equation, the perturbation equations have short-range potential and no divergent source terms for large distance. Using one of such equations which seems to be the simplest, we have computed the spectrum and the energy of gravitational radiation induced by a test particle of mass μ falling along the z-axis into a Kerr black hole of mass $M(\gg\mu)$ and angular momentum $Ma(a<M)$. It is found that the total energy radiated is $0.0170\mu c^2$ (μ/M) when $a=0.99M$, which is 1.65 times larger than that when $a=0$, i.e., the Schwarzschild case.

§ 1. Introduction

Gravitational perturbations of the Schwarzschild and the Kerr black holes have been one of the most important subjects for study; since their knowledge may give us essential information about the physics of black holes. In the case of the Schwarzschild geometry, following the Regge-Wheeler formalism,[1] Zerilli[2] gave the mathematical foundations for the problem including the source of perturbations. Using his formalism, the gravitational radiation induced by a test particle or by a dust shell falling into a Schwarzschild black hole has been studied extensively.[3),4)]

In the case of the Kerr geometry, Teukolsky[5] showed that the perturbation equations for Newman-Penrose quantities are decoupled and separable, which are finally reduced to the unique radial equation (called the Teukolsky equation). Detweiler[6] examined the energy emitted by test particles in circular orbits, by solving the Teukolsky equation. However, for a test particle of a generic trajectory in the Kerr geometry, little research has been published; the main reason is that the Teukolsky equation in its original form has a long-range potential and a divergent source function for large distance. Chandrasekhar and Detweiler,[7] and Detweiler[8] succeeded in giving several transformations of a radial wave function which bring the Teukolsky equation into the form of a wave equation with short-range potential, but they did not make the source function short-range, probably because their interests are mainly in the homogeneous equation. Unless the source is short-range, it is impossible in general to compute the spectrum and the energy induced by a test particle in the actual numerical calculations.

Recently we have shown that, in the case of the Schwarzschild geometry, the inhomogeneous Bardeen-Press(-Teukolsky) equation can be put into the form of a wave equation with the Regge-Wheeler potential and with short-range source.[9)]

In this paper, at first we generalize the above result to the case of the Kerr geometry and present a class of transformations which bring the ($s=-2$) inhomogeneous Teukolsky equation into the form similar to the Regge-Wheeler equation. Second, using one of the new perturbation equations which seems to be the simplest, we compute the spectrum and the energy of radiation induced by a test particle of mass μ falling along the z-axis into a Kerr black hole of mass $M(\gg\mu)$ and angular momentum $Ma(a<M)$.

In § 2, we derive the desired class of transformations and present the form of the resulting equations. The proof for the short-rangeness of the equations belonging to this class is given in Appendix A. One of the new perturbation equations which seems to be the simplest is explicitly given in Appendix B.

In § 3, we discuss the source term due to the presence of a test particle and give the explicit form of it in the case of a particle falling straightly along the z-axis.

In § 4, solving the perturbation equation given in Appendix B with the source term given in § 3, we present the spectrum and the energy of radiation for various values of a. A method used to construct the $s=-2$ spin-weighted spheroidal harmonics and to find their eigenvalues is given in Appendix C. A finite difference method for solving the radial equation is given in Appendix D.

§ 2. Transformation of the Teukolsky equation

As shown by Teukolsky,[5)] the decoupled perturbation equations for the Kerr geometry (with mass M and angular momentum aM ($a<M$)) are separable and the radial (master) equation of spin s ($s=0, -1, -2$) is given by[*)]

$$\Delta^{-s}\frac{d}{dr}\left(\Delta^{s+1}\frac{d_sR}{dr}\right)-{}_sV_sR=-{}_sT\,, \tag{2·1}$$

where

$${}_sV=\frac{(iK)^2}{\Delta}+\frac{isK\Delta'}{\Delta}-2isK'+{}_s\lambda \tag{2·2}$$

with

$$K=(r^2+a^2)\omega-am\,, \qquad \Delta=r^2-2Mr+a^2\,,$$

[*)] We use the units $c=G=1$ in this paper.

and m and ${}_s\lambda$ are the separation constants arising from the azimuthal function $e^{im\phi}$ and the angular eigenfunction ${}_sZ^{a\omega}_{lm}(\theta,\phi)$ (the spin-weighted spheroidal harmonic), respectively. A prime represents a derivative with respect to r. The gravitational perturbations are described by the equation with $s=-2$ and the form of the source ${}_sT$ for this case will appear in § 3. Here we only note that for a test particle falling from infinity with zero velocity, ${}_sT\sim r^{7/2}$ (or $\sim r^{5/2}$ when the orbital angular momentum is vanishing) for $r\sim+\infty$. This fact, together with the long-range nature of ${}_sV$, is the origin of difficulty.

Detweiler succeeded in giving a transformation of the $s=-2$ Teukolsky equation, Eq. (2·1), into a similar form with a short-range (real) potential,[8)]

$$\Delta^2\frac{d}{dr}\left(\frac{1}{\Delta}\frac{d\chi_d}{dr}\right)-U_d(r)\chi_d=-T_\chi\,, \tag{2·3}$$

where χ_d is related to R*) and dR/dr by

$$\chi_d=\alpha_d(r)R+\frac{\beta_d(r)}{\Delta}\frac{dR}{dr}\,, \tag{2·4}$$

in terms of certain functions α_d and β_d. Unfortunately, however, not only that the new potential derived by him does not reduce to the Regge-Wheeler (or Zerilli) potential in the limit $a\to0$, but also the new source behaves worse than the original source at $r\sim+\infty$.

In this section, we derive a class of transformations of R in which a new equation, even in the presence of source, is of short-range nature and its potential reduces to the Regge-Wheeler potential for $a=0$ (i.e., Schwarzschild limit).

Let us first consider the case when source is absent, so that R satisfies the equation

$$\Delta^2\left(\frac{1}{\Delta}R'\right)'-VR=0\,. \tag{2·5}$$

The general transformation of R which preserves the form of the linear wave equation is given by

$$\chi=\alpha(r)R+\frac{\beta(r)}{\Delta}R'\,, \tag{2·6}$$

where α and β are arbitrary functions. By taking the first and second derivatives of Eq. (2·6) with respect to r and using Eq. (2·5) one finds that χ satisfies the equation

$$\Delta^2\left(\frac{1}{\Delta}\chi'\right)'-\Delta F\chi'-U\chi=0\,, \tag{2·7}$$

*) Hereafter we consider only the case $s=-2$ and omit the suffix s in every quantity.

where

$$F=\frac{\gamma'}{\gamma}, \tag{2·8a}$$

$$U=V+\frac{\Delta^2}{\beta}\left\{\left(2\alpha+\frac{\beta'}{\Delta}\right)'-\frac{\gamma'}{\gamma}\left(\alpha+\frac{\beta'}{\Delta}\right)\right\} \tag{2·8b}$$

with

$$\gamma=\alpha\left(\alpha+\frac{\beta'}{\Delta}\right)-\frac{\beta}{\Delta}\left(\alpha'+\frac{\beta}{\Delta^2}V\right). \tag{2·8c}$$

As usually done, it is useful to introduce a new function X,

$$X=\frac{\sqrt{r^2+a^2}}{\Delta}\chi, \tag{2·9}$$

and a new radial coordinate r^*,

$$dr^*=\frac{r^2+a^2}{\Delta}dr. \tag{2·10}$$

Then, Eq. (2·7) is rewritten as

$$\frac{d^2X}{dr^{*2}}-\mathcal{F}\frac{dX}{dr^*}-\mathcal{U}X=0, \tag{2·11}$$

where

$$\mathcal{F}=\frac{\Delta F}{r^2+a^2}, \tag{2·12a}$$

$$\mathcal{U}=\frac{\Delta U}{(r^2+a^2)^2}+G^2+\frac{dG}{dr^*}-\frac{\Delta GF}{r^2+a^2} \tag{2·12b}$$

with

$$G=-\frac{\Delta'}{r^2+a^2}+\frac{r\Delta}{(r^2+a^2)^2}. \tag{2·12c}$$

Note that, since r^* varies from $-\infty$ to $+\infty$ when r varies from the horizon ($r=r_+=M+\sqrt{M^2-a^2}$) to the infinity ($r=+\infty$), the short-rangeness of Eq. (2·11) implies the condition $\mathcal{F}=O(r^{*-n})$ and $\mathcal{U}=(iK/(r^2+a^2))^2+O(r^{*-n})$ ($n\geq 2$) when $r^*\to\pm\infty$; if this condition is guaranteed, the asymptotic forms of the solutions of Eq. (2·11) become

$$X\propto\begin{cases}\exp(\pm i\omega r^*) & \text{for } r^*\to+\infty\,,\\ \exp\left(\pm i\left(\omega-\dfrac{ma}{2Mr_+}\right)r^*\right) & \text{for } r^*\to-\infty\,.\end{cases}$$

Now we impose the conditions that Eq. (2·11) should be of short-range nature and should reduce to the Regge-Wheeler equation in the limit $a\to 0$. For this purpose, consider a transformation of the form

$$\chi=\frac{f\Delta(r^2+a^2)}{gh}J_-\left[hJ_-\left(\frac{gR}{r^2+a^2}\right)\right], \tag{2·13}$$

where $J_\pm=(d/dr)\pm i(K/\Delta)$ and f, g and h are unspecified functions of r at the moment. Equation (2·13) reduces to the transformation derived by Chandrasekhar,[10] which gives the Regge-Wheeler equation, if $f\to\{\lambda(\lambda+2)-12i\omega M\}^{-1}$, $g\to$const and $h\to$const when $a\to 0$. (Actually the value of f is irrelevant; it should become a constant merely as in the cases of g and h.) By equating Eqs. (2·6) and (2·13) and using Eq. (2·5), the functions α and β are expressed in terms of f, g and h as

$$\alpha=f\left\{\frac{\Delta(r^2+a^2)}{gh}J_-\left[hJ_-\left(\frac{g}{r^2+a^2}\right)\right]+V\right\}$$

$$\equiv f\alpha_0\,, \tag{2·14a}$$

$$\beta=f\Delta^2(r^2+a^2)\left\{\frac{1}{\Delta g}J_-\left(\frac{\Delta g}{r^2+a^2}\right)+\frac{1}{gh}J_-\left(\frac{gh}{r^2+a^2}\right)\right\}$$

$$\equiv f\beta_0\,, \tag{2·14b}$$

which, by a direct calculation, lead to

$$\alpha_0=\frac{-iK}{\Delta^2}\beta_0+3iK'+\lambda+\frac{\Delta(r^2+a^2)}{gh}\left(\left(\frac{g}{r^2+a^2}\right)'h\right)', \tag{2·15a}$$

$$\beta_0=\Delta\left\{-2iK+\Delta'+\frac{\Delta(r^2+a^2)^2}{g^2h}\left(\frac{g^2h}{(r^2+a^2)^2}\right)'\right\}. \tag{2·15b}$$

It is now straightforward though a bit tedious to show that the short-range condition imposed on $\mathcal{F}$ and $\mathcal{U}$ in Eq. (2·11) is satisfied, provided that f, g and h are regular functions with no zero-points and $f=$const$+O(r^{-1})$, $g=$const $+O(r^{-2})$ and $h=$const$+O(r^{-2})$ when $r\to+\infty$ ($r^*\to+\infty$) and are $O(1)$ when $r\to r_+$ ($r^*\to-\infty$). The proof is given in Appendix A.

Thus, there is a tremendous freedom in the choice of the functions f, g and h. However, it seems that either of the following choices; (a) $f=g=h=$const, or (b) $f=h=$const and $g=(r^2+a^2)/r^2$, is simple enough for the actual numerical use. As an example, the explicit forms of $\mathcal{F}$ and $\mathcal{U}$ for the choice (b) are given in Appendix B.

Now we consider the case when source is present. In order to incorporate this case, we first note that the inverse transformation of Eq. (2·6) in case of the homogeneous equation is given by

$$R=\frac{1}{\gamma}\left\{\left(\alpha+\frac{\beta'}{\Delta}\right)\chi-\frac{\beta}{\Delta}\chi'\right\}. \tag{2·16}$$

Second, we refer to the previous result of the Schwarzschild case.[9] There we assumed that R is related to χ by

$$R=\frac{\Delta^2}{r^2}J_+J_+\left(\frac{r^2}{\Delta}\chi\right), \tag{2·17}$$

and that the equation for χ has the Regge-Wheeler form*)

$$\Delta^2\left(\frac{1}{\Delta}\chi'\right)'+(\omega^2-U_\chi)\chi=S\,. \tag{2·18}$$

In the absence of source, it is easy to show that Eq. (2·17) is just Eq. (2·16) for $a=0$ with the choice $f=\{\lambda(\lambda+2)-12i\omega M\}^{-1}$ and $g=h=1$ for α and β. We note that $\gamma=f$ in this case. Then, in the presence of source, Eq. (2·17) can be written in the form

$$R=\frac{1}{\gamma}\left\{\left(\alpha+\frac{\beta'}{\Delta}\right)\chi-\frac{\beta}{\Delta}\chi'\right\}+S\,, \tag{2·19}$$

where S is related to T by

$$T=-\frac{\Delta^2}{r^2}J_+J_+\left(\frac{r^2}{\Delta}S\right). \tag{2·20}$$

As a natural generalization of Eqs. (2·18) and (2·19) to the Kerr case, we adopt the assumption that χ satisfies the equation

$$\Delta^2\left(\frac{1}{\Delta}\chi'\right)'-\Delta F\chi'-U\chi=S\,, \tag{2·21}$$

and that R is related to χ and S by

$$R=\frac{1}{\gamma}\left\{\left(\alpha+\frac{\beta'}{\Delta}\right)\chi-\frac{\beta}{\Delta}\chi'+fS\right\} \tag{2·22}$$

with α and β given by Eqs. (2·14). Inserting Eq. (2·22) into Eq. (2·1) and using Eq. (2·21), we obtain

$$-T=\Delta^2\left(\frac{1}{\Delta}\left(\frac{fS}{\gamma}\right)'\right)'+\Delta^2\left(-\frac{\beta_0}{\Delta^3}\frac{fS}{\gamma}\right)'+(\alpha_0-V)\frac{fS}{\gamma}\,. \tag{2·23}$$

At the first glance, it is not clear yet whether Eq. (2·23) can be expressed as simple as by Eq. (2·20) so that S is well-behaved. However if one notes that α_0 and β_0 of Eqs. (2·14) can be reexpressed as

$$\alpha_0=V-\Delta^2\left[\left\{\frac{g}{\Delta(r^2+a^2)}J_+\left(\frac{r^2+a^2}{g}\right)\right\}'-\frac{g^2h}{(r^2+a^2)}J_+\left(\frac{r^2+a^2}{g}\right)J_+\left(\frac{r^2+a^2}{\Delta gh}\right)\right], \tag{2·24a}$$

*) In Ref. 9), we expressed this equation in the form of Eq. (2·11) with $\mathcal{F}=0$ and $\mathcal{U}$ being the Regge-Wheeler potential.

$$\beta_0 = -\frac{\Delta^3}{r^2+a^2}\left\{ghJ_+\left(\frac{r^2+a^2}{\Delta gh}\right)+\frac{g}{\Delta}J_+\left(\frac{r^2+a^2}{g}\right)\right\}, \tag{2·24b}$$

Eq. (2·23) reduces to

$$-T = \frac{g\Delta^2}{r^2+a^2}J_+\left[hJ_+\left(\frac{(r^2+a^2)fS}{gh\gamma\Delta}\right)\right]. \tag{2·25}$$

This is a desirable form. Introducing a new function W defined by

$$W = \frac{(r^2+a^2)fS}{gh\gamma\Delta}\exp\left(i\int\frac{K}{\Delta}dr\right), \tag{2·26}$$

we can simplify Eq. (2·25) further to

$$(hW')' = -\frac{r^2+a^2}{g\Delta^2}T\exp\left(i\int\frac{K}{\Delta}dr\right). \tag{2·27}$$

This equation can be integrated to find W with the correct boundary condition by the technique developed in Ref. 9). We note that in general $W\sim r^{1/2}$ at $r\sim+\infty$.

Finally, changing the function χ and the coordinate r into X and r^*, respectively, as defined by Eqs. (2·9) and (2·10), we arrive at an equation of the form manageable for numerical use,*)

$$\frac{d^2X}{dr^{*2}} - \mathcal{F}\frac{dX}{dr^*} - \mathcal{U}X = \mathcal{S}, \tag{2·28}$$

where $\mathcal{S}$ is defined by

$$\mathcal{S} = \frac{gh\gamma\Delta}{(r^2+a^2)^{5/2}f}W\exp\left(-i\int\frac{K}{\Delta}dr\right). \tag{2·29}$$

Because of the factor Δ and the fact $W\sim r^{1/2}$ at $r\sim+\infty$, it is apparent that $\mathcal{S}$ is indeed short-range.

§ 3. The source term induced by a test particle falling along the z-axis

The general form of the source T in the Teukolsky equation is shown in a book by Breuer[11] by means of the Geroch-Held-Penrose (GHP) formalism.[12] It is**)

$$T = 4\int d\Omega dt \rho^{-5}(\bar{\rho})^{-1}(B_2'+B_2^{*\prime})\bar{Z}^{a\omega}_{lm}(\theta,\phi)e^{i\omega t}, \tag{3·1}$$

*) It may be worthwhile to note that Eq. (2·28) can be written in the form $(d^2/d\eta^2-\tilde{\mathcal{U}})X=\tilde{\mathcal{S}}$, where $d\eta=(\gamma(r^2+a^2)/\Delta)dr$, $\tilde{\mathcal{U}}=\mathcal{U}/\gamma^2$ and $\tilde{\mathcal{S}}=\mathcal{S}/\gamma^2$. From numerical analysis, however, adopting the coordinate η leads only to more complication.

**) In this section, a prime denotes a GHP operation but not a derivative. A bar denotes complex conjugation.

where

$$B_2'=(\eth'-4\tau'-\bar{\tau})\{(\text{þ}'-2\bar{\rho}')T_{n\bar{m}}-(\eth'-\bar{\tau})T_{nn}\}, \tag{3·2a}$$

$$B_2^{*\prime}=(\text{þ}'-4\rho'-\bar{\rho}')\{(\eth'-2\bar{\tau})T_{n\bar{m}}-(\text{þ}'-\bar{\rho}')T_{\bar{m}\bar{m}}\}, \tag{3·2b}$$

and $Z^{a\omega}_{lm}(\theta,\phi)$ is a spheroidal harmonic of spin weight $s=-2$. T_{nn}, $T_{n\bar{m}}$ and $T_{\bar{m}\bar{m}}$ are the tetrad components of the source energy momentum tensor and their GHP types are $(p,q)=(-2,2)$, $(-2,0)$ and $(-2,2)$, respectively. ρ, ρ', τ and τ' are given by

$$\rho=-(r-ia\cos\theta)^{-1}, \qquad \rho'=-\rho^2\bar{\rho}\Delta/2,$$

$$\tau=ia\rho\bar{\rho}\sin\theta/\sqrt{2}, \qquad \tau'=-ia\rho^2\sin\theta/\sqrt{2}. \tag{3·3}$$

The operators þ$'$ and $\eth'$, when operating on a scalar of type (p,q), are given by

$$\text{þ}'=\frac{\rho\bar{\rho}}{2}\left[(r^2+a^2)\partial_t-\Delta\partial_r+a\partial_\phi+\frac{p+q}{2}\Delta'-\Delta(p\rho+q\bar{\rho})\right], \tag{3·4a}$$

$$\eth'=-\frac{\rho}{\sqrt{2}}\left[-ia\sin\theta\partial_t+\partial_\theta-\frac{i}{\sin\theta}\partial_\phi+\frac{p-q}{2}\cot\theta+p\rho ia\sin\theta\right], \tag{3·4b}$$

and are of type $(-1,-1)$ and $(-1,1)$, respectively.

The energy momentum tensor of a test particle of mass $\mu(\ll M)$ is given by

$$T^{\mu\nu}(x)=\mu\int d\tau\frac{dz^\mu}{d\tau}\frac{dz^\nu}{d\tau}\delta^{(4)}(x-z(\tau))$$

$$=\mu\frac{\rho\bar{\rho}}{|dr/d\tau|}\frac{dz^\mu}{d\tau}\frac{dz^\nu}{d\tau}\delta(t-t(r))\delta^{(2)}(\Omega-\Omega(r)), \tag{3·5}$$

where τ is the proper time of the particle and $t=t(r)$ and $\Omega=\Omega(r)$ represent a geodesic trajectory. In principle, as in Ref. 9), one may expand the appropriate tetrad components of Eq. (3·5) in terms of spheroidal harmonics of their respective spin weights and substitute them into Eq. (3·1) through Eqs. (3·2) in order to find the explicit expression of T for generic trajectories of the particle. However, in this paper, we only consider the simplest case in which a test particle is falling straightly along the z-axis.

First, note that the integrand of Eq. (3·1) contains no functions of t and ϕ except $e^{i\omega t-im\phi}$ and their delta functions. Thus, the derivative operators ∂_t and ∂_ϕ appearing in the operator $\eth'$ (Eq. (3·4b)) may be replaced by $-i\omega$ and im, respectively, and $\eth'$ becomes

$$\eth'=-\frac{\rho}{\sqrt{2}}\mathcal{L}_s+p\tau', \tag{3·6}$$

where $s=(p-q)/2$ and $\mathcal{L}_s$ is defined by

$$\mathcal{L}_s=\partial_\theta+\frac{m}{\sin\theta}-a\omega\sin\theta+s\cot\theta\,. \tag{3·7}$$

This operator $\mathcal{L}_s$ (as well as $\mathcal{L}_s^+$ defined by Eq. (3·12) below) is introduced by Teukolsky and Press.[13] We then have

$$\begin{aligned}&(\eth'-4\tau'-\bar{\tau})(\eth'-\bar{\tau})T_{nn}\\&=\frac{\rho}{2}[\mathcal{L}_{-1}-(7\rho+\bar{\rho})ia\sin\theta]\rho[\mathcal{L}_0-(2\rho+\bar{\rho})ia\sin\theta]T_{nn}\,.\end{aligned} \tag{3·8}$$

Furthermore, since $\rho ia\sin\theta=\rho^{-1}\partial_\theta\rho$, the r.h.s. of Eq. (3·8) reduces to

$$\frac{\rho^8}{2\bar{\rho}}\mathcal{L}_{-1}\left[\frac{1}{\rho^4}\mathcal{L}_0\left(\frac{\bar{\rho}}{\rho^2}T_{nn}\right)\right]. \tag{3·9}$$

Inserting this into Eq. (3·1) gives

$$T=-2\int d\Omega dte^{i\omega t}\bar{Z}\frac{\rho^3}{\bar{\rho}^2}\mathcal{L}_{-1}\left[\frac{1}{\rho^4}\mathcal{L}_0\left(\frac{\bar{\rho}}{\rho^2}T_{nn}\right)\right]\ , \tag{3·10}$$

where indices of the spheroidal harmonic are supressed for simplicity. Performing integrations by parts twice with respect to $d\theta$, we obtain

$$T=-2\int d\Omega dte^{i\omega t}\left(\frac{\bar{\rho}}{\rho^2}T_{nn}\right)\mathcal{L}_1^{\,+}\left[\frac{1}{\rho^4}\mathcal{L}_2^{+}\left(\frac{\rho^3}{\bar{\rho}^2}\bar{Z}\right)\right]\ , \tag{3·11}$$

where $\mathcal{L}_s^+$ is defined by[13]

$$\mathcal{L}_s^+=\partial_\theta-\frac{m}{\sin\theta}+a\omega\sin\theta+s\cot\theta\,. \tag{3·12}$$

The tetrad component T_{nn} is given by

$$\begin{aligned}T_{nn}&=T^{\mu\nu}n_\mu n_\nu\\&=\mu\frac{\rho\bar{\rho}}{|dr/d\tau|}\left(\frac{dz^\mu}{d\tau}n_\mu\right)^2\delta(t-t(r))\delta^{(2)}(\Omega-\Omega(r)),\end{aligned} \tag{3·13}$$

where the components of n_μ are

$$n_\mu=\frac{\rho\bar{\rho}}{2}(-\Delta,\ -(\rho\bar{\rho})^{-1},\ 0,\ \Delta a\sin^2\theta), \tag{3·14}$$

in the usual Boyer-Lindquist coordinates. Up to now, Eq. (3·11) with T_{nn} given by Eq. (3·13) is general except that the source T includes only the term contributed by T_{nn}.

Now we specialize Eq. (3·11). In the case of the straight trajectory along z-axis, one has $\theta=0$ and T_{nn} simplifies to

$$T_{nn}=\mu\frac{|dr/d\tau|}{4}\frac{\Delta^2}{(r^2+a^2)^3}\left(\frac{dt}{dr}+\frac{dr^*}{dr}\right)^2\delta(t-t(r))\delta^{(2)}(\Omega-\Omega(r)), \tag{3·15}$$

where the fact $\rho\bar{\rho}=|r-ia|^{-2}=(r^2+a^2)^{-1}$ is used. Then after a simple calculation by using some properties of spin-weighted spheroidal harmonics (see Appendix C), we obtain

$$T=-\frac{\mu}{2}C\left|\frac{dr}{d\tau}\right|\left(\frac{\Delta}{r^2+a^2}\right)^2(r-ia)^2\left(\frac{dv}{dr}\right)^2 e^{i\omega t(r)}, \tag{3·16}$$

where

$$C=[\mathcal{L}_1{}^+\mathcal{L}_2{}^+\bar{Z}]_{\theta=0,m=0}=[8\bar{Z}/\sin^2\theta]_{\theta=0,m=0}$$

and

$$v(r)=t(r)+r^* .$$

Inserting Eq. (3·16) into Eq. (2·27) leads to the equation

$$\frac{d}{dr}\left(h\frac{d}{dr}W\right)=\frac{\mu C}{2g(r)}\left|\frac{dr}{d\tau}\right|\left(\frac{r-ia}{r+ia}\right)\left(\frac{dv(r)}{dr}\right)^2 e^{i\omega v(r)}. \tag{3·17}$$

This equation can be integrated to give

$$W=\frac{\mu C}{(i\omega)^2}\left\{\frac{y(r)}{h(r)}e^{i\omega v(r)}+\int_r^\infty d\xi\frac{d}{d\xi}\left(\frac{y(\xi)}{h(\xi)}\right)e^{i\omega v(\xi)}\right.$$
$$\left.-i\omega\int_r^\infty d\xi\frac{1}{h(\xi)}\int_\xi^\infty d\eta\frac{d}{d\eta}\left(y(\eta)\frac{dv(\eta)}{d\eta}\right)e^{i\omega v(\eta)}\right\}, \tag{3·18}$$

where the function $y(r)$ is defined by

$$y(r)=\frac{1}{2g(r)}\left|\frac{dr}{d\tau}\right|\left(\frac{r-ia}{r+ia}\right). \tag{3·19}$$

The boundary condition employed in Eq. (3·18) arises from the same argument given in Ref. 9); i.e., that the solution X of Eq. (2·28) satisfies the outgoing wave condition at $r^*\to+\infty$ should lead to the correct (outgoing wave) boundary condition of R (for details, see Ref. 9)).

The integral (3·18) is well-defined and when it is inserted into Eq. (2·29), we easily find that the new source function $\mathcal{S}$ is sufficiently short-range.

§ 4. Numerical results

In this section, we use the perturbation equation with the choice of $\mathcal{F}$ and $\mathcal{U}$ given in Appendix B. The source term is given by Eqs. (2·29) and (3·18). A method of determining λ_l and $Z_{l0}^{a\omega}(\theta,\phi)$ is given in Appendix C. In addition to $c=G=1$, we normalize the mass unit as $M=1$. We assume the test particle has zero-velocity at infinity. Then its geodesic trajectory is determined by the equations

$$\frac{dr}{d\tau}=-\sqrt{2r/(r^2+a^2)} \tag{4·1}$$

and

$$\frac{dt}{d\tau}=(r^2+a^2)/\Delta\,. \tag{4·2}$$

Since the perturbation is axi-symmetric in the present case, we have $m=0$, and the relevant radial equation becomes

$$\left[\frac{d^2}{dr^{*2}}-\mathcal{F}(\lambda_l,\omega,a,r^*)\frac{d}{dr^*}-\mathcal{U}(\lambda_l,\omega,a,r^*)\right]X_l=\mathcal{S}_l(\lambda_l,\omega,a,r^*). \tag{4·3}$$

This should be solved under the boundary condition;

$$X_l(\omega,r^*)=\begin{cases}X_l^{\rm in}(\omega)e^{-i\omega r^*} & \text{for } r^*\to-\infty\,,\\ X_l^{\rm out}(\omega)e^{i\omega r^*} & \text{for } r^*\to+\infty\,.\end{cases} \tag{4·4}$$

Once $X_l^{\rm out}$ is known, the radial function $(R_l(\omega,r^*))$ of the Teukolsky equation for $r^*\to\infty$ is found as

$$\begin{aligned}R_l(\omega,r^*)&=-\frac{4\omega^2X_l^{\rm out}(\omega)}{[\lambda_l(\lambda_l+2)-12i\omega-12a^2\omega^2]}r^3e^{i\omega r^*}\\ &\equiv R_l^{\rm out}(\omega)r^3e^{i\omega r^*}\,.\end{aligned} \tag{4·5}$$

Then the two independent polarization modes of the metric, h_+ and $h_\times$, are given by

$$h_++ih_\times=-\frac{2}{r}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}d\omega\sum_l\frac{e^{i\omega(r^*-t)}}{\omega^2}S_l^{a\omega}(\theta)R_l^{\rm out}(\omega), \tag{4·6}$$

where $S_l^{a\omega}(\theta)$ is a spheroidal function whose definition is given in Appendix C.

We define the 2^l-pole components of h_+ and $h_\times$ by

$$h_+=\sqrt{2\pi}\int_{-\infty}^{\infty}d\omega\sum_l h_+{}^l(\omega)\,{}_{-2}Y_{l0}(\theta)e^{i\omega(r^*-t)}$$

and

$$h_\times=\sqrt{2\pi}\int_{-\infty}^{\infty}d\omega\sum_l h_\times{}^l(\omega)\,{}_{-2}Y_{l0}(\theta)e^{i\omega(r^*-t)}\,, \tag{4·7}$$

where ${}_{-2}Y_{l0}(\theta)$ are $s=-2$ spin weighted *spherical* harmonics. The energy spectrum for each mode of polarization becomes

$$\Delta E_+=\frac{\pi}{4}\int_0^\infty d\omega\omega^2\sum_l\{|h_+{}^l(\omega)|^2+|h_+{}^l(-\omega)|^2\}\equiv\int_0^\infty d\omega\sum_l\left(\frac{dE_+}{d\omega}\right)_l$$

and

$$\Delta E_\times = \frac{\pi}{4}\int_0^\infty d\omega\omega^2 \sum_l \{|h_\times{}^l(\omega)|^2 + |h_\times{}^l(-\omega)|^2\} \equiv \int_0^\infty d\omega \sum_l \left(\frac{dE_\times}{d\omega}\right)_l . \qquad (4\cdot8)$$

Only with this definition of the 2^l-pole components, the total energy spectrum $(dE^{\text{tot}}/d\omega)_l$ agrees with the sum of two modes, $(dE_+/d\omega)_l + (dE_\times/d\omega)_l$ for each l.*) We have solved Eq. (4·3) under the boundary condition of Eq. (4·4) by using a finite difference method, details of which is given in Appendix D. In the present numerical analysis, we compute up to $l=6$, which is sufficient to converge the summation in Eq. (4·6). We have computed h_+ and $h_\times$ for $a=0$, 0.35, 0.5, 0.7, 0.85 and 0.99. For each value of a and l, ω ranges from 0.02 to 1.4 with $\Delta\omega=0.02$.

In Fig. 1, we show the total energy spectrum $((dE^{\text{tot}}/d\omega)_l)$ for $a=0.99$ by solid lines and that for $a=0$ by dashed lines. For the $l=2$ component, the energy spectrum peaks at $\omega=0.37$ for $a=0.99$, while at $\omega=0.3$ for $a=0$. This is due to the fact that the resonant frequency increases with the increase of a.[14] A local maximum is present at $\omega=0.65$ for $a=0.99$, which is absent for $a=0$. This peak comes from the fact that when $a\omega\neq0$, $\int S_l^{a\omega}{}_{-2}Y_{l',0}d\Omega\neq0$ even if $l\neq l'$. For the $l=3$ and 4 components, a rather broad plateau appears near respective maxima

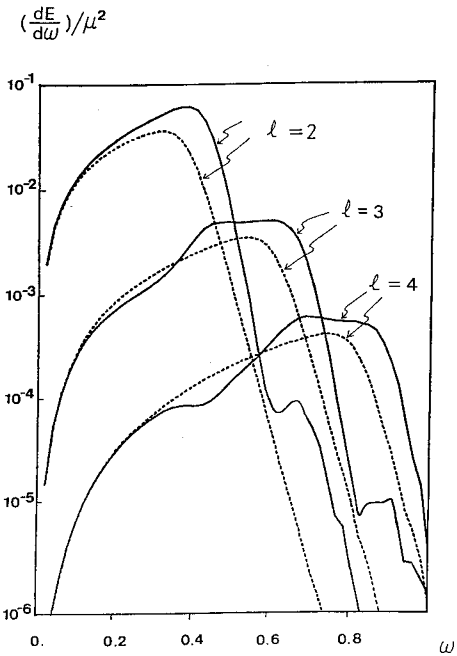

Fig. 1. The total energy spectra for 2^l-poles ($l=2, 3, 4$). Solid and dashed lines correspond to the $a=0.99$ and $a=0$ cases, respectively.

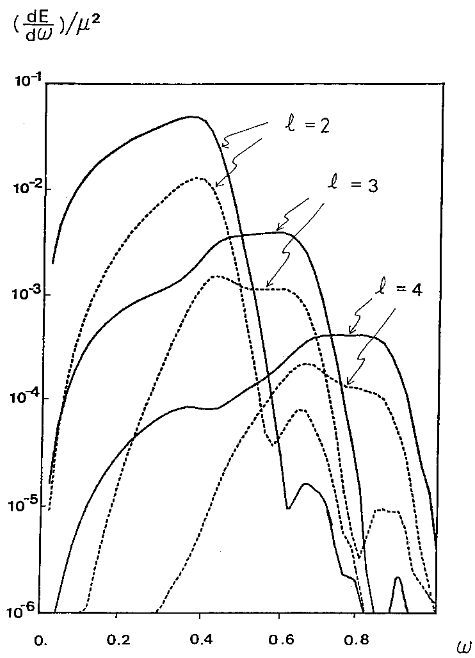

Fig. 2. The 2^l-pole energy spectra for + mode (solid lines) and × mode (dashed lines) when $a=0.99$.

*) Had we chosen the definition of the 2^l-pole components by replacing ${}_{-2}Y_{l0}(\theta)$ with $S_l^{a\omega}(\theta)$ in Eqs. (4·7), not only this statement would fail but also the very meaning of multipole moments would be altered.

for $a=0.99$ and their spectra are lower than those of $a=0$ for small ω. The second peak for $l=3$ of $a=0.99$ appears due to the same reason as that for $l=2$ of $a=0.99$

In Fig. 2, we show the 2^l-pole ($l=2, 3, 4$) energy spectra of + and × modes for $a=0.99$ by solid and dashed lines, respectively. For $l=2$, the energy spectra of both + and × modes peak at almost the same ω. However, for $l=3$ and 4, the energy spectra of × mode peak at considerably lower ω than those of + mode, though peaks themselves are rather broad.

In Fig. 3, we show the relative enhancement in the total energy radiated as a function of a for each l. Each curve can be well fitted by a quadratic function of a^2. Enhancement of the radiation decreases for each a with the increase of l. When $a=0.99$, the total radiation energy is 1.65 times larger than that when $a=0$ (i.e., Schwarzschild case).

For the Schwarzschild case, Davis et al.[3] found that the total energy contributed by each multipole falls off quickly with l obeying the empirical relation $(\Delta E)_l \propto e^{-2l}$. In Fig. 4, we show the l-dependence of radiation energy in each mode of polarization for various values of a. It is clear that this relation holds irrespective of a and of polarization.

The reason for this l-dependence is as follows: Take the $a=0.99$ case, for

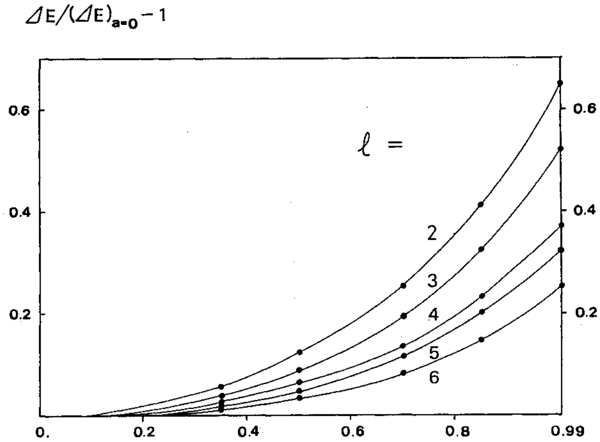

Fig. 3. The enhancement of each 2^l-pole radiation ($l=2, 3, \cdots, 6$) as a function of a. The ordinate is $(\Delta E^l(a)/\Delta E^l(a=0)-1)$.

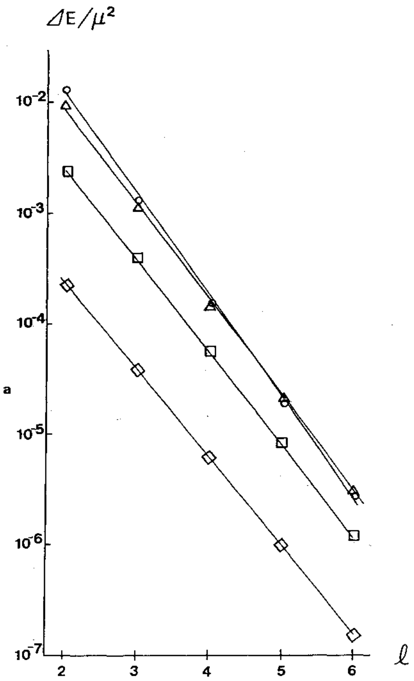

Fig. 4. The energy radiated in each mode of polarization and in each multipole for different values of a. Circles and triangles represent the + mode for $a=0.99$ and $a=0$, respectively. Squares and lozenges represent the x mode for $a=0.99$ and $a=0.35$, respectively.

example. Then $(dE^{\rm tot}/d\omega)_l$ peaks at frequency given approximately by

$$\omega_l^{\rm max}=0.19l\,, \tag{4·9}$$

which is near the resonant frequency.[14] In Figs. 5(a) and (b), we show $\mathcal{S}_l$ and $\mathrm{Re}(\mathcal{U}+\omega^2)$ when $\omega=\omega_l^{\rm max}$ for $l=2$ and 4, respectively.*) From Figs. 5(a) and (b), it is found that $\mathcal{S}_l$ for $-20<r^*<20$ can be expressed approximately as

$$\mathcal{S}_l\propto\frac{\exp(i\omega_l^{\rm max}r^*)}{(r^*+2)^2+25}\,. \tag{4·10}$$

Let $X_{\rm in}(\omega, r^*)$ and $X_{\rm out}(\omega, r^*)$ be the solutions of the homogeneous version of Eq. (4·3) under the condition as

$$X_{\rm in}(\omega, r^*)\propto e^{-i\omega r^*} \qquad \text{for } r^*\to-\infty$$

and

$$X_{\rm out}(\omega, r^*)\propto e^{i\omega r^*} \qquad \text{for } r^*\to+\infty\,, \tag{4·11}$$

respectively.

Using $X_{\rm in}$ and $X_{\rm out}$, we can express $X_l^{\rm out}(\omega)$ as

$$X_l^{\rm out}(\omega)\propto\int_{-\infty}^{\infty}dr^*\mathcal{S}_lX_{\rm in}(\omega, r^*)/W\,, \tag{4·12}$$

where

$$W=X_{\rm in}\frac{dX_{\rm out}}{dr^*}-X_{\rm out}\frac{dX_{\rm in}}{dr^*}\,. \tag{4·13}$$

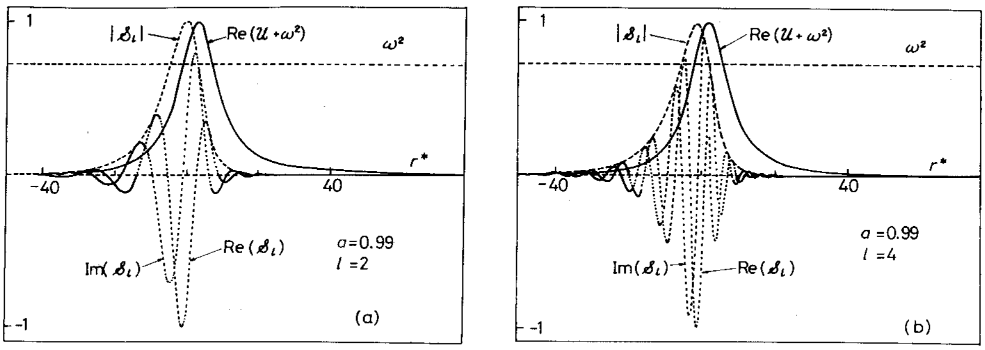

Fig. 5(a) The source term $\mathcal{S}_l$ and $\mathrm{Re}(\mathcal{U}+\omega^2)$ when $\omega=\omega_l^{\rm max}$ for $l=2$ of $a=0.99$. $|\mathcal{S}_l|$, $\mathrm{Re}(\mathcal{S}_l)$ and $\mathrm{Im}(\mathcal{S}_l)$ are normalized by the maximum value of $|\mathcal{S}_l|$. $\mathrm{Re}(\mathcal{U}+\omega^2)$ and ω^2 are normalized by the maximum value of $\mathrm{Re}(\mathcal{U}+\omega^2)$.

(b) The same as (a) for $l=4$ of $a=0.99$.

*) It is found that $\mathrm{Im}(\mathcal{U})$ is negligibly small compared with $\mathrm{Re}(\mathcal{U}+\omega^2)$. It is also found that $\mathcal{F}$ is very small. Therefore, besides $\mathcal{S}_l$, only $\mathrm{Re}(\mathcal{U}+\omega^2)$ plays an important role in the $(\varDelta E)_l\propto e^{-2l}$ relation.

When $\omega=\omega^{\max}$, since $\omega^2 \lesssim \mathrm{Re}(\mathcal{U}+\omega^2)$ for $r^* \approx 0$, $X_{\rm in}$ and $X_{\rm out}$ are expected to be slowly varying functions near $r^*=-2$ where $|\mathcal{S}_l|$ takes appreciable values. Therefore we may approximate Eq. (4·12) by

$$X_l^{\rm out}(\omega) \propto \int_{-20}^{20} dr^* \mathcal{S}_l \approx \int_{-\infty}^{\infty} dr^* \mathcal{S}_l \propto e^{-l} . \tag{4·14}$$

This leads to

$$(\varDelta E)_l \propto e^{-2l} . \tag{4·15}$$

Thus, since the behavior of $\mathcal{S}_l$ (Eq. (4·10)) is essential for deriving Eq. (4·15), we find that the important determinant of the radiation is again the phase factor of the source function as emphasized in Ref. 4), and we surely expect a different l-dependence for a different choice of sources.

Acknowledgements

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Appendix A

Here we show that the transformation defined by Eqs. (2·14) implies the short-rangeness of Eq. (2·11).

First we consider the function $\mathcal{F}$. From Eqs. (2·8a), (2·8c) and (2·12a), it is clear that, if γ is regular and $\gamma=\text{const}+O(r^{-1})$ for $r^* \to +\infty (r \to \infty)$ and $\gamma=O(1)$ for $r^* \to -\infty (r \to r_+)$, $\mathcal{F}$ satisfies the short-rangeness condition. In order to examine the behavior of γ, we express α_0 and β_0 of Eqs. (2·15) as

$$\alpha_0 = A - \frac{iK}{\varDelta} B ,$$

$$\beta_0 = \varDelta B , \tag{A·1}$$

where A and B are defined by

$$A = 3iK' + \lambda + \varDelta P , \tag{A·2a}$$

$$B = -2iK + \varDelta' + \varDelta Q \tag{A·2b}$$

with

$$P = \frac{r^2+a^2}{gh}\left(\left(\frac{g}{r^2+a^2}\right)' h\right)' \quad \text{and} \quad Q = \frac{(r^2+a^2)^2}{g^2 h}\left(\frac{g^2 h}{(r^2+a^2)^2}\right)' .$$

Inserting Eqs. (A·1) into the expression (2·8c) of γ and using Eqs. (A·2) and the explicit form of V, one finds

$$\gamma=f^2\gamma_0 \quad \text{with} \quad \gamma_0=A^2+AB'-BA'+B(BP-AQ). \tag{A·3}$$

For $r^*\to-\infty$ $(r\to r_+)$, it is apparent from Eq. (A·3) that $\gamma=O(1)$ if f, g and h are $O(1)$. For $r^*\to+\infty$ $(r\to+\infty)$, a bit more work is necessary: If we assume g and h are const$+O(r^{-2})$, P and Q become

$$P=6r^{-2}+O(r^{-4}), \qquad Q=-4r^{-1}+O(r^{-3}). \tag{A·4}$$

Then we have

$$\begin{aligned} A&=6i\omega r+\lambda+6+O(r^{-1}),\\ A'&=6i\omega+O(r^{-2}),\\ B&=-2i\omega r^2-2r+O(1),\\ B'&=-4i\omega r-2+O(r^{-2}). \end{aligned} \tag{A·5}$$

By substituting the expressions (A·4) and (A·5) into Eq. (A·3), it is easy to find that $\gamma_0=$const$+O(r^{-1})$. Thus in addition to the condition for g and h, if f $=$const$+O(r^{-1})$ one has $\gamma=$const$+O(r^{-1})$.

Next, we consider the function $\mathcal{U}$. From Eqs. (2·12), one easily sees that the third and fourth terms in the r.h.s. of Eq. (2·12b) are already of short-range, and the part of $\mathcal{U}$ we have to analyze becomes

$$\mathcal{U}_1=(\Delta U+\Delta'^2)(r^2+a^2)^{-2}. \tag{A·6}$$

Further, since

$$\begin{aligned} \frac{\Delta^2}{\beta}\left(\alpha+\frac{\beta'}{\Delta}\right)&=\frac{\Delta^2}{\beta_0}\left(\alpha_0+\frac{\beta_0'}{\Delta}\right)+\frac{f'}{f}\Delta\\ &=-iK+\Delta'+\Delta\left(\frac{f'}{f}+\frac{A+B'}{B}\right)\\ &=\begin{cases} O(1) & \text{for } r^*\to-\infty,\\ -i\omega r^2+O(r^{-3}) & \text{for } r^*\to+\infty, \end{cases} \end{aligned} \tag{A·7}$$

we find, from Eqs. (2·8b) and (A·6), that it is sufficient to examine the function

$$\mathcal{U}_2=(\Delta U_1+\Delta'^2)(r^2+a^2)^{-2}, \tag{A·8a}$$

where

$$U_1=V+\frac{\Delta^2}{\beta}\left(2\alpha+\frac{\beta'}{\Delta}\right)'. \tag{A·8b}$$

Moreover, since

$$\frac{\Delta^2}{\beta}\left(2\alpha+\frac{\beta'}{\Delta}\right)'-\frac{\Delta^2}{\beta_0}\left(2\alpha_0+\frac{\beta_0'}{\Delta}\right)'$$

$$=\frac{f'}{f}(\Delta'-2iK)+\Delta\left[\left(\frac{f'}{f}\right)+\left(\frac{f'}{f}\right)\left\{\frac{f'}{f}+\frac{2(A+B')}{B}\right\}\right]$$

$$=O(1) \qquad \text{for } r^*\to\pm\infty\,, \tag{A·9}$$

the part of $\mathcal{U}$ to be examined finally becomes

$$\mathcal{U}_3=\Delta U_2(r^2+a^2)^{-2}\,, \tag{A·10a}$$

where

$$U_2=V+\frac{\Delta^2}{\beta_0}\left(2\alpha_0+\frac{\beta_0'}{\Delta}\right)'+\frac{\Delta'^2}{\Delta}\,. \tag{A·10b}$$

Let us now evaluate Eq. (A·10b). Inserting Eq. (A·1) into Eq. (A·10b) and using Eq. (A·2), we obtain

$$U_2=V+\frac{\Delta'(\Delta'-B)}{\Delta}+2B'+\frac{\Delta}{B}(2A+B'-QB)'\,. \tag{A·11}$$

Then, from the explicit form of V (Eq. (2·2)), Eq. (A·11) reads

$$U_2=\frac{(iK)^2}{\Delta}+\lambda+4+\Delta' Q+\Delta\left\{2Q'+\frac{1}{B}(2A+B'-QB)'\right\}. \tag{A·12}$$

From this expression for U_2, it is easy to see

$$U_2=\frac{(iK)^2}{\Delta}+O(1) \qquad \text{for } r^*\to\pm\infty\,, \tag{A·13}$$

which in turn means

$$\mathcal{U}_3=(iK)^2(r^2+a^2)^{-2}+O(\Delta(r^2+a^2)^{-2})$$

$$=(iK)^2(r^2+a^2)^{-2}+O(r^{*-2}) \qquad \text{for } r^*\to\pm\infty\,. \tag{A·14}$$

This completes the proof.

As a final remark, we note that if either B (Eq. (A·2b)) or γ_0 (Eq. (A·3)) has a zero-point, the functions $\mathcal{F}$ and/or $\mathcal{U}$ become singular. Concerning B, if one chooses g and h to be real, a zero-point appears at a super-radiant mode ($0<\omega<am/2Mr_+$ and $0>\omega>am/2Mr_+$), which may have some physical meaning. As for γ_0, the position of zero-points (if they ever exist) is generally unknown. However, since these singularities seem to appear for very special choices of a, ω, l and m, we may not worry about them in practice.

Appendix B

In this appendix, we present the functions $\mathcal{F}$ and $\mathcal{U}$ with the following choice of f, g and h;

$$f=h=1 \qquad \text{and} \qquad g=(r^2+a^2)r^{-2}. \tag{B·1}$$

In this case, the functions P and Q introduced in Eqs. (A·2) are given by $P=6r^{-2}$ and $Q=-4r^{-1}$, respectively, and we have

$$\begin{aligned} A&=3iK'+\lambda+6\Delta r^{-2}, \\ B&=-2iK+\Delta'-4\Delta r^{-1}. \end{aligned} \tag{B·2}$$

Then, from Eq. (A·3) we obtain

$$\gamma_0=c_0+c_1r^{-1}+c_2r^{-2}+c_3r^{-3}+c_4r^{-4}, \tag{B·3}$$

where the constants c_i ($i=0, 1, 2, 3, 4$) are given by

$$\begin{aligned} c_0&=-12iM\omega+\lambda(\lambda+2)-12\omega(a^2\omega-am), \\ c_1&=8i\{3a^2\omega-\lambda(a^2\omega-am)\}, \\ c_2&=-24iM(a^2\omega-am)+12\{a^2-2(a^2\omega-am)^2\}, \\ c_3&=24ia^2(a^2\omega-am)-24Ma^2, \\ c_4&=12a^4. \end{aligned} \tag{B·4}$$

Since $f=$const, F (Eq. (2·8a)) is given by

$$F=\frac{\gamma_0'}{\gamma_0}=-\frac{c_1r^3+2c_2r^2+3c_3r+4c_4}{r(c_0r^4+c_1r^3+c_2r^2+c_3r+c_4)}, \tag{B·5}$$

and U (Eq. (2·8b)) has the form

$$U=U_2-\frac{\Delta'^2}{\Delta}-F\left\{-iK+\Delta'+\frac{\Delta(A+B')}{B}\right\}, \tag{B·6}$$

where Eq. (A·7) is used and U_2 (defined by Eq. (A·10b)) is written as

$$U_2=\frac{(iK)^2}{\Delta}+\lambda+4-\frac{4\Delta'}{r}+\frac{8\Delta}{r^2}\left\{1+\frac{i(a^2\omega-am)}{B}\right\}. \tag{B·7}$$

Finally, inserting Eq. (B·6) into Eq. (2·12b) and using Eq. (2·12c) lead to the expression

$$\mathcal{U}=\frac{\Delta}{(r^2+a^2)^2}(U_2+G_1+F_1), \tag{B·8}$$

where G_1 and F_1 are given by

$$G_1 = -2 + \frac{1}{r^2+a^2}\left(r\Delta' - 2\Delta + \frac{3a^2\Delta}{r^2+a^2}\right), \tag{B·9a}$$

$$F_1 = -F\left\{-iK + \Delta\left(\frac{r}{r^2+a^2} + \frac{A+B'}{B}\right)\right\}. \tag{B·9b}$$

Thus, Eqs. (2·12a) and (B·8), together with Eqs. (B·2), (B·5), (B·7) and (B·9), provide the functions $\mathcal{F}$ and $\mathcal{U}$, respectively, in the desirable forms.

Appendix C

In this appendix, we present a method of determining the eigenvalues and the eigenfunctions of Eq. (C·1) below (the $s=-2$ spin weighted spheroidal harmonics; $Z_{lm}^{a\omega}(\theta,\phi)$). Since we are concerned only with the axi-symmetric gravitational perturbation in this paper, we restrict ourselves to discuss the case $s=-2$ and $m=0$. However, the method can be extended to a general case easily.

The function $Z_{l0}^{a\omega}$ obeys the equation,

$$\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta} + a^2\omega^2\cos^2\theta \right.$$
$$\left. + 4a\omega\cos\theta - 4\cot^2\theta + \lambda - 2 - a^2\omega^2\right] S_l^{a\omega}(\theta) = 0\,, \tag{C·1}$$

where

$$S_l^{a\omega}(\theta) \equiv \sqrt{2\pi}\, Z_{l0}^{a\omega}\,.$$

From Eq. (C·1), it is easily shown that

$$S_l^{a\omega}(\theta) \propto (\sin\theta)^{\pm 2} \qquad \text{for } \theta\to 0 \text{ and } \pi\,. \tag{C·2}$$

The regularity of $S_l^{a\omega}$ demands that we should take + sign in Eq. (C·2). Then we define $f_l^{a\omega}$ by

$$f_l^{a\omega}(\theta) = S_l^{a\omega}(\theta)/\sin^2\theta\,. \tag{C·3}$$

The function $f_l^{a\omega}(\theta)$ obeys the equation

$$\left[\frac{\partial^2}{\partial\theta^2} + 5\frac{\cos\theta}{\sin\theta}\frac{\partial}{\partial\theta} + (a^2\omega^2\cos^2\theta + 4a\omega\cos\theta + \lambda - 4 - a^2\omega^2)\right] f_l^{a\omega}(\theta) = 0\,. \tag{C·4}$$

Now we approximate Eq. (C·4) by a finite difference equation. Let us introduce θ_i as

$$\theta_i = \frac{\pi}{2N}(i-1) \equiv \Delta\theta(i-1)\,. \qquad (i=1, 2, \cdots, 2N+1) \tag{C·5}$$

By defining $(f_l^{a\omega})^i = f_l^{a\omega}(\theta_i)$, the finite difference version of Eq. (C·4) is given by

$$\frac{(f_l^{a\omega})^{i+1}+(f_l^{a\omega})^{i-1}-2(f_l^{a\omega})^i}{(\Delta\theta)^2}+5\frac{\cos\theta_i}{\sin\theta_i}\frac{(f_l^{a\omega})^{i+1}-(f_l^{a\omega})^{i-1}}{2\Delta\theta}$$

$$+(a^2\omega^2\cos^2\theta_i+4a\omega\cos\theta_i+\lambda-4-a^2\omega^2)(f_l^{a\omega})^i=0$$

$$\text{for } i=2,\cdots,2N\,, \qquad (\mathrm{C}\cdot 6)$$

$$6\frac{2(f_l^{a\omega})^{i+1}-2(f_l^{a\omega})^i}{(\Delta\theta)^2}+(a^2\omega^2+4a\omega+\lambda-4-a^2\omega^2)(f_l^{a\omega})^i=0$$

$$\text{for } i=1\ (\theta=0),$$

and

$$-4\frac{2(f_l^{a\omega})^{i-1}-2(f_l^{a\omega})^i}{(\Delta\theta)^2}+(a^2\omega^2-4a\omega+\lambda-4-a^2\omega^2)(f_l^{a\omega})^i=0$$

$$\text{for } i=2N+1\ (\theta=\pi). \qquad (\mathrm{C}\cdot 7)$$

Then Eq. (C·4) is approximated by the matrix equation

$$2N+1\left\{\left[\overbrace{\begin{array}{ccc} **0 & & \\ *** & & 0 \\ 0*** & & \\ \cdots & \cdots & \cdots \\ 0 & & *** \\ & & 0** \end{array}}^{2N+1}\right]\right. \begin{bmatrix} (f_l^{a\omega})^1 \\ \vdots \\ (f_l^{a\omega})^{2N+1} \end{bmatrix} \equiv A \begin{bmatrix} (f_l^{a\omega})^1 \\ \vdots \\ (f_l^{a\omega})^{2N+1} \end{bmatrix} = 0\,. \qquad (\mathrm{C}\cdot 8)$$

Equation (C·8) has a non-trivial solution only when det $A \equiv g(\lambda, a\omega)=0$. As the matrix A is tri-diagonal, its determinant is easily computed even for large N such as $N=100$, which is typically adopted.

We first set two trial values of λ as λ^u and λ^d ($<\lambda^u$). If, for example,

$$g(\lambda^u, a\omega)>0 \quad \text{and} \quad g(\lambda^d, a\omega)<0$$

an eigenvalue λ should exist in the interval $[\lambda^d, \lambda^u]$. Therefore, we repeat this procedure until $|\lambda^d-\lambda^u|$ becomes small enough. Once the eigenvalue λ is obtained, the corresponding eigenfunction can be obtained by setting, for example, $(f_l^{a\omega})^1=1$ and solving Eq. (C·8). Once the eigenfunction is obtained, its normalization is easy to perform.

For the $a\omega=0$ case, $\lambda_l=(l+2)(l-1)$ and the eigenfunction $f_l^0(\theta)$ is well known.[11] To ensure validity of our method, we computed eigenvalues and eigenfunctions for the $a\omega=0$ case and found that the agreement is satisfactory.

Appendix D

In this appendix, we present a method of solving Eq. (4·3). We set the range of r^* by $r^*_{\min} \leqq r^* \leqq r^*_{\max}$ where $r^*_{\min}$ and $r^*_{\max}$ are defined by

$$r^*_{\min} = \mathrm{Min}(-4\pi/\omega, -50) \tag{D·1}$$

and

$$r^*_{\max} = \mathrm{Max}(20\pi/\omega, 250).$$

We define $r_i{}^*$ as

$$r_i{}^* = r^*_{\min} + (i-1)\frac{(r^*_{\max} - r^*_{\min})}{(2N+1)}. \quad (i=1, 2, \cdots, 2N+1) \tag{D·2}$$

We approximate Eq. (4·3) by a matrix equation similar to Eq. (C·8) using the boundary condition (Eq. (4·4)) at $r^* = r^*_{\min}$ and $r^* = r^*_{\max}$. Then the matrix has the form

$$\begin{pmatrix} **0 & & \\ *** & 0 & \\ 0*** & & \\ & ***0 & \\ 0 & *** & \\ & 0** & \end{pmatrix} \begin{pmatrix} X_l{}^1 \\ \vdots \\ \vdots \\ X_l^{2N+1} \end{pmatrix} = \begin{pmatrix} \mathscr{S}_l{}^1 \\ \vdots \\ \vdots \\ \mathscr{S}_l^{2N+1} \end{pmatrix}. \tag{D·3}$$

In this case, since the determinant of the matrix is non-vanishing, the solution X_l is obtained by inverting the matrix, which is rather easy to perform even for $N = 2500$ adopted in the present numerical calculations. Note that in this case the elements of the matrix as well as $\mathscr{S}_l$ are complex numbers. But this causes no difficulty.

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