Gravitational, shear and matter waves in Kantowski-Sachs cosmologies

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Abstract. A general treatment of vorticity-free, perfect fluid perturbations of Kantowski-Sachs models with a positive cosmological constant are considered within the framework of the 1+1+2 covariant decomposition of spacetime. The dynamics is encompassed in six evolution equations for six harmonic coefficients, describing gravito-magnetic, kinematic and matter perturbations, while a set of algebraic expressions determine the rest of the variables. The six equations further decouple into a set of four equations sourced by the perfect fluid, representing forced oscillations and two uncoupled damped oscillator equations. The two gravitational degrees of freedom are represented by pairs of gravito-magnetic perturbations. In contrast with the Friedmann case one of them is coupled to the matter density perturbations, becoming decoupled only in the geometrical optics limit. In this approximation, the even and odd tensorial perturbations of the Weyl tensor evolve as gravitational waves on the anisotropic Kantowski-Sachs background, while the modes describing the shear and the matter density gradient are out of phase dephased by $\pi/2$ and share the same speed of sound.

Contents

1 Introduction

The observed large scale distribution of galaxies, the fluctuations about the isotropic cosmic microwave background radiation and the late time acceleration of the universe seems to be well described by the ΛCDM model, which is based on the assumption that the geometry of the universe is given by the Robertson-Walker metric - see e.g., [1–5]. However this fit is not perfect [6–9] and because more than 95% of the matter budget needs to be described by the dark sector, it is worth exploring what effect alternative cosmological models have on the basic properties of the Universe [10–21].

It is known that anisotropies in the Hubble and deceleration parameters cannot be excluded by present observations [22], [23] and [24]. Consequently, perturbations of anisotropic cosmological models have been considered by many authors, e.g., [25–32], using both gauge dependent methods (e.g., [33]) or Bardeen's gauge invariant formalism [34]. For example the perturbations of homogeneous and anisotropic universe of the Bianchi I type was investigated in [31, 32] by using Bardeen's gauge-invariant method. However, the variables in Bardeen's theory are defined with respect to a particular coordinate system, making their geometrical and physical meaning not very transparent See the discussion in [35]. By using a covariant approach, one circumvents these problems by using the spatial curvature rather than the metric as the defining variables [36, 37]. In this way, a set of gauge-invariant perturbation variables can be easily identified as the ones that vanish on the chosen background [38–44, 50]. This feature of the covariant approach makes it a very versatile method for studying perturbations on a variety of backgrounds and physical situations and relating the results obtained in a unified way [45–49]

In this paper we present for the first time a general treatment of the vorticity-free perturbations of Kantowski-Sachs cosmologies with positive cosmological constant, extending earlier work [51], which focused only on the scalar perturbation sector. Here we present for the first time an analysis of a full scalar, vectorial and tensorial perturbations, focusing on gravitational and matter wave evolutions.

In order to achieve this, we use a covariant and gauge invariant method [52], in which spacetime is first split into a 1+3 form. The formalism has been mainly employed for computation of cosmological perturbations on a Friedmann background, applying the standard decomposition theorems [35] (see for example Refs. [38, 40, 53, 54]). If the 3-space at each point has a unique preferred direction, a further decomposition of the spacetime into a 1+(1+2) form is useful in situations where spacetime admits a spherical or Locally Rotational (LRS) symmetry. This was first employed in Ref. [55], where the spatial direction was singled out by local rotational symmetry (LRS). The formalism was developed with the purpose of investigating general gauge-invariant perturbations of the vacuum Schwarzschild spacetime [56]. With the further generalisation presented in Refs. [58] and [59], it became possible to describe gauge-invariant perturbations of LRS class II spacetimes for which the complete set of evolution and constraint equations are given in Ref. [60].

In this paper, the variables describing an almost Kantowski-Sachs spacetime are expanded into harmonics. We find that the perturbation dynamics is described by six evolution equations for six harmonic coefficients, together with a set of algebraic expressions, which determine the evolution of the rest of the variables. The evolution equations can be split into two sets - four which are sourced by the perfect fluid, representing forced oscillations and the remaining two describing damped oscillating gravito-magnetic perturbations. We further analyse the equations using the geometrical optics approximation and find that four of

the gravito-magnetic quantities evolve as gravitational waves propagating on the anisotropic Kantowski-Sachs background, while the shear and the matter density gradients, which are out of phase by $\pi/2$, share the same speed of sound.

The paper is organised as follows. In Section 2 we briefly review the 1+3 and 1+1+2 covariant approaches. In Section 3 a Kantowski-Sachs type background filled with a perfect fluid is introduced. The equations governing the linear perturbations are derived in Section 4. All type of perturbations (scalar, vector and tensor) are investigated, however they are restricted by vanishing anisotropic pressure and energy flux, which mean that we assume a perfect fluid and that the 1+3 split is done with respect to the 4-velocity also in the perturbed spacetime. For simplicity we also choose to put the vorticity to zero. This imply that the hypersurfaces perpendicular to the 4-velocity are well defined. The gauge degrees of freedom in the choice of frame are analysed in Appendix B. Applying the commutation relations given in Appendix C and the useful relations for the vector and tensor spherical harmonics given in Appendix D, the equations governing the perturbed system are derived in Appendix E. Then by fixing a frame we find that the perturbed spacetime can be described by six type of harmonic coefficients. The evolution equations for these six variables are given in Section 4. The behaviour of the perturbations in a geometrical optics approximation is discussed in Section 5, while Section 6 contains some concluding remarks.

Units where $8\pi G = 1$ and $c = 1$ are used throughout this paper.

2 The 1+3 and 1+1+2 covariant formalisms

2.1 The 1+3 covariant formalism

Let u^a be a time-like vector field obeying the usual normalisation condition $u^a u_a = -1$ and h_{ab} a spatial 3-metric satisfying $u^a h_{ab} = 0$. Then the 4-metric g_{ab} can be decomposed as

$$g_{ab} = -u_a u_b + h_{ab} \ . \tag{2.1}$$

We denote the 4-dimensional (4D) and 3-dimensional (3D) volume elements by $\eta_{abcd} = \sqrt{-g}\delta^0_{[a}\delta^1_b\delta^2_c\delta^3_{d]}$ and $\varepsilon_{abc} = \eta_{dabc}u^d$, respectively. Angular brackets $\langle \ \rangle$ on indices denote symmetrised and trace-free tensors which are projected in all indices with the metric h_{ab}. Round brackets ( ) and square brackets [ ] on indices denote the symmetric and antisymmetric parts, respectively. A dot denotes covariant derivatives along the integral curves of u^a, while D_a is the projected spatial derivative

$$\dot{T}_{b..c} = u^a \nabla_a T_{b..c} \ , \tag{2.2}$$

$$D_a T_{b..c} = h_a{}^d h_b{}^i .. h_c{}^j \nabla_d T_{i..j} \ . \tag{2.3}$$

For vanishing vorticity of u^a, D_a is the 3D covariant derivative compatible with the metric h_{ab}.

The kinematic quantities are introduced through the decomposition of the 4D covariant derivative of u^a as

$$\nabla_a u_b = \sigma_{ab} + \frac{1}{3}\Theta h_{ab} + \omega_{ab} - u_a A_b \ , \tag{2.4}$$

where $\sigma_{ab} = D_{\langle a}u_{b\rangle}$ is the shear, $\Theta = D^a u_a$ the expansion and $\omega_{ab} = D_{[a}u_{b]}$ the vorticity of u^a, finally $A_a = \dot{u}_{\langle a\rangle} = h_a{}^b \dot{u}_b$ is its acceleration. Since ω_{ab} is space-like and antisymmetric, containing 3 independent components, we introduce its (Hodge-) dual $\omega_a = \varepsilon_a{}^{bc}\omega_{bc}/2$.

The gravito-electro-magnetic quantities arise from the 1+3 covariant decomposition of the 4D Weyl tensor C_{abcd} as

$$E_{ab} = C_{acbd}u^c u^d \quad \text{and} \quad H_{ab} = \frac{1}{2}\varepsilon_a{}^{cd}C_{cdbe}u^e. \tag{2.5}$$

The quantities E_{ab} and H_{ab} are the magnetic and electric parts of C_{abcd}, respectively. The Weyl tensor is given[1] [36] by

$$\frac{1}{2}C_{abcd} = u_c u_{[a}E_{b]d} - u_d u_{[a}E_{b]c} + E_{c[a}h_{b]d} - E_{d[a}h_{b]c} - \varepsilon_{ab}{}^i H_{i[c}u_{d]} - \varepsilon_{cd}{}^i H_{i[a}u_{b]} . \tag{2.6}$$

As well-known, the Weyl tensor (posessing the symmetries of the Riemann tensor with 20 independent components, and also being traceless, meaning 10 conditions) has 10 independent components. Its electric and magnetic parts have each 5 independent components, as they are expressed by traceless and symmetric 3-tensors.

The energy-momentum tensor T_{ab} of matter fields is decomposed with respect to an observer with 4-velocity u^a in the standard way:

$$T_{ab} = \mu u_a u_b + 2q_{(a}u_{b)} + ph_{ab} + \pi_{ab} . \tag{2.7}$$

The quantities μ, q_a, p and π_{ab} are the energy density, the energy current vector, the isotropic pressure and the symmetric, trace-free anisotropic pressure tensor of matter.

The Riemann tensor can be expressed by the metric components, gravito-electro-magnetic quantities and matter variables as follow. First we use the decomposition of R_{abcd} into its Weyl and Ricci (R_{ab}) contributions

$$R_{abcd} = C_{abcd} + g_{a[c}R_{d]b} - g_{b[c}R_{d]a} - \frac{R}{3}g_{a[c}g_{d]b} , \tag{2.8}$$

where R is the Ricci scalar. Then by applying the Einstein equation

$$R_{ab} = \Lambda g_{ab} + T_{ab} - \frac{T}{2}g_{ab} , \tag{2.9}$$

with $T = g^{ab}T_{ab}$ and cosmological constant Λ, and Eqs. (2.6)-(2.7), we find

$$\begin{aligned}
R_{abcd} = {}& 2\left(u_c u_{[a}E_{b]d} - u_d u_{[a}E_{b]c} + E_{c[a}h_{b]d} - E_{d[a}h_{b]c} - \varepsilon_{ab}{}^i H_{i[c}u_{d]} - \varepsilon_{cd}{}^i H_{i[a}u_{b]}\right) \\
&- \frac{(2\Lambda - \mu - 3p)}{3}\left(u_a u_{[c}h_{d]b} - u_b u_{[c}h_{d]a}\right) + \frac{2(\Lambda + \mu)}{3}h_{a[c}h_{d]b} \\
&+ q_a u_{[c}h_{d]b} - q_b u_{[c}h_{d]a} + u_a q_{[c}h_{d]b} - u_b q_{[c}h_{d]a} + g_{a[c}\pi_{d]b} - g_{b[c}\pi_{d]a} .
\end{aligned} \tag{2.10}$$

The full set of equations arise from the Ricci identities for u^a and from the 4D Bianchi identities and can be found for instance in Refs. [52], [61].

A 3D curvature tensor can be defined in the following way (see [62]):

$$\frac{1}{2}{}^{(3)}R_{abcd}V^d = D_{[a}D_{b]}V_c - \omega_{ab}\dot{V}_{\langle c \rangle} , \tag{2.11}$$

where V^a is an arbitrary 3-vector. In the vorticity-free case (as a consequence of Frobenius's theorem) ${}^{(3)}R_{abcd}$ is the Riemann curvature of the hypersurface with metric h_{ab}. Alternatively it can be given in terms of the Gauss' equation [63]:

$$^{(3)}R_{abcd} = h_a{}^e h_b{}^f h_c{}^g h_d{}^h \, {}^{(4)}R_{abcd} - (D_c u_a)(D_d u_b) + (D_d u_a)(D_c u_b). \tag{2.12}$$

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[1]The definition of E_{ab} differs by a sign in [36].

From Eqs. (2.12), (2.4) and (2.9) the curvature scalar $^{(3)}R = h^{ac}h^{bd(3)}R_{abcd}$ is [62]:

$$\frac{^{(3)}R}{2} = \mu + \Lambda - \frac{\Theta^2}{3} + \sigma^{ab}\sigma_{ab} - \omega_{ab}\omega^{ab} \tag{2.13}$$

giving the usual Ricci scalar on a spatial hypersurface orthogonal to u^a (when $\omega_{ab} = 0$).

2.2 The 1+1+2 covariant formalism

When there is a unique preferred spatial direction at each point, it is worthwhile performing a further decomposition of spacetime into its so called 1+1+2 form. The preferred spatial direction will be singled out by a normalised vector field n^a ($n^a n_a = 1$, $n^a u_a = 0$) and this allows us to decompose the metric h_{ab} as

$$h_{ab} = n_a n_b + N_{ab} , \tag{2.14}$$

where N_{ab} is the induced 2-metric on the surface perpendicular to both n^a and u^a. The alternating Levi-Civita 2-tensor is defined as $\varepsilon_{ab} = \varepsilon_{abc}n^c$. Curly brackets { } on indices will denote symmetrised and trace-free tensors which are projected in all indices with the metric N_{ab}. A bar on vector indices will denote projection onto the 2-sphere: $v_{\bar{a}} \equiv N_{ab}v^b$. The 3D covariant derivative can then be projected into two parts as [60]

$$\widehat{T}_{b..c} = n^a D_a T_{b..c} , \tag{2.15}$$

$$\delta_a T_{b..c} = N_a{}^d N_b{}^i .. N_c{}^j D_d T_{i..j} . \tag{2.16}$$

A further decomposition of the 1+3 vector and tensor variables with respect to n^a are given according to Ref. [60]. The kinematical vectors and tensor can be decomposed as

$$A^a = \mathcal{A}n^a + \mathcal{A}^a , \tag{2.17}$$

$$\omega^a = \Omega n^a + \Omega^a , \tag{2.18}$$

$$\sigma_{ab} = \Sigma\left(n_a n_b - \frac{1}{2}N_{ab}\right) + 2\Sigma_{(a}n_{b)} + \Sigma_{ab} , \tag{2.19}$$

while the gravito-electro-magnetic variables are

$$E_{ab} = \mathcal{E}\left(n_a n_b - \frac{1}{2}N_{ab}\right) + 2\mathcal{E}_{(a}n_{b)} + \mathcal{E}_{ab} , \tag{2.20}$$

$$H_{ab} = \mathcal{H}\left(n_a n_b - \frac{1}{2}N_{ab}\right) + 2\mathcal{H}_{(a}n_{b)} + \mathcal{H}_{ab} . \tag{2.21}$$

In the above decompositions the scalars, the 2-vectors and the symmetric 2-tensors represent one, two and two independent components each, respectively. Finally, the 1+2 form of the energy current vector and anisotropic pressure tensor are

$$q_a = Q n_a + Q_a , \tag{2.22}$$

$$\pi_{ab} = \Pi\left(n_a n_b - \frac{1}{2}N_{ab}\right) + 2\Pi_{(a}n_{b)} + \Pi_{ab} . \tag{2.23}$$

Here $\mathcal{A}^a$, Ω^a, Σ^a, $\mathcal{E}^a$, $\mathcal{H}^a$, Q^a and Π^a are 2-vectors and Σ_{ab}, $\mathcal{E}_{ab}$, $\mathcal{H}_{ab}$ and Π_{ab} are trace-free, symmetric 2-tensors perpendicular to both u^a and n^a.

The additional fundamental variables of the 1+1+2 formalism arise from the projected time derivative of n^a and the 3D covariant derivative of n_a. The time derivative can be written as $\dot{n}_a = \mathcal{B}u_a + \alpha_a$, where α_a is a 2-vector (since $\dot{n}_a n^a = 0$). Now, from $u^a n_a = 0$, it follows that $u^a \dot{n}_a = -n_a A^a$. Using (2.17) one obtains $\mathcal{B} = \mathcal{A}$. Hence

$$\dot{n}_a = \mathcal{A}u_a + \alpha_a \tag{2.24}$$

($\alpha_a = \dot{n}_{\bar{a}}$). The 3D covariant derivative of n_a can be decomposed as:

$$D_a n_b = \zeta_{ab} + \frac{\phi}{2}N_{ab} + \xi\varepsilon_{ab} + n_a a_b , \tag{2.25}$$

where ϕ is the sheet expansion, $a_a = n^c D_c n_a$ is the acceleration, ς_{ab} is the shear of n^a and ξ represents its rotation in the local 3D space. Here α_a and a_a are 2-vectors and ζ_{ab} is a trace-free, symmetric 2-tensor perpendicular to both u^a and n^a.

The full set of evolution and constraint equations for perturbed LRS spacetimes are given in Ref. [60]. There are no evolution equations for $\mathcal{A}$, $\mathcal{A}_a$, α_a and there is no propagation equation for a_a. These are determined by fixing a particular frame [60].

We define the 2-dimensional (2D) curvature tensor $\mathcal{R}_{abcd}$ as

$$\frac{1}{2}\mathcal{R}_{abcd}V^d = \delta_{[a}\delta_{b]}V_c - \Omega\varepsilon_{ab}\dot{V}_{\bar{c}} + \xi\varepsilon_{ab}\widehat{V}_{\bar{c}} , \tag{2.26}$$

where V^a is an arbitrary 2-vector. Similar definitions are given in [64] for higher dimensional spacetimes. For vanishing Ω and ξ it agrees with the usual Riemann curvature tensor of N_{ab}. This definition gives (see Appendix A)

$$\mathcal{R}_{abcd} = N_a^i N_b^j N_c^k N_d^l R_{ijkl} + (\delta_a u_d)(\delta_b u_c) - (\delta_a u_c)(\delta_b u_d) - (\delta_a n_d)(\delta_b n_c) + (\delta_a n_c)(\delta_b n_d) , \tag{2.27}$$

where R_{ijkl} is the usual 4D Riemann tensor. By using Eqs. (2.10), (2.4) and (2.17)-(2.25), $\mathcal{R}_{abcd}$ is expressed as

$$\begin{aligned}
\mathcal{R}_{abcd} = & \left[\frac{2(\Lambda+\mu)}{3} - \Pi - 2\mathcal{E} - \frac{1}{2}\left(\Sigma - \frac{2\Theta}{3}\right)^2 + \frac{\phi^2}{2}\right]N_{c[a}N_{b]d} \\
& + N_{c[a}\left\{\Pi_{b]d} + 2\mathcal{E}_{b]d} + \left(\Sigma - \frac{2\Theta}{3}\right)\Sigma_{b]d} + \phi\zeta_{b]d} + \left[\Omega\left(\Sigma - \frac{2\Theta}{3}\right) + \xi\phi\right]\varepsilon_{b]d}\right\} \\
& - N_{d[a}\left\{2\Pi_{b]c} + 2\mathcal{E}_{b]c} + \left(\Sigma - \frac{2\Theta}{3}\right)\Sigma_{b]c} + \phi\zeta_{b]c} + \left[\Omega\left(\Sigma - \frac{2\Theta}{3}\right) + \xi\phi\right]\varepsilon_{b]c}\right\} \\
& + 2\varepsilon_{c[a}\left[\Omega\Sigma_{b]d} - \xi\zeta_{b]d}\right] - 2\varepsilon_{d[a}\left[\Omega\Sigma_{b]c} - \xi\zeta_{b]c}\right] \\
& + 2\left(\Omega^2 - \xi^2\right)\varepsilon_{c[a}\varepsilon_{b]d} + 2\left(\zeta_{c[a}\zeta_{b]d} - \Sigma_{c[a}\Sigma_{b]d}\right) .
\end{aligned} \tag{2.28}$$

Defining the 2D curvature scalar $\mathcal{R} = N^{ac}N^{bd}\,\mathcal{R}_{abcd}$, we find

$$\mathcal{R} = \frac{2}{3}(\mu+\Lambda) - \Pi - 2\mathcal{E} - \frac{1}{2}\left(\Sigma - \frac{2\Theta}{3}\right)^2 + \frac{\phi^2}{2} - 2\left(\Omega^2 - \xi^2\right) + \Sigma_{ab}\Sigma^{ab} - \zeta_{ab}\zeta^{ab} . \tag{2.29}$$

3 The Kantowski-Sachs background

The spatial sections of Kantowski-Sachs cosmologies have topology $R \times S^2$ and are the only spatially homogeneous cosmologies that do not fit into the Bianchi classification. This is due to that their isometry group does not admit a 3-dimensional subgroup that acts simply transitive on the hypersurfaces of homogenity. These metrics are Locally Rotationally Symmetric (LRS) and belong to the LRS class II, characterised by $\omega_{ab} = \xi = H_{ab} = 0$ [55, 65]. The square of the line-element can be written as

$$ds^2 = -dt^2 + a_1^2(t) \, dz^2 + a_2^2(t) \left(d\vartheta^2 + \sin^2 \vartheta d\varphi^2 \right) \ . \tag{3.1}$$

The 4-velocity of comoving observers is $u = \partial/\partial t$ and the direction of anisotropy is $n = a_1^{-1}\partial/\partial z$, where z is dimensionless. The coordinates ϑ and φ are the polar and azimuthal angles on S^2, respectively. The scales a_1 and a_2 have the dimension of time, a_2 being assumed to be sufficiently large to avoid periodic structures in the angles emerging in the observable Universe. Symmetry and normalisation implies [55]:

$$\hat{n}_b \equiv n^a D_a n_b = 0 \ , \quad \dot{n}_{\bar{b}} = 0 \ , \tag{3.2}$$

i.e., n_a is geodesic on local 3-space with metric h_{ab} and is Fermi propagated along the integral curves of u^a. In the spacetime given by Eq. (3.1) the non-vanishing kinematical variables of $1+1+2$ formalism are the expansion [51]:

$$\Theta = \frac{\dot{a}_1}{a_1} + 2\frac{\dot{a}_2}{a_2} \ , \tag{3.3}$$

and the scalar part

$$\Sigma = \frac{2}{3}\left(\frac{\dot{a}_1}{a_1} - \frac{\dot{a}_2}{a_2} \right) \tag{3.4}$$

of the shear σ_{ab}. Given an equation of state $p = p(\mu)$ for the pressure p and energy density μ of the perfect fluid, for any given cosmological constant, the Kantowski-Sachs models are completely determined in terms of the shear Σ, expansion Θ and μ. The electric part of 4D Weyl tensor is then determined algebraically as (see Eq. (100) in [60])

$$3\mathcal{E} = -2(\mu + \Lambda) - 3\Sigma^2 + \frac{2}{3}\Theta^2 + \Sigma\Theta \ . \tag{3.5}$$

The evolutions of Σ, Θ and μ are governed by Eqs. (96), (94) and (95) of [60]:

$$\dot{\mu} = -\Theta(\mu + p) \ , \tag{3.6}$$

$$\dot{\Theta} = -\frac{\Theta^2}{3} - \frac{3}{2}\Sigma^2 - \frac{1}{2}(\mu + 3p) + \Lambda \ , \tag{3.7}$$

$$\dot{\Sigma} = \frac{2}{3}(\mu + \Lambda) + \frac{\Sigma^2}{2} - \Sigma\Theta - \frac{2}{9}\Theta^2 \ , \tag{3.8}$$

where we have used Eq. (3.5).

For Kantowski-Sachs spacetime the 2D scalar curvature (2.29) becomes

$$\mathcal{R} = \frac{2}{3}(\mu + \Lambda) - 2\mathcal{E} - \frac{1}{2}\left(\Sigma - \frac{2\Theta}{3} \right)^2 = 2(\mu + \Lambda) + \frac{3}{2}\Sigma^2 - \frac{2\Theta^2}{3} = \frac{2}{a_2^2} \ , \tag{3.9}$$

that is two times the Gaussian curvature of the 2-spheres. Taking the time derivative of $\mathcal{R}$, and using Eqs. (3.6)-(3.8), we find

$$\dot{\mathcal{R}} = \left(\Sigma - \frac{2\Theta}{3}\right)\mathcal{R} \,. \tag{3.10}$$

One of the evolution equations (3.6)-(3.8) can be replaced by (3.10).

In summary the non-vanishing quantities on the background are given by the set

$$S^{(0)} \equiv \{\Theta, \Sigma, \mathcal{E}, \mu, p\} \tag{3.11}$$

or equivalently

$$S^{(0)} = \{\Theta, \Sigma, \mathcal{R}, \mu, p\} \,. \tag{3.12}$$

To zeroth order (on the background) $\mathcal{E}$ (or $\mathcal{R}$) are given in terms of the other quantities.

General orthogonal spatially homogeneous LRS class II spacetimes emerge as slight modifications [65]. The Kantowski-Sachs cosmologies are the only ones with $\mathcal{R} > 0$. If $\mathcal{R} < 0$ the spacetimes are of Bianchi type III and the only modifications to the above equations are to replace $\sin\vartheta$ by $\sinh\vartheta$ in equation (3.1) and to change $\mathcal{R}$ to $\mathcal{R} = -2/a_2^2$. For $\mathcal{R} = 0$ there are solutions of Bianchi type I/VII$_0$. Due to that (3.9) now becomes a (satisfied) constraint, one of the evolution equations can be dropped. There are also $\mathcal{R} = 0$ with $\Sigma = \mathcal{E} = 0$. For these the sheet expansion ϕ is in general nonzero and the system is given by equations (3.6) and (3.7) plus the constraint

$$\mu + \Lambda - \frac{1}{3}\Theta^2 + \frac{3}{4}\phi^2 = 0 \,. \tag{3.13}$$

These are flat (when $\phi = 0$) or negatively curved Friedmann models, which also fall into the Bianchi I and V classes respectively.

In this paper we consider perturbations of the $\mathcal{R} > 0$ models, but the above discussion shows that perturbations of the $\mathcal{R} < 0$ can be done in an analogous way. Perturbations of Friedmann models and Bianchi I models have been considered elsewhere [38–44].

4 Vorticity-free, perfect fluid perturbations of Kantowski-Sachs cosmologies

For simplicity, we assume the perturbed fluid is irrotational and we use a frame associated with the fluid. This requirement assigns the reference 4-velocity in the perturbed spacetime and in a 1+3 covariant formalism the frame is completely fixed. In this frame the energy current q_a and $\omega^a = \Omega n^a + \Omega^a$ vanish. Moreover, we neglect the anisotropic pressure contributions to the energy-momentum tensor, i.e., $\pi_{ab} = 0$, restricting our analysis to barotropic perfect fluids.

The variables of 1+1+2 formalism are defined with respect to the frame vectors u and n. Therefore the variables are not frame-invariant in general (see Appendix B). (Of course their combinations may result in frame-invariant quantities [56].) The frame choice does not fix completely the mapping between the perturbed and the background geometry [38], [66], [67]. The variables vanishing on the background are invariant for the remaining gauge fixing in this map according to the Stewart-Walker lemma [50]. Therefore instead of $\mathcal{E}$, Θ, Σ, μ and p we use their gradients

$$X_a = \delta_a \mathcal{E} \,, \ V_a = \delta_a \Sigma \,, \ W_a = \delta_a \Theta \,,$$
$$\mu_a = \delta_a \mu \,, \ p_a = \delta_a p \,, \tag{4.1}$$

that vanish on the background. As will be shown in section 4.2, the hat-derivatives (2.15) are related to the δ_a derivatives when the vorticity vanishes.

Hence we have the following nonzero first order quantities (that vanish on the background):

$$S^{(1)} \equiv \{X_a, V_a, W_a, \mu_a, p_a, \mathcal{A}, \mathcal{A}_a, \Sigma_a, \Sigma_{ab}, \mathcal{E}_a, \mathcal{E}_{ab}, \mathcal{H}_a, \mathcal{H}_{ab}, a_b, \phi, \xi, \zeta_{ab}\} \ . \tag{4.2}$$

From the 1+1+2 equations in [60] the following nontrivial evolution equations then hold on the perturbed Kantowski-Sachs spacetime [2]:

$$\dot{\phi} = \left(\Sigma - \frac{2\Theta}{3}\right)\left(\frac{\phi}{2} - \mathcal{A}\right) + \delta^a \alpha_a \ , \tag{4.3}$$

$$2\dot{\xi} = \left(\Sigma - \frac{2\Theta}{3}\right)\xi + \varepsilon^{ab}\delta_a \alpha_b + \mathcal{H} \ , \tag{4.4}$$

$$\dot{\mathcal{H}} = \frac{3}{2}\left(\Sigma - \frac{2\Theta}{3}\right)\mathcal{H} - \varepsilon^{ab}\delta_a \mathcal{E}_b - 3\mathcal{E}\xi \ , \tag{4.5}$$

$$\dot{\mu}_{\bar{a}} = \frac{1}{2}\left(\Sigma - \frac{2\Theta}{3}\right)\mu_a - \Theta\left(\mu_a + p_a\right) - (\mu + p)W_a + \dot{\mu}\mathcal{A}_a \ , \tag{4.6}$$

$$\dot{X}_{\bar{a}} = 2\left(\Sigma - \frac{2\Theta}{3}\right)X_a + \frac{3\mathcal{E}}{2}\left(V_a - \frac{2}{3}W_a\right) - \frac{\mu+p}{2}V_a - \frac{\Sigma}{2}\left(\mu_a + p_a\right) + \dot{\mathcal{E}}\mathcal{A}_a + \varepsilon^{bc}\delta_a\delta_b\mathcal{H}_c \ , \tag{4.7}$$

$$\dot{V}_{\bar{a}} - \frac{2}{3}\dot{W}_{\bar{a}} = \frac{3}{2}\left(\Sigma - \frac{2\Theta}{3}\right)\left(V_a - \frac{2}{3}W_a\right) - X_a + \frac{1}{3}\left(\mu_a + 3p_a\right) + \left(\dot{\Sigma} - \frac{2\dot{\Theta}}{3}\right)\mathcal{A}_a - \delta_a\delta^b\mathcal{A}_b, \tag{4.8}$$

$$\dot{\Sigma}_{\{ab\}} = \left(\Sigma - \frac{2\Theta}{3}\right)\Sigma_{ab} + \delta_{\{a}\mathcal{A}_{b\}} - \mathcal{E}_{ab} \ , \tag{4.9}$$

$$\dot{\zeta}_{\{ab\}} = \frac{1}{2}\left(\Sigma - \frac{2\Theta}{3}\right)\zeta_{ab} + \delta_{\{a}\alpha_{b\}} - \varepsilon_{c\{a}\mathcal{H}_{b\}}{}^c \ . \tag{4.10}$$

The equations containing both propagation and evolution contributions are

$$\dot{W}_{\bar{a}} - \delta_a\widehat{\mathcal{A}} = \left(\frac{\Sigma}{2} - \Theta\right)W_a - 3\Sigma V_a - \frac{1}{2}\left(\mu_a + 3p_a\right) + \dot{\Theta}\mathcal{A}_a + \delta_a\delta^b\mathcal{A}_b \ , \tag{4.11}$$

$$\widehat{\alpha}_{\bar{a}} - \dot{a}_{\bar{a}} = \left(\Sigma + \frac{\Theta}{3}\right)\left(\mathcal{A}_a + a_a\right) - \varepsilon_{ab}\mathcal{H}^b \ , \tag{4.12}$$

$$2\dot{\Sigma}_{\bar{a}} - \widehat{\mathcal{A}}_a = \delta_a\mathcal{A} - \left(\Sigma + \frac{4\Theta}{3}\right)\Sigma_a - 3\Sigma\alpha_a - 2\mathcal{E}_a \ , \tag{4.13}$$

$$\dot{\mathcal{E}}_{\bar{a}} + \frac{1}{2}\varepsilon_{ab}\widehat{\mathcal{H}}^b = \frac{3}{4}\left(\Sigma - \frac{4\Theta}{3}\right)\mathcal{E}_a + \frac{(3\mathcal{E} - 2\mu - 2p)}{4}\Sigma_a + \frac{3}{4}\varepsilon_{ab}\delta^b\mathcal{H} - \frac{3\mathcal{E}}{2}\alpha_a + \frac{1}{2}\varepsilon_{bc}\delta^b\mathcal{H}^c{}_a, \tag{4.14}$$

$$\dot{\mathcal{H}}_{\bar{a}} - \frac{1}{2}\varepsilon_{ab}\widehat{\mathcal{E}}^b = \frac{3}{4}\left(\Sigma - \frac{4\Theta}{3}\right)\mathcal{H}_a - \frac{3}{4}\varepsilon_{ab}X^b - \frac{3\mathcal{E}}{2}\varepsilon_{ab}\mathcal{A}^b + \frac{3\mathcal{E}}{4}\varepsilon_{ab}a^b - \frac{1}{2}\varepsilon_{bc}\delta^b\mathcal{E}^c{}_a, \tag{4.15}$$

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[2]There are some minor misprints in [60]. In Eq. (36) the term $(\Sigma_a - \epsilon_{ab})\Omega^b\dot{\psi}$ should probably read $-2\epsilon_{ab}\Omega^b\dot{\psi}$, in Eq. (40) $(\Sigma_a - \epsilon_{ac})\Omega^c\dot{\psi}_{\bar{b}} \to -2\epsilon_{ac}\Omega^c\dot{\psi}_{\bar{b}}$, in Eq. (52) the term $-(\frac{2}{3}\Theta + \frac{1}{2}\Sigma)\Sigma_{ab} \to -(\frac{2}{3}\Theta - \Sigma)\Sigma_{ab}$ in Eq. (53) the terms $-\xi\epsilon_{ab}\alpha^b + (\frac{1}{3}\Theta + \Sigma)(\mathcal{A}_a - a_a) \to +\xi\epsilon_{ab}\alpha^b + (\frac{1}{3}\Theta + \Sigma)(\mathcal{A}_a + a_a)$, in Eq. (76) $-\Sigma\epsilon_{ab}\mathcal{H}^b \to -\frac{3}{2}\Sigma\epsilon_{ab}\mathcal{H}^b$ and in Eq. (80) $-(\frac{1}{3}\Theta - \frac{1}{2}\Sigma)(\Sigma_a - \epsilon_{ab}\Omega^b) \to +(\frac{1}{3}\Theta - \frac{1}{2}\Sigma)(\Sigma_a - \epsilon_{ab}\Omega^b)$.

$$\dot{\mathcal{E}}_{\{ab\}} - \varepsilon_{c\{a}\widehat{\mathcal{H}}_{b\}}{}^{c} = -\frac{3}{2}\left(\Sigma + \frac{2\Theta}{3}\right)\mathcal{E}_{ab} - \varepsilon_{c\{a}\delta^{c}\mathcal{H}_{b\}} - \frac{(3\mathcal{E} + \mu + p)}{2}\Sigma_{ab},$$ (4.16)

$$\dot{\mathcal{H}}_{\{ab\}} + \varepsilon_{c\{a}\widehat{\mathcal{E}}_{b\}}{}^{c} = -\frac{3}{2}\left(\Sigma + \frac{2\Theta}{3}\right)\mathcal{H}_{ab} + \frac{3\mathcal{E}}{2}\varepsilon_{c\{a}\zeta_{b\}}{}^{c} + \varepsilon_{c\{a}\delta^{c}\mathcal{E}_{b\}} \ .$$ (4.17)

The pure propagation equations are

$$\widehat{\phi} = -\left(\Sigma - \frac{2\Theta}{3}\right)\left(\Sigma + \frac{\Theta}{3}\right) + \delta^{a}a_{a} - \mathcal{E} - \frac{2\left(\mu + \Lambda\right)}{3},$$ (4.18)

$$2\widehat{\xi} = \varepsilon^{ab}\delta_{a}a_{b} \ ,$$ (4.19)

$$\widehat{\mathcal{H}} = -\delta^{a}\mathcal{H}_{a} \ ,$$ (4.20)

$$\widehat{\mathcal{A}}_{a} = \delta_{a}\mathcal{A} \ ,$$ (4.21)

$$\widehat{p}_{\bar{a}} = -\left(\mu + p\right)\delta_{a}\mathcal{A} \ ,$$ (4.22)

$$\frac{\widehat{\mu}_{\bar{a}}}{3} - \widehat{X}_{\bar{a}} = \frac{3\mathcal{E}}{2}\delta_{a}\phi + \delta_{a}\delta^{b}\mathcal{E}_{b},$$ (4.23)

$$\frac{2}{3}\widehat{W}_{\bar{a}} - \widehat{V}_{\bar{a}} = \frac{3\Sigma}{2}\delta_{a}\phi + \delta_{a}\delta^{b}\Sigma_{b} \ ,$$ (4.24)

$$\widehat{\Sigma}_{\bar{a}} = \frac{1}{2}\left(V_{a} + \frac{4}{3}W_{a}\right) - \frac{3\Sigma}{2}a_{a} - \delta^{b}\Sigma_{ab} \ ,$$ (4.25)

$$2\widehat{\mathcal{E}}_{\bar{a}} = X_{a} - 3\mathcal{E}a_{a} - 3\Sigma\varepsilon_{ab}\mathcal{H}^{b} - 2\delta^{b}\mathcal{E}_{ab} + \frac{2}{3}\mu_{a} \ ,$$ (4.26)

$$2\widehat{\mathcal{H}}_{\bar{a}} = \delta_{a}\mathcal{H} - 3\mathcal{E}\varepsilon_{ab}\Sigma^{b} + 3\Sigma\varepsilon_{ab}\mathcal{E}^{b} - 2\delta^{b}\mathcal{H}_{ab} \ ,$$ (4.27)

$$\widehat{\Sigma}_{\{ab\}} = \delta_{\{a}\Sigma_{b\}} + \frac{3\Sigma}{2}\zeta_{ab} - \varepsilon_{c\{a}\mathcal{H}_{b\}}{}^{c} \ ,$$ (4.28)

$$\widehat{\zeta}_{\{ab\}} = \left(\Sigma + \frac{\Theta}{3}\right)\Sigma_{ab} + \delta_{\{a}a_{b\}} - \mathcal{E}_{ab} \ .$$ (4.29)

Finally, the constraints are

$$\varepsilon^{ab}\delta_{a}\mathcal{A}_{b} = 0 \ ,$$ (4.30)

$$p_{a} = -\left(\mu + p\right)\mathcal{A}_{a} \ ,$$ (4.31)

$$\varepsilon^{ab}\delta_{a}\Sigma_{b} = -3\Sigma\xi + \mathcal{H} \ ,$$ (4.32)

$$\varepsilon_{ab}\delta^{b}\xi + \delta^{b}\zeta_{ab} - \frac{\delta_{a}\phi}{2} = \frac{1}{2}\left(\Sigma - \frac{2\Theta}{3}\right)\Sigma_{a} + \mathcal{E}_{a},$$ (4.33)

$$\left(V_{a} - \frac{2}{3}W_{a}\right) + 2\delta^{b}\Sigma_{ab} = -2\varepsilon_{ab}\mathcal{H}^{b} \ .$$ (4.34)

The equations (4.3)-(4.34) were derived from the generic 1+1+2 equations given in Ref. [60], by use of the commutation relations given in Appendix C.

4.1 Harmonic expansion

Following Ref. [51] we expand the scalar perturbation variables into harmonics as

$$\Psi = \sum_{k_\parallel, k_\perp} \Psi^S_{k_\parallel k_\perp} \, P^{k_\parallel} \, Q^{k_\perp} \ . \tag{4.35}$$

The coefficients $\Psi^S_{k_\parallel k_\perp}$ depend solely of time. The function $P^{k_\parallel}$ is the eigenfunction of the Laplacian $\widehat{\Delta} = n^a \nabla_a n^b \nabla_b$ and it is constant on the $z = const$ hypersurfaces:

$$\widehat{\Delta} P^{k_\parallel} = -\frac{k_\parallel^2}{a_1^2} P^{k_\parallel} \ , \ \delta_a P^{k_\parallel} = \dot{P}^{k_\parallel} = 0 \ . \tag{4.36}$$

Here $k_\parallel$ are the constant comoving wave numbers in the direction of anisotropy and the scale factor a_1 in this direction obeys

$$\frac{\dot{a}_1}{a_1} = \Sigma + \frac{\Theta}{3} \ . \tag{4.37}$$

The harmonics are introduced on the 2-sphere as

$$\delta^2 Q^{l,m} = -\frac{l(l+1)}{a_2^2} Q^{l,m} \ , \ \widehat{Q}^{l,m} = \dot{Q}^{l,m} = 0 \ , \tag{4.38}$$

where $\delta^2 = \delta_a \delta^a$, and the second scale factor a_2 satisfies

$$\frac{\dot{a}_2}{a_2} = -\frac{1}{2} \left(\Sigma - \frac{2\Theta}{3} \right) \ . \tag{4.39}$$

For a given l value the index m runs from $-l$ to l. Due to the symmetries of the background spacetime the index m never appear explicitly, therefore we will use the following notation:

$$\delta^2 Q^{k_\perp} = -\frac{k_\perp^2}{a_2^2} Q^{k_\perp} \ , \ \widehat{Q}^{k_\perp} = \dot{Q}^{k_\perp} = 0 \ , \tag{4.40}$$

with $k_\perp^2 = l(l+1)$ comoving wave numbers in the perpendicular direction to n^a.

The vectors and tensors can be also expanded in harmonics by introducing the vector and tensor spherical harmonics [56, 57, 59]. The even (electric) and odd (magnetic) parity vector harmonics are

$$Q_a^{k_\perp} = a_2 \delta_a Q^{k_\perp} \ , \ \overline{Q}_a^{k_\perp} = a_2 \varepsilon_{ab} \delta^b Q^{k_\perp} \ , \tag{4.41}$$

and the vector Ψ_a can be expanded as

$$\Psi_a = \sum_{k_\parallel, k_\perp} P^{k_\parallel} \left(\Psi^V_{k_\parallel k_\perp} Q_a^{k_\perp} + \overline{\Psi}^V_{k_\parallel k_\perp} \overline{Q}_a^{k_\perp} \right) \ . \tag{4.42}$$

Similarly, the even and odd tensor spherical harmonics are

$$Q_{ab}^{k_\perp} = a_2^2 \delta_{\{a} \delta_{b\}} Q^{k_\perp} \ , \ \overline{Q}_{ab}^{k_\perp} = a_2^2 \varepsilon_{c\{a} \delta^c \delta_{b\}} Q^{k_\perp} \ , \tag{4.43}$$

and the tensor Ψ_{ab} can be expanded as

$$\Psi_{ab} = \sum_{k_\parallel, k_\perp} P^{k_\parallel} \left(\Psi^T_{k_\parallel k_\perp} Q_{ab}^{k_\perp} + \overline{\Psi}^T_{k_\parallel k_\perp} \overline{Q}_{ab}^{k_\perp} \right) \ . \tag{4.44}$$

This decomposition of the vectors and tensors encompasses both an expansion into spherical harmonics and into even and odd modes, similarly to the Regge-Wheeler decomposition of the perturbations of spherically symmetric spacetimes, leading to the Regge-Wheeler equation for the odd modes [68] and the Zerilli equation for the even modes [69, 70]. In our decomposition however the coefficients exhibit a (t, z) dependence, rather than (t, r). Some useful relations involving the vector and tensor spherical harmonics are enlisted in Appendix D.

4.2 Relations between harmonic coefficients from commutation rules of covariant derivatives

A Kantowski-Sachs spacetime filled with perfect fluid is characterised by the time-dependent scalars $G \equiv \{\mu, \mathcal{E}, \Sigma, \Theta, p\}$. We expand their δ_a-derivatives into harmonics on the perturbed spacetime. Thus, applying the commutation relation Eq. (C.4) for zero-order scalars and using (D.6), we find in the absence of vorticities the following relations:

$$\overline{\mu}^V_{k_\parallel k_\perp} = \overline{X}^V_{k_\parallel k_\perp} = \overline{V}^V_{k_\parallel k_\perp} = \overline{W}^V_{k_\parallel k_\perp} = \overline{p}^V_{k_\parallel k_\perp} = 0 \ . \tag{4.45}$$

However, we could also expand the anisotropic direction derivatives of G into scalar harmonics, as they are also first-order. This expansion is

$$\widehat{G} = \sum_{k_\parallel k_\perp} \widetilde{G}^S_{k_\parallel k_\perp} \ P^{k_\parallel} \ Q^{k_\perp} \ . \tag{4.46}$$

Then the even parity part of Eq. (C.3) gives

$$\frac{\widetilde{G}^S_{k_\parallel k_\perp}}{a_2} = \frac{ik_\parallel}{a_1} G^V_{k_\parallel k_\perp} \ , \tag{4.47}$$

a constraint on $\widetilde{G}^S_{k_\parallel k_\perp}$ and $G^V_{k_\parallel k_\perp}$, emerging in the absence of vorticities. Using Eq. (4.45), the odd parity part of Eq. (C.3) becomes trivial.

4.3 Full set of evolution and constraint equations for the harmonic coefficients

From the commutation rules we have found that some harmonic coefficients vanish in the absence of vorticities (see Eq. (4.45)). There is a further coefficient $\overline{\mathcal{A}}^V_{k_\parallel k_\perp} = 0$, the vanishing of which follows from Eq. (4.30). Using these relations, the perturbation equations (4.3)-(4.34) can be expanded into harmonics and are given in Appendix E. Some of these equations are first integrals of the rest. We have found 17 independent constraints for 28 variables, as presented in Appendix E. We consider adiabatic matter perturbations $p = p(\mu)$ giving $p^V_{k_\parallel k_\perp} = c^2_s \mu^V_{k_\parallel k_\perp}$ with c^2_s the square of the matter speed of sound.[3]

In the frame associated to the fluid ($q_a = 0$) the vorticity ω_a also vanishes because the fluid is irrotational. Moreover we have assumed π_{ab} is negligible. These assumptions fixes completely the frame in a 1+3 covariant formalism, however do not in a 1+1+2 description where a dyad (u^a, n^a) must be assigned (see Appendix B). The quantities q_a, ω_a and π_{ab} are invariant for some part of the infinitesimal transformations which fixes n^a. In particular, they are invariant under the infinitesimal translations given by l_a. We have 2 gauge degrees

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[3]The adiabatic assumption for the total fluid perturbation, leading to $c^2_s = \dot{p}/\dot{\mu}$ is a good approximation when one of the matter components dominates: for instance, $c^2_s = 1/3$ in the radiation dominated area and $c^2_s \approx 0$ for the dust dominated regime.

of freedom to sign n^a perpendicularly to u^a on the perturbed spacetime and to fix completely the frame. In the frame $a_a = 0$ Eqs. (E.1) and (E.24) become constraints, indicating that the fixing of l_a reduces the degrees of freedom by four. Thus, we have 6 degrees of freedom describing fully the vorticity-free perturbations in the adiabatic case with $q_a = 0 = \pi_{ab}$. These variables can be chosen as $\mu^V_{k_\parallel k_\perp}$, $\Sigma^T_{k_\parallel k_\perp}$, $\mathcal{E}^T_{k_\parallel k_\perp}$, $\overline{\mathcal{E}}^T_{k_\parallel k_\perp}$, $\mathcal{H}^T_{k_\parallel k_\perp}$ and $\overline{\mathcal{H}}^T_{k_\parallel k_\perp}$. They are invariant for the l_a-infinitesimal translation. Their evolutions are governed by two sets of decoupled equations, which follow from the evolution equations (E.20), (E.28), (E.30), (E.7), (E.31) and (E.8) of Appendix E by employing the constraints.

4.3.1 Uncoupled evolutions of gravitational perturbations

The two coefficients $\overline{\mathcal{E}}^T_{k_\parallel k_\perp}$ and $\mathcal{H}^T_{k_\parallel k_\perp}$ form a decoupled system

$$\dot{\overline{\mathcal{E}}}^T_{k_\parallel k_\perp} = -\frac{3}{2}(F + \Sigma D)\overline{\mathcal{E}}^T_{k_\parallel k_\perp} + \frac{ik_\parallel}{a_1}(1 - D)\,\mathcal{H}^T_{k_\parallel k_\perp}, \tag{4.48}$$

$$\dot{\mathcal{H}}^T_{k_\parallel k_\perp} = -\frac{a_1}{2ik_\parallel}\left(\frac{2k_\parallel^2}{a_1^2} - BC + 9\Sigma E\right)\overline{\mathcal{E}}^T_{k_\parallel k_\perp} - \frac{3}{2}(2E + F)\,\mathcal{H}^T_{k_\parallel k_\perp}, \tag{4.49}$$

with the coefficients

$$B \equiv \frac{2k_\parallel^2}{a_1^2} + \frac{k_\perp^2}{a_2^2} + \frac{9\Sigma^2}{2} + 3\mathcal{E} = \frac{2k_\parallel^2}{a_1^2} - \frac{2 - k_\perp^2}{a_2^2} + 3\Sigma\left(\Sigma + \frac{\Theta}{3}\right), \tag{4.50}$$

$$C \equiv B^{-1}\left(\frac{2 - k_\perp^2}{a_2^2} + 3\mathcal{E}\right), \tag{4.51}$$

$$D \equiv C + \frac{\mu + p}{B}, \tag{4.52}$$

$$E \equiv \frac{\Sigma}{2}\left(C - \frac{\mathcal{E}}{B}\right) + \frac{\Theta\mathcal{E}}{3B}, \tag{4.53}$$

$$F \equiv \Sigma + \frac{2\Theta}{3}. \tag{4.54}$$

The second equality in (4.50) follows from Eqs. (3.5) and (3.9).

Equivalently, the system can be rewritten as decoupled second-order linear homogeneous ordinary differential equations:

$$\ddot{\overline{\mathcal{E}}}^T_{k_\parallel k_\perp} + q_{\overline{\mathcal{E}}1}\dot{\overline{\mathcal{E}}}^T_{k_\parallel k_\perp} + q_{\overline{\mathcal{E}}0}\overline{\mathcal{E}}^T_{k_\parallel k_\perp} = 0, \tag{4.55}$$

$$\ddot{\mathcal{H}}^T_{k_\parallel k_\perp} + q_{\mathcal{H}1}\dot{\mathcal{H}}^T_{k_\parallel k_\perp} + q_{\mathcal{H}0}\mathcal{H}^T_{k_\parallel k_\perp} = 0, \tag{4.56}$$

where

$$q_{\overline{\mathcal{E}}1} = \frac{3}{2}(2E + 2F + \Sigma D) - \frac{d}{dt}\ln\frac{1 - D}{a_1}, \tag{4.57}$$

$$2q_{\overline{\mathcal{E}}0} = \frac{1 - D}{a_1}\left[\frac{2k_\parallel^2}{a_1} + a_1(9\Sigma E - BC)\right] + 3\frac{d}{dt}(F + \Sigma D)$$
$$- 3(F + \Sigma D)\left[\frac{d}{dt}\ln\frac{1 - D}{a_1} - \frac{3}{2}(2E + F)\right], \tag{4.58}$$

$$q_{\mathcal{H}1} = \frac{3}{2}(2E + 2F + \Sigma D) - \frac{d}{dt}\ln\left[\frac{2k_\parallel^2}{a_1} + a_1\left(9\Sigma E - BC\right)\right] , \qquad (4.59)$$

$$2q_{\mathcal{H}0} = \frac{1-D}{a_1}\left[\frac{2k_\parallel^2}{a_1} + a_1\left(9\Sigma E - BC\right)\right] - 3\left(2E + F\right)\frac{d}{dt}\ln\left[\frac{2k_\parallel^2}{a_1} + a_1\left(9\Sigma E - BC\right)\right]$$
$$+ \frac{9}{2}(F + \Sigma D)\left(2E + F\right) + 3\frac{d}{dt}\left(2E + F\right) . \qquad (4.60)$$

The equations (4.55)-(4.56) represent wave equations with friction. As will be shown in Section 5 of the paper, the quantities obeying these equations represent the gravitational wave degrees of freedom.

4.3.2 Evolutions with matter sources

The coefficients $\Sigma^T_{k_\parallel k_\perp}$, $\mathcal{E}^T_{k_\parallel k_\perp}$ and $\overline{\mathcal{H}}^T_{k_\parallel k_\perp}$ also form a system of differential equations coupled to the density gradient $\mu^V_{k_\parallel k_\perp}$, as follows

$$\dot{\mu}^V_{k_\parallel k_\perp} = \left[\frac{\Sigma}{2}\left(1 - 3\frac{\mu + p}{B}\right) - \frac{4\Theta}{3}\right]\mu^V_{k_\parallel k_\perp} + \frac{a_2}{2}\left(\mu + p\right)$$
$$\times\left[(1 - C)\left(B\Sigma^T_{k_\parallel k_\perp} - 3\Sigma\mathcal{E}^T_{k_\parallel k_\perp}\right) - \frac{a_1}{ik_\parallel}P\overline{\mathcal{H}}^T_{k_\parallel k_\perp}\right] , \qquad (4.61)$$

$$\dot{\Sigma}^T_{k_\parallel k_\perp} = -\frac{c_s^2}{a_2\left(\mu + p\right)}\mu^V_{k_\parallel k_\perp} + \left(\Sigma - \frac{2\Theta}{3}\right)\Sigma^T_{k_\parallel k_\perp} - \mathcal{E}^T_{k_\parallel k_\perp} , \qquad (4.62)$$

$$\dot{\mathcal{E}}^T_{k_\parallel k_\perp} = \frac{3\Sigma}{2a_2 B}\mu^V_{k_\parallel k_\perp} - \frac{\mu + p}{2}\Sigma^T_{k_\parallel k_\perp} - \frac{3}{2}\left(F + \Sigma C\right)\mathcal{E}^T_{k_\parallel k_\perp} + \frac{a_1}{2ik_\parallel}P\overline{\mathcal{H}}^T_{k_\parallel k_\perp} , \qquad (4.63)$$

$$\dot{\overline{\mathcal{H}}}^T_{k_\parallel k_\perp} = -\frac{ik_\parallel}{a_1 a_2 B}\mu^V_{k_\parallel k_\perp} - \frac{\overline{\mathcal{H}}^T_{k_\parallel k_\perp}}{S} - \frac{ik_\parallel}{a_1}\left(1 - C\right)\mathcal{E}^T_{k_\parallel k_\perp} . \qquad (4.64)$$

Here we have introduced the additional notations

$$P \equiv \frac{2k_\parallel^2}{a_1^2}\left(1 - C\right) - \frac{k_\perp^2}{a_2^2}\frac{2 - k_\perp^2}{a_2^2 B} , \qquad (4.65)$$

$$S^{-1} \equiv \frac{2}{\Sigma B}\left[\left(\mathcal{E} + \frac{3\Sigma^2}{2}\right)\frac{2 - k_\perp^2}{2a_2^2} - \mathcal{E}\frac{k_\parallel^2}{a_1^2}\right] + \frac{3}{2}F + \frac{\mathcal{E}}{\Sigma} . \qquad (4.66)$$

This system is equivalent to the ones describing scalar perturbations studied in [51]. Here the variables

$$\mathcal{D}_a \equiv a\frac{D_a\mu}{\mu} , \quad \mathcal{Z}_a \equiv aD_a\Theta , \quad \mathcal{T}_a \equiv aD_a\sigma^2 , \qquad (4.67)$$

$$\mathcal{S}_a \equiv D_a\left(\sigma^{bc}S_{bc}\right) \qquad (4.68)$$

where a is the average scale factor defined through $\Theta = 3\dot{a}/a$ and S_{ab} is the traceless part of the 3-Ricci tensor, were used. When projected onto the 2-sphere and expressed in terms of

the variables μ_a, V_a, W_a and X_a (see (4.1)) they read

$$\mathcal{D}_{\bar{a}} = a\frac{\mu_a}{\mu}, \quad \mathcal{Z}_{\bar{a}} = aW_a, \quad \mathcal{T}_{\bar{a}} = \frac{3}{2}a\Sigma V_a \tag{4.69}$$

$$\mathcal{S}_{\bar{a}} = \frac{3a}{2}\Sigma X_a + a\left(\frac{3}{2}\mathcal{E} - \Theta\Sigma + \frac{9}{4}\Sigma^2\right)V_a - \frac{a}{2}\Sigma^2 W_a. \tag{4.70}$$

Due to equation (4.45) we only have to consider the even parity components, $\mu^V_{k_\parallel k_\perp}$, $V^V_{k_\parallel k_\perp}$, $W^V_{k_\parallel k_\perp}$ and $X^V_{k_\parallel k_\perp}$, of the variables (4.1). The three latter are solved for in terms of $\Sigma^T_{k_\parallel k_\perp}$, $\mathcal{E}^T_{k_\parallel k_\perp}$, $\overline{\mathcal{H}}^T_{k_\parallel k_\perp}$ and $\mu^V_{k_\parallel k_\perp}$ in equations (E.51), (E.52) and (E.55) in appendix E. Substitution of these into equations (C.1)-(C.4) in appendix C of [51] reproduces the system (4.61)-(4.64).

We proceed with transforming Eqs. (4.62)-(4.64) into second order oscillator equations for each of the gravitational perturbations $\Sigma^T_{k_\parallel k_\perp}$, $\mathcal{E}^T_{k_\parallel k_\perp}$ and $\overline{\mathcal{H}}^T_{k_\parallel k_\perp}$, with source terms given by the matter perturbations $\mu^V_{k_\parallel k_\perp}$ and $\dot{\mu}^V_{k_\parallel k_\perp}$ which induce forced oscillations. In turn, then these gravitational perturbations act as sources for the first order evolutions of $\mu^V_{k_\parallel k_\perp}$. For the gravitational perturbations we obtain

$$\ddot{\Sigma}^T_{k_\parallel k_\perp} + q_{\Sigma 1}\dot{\Sigma}^T_{k_\parallel k_\perp} + q_{\Sigma 0}\Sigma^T_{k_\parallel k_\perp} = \frac{(1-c_s^2)}{a_2(\mu+p)}\dot{\mu}^V_{k_\parallel k_\perp} + \frac{s_{\Sigma 0}}{a_2(\mu+p)}\mu^V_{k_\parallel k_\perp}, \tag{4.71}$$

$$\ddot{\mathcal{E}}^T_{k_\parallel k_\perp} + q_{\mathcal{E}1}\dot{\mathcal{E}}^T_{k_\parallel k_\perp} + q_{\mathcal{E}0}\mathcal{E}^T_{k_\parallel k_\perp} = \frac{s_{\mathcal{E}1}}{a_2}\dot{\mu}^V_{k_\parallel k_\perp} + s_{\mathcal{E}0}\mu^V_{k_\parallel k_\perp}, \tag{4.72}$$

and

$$\ddot{\overline{\mathcal{H}}}^T_{k_\parallel k_\perp} + q_{\overline{\mathcal{H}}1}\dot{\overline{\mathcal{H}}}^T_{k_\parallel k_\perp} + q_{\overline{\mathcal{H}}0}\overline{\mathcal{H}}^T_{k_\parallel k_\perp} = \frac{ik_\parallel s_{\overline{\mathcal{H}}0}}{a_1 a_2 B}\mu^V_{k_\parallel k_\perp}, \tag{4.73}$$

with the coefficients

$$q_{\Sigma 1} = 2\Sigma + \frac{5\Theta}{3}, \tag{4.74}$$

$$q_{\Sigma 0} = \frac{(1-C)B-(\mu+p)}{2} - \frac{d}{dt}\left(\Sigma - \frac{2\Theta}{3}\right) - 3\left(\Sigma + \frac{\Theta}{3}\right)\left(\Sigma - \frac{2\Theta}{3}\right), \tag{4.75}$$

$$s_{\Sigma 0} = \frac{4\Theta}{3} - \frac{\Sigma}{2} - c_s^2(\Theta+3\Sigma) - \dot{c}_s^2 + \frac{d}{dt}\ln[a_2(\mu+p)], \tag{4.76}$$

$$q_{\mathcal{E}1} = W_1\left[\frac{1}{S}+\Theta+\frac{3}{2}\Sigma(1+C)-\frac{d}{dt}\ln(a_1 P)\right]$$
$$-W_2\left[\frac{5\Theta}{3}+\frac{\Sigma}{2}(1+3C)-\frac{d}{dt}\ln(\mu+p)\right], \tag{4.77}$$

$$q_{\mathcal{E}0} = \frac{3}{2}\frac{d}{dt}\left(\Sigma+\frac{2\Theta}{3}+\Sigma C\right) + \frac{(1-C)P}{2} - \frac{\mu+p}{2} - W_1\left(\Theta+\frac{3}{2}\Sigma(1+C)\right)$$
$$\times\left[\frac{d}{dt}\ln(a_1 P)-\frac{1}{S}\right] + W_2(\Theta+3\Sigma)\left[\Sigma-\frac{2\Theta}{3}+\frac{d}{dt}\ln(\mu+p)\right]$$
$$-W_2\frac{3\Sigma(1-C)}{2}\left[\frac{d}{dt}\ln(a_1 P)-\frac{1}{S}\right], \tag{4.78}$$

$$s_{\mathcal{E}1} = \frac{3\Sigma}{2B} + \frac{1}{(1-C)B - (\mu+p)} \left(\frac{2\Theta}{3} - \Sigma - \frac{1}{S} + \frac{d}{dt} \ln \frac{a_1 P}{\mu+p} \right) , \tag{4.79}$$

$$s_{\mathcal{E}0} = \frac{d}{dt} \left(\frac{3\Sigma}{2a_2 B} \right) - \frac{P}{2a_2 B} + \frac{c_s^2}{2a_2} + \frac{W_3 - W_4}{a_2 \left[(1-C)B - (\mu+p) \right]} , \tag{4.80}$$

and

$$q_{\overline{\mathcal{H}}1} = \frac{1}{S} + Q , \tag{4.81}$$

$$q_{\overline{\mathcal{H}}0} = \frac{Q}{S} + \left(1 - C - \frac{\mu+p}{B} \right) \frac{P}{2} + \frac{d}{dt} \frac{1}{S} , \tag{4.82}$$

$$s_{\overline{\mathcal{H}}0} = \frac{\Theta}{3} - \frac{7\Sigma}{2} + \frac{d}{dt} \ln \left[a_2 B \left(1 - C \right) \right] , \tag{4.83}$$

where we have denoted

$$W_1 = \frac{(1-C)B}{(1-C)B - (\mu+p)} , \tag{4.84}$$

$$W_2 = \frac{(\mu+p)}{(1-C)B - (\mu+p)} , \tag{4.85}$$

$$W_3 = \left[\frac{d}{dt} \ln \left(a_1 P \right) - \frac{1}{S} \right] \left[\frac{4\Theta}{3} - \frac{\Sigma (4-3C)}{2} + \frac{3\Sigma (\mu+p)}{2B} \right] , \tag{4.86}$$

$$W_4 = \left[\frac{d}{dt} \ln \left(\mu+p \right) + \left(\Sigma - \frac{2\Theta}{3} \right) \right] \left(\frac{4\Theta}{3} - \frac{\Sigma}{2} \right) , \tag{4.87}$$

$$Q = \Theta + \frac{3\Sigma}{2} \left(1 + C + \frac{\mu+p}{B} \right) - \frac{d}{dt} \ln \frac{1-C}{a_1} . \tag{4.88}$$

In the next section we analyse the high frequency limit of these equations.

5   Geometrical optics approximation

In this section we follow Isaacson's definition [71], [72] of gravitational waves on a curved background in a geometrical optics approximation. The key concept is that gravitational waves are periodic perturbations with a wavelength much shorter than the curvature radius of the background. This is known as the geometrical optics approximation, or the high frequency limit. In the notations of the present paper the physical wave numbers along z and along the spheres are $k_\parallel/a_1$ and $k_\perp/a_2$, respectively.

Then $k_\parallel, k_\perp \gg 1$, while a glance on Eqs. (3.3), (3.4), (3.5) and (3.9) implies

$$L \left(\frac{2k_\parallel^2}{a_1^2}, \frac{k_\perp^2}{a_2^2} \right) \gg \Theta^2, \Sigma^2, \mathcal{E}, \mu, p , \tag{5.1}$$

where L is any linear combination with coefficients of order unity. Implementing these in the equations we get

$$B \simeq P \simeq \frac{2k_\parallel^2}{a_1^2} + \frac{k_\perp^2}{a_2^2} , \tag{5.2}$$

and

$$D \simeq C \simeq -B^{-1}\frac{k_\perp^2}{a_2^2} \ , \quad E \simeq \frac{\Sigma}{2}C \ , \tag{5.3}$$

$$S \simeq -\Sigma B \left[\left(\mathcal{E} + \frac{3\Sigma^2}{2} \right) \frac{k_\perp^2}{a_2^2} + 2\mathcal{E}\frac{k_\parallel^2}{a_1^2} \right]^{-1} \ . \tag{5.4}$$

Thus in the geometrical optics approximation the order of the dimensionless quantities relates as $\mathcal{O}\left(a_i^2 B\right) = \mathcal{O}\left(a_i^2 P\right) \gg \mathcal{O}\left(C\right) = \mathcal{O}\left(D\right) = \mathcal{O}\left(a_i E\right) = \mathcal{O}\left(a_i F\right) = \mathcal{O}\left(a_i^{-1} S\right) = \mathcal{O}\left(1\right)$.

5.1 High frequency evolutions of the uncoupled gravitational perturbations: gravitational waves

The relevant coefficients are approximated as

$$q_{\overline{\mathcal{E}}0} \simeq q_{\mathcal{H}0} \simeq (1 - C)\left(\frac{k_\parallel^2}{a_1^2} + \frac{k_\perp^2}{2a_2^2} \right) = \frac{k_\parallel^2}{a_1^2} + \frac{k_\perp^2}{a_2^2}, \tag{5.5}$$

$$q_{\overline{\mathcal{E}}1} \simeq 3\left(F + \Sigma C\right) - \frac{d}{dt}\ln\frac{1 - C}{a_1} \ , \tag{5.6}$$

$$q_{\mathcal{H}1} \simeq 3\left(F + \Sigma C\right) - \frac{d}{dt}\ln\left[a_1 \left(\frac{2k_\parallel^2}{a_1^2} + \frac{k_\perp^2}{a_2^2} \right) \right]. \tag{5.7}$$

The damped wave equations (4.55)-(4.56) simplify to

$$\dddot{\overline{\mathcal{E}}}^T_{k_\parallel k_\perp} + q_{\overline{\mathcal{E}}1}\ddot{\overline{\mathcal{E}}}^T_{k_\parallel k_\perp} + \left(\frac{k_\parallel^2}{a_1^2} + \frac{k_\perp^2}{a_2^2} \right) \overline{\mathcal{E}}^T_{k_\parallel k_\perp} = 0 \ , \tag{5.8}$$

$$\ddot{\mathcal{H}}^T_{k_\parallel k_\perp} + q_{\mathcal{H}1}\dot{\mathcal{H}}^T_{k_\parallel k_\perp} + \left(\frac{k_\parallel^2}{a_1^2} + \frac{k_\perp^2}{a_2^2} \right) \mathcal{H}^T_{k_\parallel k_\perp} = 0 \ . \tag{5.9}$$

Both of these equations are of the form $\ddot{X} + 2\zeta\Omega\dot{X} + \Omega^2 X = 0$, where Ω represents the undamped angular frequency. For $\zeta < 1$ the oscillator is underdamped and the real angular frequency is given by $\Omega\sqrt{1 - \zeta^2}$. The propagation speed of the wave therefore is $c_w = \Omega\sqrt{1 - \zeta^2}/k_{phys}$ (with k_{phys} given by $k_\parallel/a_1$ or $k_\perp/a_2$, when the propagation is along z or along the spheres). These considerations imply that we are assuming a negligible change in the scale factors over one period.

In order to continue the analysis we define a small parameter $\varepsilon \approx (a_i k_{phys})^{-1}$ (characterising the geometrical optics approximation) and we consider perturbations along the z direction and along the sphere separately.

5.1.1 Waves propagating along the z-direction

We get $C \simeq 0$ and $F + \Sigma C = \frac{2\Theta}{3} + \Sigma$, hence the propagation equations are

$$\dddot{\overline{\mathcal{E}}}^T_{k_\parallel k_\perp} + \left(2\Theta + 3\Sigma + \frac{\dot{a}_1}{a_1} \right) \ddot{\overline{\mathcal{E}}}^T_{k_\parallel k_\perp} + \frac{k_\parallel^2}{a_1^2}\overline{\mathcal{E}}^T_{k_\parallel k_\perp} = 0 \ , \tag{5.10}$$

$$\ddot{\mathcal{H}}^T_{k_\parallel k_\perp} + \left(2\Theta + 3\Sigma + \frac{\dot{a}_1}{a_1} \right) \dot{\mathcal{H}}^T_{k_\parallel k_\perp} + \frac{k_\parallel^2}{a_1^2}\mathcal{H}^T_{k_\parallel k_\perp} = 0 \ . \tag{5.11}$$

The damping parameter turns out to be

$$\zeta_\parallel = \frac{a_1}{2k_\parallel}\left(2\Theta + 3\Sigma + \frac{\dot{a}_1}{a_1}\right) = \mathcal{O}\left(\varepsilon\right) \ , \tag{5.12}$$

and the speed of propagation of both $\overline{\mathcal{E}}^T_{k_\parallel k_\perp}$ and $\mathcal{H}^T_{k_\parallel k_\perp}$ is

$$c_\parallel = \sqrt{1-\zeta^2} \simeq 1 - \frac{\zeta^2}{2} = 1 - \mathcal{O}\left(\varepsilon^2\right) \ . \tag{5.13}$$

Thus to linear order in the geometrical optics approximation both $\overline{\mathcal{E}}^T_{k_\parallel k_\perp}$ and $\mathcal{H}^T_{k_\parallel k_\perp}$ represent gravitational waves propagating with the speed of light.

5.1.2 Waves propagating along the spheres

We get $C \simeq -1$ and $F + \Sigma C = \frac{2\Theta}{3}$, hence

$$\ddot{\overline{\mathcal{E}}}^T_{k_\parallel k_\perp} + \left(2\Theta + \frac{\dot{a}_1}{a_1}\right)\dot{\overline{\mathcal{E}}}^T_{k_\parallel k_\perp} + \frac{k_\perp^2}{a_2^2}\overline{\mathcal{E}}^T_{k_\parallel k_\perp} = 0 \ , \tag{5.14}$$

$$\ddot{\mathcal{H}}^T_{k_\parallel k_\perp} + \left(2\Theta - \frac{\dot{a}_1}{a_1} + \frac{2\dot{a}_2}{a_2}\right)\dot{\mathcal{H}}^T_{k_\parallel k_\perp} + \frac{k_\perp^2}{a_2^2}\mathcal{H}^T_{k_\parallel k_\perp} = 0 \ . \tag{5.15}$$

Then there are different damping parameters for the two fields:

$$\zeta_{\perp\overline{\mathcal{E}}} = \frac{a_2}{2k_\perp}\left(2\Theta + \frac{\dot{a}_1}{a_1}\right) = \mathcal{O}\left(\varepsilon\right) \ , \tag{5.16}$$

$$\zeta_{\perp\mathcal{H}} = \frac{a_2}{2k_\perp}\left(2\Theta - \frac{\dot{a}_1}{a_1} + \frac{2\dot{a}_2}{a_2}\right) = \mathcal{O}\left(\varepsilon\right) \ , \tag{5.17}$$

and the speeds of propagation of $\overline{\mathcal{E}}^T_{k_\parallel k_\perp}$ and $\mathcal{H}^T_{k_\parallel k_\perp}$ are also different, but only to second order:

$$c_{\perp\overline{\mathcal{E}}} \simeq 1 - \frac{\zeta_{\overline{\mathcal{E}}}^2}{2} = 1 - \mathcal{O}\left(\varepsilon^2\right) \ , \tag{5.18}$$

$$c_{\perp\mathcal{H}} \simeq 1 - \frac{\zeta_{\mathcal{H}}^2}{2} = 1 - \mathcal{O}\left(\varepsilon^2\right) \ . \tag{5.19}$$

Again, to linear order in the geometrical optics approximation both $\overline{\mathcal{E}}^T_{k_\parallel k_\perp}$ and $\mathcal{H}^T_{k_\parallel k_\perp}$ represent gravitational waves propagating with the speed of light. This is consistent with the generic theory, where the gravitational waves appear at the first order of the expansion of the Einstein equations. The second order terms, neglected in this picture can be interpreted as backreaction, leading to wavenumber-dependent dispersion.

5.2 High frequency evolutions with matter sources: gravitational, shear and matter waves

To first order in the geometrical optics approximation, the shorthand notations appearing in the evolution equations (4.71)-(4.73) simplify as follows. The coefficients of the algebraic terms of the gravitational perturbations are

$$q_{\Sigma 0} \simeq q_{\mathcal{E}0} \simeq q_{\overline{\mathcal{H}}0} \simeq \frac{k_\parallel^2}{a_1^2} + \frac{k_\perp^2}{a_2^2} \ , \tag{5.20}$$

the coefficients of the damping terms become

$$q_{\Sigma 1} \simeq 2\Sigma + \frac{5\Theta}{3} \ , \tag{5.21}$$

$$q_{\mathcal{E}1} = \Theta + \frac{\left(\mathcal{E} + \frac{3\Sigma^2}{2}\right)\frac{k_\perp^2}{a_2^2} + \left(2\mathcal{E} + 3\Sigma^2\right)\frac{k_\parallel^2}{a_1^2}}{\Sigma\left(\frac{2k_\parallel^2}{a_1^2} + \frac{k_\perp^2}{a_2^2}\right)} - \frac{d}{dt}\ln\left[a_1\left(\frac{2k_\parallel^2}{a_1^2} + \frac{k_\perp^2}{a_2^2}\right)\right] \ , \tag{5.22}$$

$$q_{\overline{\mathcal{H}}1} = \Theta + \frac{\left(\mathcal{E} + \frac{3\Sigma^2}{2}\right)\frac{k_\perp^2}{a_2^2} + \left(2\mathcal{E} + 3\Sigma^2\right)\frac{k_\parallel^2}{a_1^2}}{\Sigma\left(\frac{2k_\parallel^2}{a_1^2} + \frac{k_\perp^2}{a_2^2}\right)} - \frac{d}{dt}\ln\frac{\frac{k_\parallel^2}{a_1^2} + \frac{k_\perp^2}{a_2^2}}{a_1\left(\frac{2k_\parallel^2}{a_1^2} + \frac{k_\perp^2}{a_2^2}\right)} \ , \tag{5.23}$$

the coefficients in the algebraic source terms read

$$s_{\Sigma 0} = \frac{4\Theta}{3} - \frac{\Sigma}{2} - c_s^2\left(\Theta + 3\Sigma\right) - \dot{c}_s^2 + \frac{d}{dt}\ln\left[a_2\left(\mu + p\right)\right] \ , \tag{5.24}$$

$$s_{\mathcal{E}0} = -\frac{1 - c_s^2}{2a_2} \ , \tag{5.25}$$

$$s_{\overline{\mathcal{H}}0} = \frac{\Theta}{3} - \frac{7\Sigma}{2} + \frac{d}{dt}\ln\left[a_2\left(\frac{k_\parallel^2}{a_1^2} + \frac{k_\perp^2}{a_2^2}\right)\right] \ , \tag{5.26}$$

while the coefficient of the time derivative source term in Eq. (4.72) is

$$s_{\mathcal{E}1} = \frac{1}{2}\left(\frac{k_\parallel^2}{a_1^2} + \frac{k_\perp^2}{a_2^2}\right)^{-1}\left[\frac{d}{dt}\ln\frac{a_1\left(\frac{2k_\parallel^2}{a_1^2} + \frac{k_\perp^2}{a_2^2}\right)}{\mu + p}\right.$$

$$\left. + \frac{\left(\frac{7\Sigma^2}{2} + \frac{2\Theta\Sigma}{3} + \mathcal{E}\right)\frac{k_\perp^2}{a_2^2} + \left(2\Sigma^2 + \frac{2\Theta\Sigma}{3} + \mathcal{E}\right)\frac{2k_\parallel^2}{a_1^2}}{\Sigma\left(\frac{2k_\parallel^2}{a_1^2} + \frac{k_\perp^2}{a_2^2}\right)}\right] \ . \tag{5.27}$$

Hence the evolutions (4.71)-(4.73), to leading order simplify as

$$\ddot{\Sigma}^T_{k_\parallel k_\perp} + q_{\Sigma 1}\dot{\Sigma}^T_{k_\parallel k_\perp} + \left(\frac{k_\parallel^2}{a_1^2} + \frac{k_\perp^2}{a_2^2}\right)\Sigma^T_{k_\parallel k_\perp} = \frac{\left(1 - c_s^2\right)}{a_2\left(\mu + p\right)}\dot{\mu}^V_{k_\parallel k_\perp} \ , \tag{5.28}$$

$$\ddot{\mathcal{E}}^T_{k_\parallel k_\perp} + q_{\mathcal{E}1}\dot{\mathcal{E}}^T_{k_\parallel k_\perp} + \left(\frac{k_\parallel^2}{a_1^2} + \frac{k_\perp^2}{a_2^2}\right)\mathcal{E}^T_{k_\parallel k_\perp} = 0 \ , \tag{5.29}$$

$$\ddot{\overline{\mathcal{H}}}^T_{k_\parallel k_\perp} + q_{\overline{\mathcal{H}}1}\dot{\overline{\mathcal{H}}}^T_{k_\parallel k_\perp} + \left(\frac{k_\parallel^2}{a_1^2} + \frac{k_\perp^2}{a_2^2}\right)\overline{\mathcal{H}}^T_{k_\parallel k_\perp} = 0 \tag{5.30}$$

The fourth equation of the closed system becomes

$$\dot{\mu}^V_{k_\parallel k_\perp} = a_2\left(\mu + p\right)\left[\left(\frac{k_\parallel^2}{a_1^2} + \frac{k_\perp^2}{a_2^2}\right)\Sigma^T_{k_\parallel k_\perp} - \frac{a_1}{2ik_\parallel}\left(\frac{2k_\parallel^2}{a_1^2} + \frac{k_\perp^2}{a_2^2}\right)\overline{\mathcal{H}}^T_{k_\parallel k_\perp}\right] \ . \tag{5.31}$$

Inserting Eq. (5.31) into Eq. (5.28) we obtain

$$\ddot{\Sigma}^T_{k_\parallel k_\perp} + q_{\Sigma 1}\dot{\Sigma}^T_{k_\parallel k_\perp} + c_s^2 \left(\frac{k_\parallel^2}{a_1^2} + \frac{k_\perp^2}{a_2^2}\right)\Sigma^T_{k_\parallel k_\perp} = -\frac{a_1\left(1 - c_s^2\right)}{2ik_\parallel}\left(\frac{2k_\parallel^2}{a_1^2} + \frac{k_\perp^2}{a_2^2}\right)\overline{\mathcal{H}}^T_{k_\parallel k_\perp}\,, \qquad (5.32)$$

We comment on the system (5.29)-(5.32) as follows. The most striking feature is that the gravitational sector $\mathcal{E}^T_{k_\parallel k_\perp}$ and $\overline{\mathcal{H}}^T_{k_\parallel k_\perp}$ fully decouples from the matter density gradient $\mu^V_{k_\parallel k_\perp}$. While $\mathcal{E}^T_{k_\parallel k_\perp}$ and $\overline{\mathcal{H}}^T_{k_\parallel k_\perp}$ obey damped oscillator equations (similarly to their counterparts $\overline{\mathcal{E}}^T_{k_\parallel k_\perp}$ and $\mathcal{H}^T_{k_\parallel k_\perp}$, discussed in the previous subsection), $\Sigma^T_{k_\parallel k_\perp}$ undergoes a forced oscillation. Let us first discuss the damped oscillations in the manner of the previous subsection.

5.2.1 Gravitational waves

The undamped angular frequencies of $\overline{\mathcal{E}}^T_{k_\parallel k_\perp}$ and $\mathcal{H}^T_{k_\parallel k_\perp}$ are $\Omega = \left(\frac{k_\parallel^2}{a_1^2} + \frac{k_\perp^2}{a_2^2}\right)^{1/2}$, while the damping factors are $\zeta_\mathcal{E} = \frac{q_{\mathcal{E}1}}{2\Omega}$ and $\zeta_{\overline{\mathcal{H}}} = \frac{q_{\overline{\mathcal{H}}1}}{2\Omega}$. For the waves propagating in the z and spherical directions, respectively, we get the following damping factors of $\mathcal{O}\left(\varepsilon\right)$:

$$\zeta_{\parallel\mathcal{E}} = \zeta_{\parallel\overline{\mathcal{H}}} = \frac{a_1}{2k_\parallel}\left(\Theta + \frac{2\mathcal{E} + 3\Sigma^2}{2\Sigma} + \frac{d}{dt}\ln a_1\right) \qquad (5.33)$$

and

$$\zeta_{\perp\mathcal{E}} = \frac{a_2}{2k_\perp}\left(\Theta + \frac{2\mathcal{E} + 3\Sigma^2}{2\Sigma} - \frac{d}{dt}\ln\frac{a_1}{a_2^2}\right)\,, \qquad (5.34)$$

$$\zeta_{\perp\overline{\mathcal{H}}} = \frac{a_2}{2k_\perp}\left(\Theta + \frac{2\mathcal{E} + 3\Sigma^2}{2\Sigma} + \frac{d}{dt}\ln a_1\right)\,, \qquad (5.35)$$

also the corresponding propagation speeds:

$$c_{\parallel\mathcal{E}} \simeq 1 - \frac{\zeta_{\parallel\mathcal{E}}^2}{2} = 1 - \mathcal{O}\left(\varepsilon^2\right)\,, \qquad (5.36)$$

$$c_{\parallel\overline{\mathcal{H}}} \simeq 1 - \frac{\zeta_{\parallel\overline{\mathcal{H}}}^2}{2} = 1 - \mathcal{O}\left(\varepsilon^2\right) \qquad (5.37)$$

and

$$c_{\perp\mathcal{E}} \simeq 1 - \frac{\zeta_{\perp\mathcal{E}}^2}{2} = 1 - \mathcal{O}\left(\varepsilon^2\right)\,, \qquad (5.38)$$

$$c_{\perp\overline{\mathcal{H}}} \simeq 1 - \frac{\zeta_{\perp\overline{\mathcal{H}}}^2}{2} = 1 - \mathcal{O}\left(\varepsilon^2\right)\,. \qquad (5.39)$$

Thus, to leading order in the geometrical optics approximation the gravito-magnetic variables $\overline{\mathcal{E}}^T_{k_\parallel k_\perp}$ and $\mathcal{H}^T_{k_\parallel k_\perp}$ represent pure gravitational waves. The non-identical corrections at higher order represent backreaction.

5.2.2 Shear waves and matter density gradient waves

To leading order in the geometrical optics approximation Eq. (5.32) represents a wave for the shear $\Sigma^T_{k_\parallel k_\perp}$ propagating with the speed of sound c_s. At higher order both a damping mechanism and a force acts on this wave. The force is generated by the gravitational wave degree of freedom $\overline{\mathcal{H}}^T_{k_\parallel k_\perp}$.

Again to leading order the shear $\Sigma^T_{k_\parallel k_\perp}$ is but a time derivative of the matter density gradient $\mu^V_{k_\parallel k_\perp}$. In order to understand this claim it is necessary to remember that in the geometrical optics limit we consider wavelengths much shorter than the curvature radius, hence for the purpose of the wave propagation we can approximate the background scale factors and fluid characteristics as constants. Hence the latter also represents a wave propagating with the speed of sound c_s, but dephased with an angle $\pi/2$. This is in agreement with our assumption of an adiabatic speed of sound given by $p^V_{k_\parallel k_\perp} = c_s^2 \mu^V_{k_\parallel k_\perp}$.

To higher order it mimics the damped and forced oscillation of $\Sigma^T_{k_\parallel k_\perp}$.

5.3 The degrees of freedom in the gravitational waves

In the geometrical optics approximation we have obtained four equations representing gravitational waves propagating with the speed of light, for the even and odd modes of the 2D electric and magnetic projections of the Weyl tensor, $\mathcal{E}^T_{k_\parallel k_\perp}$, $\overline{\mathcal{E}}^T_{k_\parallel k_\perp}$, $\mathcal{H}^T_{k_\parallel k_\perp}$ and $\overline{\mathcal{H}}^T_{k_\parallel k_\perp}$. Nevertheless it is common knowledge that in general relativity gravitational waves carry only two degrees of freedom, represented by the $+$ and $\times$ polarisations. In this subsection we address this apparent mismatch in the degree of freedom counting.

We start by writing the geometrical optics limit of the uncoupled first order equations (4.48) and (4.49). By employing that at least one of the contitions $k_\parallel \gg 1$ or $k_\perp \gg 1$ holds together with $\mathcal{O}\left(a_i^2 B\right) = \mathcal{O}\left(a_i^2 P\right) \gg \mathcal{O}\left(C\right) = \mathcal{O}\left(D\right) = \mathcal{O}\left(a_i E\right) = \mathcal{O}\left(a_i F\right) = \mathcal{O}\left(a_i^{-1} S\right) = \mathcal{O}\left(1\right)$ and also the estimates (5.1), (5.2) and (5.3) we obtain

$$\dot{\overline{\mathcal{E}}}^T_{k_\parallel k_\perp} = \frac{2ik_\parallel}{a_1} \left(\frac{k_\parallel^2}{a_1^2} + \frac{k_\perp^2}{a_2^2}\right) \left(\frac{2k_\parallel^2}{a_1^2} + \frac{k_\perp^2}{a_2^2}\right)^{-1} \mathcal{H}^T_{k_\parallel k_\perp}, \tag{5.40}$$

$$\dot{\mathcal{H}}^T_{k_\parallel k_\perp} = -\frac{a_1}{2ik_\parallel} \left(\frac{2k_\parallel^2}{a_1^2} + \frac{k_\perp^2}{a_2^2}\right) \overline{\mathcal{E}}^T_{k_\parallel k_\perp}, \tag{5.41}$$

Hence we have found that $\overline{\mathcal{E}}^T_{k_\parallel k_\perp}$ and $\mathcal{H}^T_{k_\parallel k_\perp}$ are simply related, they represent the same degree of freedom. We have already shown that to leading order they obey undampened wave equations with the propagation speed of light. Again, in the geometrical optics limit the prefactors of the right hand sides can be considered constants, hence $\overline{\mathcal{E}}^T_{k_\parallel k_\perp}$ and $\mathcal{H}^T_{k_\parallel k_\perp}$ are simply the time derivatives of each other, representing the same gravitational degree of freedom. Note that this analysis could have been done also for the general case in section 4.3.1, where the same conclusions can be drawn from the system (4.48-4.49). However, since the couplings for the system in section 4.3.2 are more intricate, we have chosen to work in the geometrical optics limit throughout in this section.

Next we revisit the geometrical optics limit of the coupled first order equations (4.63) and (4.64), which simplify as

$$\dot{\mathcal{E}}^T_{k_\parallel k_\perp} = \frac{a_1}{2ik_\parallel} \left(\frac{2k_\parallel^2}{a_1^2} + \frac{k_\perp^2}{a_2^2}\right) \overline{\mathcal{H}}^T_{k_\parallel k_\perp}, \tag{5.42}$$

$$\dot{\overline{\mathcal{H}}}^T_{k_\parallel k_\perp} = -\frac{2ik_\parallel}{a_1} \left(\frac{k_\parallel^2}{a_1^2} + \frac{k_\perp^2}{a_2^2} \right) \left(\frac{2k_\parallel^2}{a_1^2} + \frac{k_\perp^2}{a_2^2} \right)^{-1} \mathcal{E}^T_{k_\parallel k_\perp} \,. \tag{5.43}$$

Thus, again, the prefactors on the right hand sides can be considered constants in the geometrical optics approximation, thus the Weyl variables $\mathcal{E}^T_{k_\parallel k_\perp}$ and $\overline{\mathcal{H}}^T_{k_\parallel k_\perp}$ satisfying undampened wave equations with the propagation speed of light are the time derivatives of each other, representing the same gravitational degree of freedom. Note that in the geometrical optics limit this second gravitational degree of freedom also decoupled from matter and all four quantities $\mathcal{Y} = \left\{ \mathcal{E}^T_{k_\parallel k_\perp}, \overline{\mathcal{E}}^T_{k_\parallel k_\perp}, \mathcal{H}^T_{k_\parallel k_\perp}, \overline{\mathcal{H}}^T_{k_\parallel k_\perp} \right\}$ obey

$$\ddot{\mathcal{Y}} + \left(\frac{k_\parallel^2}{a_1^2} + \frac{k_\perp^2}{a_2^2} \right) \mathcal{Y} = 0 \,, \tag{5.44}$$

but they represent only two degrees of freedom.

Like in the case of FLRW perturbations there are two matter degrees of freedom. One for the fluctuations of the density (via gradients of the energy conservation equation) and the other coming from the scalar part of the shear equation (representing velocity perturbations).

6   Concluding Remarks

A general treatment of vorticity-free, perfect fluid perturbations of Kantowski-Sachs models with a positive cosmological constant was considered within the framework of the 1+1+2 covariant decomposition of spacetime. We showed that the system of perturbation equations can be organised into a hierarchy of three systems, namely (i) two coupled gravito-magnetic first order differential equations, (ii) four first order differential equations for the two complementary gravito-magnetic variables, a variable describing the shear of the world-lines and the gradient of the matter density perturbation, (iii) an extended set of algebraic relations involving all variables, which provides a way of determining their evolution.

By assuming that the perturbation wavelengths is much smaller than the curvature radius of the Kantowski-Sachs background, we were able to use the geometrical optics approximation to describe the evolution of high frequency perturbations. We found that system (i) gave rise to the leading order decoupled propagation equations for gravitational waves on this background, while to the next order, damping effects make the propagation along the spheres dephased. At leading order, system (ii) gives rise to two decoupled gravitational wave propagation equations for the complementary gravito-magnetic variables, supplemented by wavelike evolutions for both the shear and matter gradient perturbations, which both propagate with the same speed of sound $c_s < 1$, out of phase by $\pi/2$. At the next order the gravito-magnetic oscillations are again damped, while the shear and matter waves obey forced oscillation wave equations.

We note that the perfect fluid is marginally stable under the vorticity-free, anisotropic pressure-avoiding perturbations at high frequency. The degrees of freedom propagating as gravitational waves in the geometrical optics approximation are exactly the even and odd tensorial perturbations of both the electric and magnetic parts of the Weyl tensor, in agreement with its generic interpretation. While we have found four such quantities obeying undampened wave equations with the propagation speed of light, the even electric and odd magnetic

Weyl projections $\left(\mathcal{E}^T_{k_\parallel k_\perp}, \overline{\mathcal{H}}^T_{k_\parallel k_\perp}\right)$ represent the same gravitational degree of freedom, while the odd electric and even magnetic Weyl projections $\left(\overline{\mathcal{E}}^T_{k_\parallel k_\perp}, \mathcal{H}^T_{k_\parallel k_\perp}\right)$ the other one.

Beyond the geometrical optics approximation we have found indications for the existence of direction dependent dispersion relations. Remarkably, the second gravitational degree of freedom $\left(\overline{\mathcal{E}}^T_{k_\parallel k_\perp}, \mathcal{H}^T_{k_\parallel k_\perp}\right)$ does not decouple from the matter density perturbation, unlike in Friedman universes.

ACKNOWLEDGMENTS

A   The relation between the 2D and 4D curvature tensors

We give here the proof for the equivalency of Eqs. (2.26) and (2.27). The second covariant derivative of any 2D dual vector field V_k projected to the 2D subspace with $N_a{}^i N_b{}^j N_c{}^k$ gives

$$N_a{}^i N_b{}^j N_c{}^k \nabla_i \nabla_j V_k = \delta_a \delta_b V_c + V^l \left[(\delta_b u_l)(\delta_a u_c) - (\delta_b n_l)(\delta_a n_c) \right]$$
$$- (\delta_a u_b) \dot{V}_{\bar{c}} + (\delta_a n_b) \widehat{V}_{\bar{c}} . \tag{A.1}$$

Then from the definition of the Riemann tensor $(\nabla_i \nabla_j - \nabla_j \nabla_i) V_k = R_{ijkl} V^l$, we find

$$N_a{}^i N_b{}^j N_c{}^k R_{ijkl} V^l = 2\delta_{[a}\delta_{b]} V_c - 2\left(\delta_{[a} u_{b]}\right) \dot{V}_{\bar{c}} + 2\left(\delta_{[a} n_{b]}\right) \widehat{V}_{\bar{c}}$$
$$- \left[(\delta_a u_l)(\delta_b u_c) - (\delta_a u_c)(\delta_b u_l) \right.$$
$$\left. - (\delta_a n_l)(\delta_b n_c) + (\delta_a n_c)(\delta_b n_l) \right] V^l . \tag{A.2}$$

With $\delta_{[a} u_{b]} = \Omega \varepsilon_{ab}$ and $\delta_{[a} n_{b]} = \xi \varepsilon_{ab}$, the above identity reduces to

$$\delta_{[a}\delta_{b]} V_c - \Omega \varepsilon_{ab} \dot{V}_{\bar{c}} + \xi \varepsilon_{ab} \widehat{V}_{\bar{c}} = \frac{V^d}{2} \left[N_a{}^i N_b{}^j N_c{}^k N_d{}^l R_{ijkl} + (\delta_a u_d)(\delta_b u_c) \right.$$
$$\left. - (\delta_a u_c)(\delta_b u_d) - (\delta_a n_d)(\delta_b n_c) + (\delta_a n_c)(\delta_b n_d) \right] . \tag{A.3}$$

The square bracket on the right hand side is the 2D curvature tensor $\mathcal{R}_{abcd}$, as can be seen by comparing the left hand side with the definition (2.26).

B   Infinitesimal frame transformations on the Kantowski-Sachs background filled with perfect fluid

An infinitesimal frame transformation from the dyad (u^a, n^a) to the dyad $(\overline{u}^a, \overline{n}^a)$ can be defined as (see for higher dimensional spacetime [64]):

$$\overline{u}_a = u_a + v_a + \nu n_a , \text{ with } u^a v_a = n^a v_a = 0, \tag{B.1}$$
$$\overline{n}_a = n_a + l_a + m u_a , \text{ with } u^a l_a = n^a l_a = 0, \tag{B.2}$$

where v_a, l_a, ν, m are all first order. We will neglect the second order contributions. The new dyad also obeys

$$\overline{u}^a \overline{u}_a = -1 , \ \overline{n}^a \overline{n}_a = 1 , \ \overline{u}^a \overline{n}_a = 0 , \tag{B.3}$$

which implies

$$\nu = m \ . \tag{B.4}$$

There are five gauge degrees of freedom to fix the frame on the perturbed spacetime. The transformations with $v_a = l_a = 0$ represent 2D infinitesimal Lorentz boosts, while the parameters v_a and l_a are related to infinitesimal translations.

The fundamental algebraic tensors N_{ab} and ε_{ab} change accordingly:

$$\overline{N}_{ab} = N_{ab} + 2u_{(a}v_{b)} - 2n_{(a}l_{b)} \ , \tag{B.5}$$

$$\overline{\varepsilon}_{ab} = \varepsilon_{ab} + 2n_{[a}\varepsilon_{b]c}l^c - 2u_{[a}\varepsilon_{b]c}v^c \ . \tag{B.6}$$

The new 2-metric obeys $\overline{N}_{ab}\overline{n}^a = \overline{N}_{ab}\overline{u}^a = 0$.

The kinematic quantities defined for the new dyad vectors arise from the decomposition of the covariant derivatives of $\overline{u}_a$ and $\overline{n}_a$ similarly to that given in Section 2. This implies the following transformations rules on Kantowski-Sachs background for the kinematic quantities:

$$\overline{\Theta} = \Theta + \widehat{\nu} + \delta_a v^a \ , \tag{B.7}$$

$$\overline{\mathcal{A}} = \mathcal{A} + \dot{\nu} + \left(\Sigma + \frac{\Theta}{3}\right)\nu \ , \tag{B.8}$$

$$\overline{\Omega} = \Omega + \frac{1}{2}\varepsilon^{ab}\delta_a v_b \ , \tag{B.9}$$

$$\overline{\Sigma} = \Sigma + \frac{2}{3}\widehat{\nu} - \frac{1}{3}\delta_a v^a \ , \tag{B.10}$$

$$\overline{\phi} = \phi + \delta_a l^a - \left(\Sigma - \frac{2\Theta}{3}\right)\nu \ , \tag{B.11}$$

$$\overline{\xi} = \xi + \varepsilon^{ab}\delta_a l_b \ , \tag{B.12}$$

$$\overline{\mathcal{A}}_a = \mathcal{A}_a + \dot{v}_{\bar{a}} - \frac{1}{2}\left(\Sigma - \frac{2\Theta}{3}\right)v_a \ , \tag{B.13}$$

$$\overline{\Omega}_a = \Omega_a - \frac{1}{2}\varepsilon_{ab}\left(\widehat{v}^b - \delta^b\nu\right) \ , \tag{B.14}$$

$$\overline{\Sigma}_a = \Sigma_a + \frac{1}{2}\left(\widehat{v}_{\bar{a}} + \delta_a\nu - 3\Sigma l_a\right) \ , \tag{B.15}$$

$$\overline{a}_a = a_a + \widehat{l}_{\bar{a}} - \left(\Sigma + \frac{\Theta}{3}\right)v_a \ , \tag{B.16}$$

$$\overline{\alpha}_a = \alpha_a + \dot{l}_{\bar{a}} \ , \tag{B.17}$$

$$\overline{\Sigma}_{ab} = \Sigma_{ab} + \delta_{\{a}v_{b\}} \ , \tag{B.18}$$

$$\overline{\zeta}_{ab} = \zeta_{ab} + \delta_{\{a}l_{b\}} \ , \tag{B.19}$$

The gravito-electro-magnetic quantities $\mathcal{E}$, $\mathcal{H}$, $\mathcal{E}_{ab}$ and $\mathcal{H}_{ab}$ are invariant under the infinitesimal frame change, while the transformation laws of $\mathcal{E}_a$ and $\mathcal{H}_a$ are

$$\overline{\mathcal{E}}_a = \mathcal{E}_a - \frac{3\mathcal{E}}{2}l_c \ , \tag{B.20}$$

$$\overline{\mathcal{H}}_a = \mathcal{H}_a - \frac{3\mathcal{E}}{2}\varepsilon_{ab}v^b \ . \tag{B.21}$$

The matter variables μ, p, Π, Π_a and Π_{ab} are invariant under the infinitesimal frame change, while Q and Q_a describing the energy current transform as

$$\overline{Q} = Q - (\mu + p)\, \nu \,, \tag{B.22}$$

$$\overline{Q}_a = Q_a - (\mu + p)\, \upsilon_a \,. \tag{B.23}$$

The gauge-invariant variables defined by Eq. (4.1) transform as

$$G_a = G_a + \dot{G}\upsilon_a \,, \tag{B.24}$$

where $G_a \equiv \{\mu_a, X_a, V_a, W_a, p_a\}$ and $G \equiv \{\mu, \mathcal{E}, \Sigma, \Theta, p\}$, respectively.

C Commutation relations

The commutation relations of covariant derivatives of the scalar field Ψ on Kantowski-Sachs background, to first order are

$$\widehat{\dot{\Psi}} - \dot{\widehat{\Psi}} = -\mathcal{A}\dot{\Psi} + \left(\Sigma + \frac{\Theta}{3}\right)\widehat{\Psi} \,, \tag{C.1}$$

$$\delta_a \dot{\Psi} - N_a{}^b \left(\delta_b \Psi\right)^{\cdot} = -\mathcal{A}_a \dot{\Psi} - \frac{1}{2}\left(\Sigma - \frac{2\Theta}{3}\right)\delta_a \Psi \,, \tag{C.2}$$

$$\delta_a \widehat{\Psi} - N_a{}^b \left(\widehat{\delta_b \Psi}\right) = -2\varepsilon_{ab}\Omega^b \dot{\Psi} \,, \tag{C.3}$$

$$\delta_{[a}\delta_{b]}\Psi = \varepsilon_{ab}\Omega\dot{\Psi} \,. \tag{C.4}$$

Similar relations hold for the first order 2-vector Ψ_a:

$$\widehat{\dot{\Psi}}_{\bar{a}} - \dot{\widehat{\Psi}}_{\bar{a}} = \left(\Sigma + \frac{\Theta}{3}\right)\widehat{\Psi}_{\bar{a}} \,, \tag{C.5}$$

$$\delta_a \dot{\Psi}_b - N_a{}^c N_b{}^d \left(\delta_c \Psi_d\right)^{\cdot} = -\frac{1}{2}\left(\Sigma - \frac{2\Theta}{3}\right)\delta_a \Psi_b \,, \tag{C.6}$$

$$\delta_a \widehat{\Psi}_b - N_a{}^c N_b{}^d \left(\widehat{\delta_c \Psi_d}\right) = 0 \,, \tag{C.7}$$

$$\delta_{[a}\delta_{b]}\Psi_c = 2\mathcal{R}N_{c[a}\Psi_{b]} \,, \tag{C.8}$$

and for the first order symmetric, trace-free 2-tensor Ψ_{ab}:

$$\widehat{\dot{\Psi}}_{\{ab\}} - \dot{\widehat{\Psi}}_{\{ab\}} = \left(\Sigma + \frac{\Theta}{3}\right)\widehat{\Psi}_{\bar{a}} \,, \tag{C.9}$$

$$\delta_a \dot{\Psi}_{bc} - N_a{}^d N_b{}^e N_c{}^f \left(\delta_d \Psi_{ef}\right)^{\cdot} = -\frac{1}{2}\left(\Sigma - \frac{2\Theta}{3}\right)\delta_a \Psi_{bc} \,, \tag{C.10}$$

$$\delta_a \widehat{\Psi}_{bc} - N_a{}^d N_b{}^e N_c{}^f \left(\widehat{\delta_d \Psi_{ef}}\right) = 0 \,, \tag{C.11}$$

$$2\delta_{[a}\delta_{b]}\Psi_{cd} = \mathcal{R}\left(N_{c[a}\Psi_{b]d} + N_{d[a}\Psi_{b]c}\right) \,. \tag{C.12}$$

where $\mathcal{R}$ is given by Eq. (3.9).

D   Properties of vector and tensor spherical harmonics

In this Appendix we enlist a set of identities for the even $Q_a^{k_\perp}$ and odd $\overline{Q}_a^{k_\perp}$ vector spherical harmonics, including the orthogonality relations

$$N^{ab}Q_a^{k_\perp}\overline{Q}_b^{k_\perp} = 0 \ , \tag{D.1}$$

the algebraic relations

$$Q_a^{k_\perp} = -\varepsilon_a{}^b\overline{Q}_b^{k_\perp} \ , \quad \overline{Q}_a^{k_\perp} = \varepsilon_a{}^b Q_b^{k_\perp} \ , \tag{D.2}$$

and the differential relations

$$\dot{Q}_a^{k_\perp} = \widehat{Q}_a^{k_\perp} = 0 \ , \quad \dot{\overline{Q}}_a^{k_\perp} = \widehat{\overline{Q}}_a^{k_\perp} = 0 \ , \tag{D.3}$$

$$\delta^2 Q_a^{k_\perp} = \frac{1 - k_\perp^2}{a_2^2}Q_a^{k_\perp} \ , \quad \delta^2\overline{Q}_a^{k_\perp} = \frac{1 - k_\perp^2}{a_2^2}\overline{Q}_a^{k_\perp} \ , \tag{D.4}$$

$$\delta^a Q_a^{k_\perp} = -\frac{k_\perp^2}{a_2}Q^{k_\perp} \ , \quad \delta^a\overline{Q}_a^{k_\perp} = 0 \ , \tag{D.5}$$

$$\varepsilon^{ab}\delta_a Q_b^{k_\perp} = 0 \ , \quad \varepsilon^{ab}\delta_a\overline{Q}_b^{k_\perp} = \frac{k_\perp^2}{a_2}Q^{k_\perp} \ . \tag{D.6}$$

The even and odd tensor spherical harmonics obey in turn the orthogonality relations

$$N^{ab}N^{cd}Q_{ac}^{k_\perp}\overline{Q}_{bd}^{k_\perp} = 0 \ , \tag{D.7}$$

the algebraic relations

$$Q_{ab}^{k_\perp} = \varepsilon_{\{a}{}^c\overline{Q}_{b\}c}^{k_\perp} \ , \quad \overline{Q}_{ab}^{k_\perp} = -\varepsilon_{\{a}{}^c Q_{b\}c}^{k_\perp} \ , \tag{D.8}$$

and the differential relations

$$\dot{Q}_{ab}^{k_\perp} = \widehat{Q}_{ab}^{k_\perp} = 0 \ , \quad \dot{\overline{Q}}_{ab}^{k_\perp} = \widehat{\overline{Q}}_{ab}^{k_\perp} = 0 \ , \tag{D.9}$$

$$\delta^2 Q_{ab}^{k_\perp} = \frac{4 - k_\perp^2}{a_2^2}Q_{ab}^{k_\perp} \ , \quad \delta^2\overline{Q}_{ab}^{k_\perp} = \frac{4 - k_\perp^2}{a_2^2}\overline{Q}_{ab}^{k_\perp} \ , \tag{D.10}$$

$$\delta^b Q_{ab}^{k_\perp} = \frac{2 - k_\perp^2}{2a_2}Q_a^{k_\perp} \ , \quad \delta^b\overline{Q}_{ab}^{k_\perp} = -\frac{2 - k_\perp^2}{2a_2}\overline{Q}_a^{k_\perp} \ , \tag{D.11}$$

$$\varepsilon_a{}^c\delta^b Q_{bc}^{k_\perp} = \frac{2 - k_\perp^2}{2a_2}\overline{Q}_a^{k_\perp} \ , \quad \varepsilon_a{}^c\delta^b\overline{Q}_{bc}^{k_\perp} = \frac{2 - k_\perp^2}{2a_2}Q_a^{k_\perp} \ , \tag{D.12}$$

$$\varepsilon^{bc}\delta_b Q_{ac}^{k_\perp} = \frac{2 - k_\perp^2}{2a_2}\overline{Q}_a^{k_\perp} \ , \quad \varepsilon^{bc}\delta_b\overline{Q}_{ac}^{k_\perp} = \frac{2 - k_\perp^2}{2a_2}Q_a^{k_\perp} \ . \tag{D.13}$$

For reviews of various types of harmonics used in relativity see, e.g., [73, 74].

E   Harmonic expansion of the vorticity-free perturbation equations

We give here the harmonic decomposition of Eqs. (4.3)-(4.34), employing Eq. (4.45) and $\overline{\mathcal{A}}_{k_\parallel k_\perp}^V = 0$. The perturbation equations decouple into two sets, describing the even and odd parity sectors.

E.1 Odd parity sector

The evolution equations for gauge-invariant 2-vector perturbation variables with odd parity are

$$\dot{\overline{a}}^V_{k_\parallel k_\perp} = \frac{ik_\parallel}{a_1}\overline{\alpha}^V_{k_\parallel k_\perp} + \mathcal{H}^V_{k_\parallel k_\perp} - \left(\Sigma + \frac{\Theta}{3}\right)\overline{a}^V_{k_\parallel k_\perp}, \tag{E.1}$$

$$2\dot{\overline{\Sigma}}^V_{k_\parallel k_\perp} = -\left(\Sigma + \frac{4\Theta}{3}\right)\overline{\Sigma}^V_{k_\parallel k_\perp} - 3\Sigma\overline{\alpha}^V_{k_\parallel k_\perp} - 2\overline{\mathcal{E}}^V_{k_\parallel k_\perp}, \tag{E.2}$$

$$\dot{\overline{\mathcal{E}}}^V_{k_\parallel k_\perp} = -\frac{ik_\parallel}{2a_1}\mathcal{H}^V_{k_\parallel k_\perp} + \frac{3}{4}\left(\Sigma - \frac{4\Theta}{3}\right)\overline{\mathcal{E}}^V_{k_\parallel k_\perp} + \frac{3}{4a_2}\mathcal{H}^S_{k_\parallel k_\perp}$$
$$+ \frac{(3\mathcal{E} - 2\mu - 2p)}{4}\overline{\Sigma}^V_{k_\parallel k_\perp} + \frac{2 - k_\perp^2}{4a_2}\mathcal{H}^T_{k_\parallel k_\perp} - \frac{3\mathcal{E}}{2}\overline{\alpha}^V_{k_\parallel k_\perp}, \tag{E.3}$$

$$\dot{\overline{\mathcal{H}}}^V_{k_\parallel k_\perp} = \frac{ik_\parallel}{2a_1}\mathcal{E}^V_{k_\parallel k_\perp} + \frac{3}{4}\left(\Sigma - \frac{4\Theta}{3}\right)\overline{\mathcal{H}}^V_{k_\parallel k_\perp} - \frac{3}{4}X^V_{k_\parallel k_\perp} - \frac{3\mathcal{E}}{2}\mathcal{A}^V_{k_\parallel k_\perp} + \frac{3\mathcal{E}}{4}a^V_{k_\parallel k_\perp} - \frac{2 - k_\perp^2}{4a_2}\mathcal{E}^T_{k_\parallel k_\perp}, \tag{E.4}$$

The evolution equations for the odd parity tensor perturbations are

$$\dot{\overline{\Sigma}}^T_{k_\parallel k_\perp} = \left(\Sigma - \frac{2\Theta}{3}\right)\overline{\Sigma}^T_{k_\parallel k_\perp} - \overline{\mathcal{E}}^T_{k_\parallel k_\perp}, \tag{E.5}$$

$$\dot{\overline{\zeta}}^T_{k_\parallel k_\perp} = \frac{1}{2}\left(\Sigma - \frac{2\Theta}{3}\right)\overline{\zeta}^T_{k_\parallel k_\perp} - \frac{\overline{\alpha}^V_{k_\parallel k_\perp}}{a_2} - \mathcal{H}^T_{k_\parallel k_\perp}, \tag{E.6}$$

$$\dot{\overline{\mathcal{E}}}^T_{k_\parallel k_\perp} = \frac{ik_\parallel}{a_1}\mathcal{H}^T_{k_\parallel k_\perp} - \frac{3}{2}\left(\Sigma + \frac{2\Theta}{3}\right)\overline{\mathcal{E}}^T_{k_\parallel k_\perp} - \frac{1}{a_2}\mathcal{H}^V_{k_\parallel k_\perp} - \frac{(3\mathcal{E} + \mu + p)}{2}\overline{\Sigma}^T_{k_\parallel k_\perp}, \tag{E.7}$$

$$\dot{\overline{\mathcal{H}}}^T_{k_\parallel k_\perp} = -\frac{ik_\parallel}{a_1}\mathcal{E}^T_{k_\parallel k_\perp} - \frac{3}{2}\left(\Sigma + \frac{2\Theta}{3}\right)\overline{\mathcal{H}}^T_{k_\parallel k_\perp} + \frac{3\mathcal{E}}{2}\zeta^T_{k_\parallel k_\perp} + \frac{1}{a_2}\mathcal{E}^V_{k_\parallel k_\perp}. \tag{E.8}$$

They obey the constraints

$$\frac{\mathcal{A}^S_{k_\parallel k_\perp}}{a_2} = \frac{ik_\parallel}{a_1}\mathcal{A}^V_{k_\parallel k_\perp}, \tag{E.9}$$

$$\frac{ik_\parallel}{a_1}\overline{\Sigma}^V_{k_\parallel k_\perp} - \frac{3\Sigma}{2}\overline{a}^V_{k_\parallel k_\perp} + \frac{2 - k_\perp^2}{2a_2}\overline{\Sigma}^T_{k_\parallel k_\perp}, \tag{E.10}$$

$$\frac{2ik_\parallel}{a_1}\overline{\mathcal{E}}^V_{k_\parallel k_\perp} = -3\mathcal{E}\overline{a}^V_{k_\parallel k_\perp} - 3\Sigma\mathcal{H}^V_{k_\parallel k_\perp} + \frac{2 - k_\perp^2}{a_2}\overline{\mathcal{E}}^T_{k_\parallel k_\perp}, \tag{E.11}$$

$$\frac{2ik_\parallel}{a_1}\overline{\mathcal{H}}^V_{k_\parallel k_\perp} = -3\mathcal{E}\Sigma^V_{k_\parallel k_\perp} + \frac{2 - k_\perp^2}{a_2}\overline{\mathcal{H}}^T_{k_\parallel k_\perp} + 3\Sigma\mathcal{E}^V_{k_\parallel k_\perp}, \tag{E.12}$$

$$\frac{ik_\parallel}{a_1}\overline{\Sigma}^T_{k_\parallel k_\perp} = -\frac{\overline{\Sigma}^V_{k_\parallel k_\perp}}{a_2} + \frac{3\Sigma}{2}\overline{\zeta}^T_{k_\parallel k_\perp} - \mathcal{H}^T_{k_\parallel k_\perp}, \tag{E.13}$$

$$\frac{ik_\parallel}{a_1}\overline{\zeta}^T_{k_\parallel k_\perp} = \left(\Sigma + \frac{\Theta}{3}\right)\overline{\Sigma}^T_{k_\parallel k_\perp} - \frac{1}{a_2}\overline{a}^V_{k_\parallel k_\perp} - \overline{\mathcal{E}}^T_{k_\parallel k_\perp}, \tag{E.14}$$

$$\frac{\xi^S_{k_\parallel k_\perp}}{a_2} - \frac{2 - k_\perp^2}{2a_2}\overline{\zeta}^T_{k_\parallel k_\perp} = \left(\Sigma - \frac{2\Theta}{3}\right)\frac{\overline{\Sigma}^V_{k_\parallel k_\perp}}{2} + \overline{\mathcal{E}}^V_{k_\parallel k_\perp}, \tag{E.15}$$

$$2\mathcal{H}^V_{k_\parallel k_\perp} = \frac{2 - k_\perp^2}{a_2}\overline{\Sigma}^T_{k_\parallel k_\perp}. \tag{E.16}$$

E.2 Even parity sector

The evolution equations governing the gauge-invariant scalar perturbation variables with even parity are

$$\dot{\phi}^S_{k_\parallel k_\perp} = \left(\Sigma - \frac{2\Theta}{3}\right)\left(\frac{\phi^S_{k_\parallel k_\perp}}{2} - \mathcal{A}^S_{k_\parallel k_\perp}\right) - \frac{k_\perp^2}{a_2}\alpha^V_{k_\parallel k_\perp}, \tag{E.17}$$

$$2\dot{\xi}^S_{k_\parallel k_\perp} = \left(\Sigma - \frac{2\Theta}{3}\right)\xi^S_{k_\parallel k_\perp} + \mathcal{H}^S_{k_\parallel k_\perp} + \frac{k_\perp^2}{a_2}\bar{\alpha}^V_{k_\parallel k_\perp}, \tag{E.18}$$

$$\dot{\mathcal{H}}^S_{k_\parallel k_\perp} = \frac{3}{2}\left(\Sigma - \frac{2\Theta}{3}\right)\mathcal{H}^S_{k_\parallel k_\perp} - \frac{k_\perp^2}{a_2}\bar{\mathcal{E}}^V_{k_\parallel k_\perp} - 3\mathcal{E}\xi^S_{k_\parallel k_\perp}. \tag{E.19}$$

The evolution of even parity 2-vector perturbations are given by

$$\dot{\mu}^V_{k_\parallel k_\perp} = \left(\frac{\Sigma}{2} - \frac{4\Theta}{3}\right)\mu^V_{k_\parallel k_\perp} - (\mu + p)W^V_{k_\parallel k_\perp} - \Theta p^V_{k_\parallel k_\perp} + \dot{\mu}\mathcal{A}^V_{k_\parallel k_\perp}, \tag{E.20}$$

$$\begin{aligned}
\dot{X}^V_{k_\parallel k_\perp} &= 2\left(\Sigma - \frac{2\Theta}{3}\right)X^V_{k_\parallel k_\perp} - \frac{\mu + p}{2}V^V_{k_\parallel k_\perp} + \frac{3\mathcal{E}}{2}\left(V^V_{k_\parallel k_\perp} - \frac{2}{3}W^V_{k_\parallel k_\perp}\right) \\
&\quad + \dot{\mathcal{E}}\mathcal{A}^V_{k_\parallel k_\perp} - \frac{\Sigma}{2}\left(\mu^V_{k_\parallel k_\perp} + p^V_{k_\parallel k_\perp}\right) + \frac{k_\perp^2}{a_2^2}\bar{\mathcal{H}}^V_{k_\parallel k_\perp},
\end{aligned} \tag{E.21}$$

$$\begin{aligned}
\dot{V}^V_{k_\parallel k_\perp} - \frac{2}{3}\dot{W}^V_{k_\parallel k_\perp} &= \left(\dot{\Sigma} - \frac{2\dot{\Theta}}{3}\right)\mathcal{A}^V_{k_\parallel k_\perp} + \frac{k_\perp^2}{a_2^2}\mathcal{A}^V_{k_\parallel k_\perp} + \frac{3}{2}\left(\Sigma - \frac{2\Theta}{3}\right) \\
&\quad \times \left(V^V_{k_\parallel k_\perp} - \frac{2}{3}W^V_{k_\parallel k_\perp}\right) + \frac{1}{3}\left(\mu^V_{k_\parallel k_\perp} + 3p^V_{k_\parallel k_\perp}\right) - X^V_{k_\parallel k_\perp},
\end{aligned} \tag{E.22}$$

$$\begin{aligned}
\dot{W}^V_{k_\parallel k_\perp} &= \frac{ik_\parallel}{a_1 a_2}\mathcal{A}^S_{k_\parallel k_\perp} + \left(\frac{\Sigma}{2} - \Theta\right)W^V_{k_\parallel k_\perp} + \dot{\Theta}\mathcal{A}^V_{k_\parallel k_\perp} \\
&\quad - \frac{1}{2}\left(\mu^V_{k_\parallel k_\perp} + 3p^V_{k_\parallel k_\perp}\right) - 3\Sigma V^V_{k_\parallel k_\perp} - \frac{k_\perp^2}{a_2^2}\mathcal{A}^V_{k_\parallel k_\perp},
\end{aligned} \tag{E.23}$$

$$\dot{a}^V_{k_\parallel k_\perp} = \frac{ik_\parallel}{a_1}\alpha^V_{k_\parallel k_\perp} - \bar{\mathcal{H}}^V_{k_\parallel k_\perp} - \left(\Sigma + \frac{\Theta}{3}\right)\left(\mathcal{A}^V_{k_\parallel k_\perp} + a^V_{k_\parallel k_\perp}\right), \tag{E.24}$$

$$2\dot{\Sigma}^V_{k_\parallel k_\perp} = \frac{ik_\parallel}{a_1}\mathcal{A}^V_{k_\parallel k_\perp} + \frac{1}{a_2}\mathcal{A}^S_{k_\parallel k_\perp} - 3\Sigma\alpha^V_{k_\parallel k_\perp} - \left(\Sigma + \frac{4\Theta}{3}\right)\Sigma^V_{k_\parallel k_\perp} - 2\mathcal{E}^V_{k_\parallel k_\perp}, \tag{E.25}$$

$$\begin{aligned}
\dot{\mathcal{E}}^V_{k_\parallel k_\perp} &= \frac{3}{4}\left(\Sigma - \frac{4\Theta}{3}\right)\mathcal{E}^V_{k_\parallel k_\perp} + \frac{(3\mathcal{E} - 2\mu - 2p)}{4}\Sigma^V_{k_\parallel k_\perp} \\
&\quad + \frac{ik_\parallel}{2a_1}\bar{\mathcal{H}}^V_{k_\parallel k_\perp} - \frac{3\mathcal{E}}{2}\alpha^V_{k_\parallel k_\perp} + \frac{2 - k_\perp^2}{4a_2}\bar{\mathcal{H}}^T_{k_\parallel k_\perp},
\end{aligned} \tag{E.26}$$

$$\dot{\mathcal{H}}^V_{k_\parallel k_\perp} = -\frac{ik_\parallel}{2a_1}\bar{\mathcal{E}}^V_{k_\parallel k_\perp} + \frac{3}{4}\left(\Sigma - \frac{4\Theta}{3}\right)\mathcal{H}^V_{k_\parallel k_\perp} - \frac{3\mathcal{E}}{4}\bar{a}^V_{k_\parallel k_\perp} - \frac{2 - k_\perp^2}{4a_2}\bar{\mathcal{E}}^T_{k_\parallel k_\perp}, \tag{E.27}$$

The evolution equations for the even parity 2-tensor perturbations are

$$\dot{\Sigma}^T_{k_\parallel k_\perp} = \left(\Sigma - \frac{2\Theta}{3}\right)\Sigma^T_{k_\parallel k_\perp} + \frac{1}{a_2}\mathcal{A}^V_{k_\parallel k_\perp} - \mathcal{E}^T_{k_\parallel k_\perp},$$

(E.28)

$$\dot{\zeta}^T_{k_\parallel k_\perp} = \frac{1}{2}\left(\Sigma - \frac{2\Theta}{3}\right)\zeta^T_{k_\parallel k_\perp} + \frac{1}{a_2}\alpha^V_{k_\parallel k_\perp} + \overline{\mathcal{H}}^T_{k_\parallel k_\perp},$$

(E.29)

$$\dot{\mathcal{E}}^T_{k_\parallel k_\perp} = -\frac{ik_\parallel}{a_1}\overline{\mathcal{H}}^T_{k_\parallel k_\perp} - \frac{3}{2}\left(\Sigma + \frac{2\Theta}{3}\right)\mathcal{E}^T_{k_\parallel k_\perp} - \frac{(3\mathcal{E} + \mu + p)}{2}\Sigma^T_{k_\parallel k_\perp} - \frac{1}{a_2}\overline{\mathcal{H}}^V_{k_\parallel k_\perp},$$

(E.30)

$$\dot{\mathcal{H}}^T_{k_\parallel k_\perp} = \frac{ik_\parallel}{a_1}\overline{\mathcal{E}}^T_{k_\parallel k_\perp} - \frac{3}{2}\left(\Sigma + \frac{2\Theta}{3}\right)\mathcal{H}^T_{k_\parallel k_\perp} - \frac{3\mathcal{E}}{2}\overline{\zeta}^T_{k_\parallel k_\perp} + \frac{1}{a_2}\overline{\mathcal{E}}^V_{k_\parallel k_\perp} .$$

(E.31)

The constraint equations for the even parity sector are

$$\frac{ik_\parallel}{a_1 a_2}\phi^S_{k_\parallel k_\perp} = \frac{1}{3}\left(\Sigma + \frac{4\Theta}{3}\right)W^V_{k_\parallel k_\perp} - \left(2\Sigma - \frac{\Theta}{3}\right)V^V_{k_\parallel k_\perp} - \frac{k_\perp^2}{a_2^2}a^V_{k_\parallel k_\perp} - X^V_{k_\parallel k_\perp} - \frac{2}{3}\mu^V_{k_\parallel k_\perp},$$

(E.32)

$$\frac{ik_\parallel}{a_1}\xi^S_{k_\parallel k_\perp} = \frac{k_\perp^2}{2a_2}\overline{a}^V_{k_\parallel k_\perp},$$

(E.33)

$$\frac{ik_\parallel}{a_1}\mathcal{H}^S_{k_\parallel k_\perp} = \frac{k_\perp^2}{a_2}\mathcal{H}^V_{k_\parallel k_\perp},$$

(E.34)

$$\frac{ik_\parallel}{a_1}p^V_{k_\parallel k_\perp} = -\frac{\mu + p}{a_2}\mathcal{A}^S_{k_\parallel k_\perp},$$

(E.35)

$$\frac{ik_\parallel}{a_1}\left(X^V_{k_\parallel k_\perp} - \frac{\mu^V_{k_\parallel k_\perp}}{3}\right) = \frac{k_\perp^2}{a_2^2}\mathcal{E}^V_{k_\parallel k_\perp} - \frac{3\mathcal{E}}{2a_2}\phi^S_{k_\parallel k_\perp},$$

(E.36)

$$\frac{ik_\parallel}{a_1}\left(V^V_{k_\parallel k_\perp} - \frac{2}{3}W^V_{k_\parallel k_\perp}\right) = \frac{k_\perp^2}{a_2^2}\Sigma^V_{k_\parallel k_\perp} - \frac{3\Sigma}{2a_2}\phi^S_{k_\parallel k_\perp},$$

(E.37)

$$\frac{ik_\parallel}{a_1}\Sigma^V_{k_\parallel k_\perp} = -\frac{3\Sigma}{2}a^V_{k_\parallel k_\perp} - \frac{2 - k_\perp^2}{2a_2}\Sigma^T_{k_\parallel k_\perp} + \frac{1}{2}\left(V^V_{k_\parallel k_\perp} + \frac{4}{3}W^V_{k_\parallel k_\perp}\right),$$

(E.38)

$$\frac{2ik_\parallel}{a_1}\mathcal{E}^V_{k_\parallel k_\perp} = X^V_{k_\parallel k_\perp} - 3\mathcal{E}a^V_{k_\parallel k_\perp} + \frac{2}{3}\mu^V_{k_\parallel k_\perp} + 3\Sigma\overline{\mathcal{H}}^V_{k_\parallel k_\perp} - \frac{2 - k_\perp^2}{a_2}\mathcal{E}^T_{k_\parallel k_\perp},$$

(E.39)

$$\frac{2ik_\parallel}{a_1}\mathcal{H}^V_{k_\parallel k_\perp} = \frac{1}{a_2}\mathcal{H}^S_{k_\parallel k_\perp} - \frac{2 - k_\perp^2}{a_2}\mathcal{H}^T_{k_\parallel k_\perp} - 3\Sigma\overline{\mathcal{E}}^V_{k_\parallel k_\perp} + 3\mathcal{E}\overline{\Sigma}^V_{k_\parallel k_\perp},$$

(E.40)

$$\frac{ik_\parallel}{a_1}\Sigma^T_{k_\parallel k_\perp} = \frac{\Sigma^V_{k_\parallel k_\perp}}{a_2} + \frac{3\Sigma}{2}\zeta^T_{k_\parallel k_\perp} + \overline{\mathcal{H}}^T_{k_\parallel k_\perp},$$

(E.41)

$$\frac{ik_\parallel}{a_1}\zeta^T_{k_\parallel k_\perp} = \left(\Sigma + \frac{\Theta}{3}\right)\Sigma^T_{k_\parallel k_\perp} + \frac{1}{a_2}a^V_{k_\parallel k_\perp} - \mathcal{E}^T_{k_\parallel k_\perp},$$

(E.42)

$$p^V_{k_\parallel k_\perp} = -(\mu + p)\mathcal{A}^V_{k_\parallel k_\perp},$$

(E.43)

$$\frac{k_\perp^2}{a_2}\overline{\Sigma}^V_{k_\parallel k_\perp} = -3\Sigma\xi^S_{k_\parallel k_\perp} + \mathcal{H}^S_{k_\parallel k_\perp},$$

(E.44)

$$\frac{2-k_\perp^2}{a_2}\zeta^T_{k_\parallel k_\perp} - \frac{\phi^S_{k_\parallel k_\perp}}{a_2} = \left(\Sigma - \frac{2\Theta}{3}\right)\Sigma^V_{k_\parallel k_\perp} + 2\mathcal{E}^V_{k_\parallel k_\perp}, \tag{E.45}$$

$$2\overline{\mathcal{H}}^V_{k_\parallel k_\perp} = V^V_{k_\parallel k_\perp} - \frac{2}{3}W^V_{k_\parallel k_\perp} + \frac{2-k_\perp^2}{a_2}\Sigma^T_{k_\parallel k_\perp}. \tag{E.46}$$

We have checked the consistency of the evolution equations and constraints (Eqs. (E.35), (E.9) and (E.43), containing no evolutions for $\mathcal{A}^S_{k_\parallel k_\perp}$, $\mathcal{A}^V_{k_\parallel k_\perp}$ and $p^V_{k_\parallel k_\perp}$) by showing that the time derivatives of the latter are identities. In fact not all algebraic relations are independent, their relations are summarised as follows: 1) Eq. (E.35) follows from Eqs. (E.9) and (E.43); 2) Eq. (E.10) follows from Eqs. (E.16), (E.33), (E.34) and (E.44); 3) Eq. (E.45) follows from Eqs. (E.12), (E.37), (E.41) and (E.46); 4) Eq. (E.40) follows from Eqs. (E.13), (E.15), (E.16) and (E.44); 5) Eq. (E.15) is consequence of Eqs. (E.11), (E.14), (E.16), (E.33), (E.34) and (E.44); and finally 6) Eq. (E.32) is consequence of Eqs. (E.12), (E.37), (E.38), (E.39), (E.41), (E.42) and (E.46). Thus there are 17 constraints for the 28 variables ($\mathcal{A}^S_{k_\parallel k_\perp}$, $\mathcal{H}^S_{k_\parallel k_\perp}$, $\phi^S_{k_\parallel k_\perp}$, $\xi^S_{k_\parallel k_\perp}$, $\mu^V_{k_\parallel k_\perp}$, $p^V_{k_\parallel k_\perp}$, $\mathcal{A}^V_{k_\parallel k_\perp}$, $V^V_{k_\parallel k_\perp}$, $W^V_{k_\parallel k_\perp}$, $X^V_{k_\parallel k_\perp}$, $\Sigma^V_{k_\parallel k_\perp}$, $\overline{\Sigma}^V_{k_\parallel k_\perp}$, $a^V_{k_\parallel k_\perp}$, $\overline{a}^V_{k_\parallel k_\perp}$, $\alpha^V_{k_\parallel k_\perp}$, $\overline{\alpha}^V_{k_\parallel k_\perp}$, $\mathcal{E}^V_{k_\parallel k_\perp}$, $\overline{\mathcal{E}}^V_{k_\parallel k_\perp}$, $\mathcal{H}^V_{k_\parallel k_\perp}$, $\overline{\mathcal{H}}^V_{k_\parallel k_\perp}$, $\Sigma^T_{k_\parallel k_\perp}$, $\overline{\Sigma}^T_{k_\parallel k_\perp}$, $\zeta^T_{k_\parallel k_\perp}$, $\overline{\zeta}^T_{k_\parallel k_\perp}$, $\mathcal{E}^T_{k_\parallel k_\perp}$, $\overline{\mathcal{E}}^T_{k_\parallel k_\perp}$, $\mathcal{H}^T_{k_\parallel k_\perp}$, $\overline{\mathcal{H}}^T_{k_\parallel k_\perp}$).

Imposing an equation of state $p = p(\mu)$ gives $p^V_{k_\parallel k_\perp} = c_s^2 \mu^V_{k_\parallel k_\perp}$ where $c_s^2 = dp/d\mu$ is the adiabatic speed of sound squared. With the freedom in the choice of frame (Appendix B) the condition $a_a = 0$ can be set. In this case there are 19 constraints for 25 variables. By the frame choice and Eq. (E.33) we have $\xi^S_{k_\parallel k_\perp} = 0$. The evolution of all other harmonic coefficient follows from the evolutions of $\mu^V_{k_\parallel k_\perp}$, $\Sigma^T_{k_\parallel k_\perp}$, $\mathcal{E}^T_{k_\parallel k_\perp}$, $\overline{\mathcal{E}}^T_{k_\parallel k_\perp}$, $\mathcal{H}^T_{k_\parallel k_\perp}$ and $\overline{\mathcal{H}}^T_{k_\parallel k_\perp}$ by (E.1) and the constraints. The perturbation variables coupled to $\mu^V_{k_\parallel k_\perp}$, $\Sigma^T_{k_\parallel k_\perp}$, $\mathcal{E}^T_{k_\parallel k_\perp}$, $\overline{\mathcal{H}}^T_{k_\parallel k_\perp}$ are

$$\mathcal{A}^V_{k_\parallel k_\perp} = \frac{a_1}{ik_\parallel}\frac{\mathcal{A}^S_{k_\parallel k_\perp}}{a_2} = -\frac{c_s^2}{\mu + p}\mu^V_{k_\parallel k_\perp}, \tag{E.47}$$

$$\frac{ik_\parallel}{a_1}\zeta^T_{k_\parallel k_\perp} = \left(\Sigma + \frac{\Theta}{3}\right)\Sigma^T_{k_\parallel k_\perp} - \mathcal{E}^T_{k_\parallel k_\perp}, \tag{E.48}$$

$$\frac{ik_\parallel}{a_1 a_2}\Sigma^V_{k_\parallel k_\perp} = -\left(B + \frac{2-k_\perp^2}{a_2^2}\right)\frac{\Sigma^T_{k_\parallel k_\perp}}{2} + \frac{3}{2}\Sigma\mathcal{E}^T_{k_\parallel k_\perp} - i\frac{k_\parallel}{a_1}\overline{\mathcal{H}}^T_{k_\parallel k_\perp}, \tag{E.49}$$

$$\frac{2\overline{\mathcal{H}}^V_{k_\parallel k_\perp}}{3a_2} = -\frac{\Sigma}{a_2 B}\mu^V_{k_\parallel k_\perp} + \Sigma C\mathcal{E}^T_{k_\parallel k_\perp} - \mathcal{E}\Sigma^T_{k_\parallel k_\perp} - \frac{ik_\parallel}{a_1}J\frac{\overline{\mathcal{H}}^T_{k_\parallel k_\perp}}{3}, \tag{E.50}$$

$$\frac{V^V_{k_\parallel k_\perp}}{a_2} = -\frac{2\Sigma}{a_2 B}\mu^V_{k_\parallel k_\perp} - \frac{2B}{3}(2+C)\Sigma^T_{k_\parallel k_\perp} + \Sigma(1+2C)\mathcal{E}^T_{k_\parallel k_\perp} - \frac{2}{3}\frac{ik_\parallel}{a_1}(1+J)\overline{\mathcal{H}}^T_{k_\parallel k_\perp}, \tag{E.51}$$

$$\frac{W^V_{k_\parallel k_\perp}}{a_2} = \frac{3\Sigma}{2a_2 B}\mu^V_{k_\parallel k_\perp} + \left(3\mathcal{E} + \frac{2-k_\perp^2}{a_2^2} - B\right)\frac{\Sigma^T_{k_\parallel k_\perp}}{2}$$
$$+ \frac{3\Sigma}{2}(1-C)\mathcal{E}^T_{k_\parallel k_\perp} - \frac{ik_\parallel}{2a_1}(2-J)\overline{\mathcal{H}}^T_{k_\parallel k_\perp}, \tag{E.52}$$

$$\frac{ik_\parallel}{a_1}\frac{\phi^S_{k_\parallel k_\perp}}{a_2^2} = -\frac{k_\parallel^2}{a_1^2}\frac{2}{a_2 B}\mu^V_{k_\parallel k_\perp} - \frac{BL}{3\Sigma}\Sigma^T_{k_\parallel k_\perp} + \left(L - \frac{2-k_\perp^2}{a_2^2 B}\frac{k_\perp^2}{a_2^2}\right)\mathcal{E}^T_{k_\parallel k_\perp} - \frac{ik_\parallel}{a_1}\frac{2L}{3\Sigma}\overline{\mathcal{H}}^T_{k_\parallel k_\perp}, \tag{E.53}$$

$$\frac{ik_\parallel}{a_1 a_2}\alpha^V_{k_\parallel k_\perp} = -\left[\frac{3\Sigma}{2B} + \left(\Sigma + \frac{\Theta}{3}\right)\frac{c_s^2}{\mu+p}\right]\frac{\mu^V_{k_\parallel k_\perp}}{a_2}$$
$$-\frac{3\mathcal{E}}{2}\Sigma^T_{k_\parallel k_\perp} + \frac{3\Sigma}{2}C\mathcal{E}^T_{k_\parallel k_\perp} - \frac{ik_\parallel}{2a_1}J\overline{\mathcal{H}}^T_{k_\parallel k_\perp} , \tag{E.54}$$

$$\frac{X^V_{k_\parallel k_\perp}}{a_2} = \left[1 - \frac{3}{B}\left(\frac{k_\perp^2}{a_2^2} + 3\mathcal{E}\right)\right]\frac{\mu^V_{k_\parallel k_\perp}}{3a_2} - \frac{\mathcal{E}}{\Sigma}\left(\frac{2}{a_2^2} + 3\mathcal{E}\right)\Sigma^T_{k_\parallel k_\perp} + C\left(\frac{k_\perp^2}{a_2^2} + 3\mathcal{E}\right)\mathcal{E}^T_{k_\parallel k_\perp}$$
$$+\frac{a_1}{ik_\parallel}\left[\mathcal{E}\left(\mathcal{E} + \frac{2}{3a_2^2}\right)\frac{k_\parallel^2}{a_1^2} - \Sigma^2\frac{2-k_\perp^2}{4a_2^2}\frac{k_\perp^2}{a_2^2}\right]\frac{6\overline{\mathcal{H}}^T_{k_\parallel k_\perp}}{\Sigma B} , \tag{E.55}$$

$$\frac{ik_\parallel}{a_1 a_2}\mathcal{E}^V_{k_\parallel k_\perp} = \frac{k_\parallel^2}{a_1^2}\frac{\mu^V_{k_\parallel k_\perp}}{a_2 B} - \frac{3\mathcal{E}}{2}\left(\Sigma + \frac{\Theta}{3}\right)\Sigma^T_{k_\parallel k_\perp} + \left(\frac{3\mathcal{E}}{2} - C\frac{k_\parallel^2}{a_1^2}\right)\mathcal{E}^T_{k_\parallel k_\perp}$$
$$-\frac{ik_\parallel}{a_1}\frac{3\overline{\mathcal{H}}^T_{k_\parallel k_\perp}}{2B}\left[2\mathcal{E}\left(\Sigma + \frac{\Theta}{3}\right) + \Sigma\frac{2-k_\perp^2}{a_2^2}\right] , \tag{E.56}$$

and which are coupled to $\overline{\mathcal{E}}^T_{k_\parallel k_\perp}$, $\mathcal{H}^T_{k_\parallel k_\perp}$ are

$$\mathcal{H}^V_{k_\parallel k_\perp} = \frac{ik_\parallel}{a_1}\frac{a_2}{k_\perp^2}\mathcal{H}^S_{k_\parallel k_\perp} = -\frac{ik_\parallel}{a_1}\overline{\alpha}^V_{k_\parallel k_\perp} = \frac{ik_\parallel}{a_1}\overline{\Sigma}^V_{k_\parallel k_\perp}$$
$$= \frac{2-k_\perp^2}{2a_2}\overline{\Sigma}^T_{k_\parallel k_\perp} = \frac{2-k_\perp^2}{a_2 B}\left(\frac{3\Sigma}{2}\overline{\mathcal{E}}^T_{k_\parallel k_\perp} + \frac{ik_\parallel}{a_1}\mathcal{H}^T_{k_\parallel k_\perp}\right) , \tag{E.57}$$

$$\frac{B}{2}\overline{\zeta}^T_{k_\parallel k_\perp} = \left(\Sigma + \frac{\Theta}{3}\right)\mathcal{H}^T_{k_\parallel k_\perp} - \frac{a_1}{ik_\parallel}\left(\frac{k_\parallel^2}{a_1^2} - \frac{2-k_\perp^2}{2a_2^2}\right)\overline{\mathcal{E}}^T_{k_\parallel k_\perp} , \tag{E.58}$$

$$\overline{\mathcal{E}}^V_{k_\parallel k_\perp} = \frac{2-k_\perp^2}{2a_2}\left[\frac{a_1}{ik_\parallel}\left(1 - \frac{9\Sigma^2}{2B}\right)\overline{\mathcal{E}}^T_{k_\parallel k_\perp} - \frac{3\Sigma}{B}\mathcal{H}^T_{k_\parallel k_\perp}\right] , \tag{E.59}$$

where B and C are defined in Eqs. (4.50) and (4.51), respectively, and

$$L \equiv 2C\frac{k_\parallel^2}{a_1^2} + \frac{k_\perp^2}{a_2^2}\left(1 + \frac{2-k_\perp^2}{a_2^2 B}\right) , \tag{E.60}$$

$$J \equiv \frac{(2-k_\perp^2)k_\perp^2 a_1^2}{k_\parallel^2 a_2^4 B} + 2C . \tag{E.61}$$

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