Geophys. J. Int. (2000) **142,** 351–360

Gravitational stability of spherical self-gravitating relaxation models

L. L. A. Vermeersen[1,]* and J. X. Mitrovica[2]

[1] *Dipartimento di Scienze della Terra, Sezione Geofisica, Università di Milano, Via L. Cicognara* 7, I-20129 *Milano, Italy*

[2] *Department of Physics, University of Toronto,* 60 *St. George Street, Toronto, Ontario* M5S 1A7, *Canada. E-mail: jxm@physics.utoronto.ca*

Accepted 2000 March 1. Received 2000 February 24; in original form 1998 December 6

SUMMARY

The gravitational stability of spherical, self-gravitating, hydrostatically pre-stressed planetary models remains a subject of active interest. Love (1907, 1911) was the first to show that purely elastic models can become unstable when values of rigidity and bulk modulus are insufficient to counteract self-gravitational collapse. We revisit his calculations and extend his work to show that so-called dilatational (or 'D') modes of a viscoelastic sphere can also become unstable to self-gravitation in a specific region of Lamé parameter space. As an example, we derive a marginal stability curve for the dilatational modes of a homogeneous planetary model at spherical harmonic degree two. We demonstrate that the stability conditions are independent of viscosity and that the instability will occur only when the homogeneous earth model is already unstable to the elastic instability described by Love (1907, 1911). Finally, we also consider a class of Rayleigh–Taylor (or 'RT') instabilities related to unstable density stratification in planetary models. This convective instability is explored using both a homogeneous Maxwell viscoelastic sphere (which has an unstable layering at all depths) and a suite of Maxwell earth models that adopt the elastic and density structure of the seismic model PREM (which has regions of unstable density stratification within the upper mantle). We argue that previous studies have significantly overestimated the potential importance of these modes to Earth evolution. For example, suggestions that the timescale of the RT modes is short relative to the age of the Earth face the fundamental problem that the ensuing convective instability would have long ago destroyed the unstable layering and produced an adiabatic profile. We predict that at low degrees the RT instabilities for a PREM density profile and realistic viscosity stratification have timescales comparable to the age of the Earth. It is unclear, in any event, whether the unstable density layering in the PREM upper mantle is robust.

Key words: earth models, gravitational stability.

INTRODUCTION

The planform of mantle convection remains a source of active debate in geophysics; however, the existence of convective motions in this region is well established. Both direct and indirect evidence for flow-induced mantle heterogeneity exist. The former include, for example, plate tectonic motions and the increasingly high-resolution tomographic images of seismic velocity variations.

Despite the presence of convective motions, there are a range of geodynamic observables that can be modelled under the assumption that the Earth is stably stratified. Examples include (post-)seismic deformations and a suite of anomalies linked to postglacial adjustment. These models are commonly termed 'relaxation models'. In principle, they describe the complete or partial recovery of the perturbed earth due to forcings (for example, ice melt or growth, sea-level changes, atmospheric pressure variations, earthquakes, tidal interactions, changes in centrifugal potential, etc.) as a function of both the parameters defining the earth model and the space–time geometry of the forcing.

Stable stratification is a central assumption in the application of (linearized) relaxation models. Accordingly, the question of whether these linearized models are stable to perturbing forcings is a problem of long-standing interest. An important early contribution to the discussion of stability of elastic

* Now at: DEOS, Fac. Aerospace Engineering, Delft University of Technology, Kluyverweg 1, NL-2629 HS Delft, the Netherlands. E-mail: b.vermeersen@lr.tudelft.nl

spherical relaxation models was provided by Love (1907, 1911). He derived analytical expressions for the stability criteria of homogeneous compressible elastic spheres for low-degree (i.e. long-wavelength) harmonics. Elastic models are unstable whenever the rigidity and incompressibility are insufficient to prevent collapse of the sphere due to self-gravitation. Practical considerations limited Love (1911) to presenting expressions for a small number of points situated on the curves of marginal stability in the Lamé parameter space.

Stability conditions for spherical viscoelastic models have been the subject of recent interest. In particular, two classes of possible gravitational instability in such models have been discussed. The first relates to so-called dilatation modes (henceforth D modes), which occur in viscoelastic models with some level of compressibility. These modes have been discussed by a number of authors (e.g. Han & Wahr 1995; Vermeersen *et al.* 1996), and Vermeersen & Mitrovica (1998) have recently shown that there exist regions of Lamé parameter space in which the modes are unstable (that is, regions where the modal excitation increases exponentially with time). We will explore this issue in detail below. In particular, we consider the marginal stability curve for the D modes at low degree for a self-gravitating, homogeneous Maxwell viscoelastic sphere. Furthermore, we compare this result with a plot of Love's (1911) analytic expressions for the stability of elastic relaxation at low degree.

Unstable layering of the radial density profile gives rise to a second class of instability in viscoelastic earth models, as discussed in a series of recent studies (Plag & Jüttner 1995; Vermeersen & Sabadini 1997; Vermeersen & Mitrovica 1998; Hanyk *et al.* 1999). The stability condition for incompressible models is simple; namely, instabilities occur whenever the radial density profile shows inversions (i.e. high density over low density). (We note, once again, that D modes are entirely absent in incompressible models.) In a viscoelastic compressible rheology, the Williamson–Adams (WA) criterion determines whether the radial density stratification is stable or not: if the square of the Brunt–Väisälä frequency is negative, then the layering is unstable in a compressible model. The WA criterion implies that the density must be increasing with depth to some extent for a compressible viscoelastic sphere to be stable. Instabilities associated with 'unstable layering' of viscoelastic models are of the Rayleigh–Taylor (henceforth RT, after Plag & Jüttner 1995) type.

In this paper we revisit the issue of unstable RT modes by describing in detail results presented by Vermeersen & Mitrovica (1998) for spherical, Maxwell viscoelastic earth models. Specifically, we consider both a homogeneous model and a model with radial profiles of density, rigidity and bulk modulus specified by PREM (Dziewonski & Anderson 1981). The homogeneous compressible model is subject to RT instabilities. This is true also for the second model, since the square of the Brunt–Väisälä frequency is negative in PREM within a small layer just above 670 km depth and for all regions above the transition zone (i.e. depths less than 400 km).

Previous analyses of RT modes in viscoelastic models have led to some rather strong claims. For example, Plag & Jüttner (1995) adopted PREM and concluded that 'The excitation amplitudes of the [RT] modes are of the same order of magnitude as those for stable eigenmodes .... For typical viscosity profiles derived from post-glacial rebound studies, the characteristic times are of the order of 10^7 to 10^8 y, while for a profile with a very low viscosity in the asthenosphere the characteristic times found here are as low as 6×10^3 y ... [and] these modes could affect the evolution of the planet' (p. 267). Hanyk *et al.* (1999) adopted a homogeneous compressible model, and they predicted fastest 'growth times on the order of ten thousand years ... for the longest wavelength' (p. 557). They furthermore suggested that these modes may play a role in 'large-amplitude rotational instabilities' (p. 560), and as an example they cited an inertial interchange true polar wander event (timescale $\sim10^7$ yr) recently invoked for the Cambrian.

There are a number of reasons to be skeptical of these claims. Hanyk *et al.* (1999) expressed 'surprise' (p. 557) that they found RT modes in their homogeneous, compressible earth model; however, that such a model must be unstable has been known for at least four decades (see e.g. Longman 1962, p. 486). Indeed, the entire homogeneous sphere would be gravitationally unstable in their calculations, and hence any application of the results to the Earth would be suspect. Results based on the adoption of PREM raise two further issues. First, it is unclear to what extent departures from the WA condition in the PREM upper mantle are constrained. Second, the timescales cited above by Plag & Jüttner (1995) are short relative to the age of the Earth and therefore the unstable layering in PREM could not have been sustained for any significant fraction of Earth history. That is, RT (convective) instabilities would have quickly acted to produce an adiabatic profile in the region in question. This physics appears to us to be a rather fundamental objection to these quoted timescales, and it implies that either the timescales are too short, or the unstable layering in PREM is not a robust feature of that model.

ELASTIC INSTABILITIES

In the case of an elastic, homogeneous, spherical, self-gravitating model, the only instability that can arise occurs when the rigidity and resistance to compression of the model are insufficient to prevent self-gravitational collapse. Love (1907) discussed this mechanism and modelled it under the assumption that the pre-stress of the body was not advected in deformation. Love (1911) subsequently corrected his theory by setting the Lagrangian time-derivative of pre-stress to be zero. In the following we will discuss the results obtained by Love (1911).

In Chapter IX of Love (1911), an analysis is given for the gravitational stability of a hydrostatically pre-stressed, spherical, elastic, self-gravitating, homogeneous sphere. The condition of marginal stability in the (λ, μ)-plane that Love (1911) derived for harmonic degree two is given by his eq. (21) (Section 138, p. 119):

$$\frac{\left[10(3\alpha^2+\beta^2)-\dfrac{3\alpha^2-\beta^2}{\alpha^2-\beta^2}\,\alpha^2\beta^2R^2\right]\psi_2+2(3\alpha^2+\beta^2)\psi_1}{[4\alpha^2\beta^2R^2-25(3\alpha^2+\beta^2)]\psi_2-4(3\alpha^2-\beta^2)\psi_1}+\frac{\left[10(3\beta^2+\alpha^2)-\dfrac{3\beta^2-\alpha^2}{\alpha^2-\beta^2}\,\alpha^2\beta^2R^2\right]\chi_2-2(3\beta^2+\alpha^2)\chi_1}{[4\alpha^2\beta^2R^2+25(3\beta^2+\alpha^2)]\chi_2-4(3\beta^2-\alpha^2)\chi_1}=0\,, \tag{1}$$

in which (see Section 124, p. 108 of Love 1911)

$$\alpha^2=\frac{2g\rho}{(\lambda+2\mu)R}\left(1+2\sqrt{1+\frac{3}{8}\frac{\lambda}{\mu}}\right), \tag{2a}$$

$$\beta^2=\frac{2g\rho}{(\lambda+2\mu)R}\left(-1+2\sqrt{1+\frac{3}{8}\frac{\lambda}{\mu}}\right), \tag{2b}$$

and, according to eq. (17) of Love (1911; Section 125, p. 109),

$$\psi_1=\psi_1(\alpha R)=\frac{\alpha R\cos(\alpha R)-\sin(\alpha R)}{(\alpha R)^3}, \tag{3a}$$

$$\psi_2=\psi_2(\alpha R)=\frac{[3-(\alpha R)^2]\sin(\alpha R)-3\alpha R\cos(\alpha R)}{(\alpha R)^5}, \tag{3b}$$

$$\chi_1=\chi_1(\beta R)=\frac{\beta R\cosh(\beta R)-\sinh(\beta R)}{(\beta R)^3}, \tag{3c}$$

$$\chi_2=\chi_2(\beta R)=\frac{[3+(\beta R)^2]\sinh(\beta R)-3\beta R\cosh(\beta R)}{(\beta R)^5}. \tag{3d}$$

In eqs (1)–(3), λ and μ are the usual Lamé parameters, R is the mean earth radius and g the surface gravity for a homogeneous earth with density ρ. In our homogeneous earth models we will use the values that Love (1911) used, viz. $R=6370$ km, $\rho=5500$ kg m^{-3}, while $g=(4/3)\pi G\rho R$, where G is the gravitational constant. The parameters α and β have the positive definite values of their respective quadratic forms in eq. (2).

We can rewrite eq. (1) in the following simple way:

$$\frac{A}{C}+\frac{B}{D}=\frac{AD+BC}{CD}=0, \tag{4}$$

with

$$A=\left[10(3\alpha^2+\beta^2)-\frac{3\alpha^2-\beta^2}{\alpha^2-\beta^2}\alpha^2\beta^2R^2\right]\psi_2+2(3\alpha^2+\beta^2)\psi_1, \tag{5a}$$

$$B=\left[10(3\beta^2+\alpha^2)-\frac{3\beta^2-\alpha^2}{\alpha^2-\beta^2}\alpha^2\beta^2R^2\right]\chi_2-2(3\beta^2+\alpha^2)\chi_1, \tag{5b}$$

$$C=[4\alpha^2\beta^2R^2-25(3\alpha^2+\beta^2)]\psi_2-4(3\alpha^2-\beta^2)\psi_1, \tag{5c}$$

$$D=[4\alpha^2\beta^2R^2+25(3\beta^2+\alpha^2)]\chi_2-4(3\beta^2-\alpha^2)\chi_1. \tag{5d}$$

From eq. (2) it follows that

$$\alpha^2-\beta^2=\frac{4g\rho}{(\lambda+2\mu)R} \tag{6}$$

and thus, since λ and μ are positive definite, both the numerator $AD+BC$ and the denominator CD in eq. (4) have no singularities.

Love (1911) noted that if the ratio λ/μ was specified then the ratio α/β was known from eq. (2) (or vice versa). In this case, eq. (1) (or, equivalently, eq. 4) can be rewritten with α as the single unknown. There are several values of α that will satisfy this equation, and each of these is associated (once again via eq. 2) with specific values of λ and μ. Indeed, the largest values of λ and μ determined in this fashion represent the condition of marginal stability for the ratio λ/μ being considered.

As an example, consider the results cited by Love (1911). Love (1911) treated the two cases $\alpha/\beta=3/2$ and 18/17. These cases correspond to ratios $\lambda/\mu=46/25$ and 247246/1225. Love (1911) found that the former led to the solution $\alpha R=4.56$ and, ultimately, to (λ, μ) equal to (5.7, 3.1) in units of 10^{10} Pa. Analogous values for the latter case were $\alpha R=5.45$, $\lambda=42.38\times10^{10}$ Pa and $\mu=0.21\times10^{10}$ Pa.

We have revisited this calculation by searching for all solutions to eq. (4) for a wide range of *a priori* ratios α/β. The results (Fig. 1a) clearly indicate the presence of a family of solutions, with the curve of 'marginal stability' given by the upper line. We note that Love's (1911) case of $\alpha/\beta=3/2$ lies

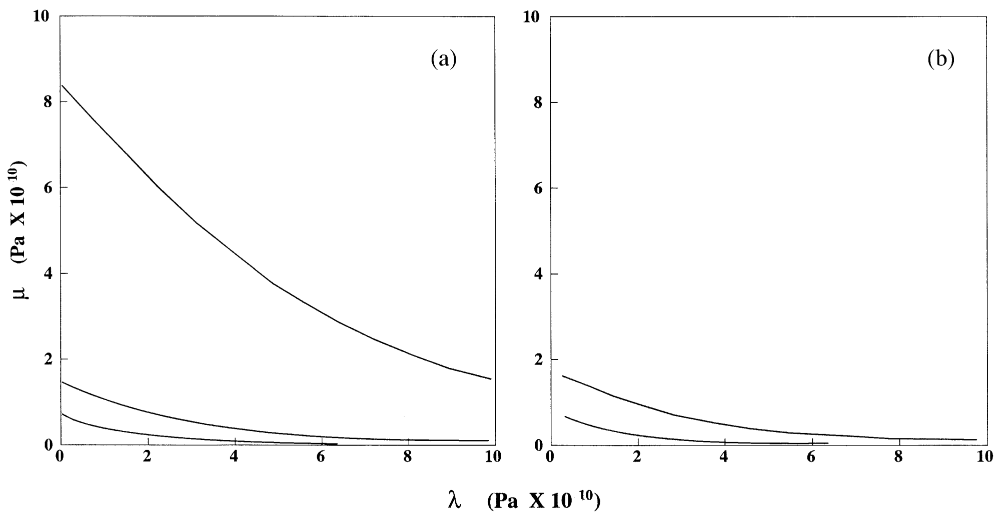

Figure 1. Stability curves for a homogeneous compressible elastic sphere (see text for details) at spherical harmonic degree 2 in the (λ, μ) parameter space, computed from expressions derived by Love (1911). (a) Plot of curves satisfying $AD+BC=0$ (see eq. 4); (b) plot of curves satisfying $CD=0$ (see eq. 4).

(almost) on this curve. In Fig. 1(b) we consider cases where the denominator of eq. (4) becomes zero. This family of curves falls below the line of marginal stability in Fig. 1(a), so we can take the latter to represent the boundary between stable and unstable regions of model space.

The elastic instabilities considered here will not exist in incompressible models (since the Lamé parameter, λ, is infinite in this case). These instabilities may, however, exist in either compressible elastic or viscoelastic models. The predictions appearing in the next two sections (dealing with D and RT modes) are based on Maxwell viscoelastic earth models. These models are not compressible at all timescales; indeed, a Maxwell body behaves as an incompressible inviscid fluid in the long-term (or so-called fluid) limit and it is commonly described as 'viscously incompressible' (Peltier 1976; Wu & Peltier 1982; Mitrovica & Peltier 1992). In this context, our use of the term 'compressible viscoelastic' in describing the predictions below refers to our adoption of Maxwell earth models with finite Lamé parameter λ. A purely incompressible viscoelastic rheology (i.e. incompressible at all timescales) may be obtained from such models by considering the limit $\lambda \rightarrow \infty$.

VISCOELASTIC INSTABILITIES: D MODES

For a homogeneous earth model with volume-averaged parameters for the density, rigidity and compressibility, one set of D modes is triggered in the negative s-domain (that is, the strengths of these modes decay exponentially with time). Vermeersen *et al.* (1996, eq. 46) derived an analytical approximation formula for the s-values of these modes:

$$s_m = -\frac{(m\pi/R)^2\kappa - 4\rho g/R}{(m\pi/R)^2(\lambda+2\mu) - 4\rho g/R}\,\frac{1}{\tau_M}\,, \tag{7}$$

where $\kappa = \lambda + 2\mu/3$ is the bulk modulus and $\tau_M = \nu/\mu$ is the Maxwell time. Notice that for an incompressible viscoelastic model ($\lambda \rightarrow \infty$), the s_m all collapse to the inverse Maxwell time, and thus the D modes disappear (Vermeersen *et al.* 1996).

In Vermeersen *et al.* (1996) the set of D modes was indexed as $m = 1, 2, 3, \ldots$; however, Vermeersen & Sabadini (1997) have shown that the parameter $C(s)$ discussed by Vermeersen *et al.* (1996, eq. 33) should have been split into two distinct cases. This oversight has several consequences. First, the numerical results presented by Vermeersen *et al.* (1996, Tables 1, 2 and 3 and Fig. 1) are slightly altered (Hanyk *et al.* 1999). Second, the mode $m = 1$ does not exist, so that the indexing in eq. (7) is properly started at $m = 2$. Finally, Vermeersen & Sabadini (1998) showed that the so-called D0 mode (with inverse relaxation time s_0) is analytically associated with the Maxwell time.

For Lamé parameters characteristic of Earth-like values, the inverse relaxation times s_m of the D modes are negative. In previous discussions of these D modes, however, it has not been recognized that there are regions of (λ, μ) space in which the eigenvalues s_m are positive. In this case, the modal strength grows with time and the D modes are unstable. To consider this case, let us rewrite eq. (7) as

$$s_m = -\frac{\Gamma_m(\lambda+2\mu/3) - 1}{\Gamma_m(\lambda+2\mu) - 1}\,\frac{1}{\tau_M}\,, \tag{8}$$

with $\Gamma_m = m^2\pi^2/(4\rho g R)$. One can easily show that s_m becomes positive when $\Gamma_m^{-1} - 2\mu < \lambda < \Gamma_m^{-1} - 2\mu/3$. The relaxation times of the D modes are dependent on viscosity through the Maxwell time, τ_M; however, the condition for instability is *not* dependent on viscosity. Furthermore, the region of stability (or instability) is dependent on the density, but not on the *radial density gradient* (which is equal to zero in a homogeneous model).

In Fig. 2 we consider the D-mode stability for a homogeneous Maxwell viscoelastic earth model characterized (as in the last section) by $R = 6370$ km, $\rho = 5500$ kg m^{-3} and $g = (4/3)\pi G\rho R$. For $m = 2$, s_m becomes positive in the wedge-shaped region between the lines $s = 0$ and $s = \infty$. These two lines are defined by cases where the numerator and denominator of eq. (8) are zero, respectively. The two lines intersect at $(\lambda, \mu) = (\Gamma_m^{-1}, 0)$. For values of m larger than 2, the wedge-shaped region would be below the line $s = 0$ in Fig. 2, so the $m = 2$, $s = 0$ case provides the marginal stability curve; that is, this curve defines a lower bound for the Lamé parameters necessary for the gravitational stability of the dilatational modes of the homogeneous, Maxwell viscoelastic model.

A comparison of Figs 1(a) and 2 indicates that the marginal stability curve for the D modes is situated at smaller values of the Lamé parameters than the stability curve of the purely elastic model. This suggests that the D modes in compressible viscoelastic models do not (at least in the cases we have considered) determine the critical values of the Lamé parameters for which self-gravitational collapse of the earth model occurs. Consequently, the same condition that Love (1911) derived for the critical values of the elastic parameters necessary to counteract self-gravitation in an elastic homogeneous compressible model applies to a homogeneous viscoelastic compressible model. The only caveat to this conclusion is that the latter models are also subject to RT instabilities when the density profile includes regions of unstable layering. We consider this case next.

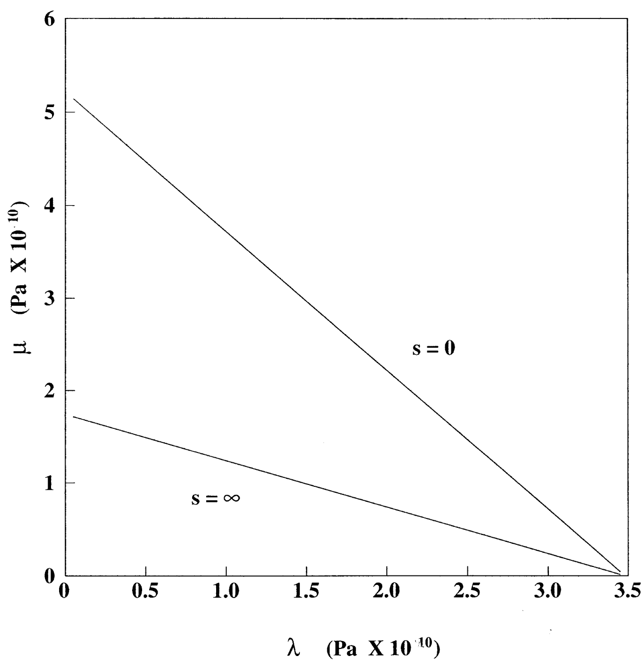

Figure 2. Curves defining the stability of D modes of a homogeneous compressible viscoelastic sphere (see text for details) at spherical harmonic degree 2 in the (λ, μ) parameter space. The curve $s = 0$ is the case where the numerator of eq. (8) is equal to zero; the curve $s = \infty$ is the case where the denominator of eq. (8) is equal to zero.

VISCOELASTIC INSTABILITIES: RT MODES

PREM (Dziewonski & Anderson 1981) is widely used in seismological and geodynamic modelling for its radial density, rigidity and bulk modulus profiles. Between 24.4 and 220.0 km depth, PREM shows an inversion of the density profile; that is, the density *decreases* with depth (Dziewonski & Anderson 1981, Tables 2 and 3). The influence of the density inversions of PREM on incompressible viscoelastic relaxation models was studied by Vermeersen & Sabadini (1997, Section 6). They concluded that the density inversions can have a significant impact on the Love numbers when the lithosphere is taken to be viscoelastic instead of purely elastic. They also concluded that the shallow density inversions in PREM have more impact on surface load forcings than on tidal (gravitational) forcings.

The square of the Brunt–Väisälä frequency is given by

$$N^2 = -\frac{g}{\rho}\left(\frac{d\rho}{dr} + \frac{\rho^2 g}{\kappa}\right). \tag{9}$$

A density layering is unstable in a compressible model anywhere where N^2 is negative, and this occurs in two radial regions within PREM: the surface down to 400 km depth, and in a small region of the transition zone just above the 670 km discontinuity. These regions give rise to RT instabilities in compressible viscoelastic models that adopt PREM.

In the following subsections we will present results for Maxwell viscoelastic surface load Love numbers h, ℓ and k. These numbers govern the impulse response of the viscoelastic earth model and, in the time domain, they have the forms (Peltier 1974)

$$h_l(t) = h_l^{\mathrm{E}}\delta(t) + \sum_{j=1}^{N} r_j^l \exp(s_j^l t), \tag{10}$$

$$\ell_l(t) = \ell_l^{\mathrm{E}}\delta(t) + \sum_{j=1}^{N} r_j'^l \exp(s_j^l t), \tag{11}$$

$$k_l(t) = k_l^{\mathrm{E}}\delta(t) + \sum_{j=1}^{N} r_j''^l \exp(s_j^l t), \tag{12}$$

where the first term on the right-hand side of each equation is the immediate elastic response to the impulse load (hence the superscript E), and the second term is the non-elastic response. The latter is formed from a superposition of N modes of exponential decay defined by a common set of eigenfrequencies s_j^l, and a distinct set of modal amplitudes (r_j^l, $r_j'^l$, $r_j''^l$). The symbols t and δ represent time and the Dirac delta function. The h, ℓ, and k Love numbers are coefficients in Legendre polynomial expansions for the radial and tangential surface displacements and the gravitational potential perturbation, respectively. If a given eigenfrequency is negative, then the associated mode is stable and the inverse eigenfrequency can be termed the decay time of that mode. However, if an eigenfrequency is positive the mode is unstable and we will simply refer to the inverse eigenfrequency as the 'timescale' of the mode. Finally, the ratio of the modal amplitude to the associated eigenfrequency is commonly termed the 'modal strength' (Wu & Peltier 1982).

Homogeneous compressible model

Fig. 3 shows the modal eigenfrequencies and strengths of a homogeneous compressible viscoelastic sphere with parameters taken from Vermeersen *et al.* (1996): $\nu = 10^{21}$ Pa s, $\rho = 5517$ kg m^{-3}, $R = 6371$ km, $\lambda = 3.5288 \times 10^{11}$ Pa and $\mu = 1.4519 \times 10^{11}$ Pa. The same homogeneous model was recently considered by Vermeersen & Mitrovica (1998) and Hanyk *et al.* (1999) in their analyses of RT modes. The plots do not include the D modes, which in this case are stable, and therefore the only stable modal branch is the so-called fundamental set, labelled M0. The open circles represent the unstable (eigenfrequencies greater than zero) RT modes, and, following Plag & Jüttner (1995), we denote branches associated with these modes by RTn ($n = 1, 2, \ldots$). The label RT1 is the branch with the highest strength, RT2 is the next highest, and so on. Although we show eigenfrequencies for at least a portion of seven RT branches in Fig. 3(a), the plots of modal strength only include the RT1, RT2 and RT3 branches.

The eigenfrequencies of the unstable RT modes are at least one order of magnitude smaller than the stable M0 modes. The shortest inverse eigenfrequency (or timescale) for the RT modes is approximately 10 kyr for the RT1 modal branch at degree 2. This timescale grows with increasing spherical harmonic degree (or decreasing wavelength) along the RT1 branch, so that by degree 50 it reaches ~ 500 kyr. The variation of eigenfrequencies with harmonic degree for the remaining RT modal families (RTn, $n = 2, 3, \ldots$) is not monotonic; indeed, these mode branches obtain a maximum eigenfrequency for degrees higher than 2, and this maximum appears to shift towards higher degrees as n increases.

In the case of the h and k Love numbers, the modal strengths for the stable M0 modes are about one order of magnitude greater than the RT1 strengths at low degrees. By degree 50 this discrepancy increases to over two orders of magnitude. This behaviour contrasts with the ℓ case, where the modal strengths for all three unstable mode branches are significantly larger than the M0 strengths. We conclude that horizontal motions on a homogeneous, compressible viscoelastic earth model loaded by surface masses are exceedingly unstable.

A common procedure for checking whether all viscoelastic normal modes have been found in the numerical procedure has been to invoke the so-called 'isostatic limit' (Wu & Peltier 1982). In practice, one first performs an independent calculation in which the limit is determined by assuming that all non-elastic regions are inviscid, and, second, one compares this prediction to the limit determined from the normal modes. As an example of the latter, let us convolve eq. (10) with a Heaviside load applied at $t = 0$:

$$h_l^{\mathrm{H}}(t > 0) = h_l^{\mathrm{E}} + \sum_{j=1}^{N} \frac{r_j^l}{s_j^l}\left[1 - \exp(s_j^l t)\right], \tag{13}$$

where the superscript H denotes the Heaviside response. Now, *if all modes are stable*, the long-time limit of this response can be written as

$$h_l^{\mathrm{H}}(t = \infty) = h_l^{\mathrm{E}} + \sum_{j=1}^{N} \frac{r_j^l}{s_j^l}. \tag{14}$$

Hanyk *et al.* (1999) observed that expression (14), when both stable *and* unstable modes are included in the sum, yields a match to the analytically determined value for the inviscid limit of a homogeneous earth. This intriguing result, which we have confirmed, has implications that extend beyond the simple homogeneous model considered in this section to the radially stratified models of the next section. We will explore this issue in detail in future work.

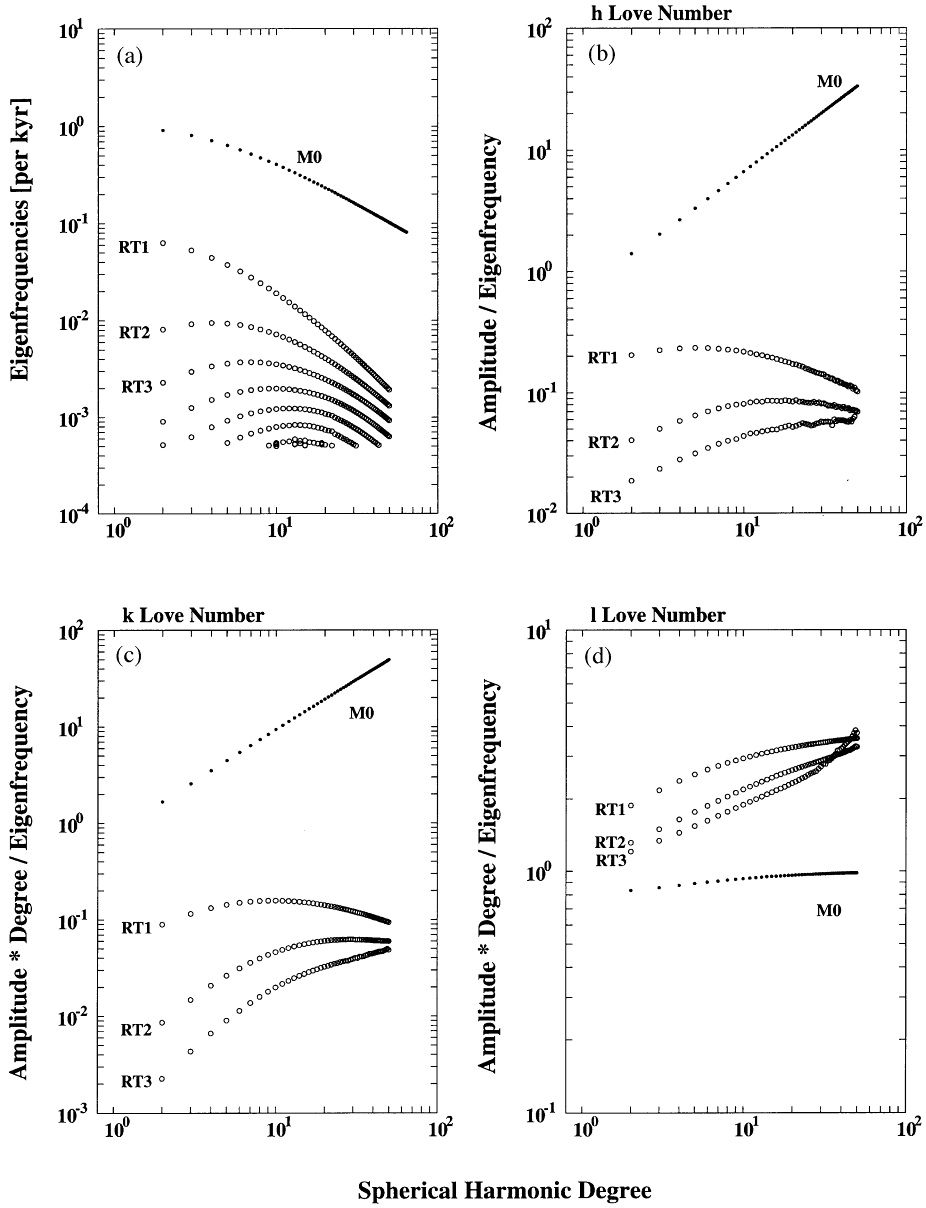

Figure 3. Eigenfrequencies (a) and modal strengths (b–d) of the normal modes of a homogeneous compressible viscoelastic sphere. The parameters defining the viscoelastic structure of the model are given in the text. The dots refer to stable (eigenfrequencies less than zero) modes. We do not show the D modes, hence the only stable mode family is the fundamental (M0) branch. The open circles depict unstable (positive eigenfrequency) RT modes. (a) shows a number of the RT mode families, but only the three with greatest strengths (RT1, RT2 and RT3) are shown in (b)–(d). Note that the k and ℓ Love number strengths have been scaled by the associated harmonic degree.

PREM

In this section we turn our attention to predictions of viscoelastic normal modes generated using a model with the density and elastic structure given by the model PREM (Dziewonski & Anderson 1981). Plag & Jüttner (1995) were the first to detect positive eigenfrequencies in models based on PREM. In particular, they considered four earth models: (1) uniform viscosity of 10^{21} Pa s and no lithosphere, as studied earlier by Peltier (1974); (2) 80-km-thick elastic lithosphere, and upper and lower mantle viscosities of 10^{21} and 2×10^{21} Pa s; (3) 80-km-thick elastic lithosphere, and upper and lower mantle viscosities of 5×10^{20} and 5×10^{21} Pa s [both model 2 and model 3 are adopted from Mitrovica & Peltier (1992)]; and (4) a radial profile of mantle viscosity varying from below 10^{17} Pa s in the asthenosphere (that is, the region characterized by a negative value of N^2 in PREM) to $\sim10^{27}$ Pa s close to the core–mantle boundary (CMB). We present a detailed analysis of a set of models similar to models (1)–(3) considered by Plag & Jüttner (1995).

In Fig. 4 we consider the eigenfrequencies and modal strengths for a model consisting of an inviscid core and an isoviscous 10^{21} Pa s mantle. Of the numerous stable normal modes we only plot mode branches associated with the deflection of density discontinuities at the surface (the fundamental mode, M0), the CMB (C0), 670 km depth (M1) and 400 km depth (M2). Once again, the open circles represent unstable RT modes. A very large number of these are shown on the eigenfrequency plot, but only the modal branch with the largest modal strength (RT1) is plotted in the remaining frames. We note that

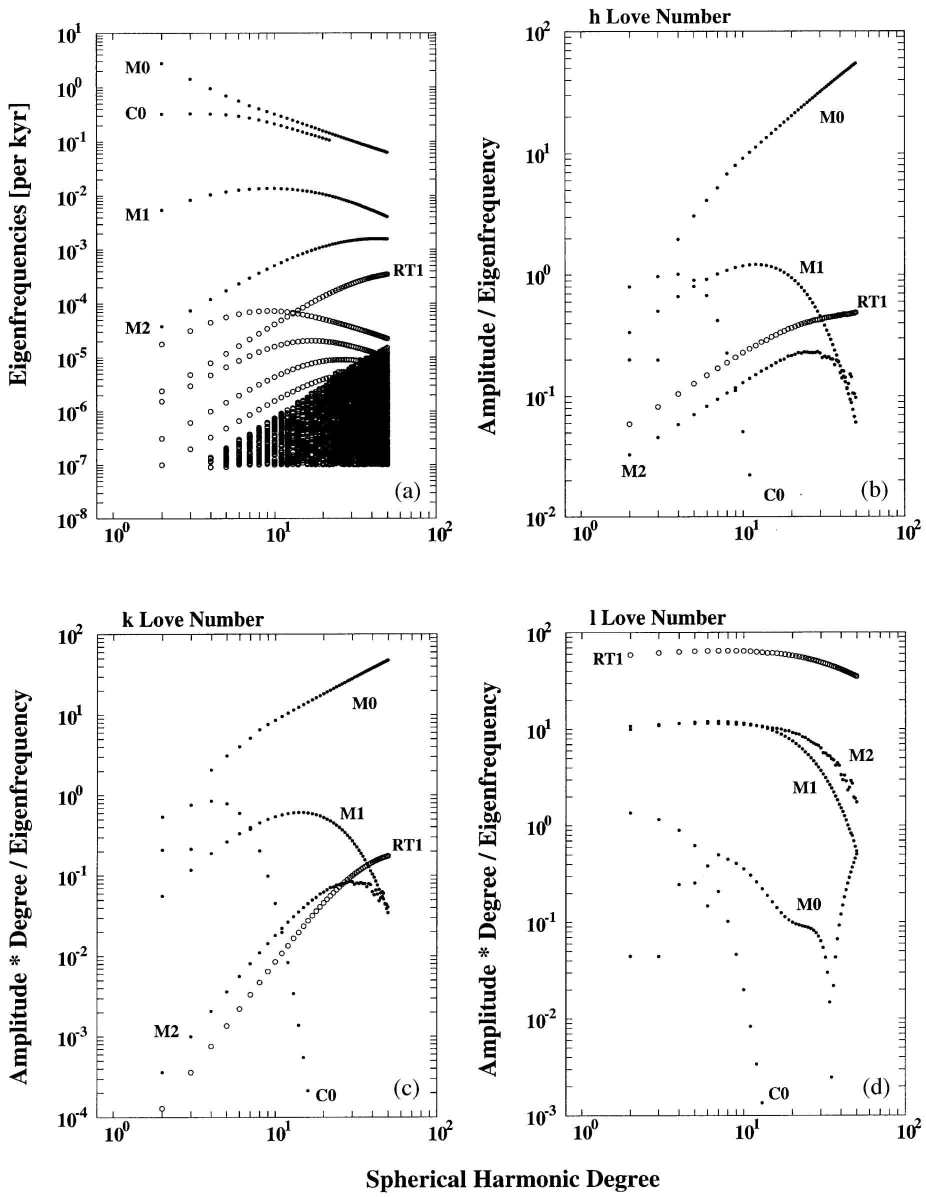

Figure 4. As in Fig. 3 except for eigenfrequencies (a) and strengths (b–d) of an earth model with elastic and density structure given by the model PREM (Dziewonski & Anderson 1981). The model has no lithosphere and a uniform mantle viscosity of 10^{21} Pa s. Of the many stable mode branches, only the M0, C0, M1 and M2 mode families are plotted. (a) shows a very large number of RT modes, but only the family with greatest strength (RT1) is shown in (b)–(d).

the eigenfrequencies of the C0 mode are not plotted above degree 22 because the numerical propagation procedure used to compute the normal-mode parameters begins at a radius above the CMB in this degree range. Furthermore, we have not plotted modes with eigenfrequencies less than 9×10^{-8} kyr^{-1} (an eigenfrequency of 2×10^{-7} kyr^{-1} coincides with the age of the Earth).

The dependence of the eigenfrequency of the RT1 mode branch on harmonic degree is clearly distinct from both the variation apparent in other RT families in Fig. 4 and the variation evident in all RT branches in Fig. 3. In particular, the timescale of the RT1 modes in Fig. 4 increases monotonically with spherical harmonic degree (at least in the range shown in the figure). At degree 2 this timescale is $\sim 10^9$ yr, while at degree 50 it is $\sim 3 \times 10^6$ yr (see also Plag & Jüttner 1995). As a consequence, and in contrast to the homogeneous case (Fig. 3), the timescale of the RT1 mode at degree 2 is six orders of magnitude longer than the decay time of the fundamental mode at that degree. Indeed, all unstable modes have timescales that are longer than the M0, C0, M1 and M2 modes at all harmonic degrees depicted in the figure.

For the h and k Love numbers the strengths of the RT1 modes are comparable to the stable M2 modes and far smaller (generally by several orders of magnitude or more) than the mode with greatest strength at each degree. As in Fig. 3, the ℓ Love number results are distinct; in this case, the strengths of the RT1 modes are larger, by an order of magnitude, than the stable mode with greatest strength.

In Fig. 5 we show results for an earth model identical to the one adopted in constructing Fig. 4 with the exception that we include an 80 km elastic lithosphere. (In practice, this feature is generated by setting the viscosity in the top 80 km of the model

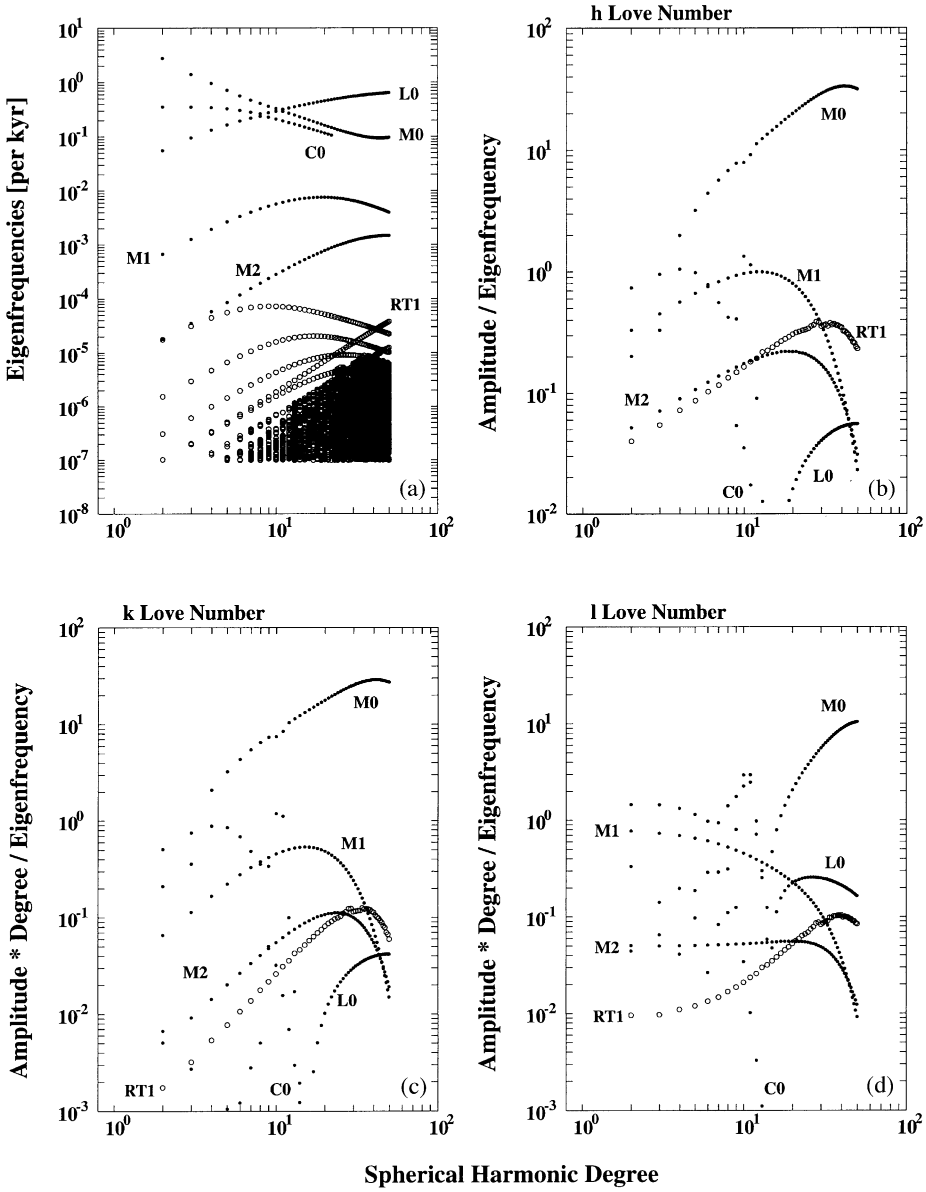

Figure 5. Same as Fig. 4, but now with an 80 km thick lithosphere added to the model. The L0 mode family is a stable branch associated with the contrast in rheology at the base of the lithosphere.

to an exceedingly high value.) The inclusion of an elastic lithosphere has several important effects on the normal modes of the viscoelastic earth model. First, the set of stable modes now includes the L0 mode, associated with the contrast in rheology at the base of the lithosphere (Wu & Peltier 1982; Wolf 1985). The eigenfrequencies and modal strengths of the remaining stable mode branches (M0, M1, M2 and, to a lesser extent, C0) are also perturbed by the presence of an elastic plate. The timescale of the stable RT1 modes increases by about an order of magnitude by the addition of an 80 km lithosphere (as first demonstrated by Plag & Jüttner 1995); in contrast, the eigenfrequencies for the remaining set of unstable modes shown in the figure appear to be insensitive to this aspect of the model. The timescale of the RT1 modes now ranges from $\sim 10^{10}$ yr at degree 2 (twice the age of the Earth) to 2.5×10^{7} yr at degree 50. For the *h* and *k* Love numbers, the modal strengths of the RT1 branch are moderately reduced, and they remain comparable to strengths associated with the stable M2 modes. The greatest reduction in strength is apparent at high degrees, where the influence of the elastic lithosphere should be greatest. The addition of an elastic lithosphere clearly has the most significant impact on horizontal deformations. Indeed, the ℓ Love number strengths are reduced by three to four orders of magnitude from Fig. 4 to Fig. 5 (see also Plag & Jüttner 1995). Not surprisingly, the elastic plate acts to stabilize surface horizontal deformations of the earth model associated with the RT1 modes. Overall, the modal strengths for the RT1 branch are one to three orders of magnitude smaller than the greatest strength of the stable mode set.

Finally, in Fig. 6 we consider an earth model with an upper mantle viscosity of 5×10^{20} Pa s and a lower mantle viscosity of 5×10^{21} Pa s. A comparison of Figs 5 and 6 shows relatively moderate changes. The eigenfrequencies of the RT1 modes increase by roughly a factor of two. A similar increase is

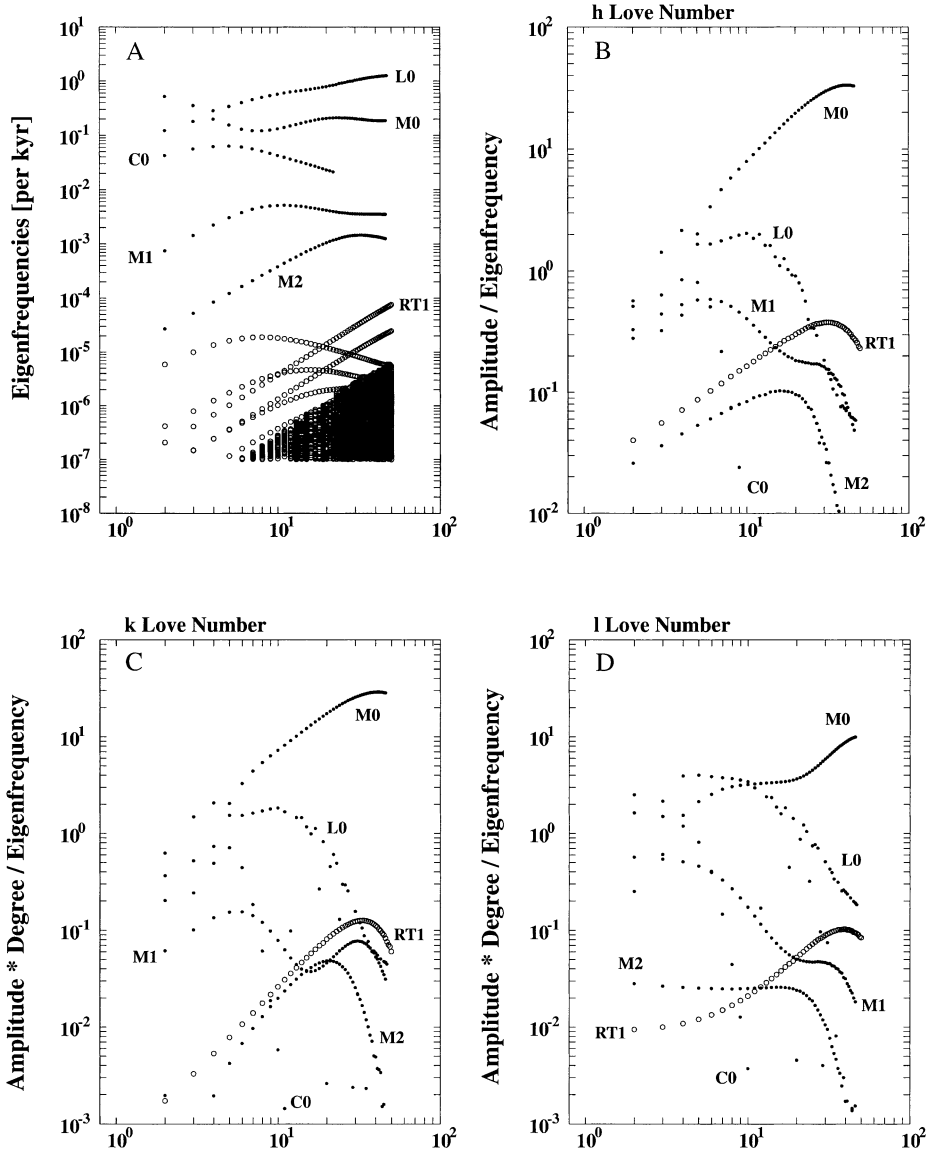

Figure 6. Same as Fig. 5, but now with an upper mantle viscosity of 5×10^{20} Pa s and a lower mantle viscosity of 5×10^{21} Pa s.

evident in a second RT branch that emerges (relative to Fig. 5) from the dense region of unstable modes at the bottom right of the eigenfrequency plot. This second branch has a similar monotonic increase of eigenfrequency with harmonic degree as is evident in the RT1 case. The slight increase in eigenfrequency, or decrease in timescale, is expected since the upper mantle regions of unstable density stratification in PREM are subject to a lower viscosity in Fig. 6 than in Fig. 5. The modal strengths of the RT1 branch appear to be relatively insensitive to this change in the radial profile of mantle viscosity.

Most of our model results compare well with those obtained by Plag & Jüttner (1995), although we have not considered a model similar to their model 4 (characterized, among other things, by a very weak asthenosphere). However, our conclusions differ from theirs in several significant ways. First, they argued for the strengths that 'we find the RT modes to be of the same order as most of the stable modes .... Only the dominant stable modes are an order of magnitude larger than the RT modes' (p. 284). Our results including an elastic lithosphere indicate that, in general, the stable modes have significantly greater strength, and that the difference between the strengths of the dominant stable and unstable modes is generally two orders of magnitude. Second, they concluded that 'For low spherical degrees, the characteristic times are of the order of 10^8 years ... except for (model #4) where they are as low as 5×10^5 y' (p. 284). For realistic earth models, in which the PREM elastic and density structure is adopted, the timescale of the RT1 modes at the lowest degrees is actually two orders of magnitude greater than the upper bound cited by Plag & Jüttner (1995) and comparable to the age of the Earth. This is an important point, since Hanyk *et al.* (1999) have, for example, suggested that RT modes may be implicated in relatively rapid and dramatic rotational (i.e. degree two) instabilities. Our results suggest that this link is highly unlikely.

The lower bound on the timescale cited by Plag & Jüttner (1995) for low-degree deformations (5×10^5 yr) is based on results generated using a model in which the regions of unstable density stratification in PREM coincide with a region of very low ($< 10^{17}$ Pa s) viscosity. In this case, Plag & Jüttner (1995) found RT instability timescales of $\sim 10^4$ yr by degree 30 and $\sim 10^3$ yr by degree 100. As we discussed in the Introduction, an RT instability proceeding at this timescale would quickly evolve any unstable density stratification into an adiabatic profile, and it is unclear how such stratifications could be argued to persist over geological timescales.

ACKNOWLEDGMENTS

This work was begun while BV was at the Geodetic Institute Stuttgart and JXM at the University of Toronto, and was completed while both were visiting the Department of Earth Sciences at the University of Milan. We wish to thank R. Sabadini, R. O'Connell and L. Hanyk for many useful discussions, and an anonymous reviewer for constructive comments. BV thanks the Alexander von Humboldt Foundation and ASI for partial financial support, and JXM acknowledges funding from the Natural Sciences and Engineering Research Council of Canada and the Canadian Institute for Advanced Research—Earth Systems Evolution Program.

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