Gravito-electromagnetic analogies

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Abstract

We reexamine and further develop different gravito-electromagnetic (GEM) analogies found in the literature, and clarify the connection between them. Special emphasis is placed in two exact physical analogies: the analogy based on inertial fields from the so-called "1+3 formalism", and the analogy based on tidal tensors. Both are reformulated, extended and generalized. We write in both formalisms the Maxwell and the full exact Einstein field equations with sources, plus the algebraic Bianchi identities, which are cast as the source-free equations for the gravitational field. New results within each approach are unveiled. The well known analogy between linearized gravity and electromagnetism in Lorentz frames is obtained as a limiting case of the exact ones. The formal analogies between the Maxwell and Weyl tensors, and the related issue of super-energy, are also discussed, and the physical insight from the tidal tensor formalism is seen to yield a suggestive interpretation of the phenomenon of gravitational radiation. The precise conditions under which a similarity between gravity and electromagnetism occurs are discussed, and we conclude by summarizing the main outcome of each approach.

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1 Introduction

This work has two main goals: one is to establish the connection between the several gravito-electromagnetic analogies existing in the literature, summarizing the main results and insights offered by each of them; the second is to further develop and extend some of these analogies.

In an earlier work by one of the authors [1, 2], a gravito-electromagnetic analogy based on tidal tensors was presented, and its relationship with 1) the well known analogy between linearized gravity and electromagnetism, 2) the mapping, via the Klein-Gordon equation, between ultrastationary spacetimes and magnetic fields in curved spacetimes, and 3) the formal analogies between the Weyl and Maxwell tensors (their decomposition into electric and magnetic parts, the quadratic scalar invariants they form, and the field equations they obey) was discussed.

Building up on the work in [2], another approach is herein added to the discussion: the exact analogy based on the fields of inertial forces, arising in the context of the 1+3 splitting of spacetime. This approach, which is herein reformulated and suitably generalized, is still not very well known, but very far reaching. It is therefore important to understand how it relates with the other known analogies, and in particular with the (also exact) approach based on tidal tensors.

Each of the analogies discussed here are also further developed, and some new results within each of them are presented. We start in Sec. 2 by revisiting the analogy based on tidal tensors introduced in [1] (and partly reviewed in [3]). In [1] it was shown that there is an exact, physical gravito-electromagnetic analogy relating the so-called electric ($\mathbb{E}_{\alpha\beta}$) and magnetic ($\mathbb{H}_{\alpha\beta}$) parts of the Riemann tensor with the electromagnetic tidal tensors ($E_{\alpha\beta},\ B_{\alpha\beta}$) defined therein, which is manifest the "relative acceleration" between two nearby test particles, and in the forces exerted on spinning (pole-dipole) particles. In this work we unveil another exact physical analogy involving $B_{\alpha\beta}$ and $\mathbb{H}_{\alpha\beta}$, manifest in the equations for the relative precession of spinning particles: both in electromagnetism and gravity, the relative precession of a test gyroscope/magnetic dipole relative to a system of axes attached to comoving guiding gyroscopes/magnetic dipoles at a neighboring point, is given by a contraction of $B_{\alpha\beta}/\mathbb{H}_{\alpha\beta}$ with the separation vector δx^{β}. This gives a physical interpretation for the tensors $B_{\alpha\beta}$ and $\mathbb{H}_{\alpha\beta}$ complementary to the one given by the forces on the spinning particles. The expression for the differential precession of gyroscopes in terms of $\mathbb{H}_{\alpha\beta}$ was originally found in a recent work [4]; herein (Sec. 2.3) we re-derive this result through a different procedure (which, we believe, is more clear), and we obtain its electromagnetic analogue.

The analogy based on tidal tensors extends to the field equations of both theories; in [1] it was shown that by taking the traces and antisymmetric parts of the electromagnetic tidal tensors one obtains the Maxwell equations, and performing the same operations in the gravitational tidal tensors leads to a strikingly similar set of equations, which turn out to be some projections of the gravitational field equations. Herein, building on that work, we extend this formalism to the full gravitational field equations (Einstein equations with sources plus the algebraic Bianchi identities); we show that they can be decomposed in a set of equations involving only the sources, the gravitoelectric $\mathbb{E}_{\alpha\beta}$ and gravitomagnetic $\mathbb{H}_{\alpha\beta}$ tidal tensors, plus a third spatial tensor $\mathbb{F}_{\alpha\beta}$, introduced by Bel [5], which has no electromagnetic analogue. More precisely, making a full 1+3 covariant splitting of the gravitational field equations, one obtains a subset of four equations (involving the temporal part of the curvature), which are formally similar to the Maxwell equations written in this formalism (as found in [1]), *both being algebraic equations involving only tidal tensors and sources*, plus an additional pair of equations involving the purely spatial curvature (encoded in $\mathbb{F}_{\alpha\beta}$), which have no electromagnetic counterpart. It is discussed how this approach, by bringing together the two theories into a single formalism, is especially suited for a transparent comparison between the two interactions, which reduces to comparing the tidal tensors of both sides (which is straightforward in this framework). Fundamental differences between the two interactions are encoded in the symmetries and time projections of the tidal tensors (as already pointed out in [1]); herein we explore the consequences in terms of the worldline deviation of (monopole) test particles; and in the companion paper [6], in terms of the dynamics of spinning multipole test particles.

In Sec. 3, we discuss another exact gravito-electromagnetic analogy, the one drawing a parallelism between spatial inertial forces – described by the gravitoelectromagnetic (GEM) fields – and the electromagnetic fields. GEM fields are best known from linearized theory, e.g. [7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17]. Less well known are the exact analogies based on inertial fields that arise in the splitting of spacetime into time+space with respect to two preferred congruences of observers: time-like Killing congruences in stationary spacetimes (usually called the "1+3" splitting), introduced by Landau-Lifshitz [18], and further worked out by other authors [19, 20, 21, 22, 23, 24], which leads to the so-called "quasi-Maxwell" analogy; and hypersurface orthogonal observers, e.g. [25, 26] (usually called the "3+1" splitting, mostly used also in stationary spacetimes), which also leads to a GEM analogy based on exact equations, albeit not as close as in the former case. Lesser known still is the existence of an exact formulation applying to *arbitrary observer congruences* in *arbitrary spacetimes* [27, 28], which performs a splitting in terms of a time direction parallel to the worldlines of arbitrary observers, and the local rest spaces orthogonal to them. In this latter framework, a close gravito-electromagnetic analogy (in the sense of a one to one correspondence) is, for most effects, only recovered in the case of rigid observer congruences in stationary spacetimes (exceptions are the case of the spin precession, and the hidden momentum of a spinning particle, discussed in [6]). Herein we reformulate this approach in a slightly more general form, in the sense that we use an arbitrary frame, which splits the gravitomagnetic field $\vec{H}$ into its two constituent parts, of different mathematical origin: the vorticity $\vec{\omega}$ of the observer congruence and the rotation $\vec{\Omega}$ of the local tetrads (associated to each local observer) relative to local Fermi-Walker transport. This degree of generality allows the description of the inertial forces of any frame, encompassing the many different gravitomagnetic fields that have been defined in the literature; it is also of use both in the companion paper [6], and in [29]. An important result of this approach is an exact expression for the geodesic equation, written in the language of GEM fields, that is fully general (valid for arbitrary fields, and formulated in terms of an arbitrary frame); the approximate GEM expressions from the post-Newtonian and linearized theory in the literature, as well as the exact expressions of the quasi-Maxwell formalism for stationary spacetimes, are just special cases of this general equation. The exact, mathematically rigorous formulation yields a powerful formalism with a very broad spectrum of applications, and an adequate account of some subtleties involved, which are overlooked in the more common linear approaches. Indeed, the inertial GEM fields in this formulation — the gravitoelectric field $\vec{G}$, which is but *minus* the acceleration of the congruence observers, and the gravitomagnetic field $\vec{H} = \vec{\omega} + \vec{\Omega}$ — can be regarded as the general, exact form of the GEM 3-vectors fields of the usual linearized theory, which (in the way they a are usually presented) are somewhat naively derived from the temporal components of the metric tensor (drawing a parallelism with the electromagnetic potentials), without making apparent their status as artifacts of the reference frame, and in particular their relation with the kinematical quantities associated to the observer's congruence. Problems which in the approximate descriptions usually end up being treated superficially, overlooking the complicated underlying issue, are the effects concerning gyroscope "precession" relative to the "distant stars" (such as the Lense-Thirring and the geodetic precessions) — the question arising: how can one talk about the "precession" of a local gyroscope relative to the distant stars, as it amounts to comparing systems of vectors at different points in a curved spacetime? Such comparison can be done in a certain class of spacetimes, and the mathematical basis for it is discussed in Secs. 3.1 and 3.3.

Drawing a parallelism with what is done in Sec. 2, we express the Maxwell equations, and the full Einstein equations and algebraic Bianchi identities, in this formalism. Again, a set of four equations are produced which exhibit many similarities with their electromagnetic counterparts, plus two additional equations which have no electromagnetic analogue. We show that an exact GEM analogy relating the force on a gyroscope with the force on a magnetic dipole exists also in this formalism. In an earlier work by one of the authors [20], the force exerted on a gyroscope, at rest with respect to a rigid frame in a stationary spacetime, was written in terms of the GEM fields associated to that frame; herein we obtain the electromagnetic counterpart, the force exerted on a magnetic dipole at rest in a rigid, arbitrarily accelerated frame, in terms of the electromagnetic fields associated to that frame, and show there is a one to one correspondence with the gravitational analogue. And we clarify the relationship between the inertial fields and the tidal tensors of Sec. 2 (a particularly important result in the context of this work).

In Sec. 4 we discuss a special class of spacetimes admitting global rigid geodesic congruences, the "ultra-stationary" spacetimes. They have interesting properties in the context of GEM, which were discussed in [1]: they are exactly mapped, via the Klein-Gordon equation, into magnetic fields in curved spacetimes, and have a linear gravitomagnetic tidal tensor, matching the electromagnetic analogue. Herein we revisit those spacetimes in the framework of the GEM inertial fields of Sec. 3, which sheds light on their properties: these metrics are characterized by a vanishing gravitoelectric field and a non-vanishing gravitomagnetic field which is *linear in the metric tensor*; these two properties together explain the above mentioned exact mapping, and the linearity of the gravitomagnetic tidal tensor. We also interpret the non-vanishing of the gravitoelectric tidal tensor, a question left open in [1].

In Sec. 5 we explain the relation between the exact approach based in the inertial GEM fields of Sec. 3, and the popular gravito-electromagnetic analogy based on linearized theory, e.g. [7, 8, 10, 9]. The latter is obtained as a special limit of the exact equations of the former. Taking this route gives, as explained above, a clearer account of the physical meaning of the GEM fields and other kinematical quantities involved. It is also a procedure for obtaining the field equations in terms of (physically meaningful) GEM fields that does not rely on choosing the harmonic gauge condition and its inherent subtleties (which have been posing some difficulties in the literature, see e.g. [9, 30, 10, 2]). The usual expression for the force on a gyroscope in the literature (e.g. [13, 11, 7]) is also seen to be a (very) limiting case of the exact equation given in the tidal tensor formalism of Sec. 2.

In Sec. 6 we discuss the analogy between the electric ($\mathcal{E}_{\alpha\beta}$) and magnetic ($\mathcal{H}_{\alpha\beta}$) parts of the Weyl tensor, and the electric (E^{α}) and magnetic (B^{α}) fields, e.g. [31, 32, 33, 34, 35, 36, 37, 38, 39, 40, 41]. As already discussed in [2], this is a purely formal analogy, as it deals with tidal tensors, in the gravitational side, and vector fields (not tidal tensors, which are one order higher in differentiation) in electromagnetism. One of the major results of this approach is that the differential Bianchi identities (the higher order field equations), when expressed in terms of $\mathcal{E}_{\alpha\beta}$ and $\mathcal{H}_{\alpha\beta}$, exhibit some formal similarities with the Maxwell equations written in terms of E^{α}, B^{α}, measured with respect to an arbitrarily shearing, rotating and expanding congruence. In the case of vacuum, in the linear regime, they become Matte's equations — a set of equations formally similar to Maxwell's equations in a Lorentz frame, only with the gravitational tidal tensors ($\mathcal{E}_{ij} = \mathbb{E}_{ij}$, $\mathcal{H}_{ij} = \mathbb{H}_{ij}$) in place of the electromagnetic fields. We arrive at a suggestive interpretation of gravitational radiation as a pair of traveling orthogonal tidal tensors, propagating by mutually inducing each other (just like $\vec{E}$ and $\vec{B}$ in the electromagnetic waves), and whose effects on test particles are readily established via the tidal tensor approach of Sec. 2. This analogy has been used to address the problem of the definition of local quantities for the gravitational field (the "super-energy" scalar, and the "super-Poynting" vector) analogous to the electromagnetic field energy and Poynting vector, having motivated the definition of the Bel-Robinson tensor (the so-called "super-energy" tensor). The viewpoint that gravitational waves should be thought of as carrying superenergy fits well with the interpretation above, which is briefly discussed.

1.1 Notation and conventions

1. *Signature and signs.* We use the signature $-+++$; $\epsilon_{\alpha\beta\sigma\gamma} \equiv \sqrt{-g}[\alpha\beta\gamma\delta]$ denotes the Levi-Civita tensor, and we follow the orientation $[1230] = 1$ (i.e., in flat spacetime $\epsilon_{1230} = 1$). $\epsilon_{ijk} \equiv \epsilon_{ijk0}$ is the 3-D alternating tensor. $\star$ denotes the Hodge dual.

2. *Time and space projectors.* $(\top^u)^\alpha_{\ \beta} \equiv -u^\alpha u_\beta$, $(h^u)^\alpha_{\ \beta} \equiv u^\alpha u_\beta + g^\alpha_{\ \beta}$ are, respectively, the projectors parallel and orthogonal to a unit time-like vector u^α; may be interpreted as the time and space projectors in the local rest frame of an observer of 4-velocity u^α. $\langle\alpha\rangle$ denotes the index of a spatially projected tensor: $A^{\langle\alpha\rangle\beta\ldots} \equiv (h^u)^\alpha_{\ \beta} A^{\mu\beta\ldots}$.

3. $\rho_c = -j^\alpha u_\alpha$ and j^α are, respectively, the charge density and current 4-vector; $\rho = T_{\alpha\beta} u^\alpha u^\beta$ and $J^\alpha = -T^\alpha_{\ \beta} u^\beta$ are the mass/energy density and current (quantities measured by the observer of 4-velocity u^α); $T_{\alpha\beta} \equiv$ energy-momentum tensor.

4. $S^\alpha \equiv$ spin 4-vector; $\mu^\alpha \equiv$ magnetic dipole moment; defined such that their components in the particle's proper frame are $S^\alpha = (0, \vec{S})$, $\vec{\mu} = (0, \vec{\mu})$. For their precise definition in terms of moments of $T^{\alpha\beta}$ and j^α, see [6].

5. *Tensors resulting from a measurement process.* $(A^u)^{\alpha_1 .. \alpha_n}$ denotes the tensor $\mathbf{A}$ as measured by an observer $\mathcal{O}(u)$ of 4-velocity u^α. For example, $(E^u)^\alpha \equiv F^\alpha_{\ \beta} u^\beta$, $(E^u)_{\alpha\beta} \equiv F_{\alpha\gamma;\beta} u^\gamma$ and $(\mathbb{E}^u)_{\alpha\beta} \equiv R_{\alpha\nu\beta\nu} u^\nu u^\mu$ denote, respectively, the electric field, electric tidal tensor, and gravitoelectric tidal tensor as measured by $\mathcal{O}(u)$. Analogous forms apply to their magnetic/gravitomagnetic counterparts.
   For 3-vectors we use notation $\vec{A}(u)$; for example, $\vec{E}(u)$ denotes the electric 3-vector field as measured by $\mathcal{O}(u)$ (i.e., the space part of $(E^u)^\alpha$, *written in a frame* where $u^i = 0$). Often we drop the superscript (e.g. $(E^U)^\alpha \equiv E^\alpha$), or the argument of the 3-vector: $\vec{E}(U) \equiv \vec{E}$, when the meaning is clear.

6. *Electromagnetic field.* The Maxwell tensor $F^{\alpha\beta}$ and its Hodge dual $\star F^{\alpha\beta} \equiv \epsilon_{\mu\nu}^{\ \ \alpha\beta} F_{\mu\nu}/2$ decompose in terms of the electric $(E^u)^\alpha \equiv F^\alpha_{\ \beta} u^\beta$ and magnetic $(B^u)^\alpha \equiv \star F^\alpha_{\ \beta} u^\beta$ fields measured by an observer of 4-velocity u^α as

$$F_{\alpha\beta} = 2u_{[\alpha}(E^u)_{\beta]} + \epsilon_{\alpha\beta\gamma\delta} u^\delta (B^u)^\gamma \quad \text{(a)} \qquad \star F_{\alpha\beta} = 2u_{[\alpha}(B^u)_{\beta]} - \epsilon_{\alpha\beta\gamma\sigma} u^\sigma (E^u)^\gamma \quad \text{(b)} \quad (1)$$

2 The gravito-electromagnetic analogy based on tidal tensors

The rationale behind the tidal tensor gravito-electromagnetic analogy is to make a comparison between the two interactions based on physical forces present in both theories. The electromagnetic

Lorentz force has no physical counterpart in gravity, as monopole point test particles in a gravitational field move along geodesics, without any force being exerted on them. In this sense, the analogy drawn in Sec. 3.2 between Eqs. (54) and (52) is a comparison of a physical electromagnetic force to an artifact of the reference frame. Tidal forces, by their turn, are covariantly present in both theories, and their mathematical description in terms of objects called "tidal tensors" is the basis of this approach. Tidal forces manifest themselves in essentially two basic effects: the relative acceleration of two nearby monopole test particles, and in the net force exerted on dipoles. The notions of multipole moments arise from a description of the *test* bodies in terms of the fields they *would* produce. In electromagnetism they are the multipole expansions of the 4-current density vector $j^\alpha = (\rho_c, \vec{j})$, rigorously established in [48], and well known in textbooks as the moments of the charge and current densities. In gravity they are are the moments of the energy momentum tensor $T_{\alpha\beta}$, the so called [47] "gravitational skeleton", of which only the moments of the 4-current density $J^\alpha = -T^{\alpha\beta}U_\beta$ have an electromagnetic counterpart. Monopole particles in the context of electromagnetism are those whose only non-vanishing moment is the total charge; dipole particles are particles with non-vanishing electric and magnetic dipole moments (i.e., respectively the dipole moments of ρ_c and $\vec{j}$); see [48] and companion paper [6] for precise definitions of these moments. Monopole particles in gravity are particles whose only non-vanishing moment of $T^{\alpha\beta}$ is the mass, and correspond to the usual notion of *point test particle*, which moves along geodesics. There is no gravitational analogue of the intrinsic electric dipole, as there are no negative masses; but there is an analogue of the magnetic dipole moment, which is the "intrinsic" angular momentum (i.e. the angular momentum about the particle's center of mass), usually dubbed spin vector/tensor. Sometimes we will also call it, for obvious reasons, the "gravitomagnetic dipole moment". A particle possessing only pole-dipole gravitational moments corresponds to the notion of an ideal gyroscope. We thus have two physically analogous effects suited to compare gravitational and electromagnetic tidal forces: worldline deviation of nearby monopole test particles, and the force exerted on magnetic dipoles/gyroscopes. An exact gravito-electromagnetic analogy, summarized in Table 1, emerges from this comparison.

Eqs. (1.1) are the worldline deviations for nearby test particles with the same[1] tangent vector (and the same ratio charge/mass in the electromagnetic case), separated by the infinitesimal vector δx^α. They tell us that the so-called (e.g. [45]) electric part of the Riemann tensor $\mathbb{E}^\alpha{}_\beta \equiv R^\alpha{}_{\mu\beta\nu}U^\mu U^\nu$ plays in the geodesic deviation equation (1.1b) the same physical role as the tensor $E_{\alpha\beta} \equiv F_{\alpha\gamma;\beta}U^\gamma$ in the electromagnetic worldline deviation (1.1a): in a gravitational field the "relative acceleration" between two nearby test particles, with the *same 4-velocity* U^α, is given by a contraction of $\mathbb{E}_{\alpha\beta}$ with the separation vector δx^β; just like in an electromagnetic field the "relative acceleration" between two nearby charged particles (with the same U^α and ratio q/m) is given by a contraction of the electric tidal tensor $E_{\alpha\beta}$ with δx^α. $E_{\alpha\beta}$ measures the tidal effects produced by the electric field

[1] We want to emphasize this point, which, even today, is not clear in the literature. Eqs. (1.1) apply to the *instant* where the two particles have the same (or infinitesimally close, in the gravitational case) tangent vector. When the particles have arbitrary velocities, both in electromagnetism and gravity, their relative acceleration is not given by a simple contraction of a tidal tensor with a separation vector; the equations include more terms, see [1, 2, 54]. There is however a difference: whereas Eq. (1.1a) requires strictly $\delta\mathbf{U} = \mathbf{U}_2 - \mathbf{U}_1 = 0$, see [115, 1, 2], Eq. (1.1b) allows for an infinitesimal $\delta U \propto \delta x$, as can be seen from Eq. (6) of [54]. That means that (1.1b) holds for *infinitesimally close* curves belonging to an arbitrary geodesic *congruence* (it is in this sense that in e.g. [53, 114] δU is portrayed as "arbitrary" — it is understood to be infinitesimal therein, as those treatments deal with congruences of curves).

$E^\alpha = F^\alpha{}_\gamma U^\gamma$ *as measured by the test particle* of 4-velocity U^α. We can define it as a covariant

Table 1: The gravito-electromagnetic analogy based on tidal tensors.

Electromagnetism		Gravity	
Worldline deviation: $\frac{D^2\delta x^\alpha}{d\tau^2} = \frac{q}{m}E^\alpha{}_\beta\delta x^\beta,\ \ E^\alpha{}_\beta \equiv F^\alpha{}_{\mu;\beta}U^\mu$	(1.1a)	Geodesic deviation: $\frac{D^2\delta x^\alpha}{d\tau^2} = -\mathbb{E}^\alpha{}_\beta\delta x^\beta,\ \ \mathbb{E}^\alpha{}_\beta \equiv R^\alpha{}_{\mu\beta\nu}U^\mu U^\nu$	(1.1b)
Force on magnetic dipole: $F^\beta_{EM} = B_\alpha{}^\beta\mu^\alpha,\ \ B^\alpha{}_\beta \equiv \star F^\alpha{}_{\mu;\beta}U^\mu$	(1.2a)	Force on gyroscope: $F^\beta_G = -\mathbb{H}_\alpha{}^\beta S^\alpha,\ \ \mathbb{H}^\alpha{}_\beta \equiv \star R^\alpha{}_{\mu\beta\nu}U^\mu U^\nu$	(1.2b)
Differential precession of magnetic dipoles: $\delta\Omega^i_{\mathrm{EM}} = -\sigma B^i{}_\beta\delta x^\beta$	(1.3a)	Differential precession of gyroscopes: $\delta\Omega^i_{\mathrm{G}} = \mathbb{H}^i{}_\beta\delta x^\beta$	(1.3b)
Maxwell Source Equations $F^{\alpha\beta}{}_{;\beta} = 4\pi J^\beta$		Einstein Equations $R_{\mu\nu} = 8\pi\left(T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T^\alpha{}_\alpha\right)$	
• Time Projection: $E^\alpha{}_\alpha = 4\pi\rho_c$	(1.4a)	• Time-Time Projection: $\mathbb{E}^\alpha{}_\alpha = 4\pi\left(2\rho + T^\alpha{}_\alpha\right)$	(1.4b)
• Space Projection: $B_{[\alpha\beta]} = \frac{1}{2}\star F_{\alpha\beta;\gamma}U^\gamma - 2\pi\epsilon_{\alpha\beta\sigma\gamma}j^\sigma U^\gamma$	(1.5a)	• Time-Space Projection: $\mathbb{H}_{[\alpha\beta]} = -4\pi\epsilon_{\alpha\beta\sigma\gamma}J^\sigma U^\gamma$	(1.5b)
No electromagnetic analogue		• Space-Space Projection: $\mathbb{F}^\alpha{}_\beta + \mathbb{E}^\alpha{}_\beta - \mathbb{F}^\sigma{}_\sigma h^\alpha{}_\beta = 8\pi\left[\frac{1}{2}T^\gamma{}_\gamma h^\alpha{}_\beta - T^{\langle\alpha\rangle}{}_{\langle\beta\rangle}\right]$	(1.6)
Bianchi Identity $\star F^{\alpha\beta}{}_{;\beta} = 0 \quad (\Leftrightarrow F_{[\alpha\beta;\gamma]} = 0\ )$		Algebraic Bianchi Identity $\star R^{\gamma\alpha}{}_{\gamma\beta} = 0 \quad (\Leftrightarrow R_{[\alpha\beta\gamma]\delta} = 0)$	
• Time Projection: $B^\alpha{}_\alpha = 0$	(1.7a)	• Time-Time (or Space-Space) Proj.: $\mathbb{H}^\alpha{}_\alpha = 0$	(1.7b)
• Space Projection: $E_{[\alpha\beta]} = \frac{1}{2}F_{\alpha\beta;\gamma}U^\gamma$	(1.8a)	• Space-Time Projection: $\mathbb{E}_{[\alpha\beta]} = 0$	(1.8b)
No electromagnetic analogue		• Time-Space Projection: $\mathbb{F}_{[\alpha\beta]} = 0$	

derivative of the electric field as measured in the inertial frame momentarily comoving with the particle: $E_{\alpha\beta} = E_{\alpha;\beta}|_{U=const}$. Hence we dub it the "electric tidal tensor", and its gravitational counterpart the "gravitoelectric tidal tensor".

Eqs (1.2) are, respectively, the electromagnetic force on a magnetic dipole [6], and the Mathisson-Papapetrou equation [47, 49] for the gravitational force exerted on a gyroscope (supplemented by the Mathisson-Pirani spin condition [47, 50]; see [6] for more details). They tell us that the magnetic part of the Riemann tensor $\mathbb{H}^\alpha{}_\beta \equiv \star R^\alpha{}_{\mu\beta\nu}U^\mu U^\nu$ plays in the gravitational force (1.2b) the same physical role as the tensor $B_{\alpha\beta} \equiv \star F_{\alpha\gamma;\beta}U^{\gamma}$ in the electromagnetic force (1.2a): the gravitational force exerted on a spinning particle of 4-velocity U^{α} is exactly given by a contraction of $\mathbb{H}_{\alpha\beta}$ with the spin vector S^{α} (the "gravitomagnetic dipole moment"), just like its electromagnetic counterpart is exactly given by a contraction of the magnetic tidal tensor $B_{\alpha\beta}$ with the magnetic dipole moment μ^{α}. $B_{\alpha\beta}$ measures the tidal effects produced by the magnetic field $B^{\alpha} = \star F^{\alpha}_{\ \gamma}U^{\gamma}$ *as measured by the particle* of 4-velocity U^{α}; for this reason we dub it the "magnetic tidal tensor", and its gravitational analogue $\mathbb{H}_{\alpha\beta}$ the "gravitomagnetic tidal tensor".

2.1 Tidal tensor formulation of Maxwell and Einstein equations

Taking time and space projections, Maxwell's and Einstein's equations can be expressed in tidal tensor formalism; that makes explicit a striking aspect of the analogy: Maxwell's equations (the source equations plus the Bianchi identity) may be cast as a set of *algebraic* equations involving only tidal tensors and source terms (the charge current 4-vector); and the gravitational field equations (Einstein's source equations plus the algebraic Bianchi identity) as a set of five independent equations, consisting of two parts: i) a subset of four equations formally very similar to Maxwell's, that are likewise algebraic equations involving only tidal tensors and sources (the mass-energy current vector), and ii) a fifth equation involving the space part of $T^{\alpha\beta}$ and a spatial rank 2 tensor which has no electromagnetic analogue. This is what we are going to show next. For that, we first introduce the time and space projectors with respect to a unit time-like vector U^{α} (i.e., the projectors parallel and orthogonal to U^{α}):

$$\top^{\alpha}_{\ \beta} \equiv (\top^{U})^{\alpha}_{\ \beta} = -U^{\alpha}U_{\beta}; \qquad h^{\alpha}_{\ \beta} \equiv (h^{U})^{\alpha}_{\ \beta} = U^{\alpha}U_{\beta} + \delta^{\alpha}_{\ \beta}. \tag{2}$$

A vector A^{α} can be split in its time and space projections with respect to U^{α}; and an arbitrary rank n tensor can be completely decomposed taking time and space projections in each of its indices (e.g. [27]):

$$A^{\alpha} = \top^{\alpha}_{\ \beta}A^{\beta} + h^{\alpha}_{\ \beta}A^{\beta}; \qquad A^{\alpha_1\ldots\alpha_n} = \left(\top^{\alpha_1}_{\ \ \beta_1} + h^{\alpha_1}_{\ \ \beta_1}\right)\ldots\left(\top^{\alpha_n}_{\ \ \beta_n} + h^{\alpha_n}_{\ \ \beta_n}\right)A^{\beta_1\ldots\beta_n}. \tag{3}$$

Instead of using $h^{\mu}_{\ \sigma}$, one can also, if convenient, spatially project an index of a tensor $A^{\sigma\cdots}$ contracting it with the spatial 3-form $\epsilon_{\alpha\beta\sigma\gamma}U^{\gamma}$; for instance, for the case of vector A^{σ}, one obtains the spatial 2-form $\epsilon_{\alpha\beta\sigma\gamma}U^{\gamma}A^{\sigma} = \star A_{\alpha\beta\gamma}U^{\gamma}$, which contains precisely the same information as the spatial vector $A^{\mu}h^{\sigma}_{\ \mu} \equiv A^{\langle\sigma\rangle}$ (the former is the spatial dual of the latter). New contraction with $\epsilon^{\alpha\beta}_{\ \ \mu\nu}U^{\nu}$ yields $A^{\langle\sigma\rangle}$ again. Indeed we may write

$$h^{\mu}_{\sigma} = \frac{1}{2}\epsilon^{\alpha\beta\mu\nu}U_{\nu}\epsilon_{\alpha\beta\sigma\gamma}U^{\gamma}.$$

Another very useful relation is the following. The space projection $h^{\mu}_{\ \alpha}h^{\nu}_{\ \beta}F_{\mu\nu} \equiv F_{\langle\alpha\rangle\langle\beta\rangle}$ of a 2-form $F_{\alpha\beta} = F_{[\alpha\beta]}$ is equivalent to the tensor $\epsilon_{\mu\nu\alpha\beta}F^{\alpha\beta}U^{\nu} = 2\star F_{\mu\nu}U^{\nu}$ (i.e., spatially projecting $F_{\alpha\beta}$ is equivalent to time-projecting its Hodge dual). We have:

$$F_{\langle\alpha\rangle\langle\beta\rangle} = \frac{1}{2}\epsilon_{\mu\alpha\beta\lambda}U^{\lambda}\epsilon^{\mu}_{\ \nu\sigma\delta}U^{\nu}F^{\sigma\delta} = \epsilon_{\mu\alpha\beta\lambda}U^{\lambda}\star F^{\mu}_{\ \nu}U^{\nu}. \tag{4}$$

Now let $F_{\gamma_1\ldots\gamma_n\alpha\beta\delta_1\ldots\delta_m} = F_{\gamma_1\ldots\gamma_n[\alpha\beta]\delta_1\ldots\delta_m}$, be some tensor antisymmetric in the pair α, β; an equality similar to the one above applies:

$$F_{\gamma_1\ldots\gamma_n\langle\alpha\rangle\langle\beta\rangle\delta_1\ldots\delta_m} = \frac{1}{2}\epsilon_{\mu\alpha\beta\lambda}U^{\lambda}\epsilon^{\mu}_{\ \nu\sigma\delta}U^{\nu}F_{\gamma_1\ldots\gamma_n}{}^{\sigma\delta}{}_{\delta_1\ldots\delta_m}\,. \tag{5}$$

2.1.1 Maxwell's equations

Maxwell's equations are given in tensor form by the pair of equations:

$$F^{\alpha\beta}{}_{;\beta} = 4\pi j^{\alpha} \quad \text{(a);} \qquad \star F^{\alpha\beta}{}_{;\beta} = 0 \quad \text{(b).} \tag{6}$$

Here (6a) are the Maxwell source equations, and (6b) are the source-free equations, equivalent to $F_{[\alpha\beta;\gamma]} = 0$, and commonly called the Bianchi identity for $F_{\alpha\beta}$. These equations can be expressed in terms of tidal tensors using the decompositions

$$F_{\alpha\beta;\gamma} = 2U_{[\alpha}E_{\beta]\gamma} + \epsilon_{\alpha\beta\mu\sigma}U^{\sigma}B^{\mu}{}_{\gamma} \; ; \tag{7}$$

$$\star F_{\alpha\beta;\gamma} = 2U_{[\alpha}B_{\beta]\gamma} - \epsilon_{\alpha\beta\mu\sigma}U^{\sigma}E^{\mu}{}_{\gamma} \; . \tag{8}$$

These expressions are obtained decomposing the tensors $F_{\alpha\beta;\gamma}$ and $\star F_{\alpha\beta;\gamma}$ in their time and space projections in the first two indices, using Eq. (4) to project spatially. Taking the time projection of (6a), we obtain Eq. (1.4a) of Table 1; taking the space projection, by contracting with the spatial 3-form $\epsilon_{\mu\nu\alpha\sigma}U^{\sigma}$, yields Eq. (1.5a). The same procedure applied to Eq. (6b) yields Eqs. (1.7a) and (1.8a) as time and space projections, respectively.

Hence, in this formalism, Maxwell's equations are cast as the equations for the traces and antisymmetric parts of the electromagnetic tidal tensors; and they involve only tidal tensors and sources, which is easily seen substituting the decompositions (7)-(8) in Eqs (1.8a) and (1.5a), leading to the equivalent set:

$$E^{\alpha}{}_{\alpha} = 4\pi\rho_c \; ; \tag{9}$$

$$E_{[\alpha\beta]} = U_{[\alpha}E_{\beta]\gamma}U^{\gamma} + \frac{1}{2}\epsilon_{\alpha\beta\mu\sigma}U^{\sigma}B^{\mu\gamma}U_{\gamma} \; ; \tag{10}$$

$$B^{\alpha}{}_{\alpha} = 0 \; ; \tag{11}$$

$$B_{[\alpha\beta]} = U_{[\alpha}B_{\beta]\gamma}U^{\gamma} - \frac{1}{2}\epsilon_{\alpha\beta\mu\sigma}U^{\sigma}E^{\mu\gamma}U_{\gamma} - 2\pi\epsilon_{\alpha\beta\sigma\gamma}j^{\sigma}U^{\gamma} \; . \tag{12}$$

Here j^{α} denotes the current 4-vector, and $\rho_c \equiv -j^{\alpha}U_{\alpha}$ the charge density as measured by an observer of 4-velocity U^{α}. The pair of Eqs. (10) and (12) can be condensed in the equivalent pair

$$\epsilon^{\beta\gamma}{}_{\alpha\delta}U^{\delta}E_{[\gamma\beta]} = -B_{\alpha\beta}U^{\beta}; \quad \text{(a)} \qquad \epsilon^{\beta\gamma}{}_{\alpha\delta}U^{\delta}B_{[\gamma\beta]} = E_{\alpha\beta}U^{\beta} + 4\pi j_{\alpha}. \quad \text{(b)} \tag{13}$$

In a Lorentz frame in flat spacetime, since $U^{\alpha}{}_{;\beta} = U^{\alpha}{}_{,\beta} = 0$, we have $E_{\gamma\beta} = E_{\gamma;\beta}$, $B_{\gamma\beta} = B_{\gamma;\beta}$; and (using $U^{\alpha} = \delta^{\alpha}_{0}$) Eqs. (13) can be written in the familiar vector forms $\nabla \times \vec{E} = -\partial\vec{B}/\partial t$ and $\nabla \times \vec{B} = \partial\vec{E}/\partial t + 4\pi\vec{j}$, respectively. Likewise, Eqs. (9) and (11) reduce in this frame to the familiar forms $\nabla \cdot \vec{E} = 4\pi\rho_c$ and $\nabla \cdot \vec{B} = 0$, respectively.

2.1.2 Einstein's equations

Equations (14a) below are the Einstein source equations for the gravitational field; Eqs (14b) are the algebraic Bianchi identity, equivalent to $R_{[\alpha\beta\gamma]\delta} = 0$:

$$R^{\gamma}{}_{\alpha\gamma\beta} \equiv R_{\alpha\beta} = 8\pi\left(T_{\alpha\beta} - \frac{1}{2}g_{\alpha\beta}T^{\gamma}{}_{\gamma}\right) \quad \text{(a);} \qquad \star R^{\gamma\alpha}{}_{\gamma\beta} = 0 \quad \text{(b).} \tag{14}$$

In order to express these equations in the tidal tensor formalism we will decompose the Riemann tensor in its time and space projections (in each of its indices) with respect to a unit time-like vector U^{α}, $R^{\alpha\beta\gamma\delta} = \left(\top^{\alpha}{}_{\rho} + h^{\alpha}_{\ \rho}\right) ... \left(\top^{\delta}{}_{\sigma} + h^{\delta}_{\ \sigma}\right) R^{\rho..\sigma}$, cf. Eq. (3); we obtain[2]

$$\begin{aligned} R^{\alpha\beta}{}_{\gamma\delta} &= 4\mathbb{E}^{[\alpha}{}_{[\gamma}U_{\delta]}U^{\beta]} + 2\left\{\epsilon^{\mu\chi}{}_{\gamma\delta}U_{\chi}\mathbb{H}_{\mu}{}^{[\beta}U^{\alpha]} + \epsilon^{\mu\alpha\beta\chi}U_{\chi}\mathbb{H}_{\mu[\delta}U_{\gamma]}\right\} \\ &\quad + \epsilon^{\alpha\beta\phi\psi}U_{\psi}\epsilon^{\mu\nu}{}_{\gamma\delta}U_{\nu}\mathbb{F}_{\phi\mu}, \end{aligned} \tag{15}$$

where we made use of the identity (4) to project spatially an antisymmetric pair of indices, noting that $R_{\alpha\beta\gamma\delta}$ can be regarded as a double 2-form. This equation tells us that the Riemann tensor decomposes, with respect to U^{α}, in three *spatial* tensors: the gravitoelectric tidal tensor $\mathbb{E}_{\alpha\beta}$, the gravitomagnetic tidal tensor $\mathbb{H}_{\alpha\beta}$, plus a third tensor

$$\mathbb{F}_{\alpha\beta} \equiv \star R \star_{\alpha\gamma\beta\delta} U^{\gamma}U^{\delta} = \epsilon^{\mu\nu}{}_{\alpha\gamma}\epsilon^{\lambda\tau}{}_{\beta\delta}R_{\mu\nu\lambda\tau}U^{\gamma}U^{\delta} \ ,$$

introduced by Bel [5], which encodes the purely spatial curvature with respect to U^{α}, and has no electromagnetic analogue. In order to obtain Eq. (15), we made use of the symmetries $R^{\alpha\beta\gamma\delta} = R^{[\alpha\beta][\gamma\delta]}$, and in the case of the terms involving $\mathbb{H}_{\alpha\beta}$ (and *only for these terms*) we also assumed the pair exchange symmetry $R^{\alpha\beta\gamma\delta} = R^{\gamma\delta\alpha\beta}$. $\mathbb{E}_{\alpha\beta}$ and $\mathbb{F}_{\alpha\beta}$ are symmetric (and spatial), and therefore have 6 independent components each; $\mathbb{H}_{\alpha\beta}$ is traceless (and spatial), and so has 8 independent components. Therefore these three tensors together encode the 20 independent components of the Riemann tensor.

In what follows we will need also the Hodge dual, in the first two indices, of the decomposition (15):

$$\begin{aligned} \star R^{\alpha\beta}{}_{\gamma\delta} &= 2\epsilon^{\alpha\beta}{}_{\lambda\tau}\mathbb{E}^{\lambda}{}_{[\gamma}U_{\delta]}U^{\tau} + 4U^{[\alpha}\mathbb{H}^{\beta]}{}_{[\delta}U_{\gamma]} + \epsilon^{\alpha\beta}{}_{\lambda\tau}\epsilon^{\mu\nu}{}_{\gamma\delta}U_{\nu}\mathbb{H}_{\mu}{}^{\tau}U^{\lambda} \\ &\quad -2U^{[\alpha}\mathbb{F}^{\beta]}{}_{\mu}\epsilon^{\mu\nu}{}_{\gamma\delta}U_{\nu} \ . \end{aligned} \tag{16}$$

The Ricci tensor $R^{\beta}_{\ \delta} = R^{\alpha\beta}{}_{\alpha\delta}$ and the tensor $\star R^{\mu}{}_{\alpha\mu\beta}$ follow as:

$$R^{\beta}_{\ \delta} = -\epsilon^{\alpha\beta\mu\nu}\mathbb{H}_{\mu\alpha}U_{\delta}U_{\nu} - \epsilon_{\alpha\delta\mu\nu}\mathbb{H}^{\mu\alpha}U^{\beta}U^{\nu} - \mathbb{F}^{\beta}_{\ \delta} - \mathbb{E}^{\beta}_{\ \delta} + \mathbb{E}^{\sigma}_{\ \sigma}U^{\beta}U_{\delta} + \mathbb{F}^{\sigma}_{\ \sigma}h^{\beta}_{\ \delta}, \tag{17}$$

$$\star R^{\alpha\beta}{}_{\alpha\delta} = \epsilon^{\alpha\beta}{}_{\lambda\tau}\mathbb{E}^{\lambda}{}_{\alpha}U_{\delta}U^{\tau} - \delta^{\beta}_{\ \delta}\mathbb{H}^{\alpha}_{\ \alpha} + U^{\beta}\mathbb{F}^{\alpha}{}_{\mu}\epsilon^{\mu\nu}{}_{\alpha\delta}U_{\nu}. \tag{18}$$

Substituting (17) in (14a), and (18) in (14b), we obtain Einstein's equations and the algebraic Bianchi identities in terms of the tensors $\mathbb{E}_{\alpha\beta}$, $\mathbb{H}_{\alpha\beta}$, $\mathbb{F}_{\alpha\beta}$. Now let us make the time-space splitting of these equations. Eq. (14a) is symmetric, hence it only has 3 non-trivial projections: time-time, time-space, and space-space. The time-time projection yields

$$\mathbb{E}^{\alpha}_{\ \alpha} = 4\pi\left(2\rho + T^{\alpha}_{\ \alpha}\right), \tag{19}$$

where $\rho \equiv T^{\alpha\beta}U_{\beta}U_{\alpha}$ denotes the mass-energy density as measured by an observer of 4-velocity U^{α}. Contraction of (17) and (14a) with the time-space projector $\top^{\theta}{}_{\beta}\epsilon^{\delta}{}_{\sigma\tau\gamma}U^{\gamma}$ yields:

$$\mathbb{H}_{[\sigma\tau]} = -4\pi\epsilon_{\lambda\sigma\tau\gamma}J^{\lambda}U^{\gamma}\,, \tag{20}$$

[2]The characterization of the Riemann tensor by these three spatial rank 2 tensors is known as the "Bel decomposition", even though the explicit decomposition (15) is not presented in any of Bel's papers (e.g. [5]). To the author's knowledge, an equivalent expression (Eq. (4.6) therein) can only be found at [104].

where $J^{\alpha} \equiv -T^{\alpha\beta}U_{\beta}$ is the mass/energy current as measured by an observer of 4-velocity U^{α}. The space-space projection yields:

$$\mathbb{F}^{\lambda}{}_{\theta} + \mathbb{E}^{\lambda}{}_{\theta} - \mathbb{F}^{\sigma}{}_{\sigma} h^{\lambda}{}_{\theta} = 8\pi \left[h^{\lambda}{}_{\theta} \frac{1}{2} T^{\alpha}{}_{\alpha} - T_{\langle\theta\rangle}^{\ \langle\lambda\rangle} \right]. \tag{21}$$

where $T_{\langle\theta\rangle}^{\ \langle\lambda\rangle} \equiv h^{\lambda}{}_{\delta} h^{\beta}{}_{\theta} T_{\beta}{}^{\delta}$.

Since the tensor $\star R^{\gamma\alpha}{}_{\gamma\beta}$ is not symmetric, Eq. (14b) seemingly splits into four parts: a time-time, time-space, space-time, and space-space projections. However, the time-time and space-space projections yield the same equation. Substituting decomposition (18) in Eq. (14b), and taking the time-time (or space-space), time-space, and space-time projections, yields, respectively:

$$\mathbb{H}^{\alpha}{}_{\alpha} = 0; \quad \text{(a)} \qquad \mathbb{F}_{[\alpha\beta]} = 0; \quad \text{(b)} \qquad \mathbb{E}_{[\alpha\beta]} = 0 \quad \text{(c)}. \tag{22}$$

Note however that Eqs. (19)-(22) are not a set of six *independent* equations (only five), as Eqs. (22b), (22c) and (21) are not independent; using the latter, together with (22b)/(22c), one can obtain the remaining one, (22c)/(22b).

The gravitational field equations are summarized and contrasted with their electromagnetic counterparts in Table 1. Eqs. (1.4b)-(1.5b), (1.7b)-(1.8b) are very similar in form to Maxwell Eqs. (1.4a)-(1.8a); they are their *physical* gravitational analogues, since both are the traces and antisymmetric parts of the tensors $\{E_{\alpha\beta},\ B_{\alpha\beta}\} \leftrightarrow \{\mathbb{E}_{\alpha\beta},\ \mathbb{H}_{\alpha\beta}\}$, which we know, from equations (1.1) and (1.2), to play analogous physical roles in the two theories. Note this interesting aspect of the analogy: if one replaces, in Eqs. (9)-(12), the electromagnetic tidal tensors ($E_{\alpha\beta}$ and $B_{\alpha\beta}$) by the gravitational ones ($\mathbb{E}_{\alpha\beta}$ and $\mathbb{H}_{\alpha\beta}$), and the charges by masses (i.e., density ρ_c and current j^{α} of charge by density ρ and current J^{α} of mass), one *almost* obtains Eqs. (1.4b)-(1.5b), (1.7b)-(1.8b), apart from a factor of 2 in the source term in (1.5b) and the difference in the source of Eq. (9). This happens because, since $\mathbb{E}_{\alpha\beta}$ and $\mathbb{H}_{\alpha\beta}$ are spatial tensors, all the contractions with U^{α} present in Eqs. (10) and (12) vanish. In the case of vacuum, the four gravitational equations which are analogous to Maxwell's are thus *exactly* obtained from the latter by simply replacing $\{E_{\alpha\beta}, B_{\alpha\beta}\} \to \{\mathbb{E}_{\alpha\beta}, \mathbb{H}_{\alpha\beta}\}$.

Eqs. (22b) and (21), involving $\mathbb{F}_{\alpha\beta}$, have no electromagnetic analogue. Eq. (21) involves also, as a source, the space-space part of the energy momentum tensor, $T^{\langle\alpha\rangle\langle\beta\rangle}$, which, unlike the energy current 4-vector $J^{\alpha} = -T^{\alpha\beta}U_{\beta}$ (analogous to the charge current 4-vector j^{α}), has no electromagnetic counterpart. It is worth discussing this equation in some detail. It has a fundamental difference[3] with respect to the other gravitational field equations in Table 1, and with their electromagnetic analogues: the latter are algebraic equations involving only the traces and antisymmetric parts of the tidal tensors (or of $\mathbb{F}_{\alpha\beta}$), plus the source terms; they impose no condition on the symmetric parts. In electromagnetism, this is what allows the field to be dynamical, and waves to exist (their tidal tensors are described, in an inertial frame, by Eqs. (156)-(157) below); were there additional independent algebraic equations for the traceless symmetric part of the tidal tensors, and these fields would be fixed. But Eq. (21), by contrast, is an equation for the symmetric parts of the tensors $\mathbb{E}_{\alpha\beta}$ and $\mathbb{F}_{\alpha\beta}$. It can be split into two parts. Taking the trace, and using (19), one obtains the source equation for $\mathbb{F}_{\alpha\beta}$:

$$\mathbb{F}^{\sigma}{}_{\sigma} = 8\pi\rho \ ; \tag{23}$$

[3] We thank João Penedones for drawing our attention to this point.

substituting back in (21) we get:

$$\mathbb{F}^{\alpha}_{\ \beta}+\mathbb{E}^{\alpha}_{\ \beta}=8\pi\left[h^{\alpha}_{\ \beta}\left(\frac{1}{2}T^{\gamma}_{\ \gamma}+\rho\right)-T^{\langle\alpha\rangle}_{\ \ \langle\beta\rangle}\right]. \tag{24}$$

This equation tells us that the tensor $\mathbb{F}^{\alpha}_{\ \beta}$ is not an extra (comparing with electrodynamics) *independent* object; given the sources and the gravitoelectric tidal tensor $\mathbb{E}_{\alpha\beta}$, $\mathbb{F}_{\alpha\beta}$ is completely determined by (24).

In vacuum ($T^{\alpha\beta}=0$, $j^{\alpha}=0$), the Riemann tensor becomes the Weyl tensor: $R_{\alpha\beta\gamma\delta}=C_{\alpha\beta\gamma\delta}$; due to the self duality property of the latter: $C_{\alpha\beta\gamma\delta}=-\star C\star_{\alpha\beta\gamma\delta}$, it follows that $\mathbb{F}_{\alpha\beta}=-\mathbb{E}_{\alpha\beta}$.

Eqs. in Table 1, have the status of constraints for the *tidal* fields. They are especially suited to compare the tidal dynamics (i.e., Eqs. (1.1) and (1.2)) of the two interactions, which is discussed in the next section. But they do not tell us about the dynamics of the fields *themselves*. To obtain dynamical field equations, one possible route is to take one step back and express the tidal tensors in terms of gauge fields (such as the GEM inertial fields $\vec{G}$, $\vec{H}$ and the shear $K_{(\alpha\beta)}$ of the 1+3 formalism of Sec. 3; the general expressions of Einstein equations in terms of these fields is given Sec. 3.4.2 below; but is also possible to write the equations for the dynamics of the tidal tensors (the *physical* fields); that is done not through Einstein equations (14), but through the differential Bianchi identity $R_{\sigma\tau[\mu\nu;\alpha]}=0$, together with decomposition (15), and using (14) to substitute $R^{\alpha\beta}$ by the source terms. The resulting equations, for the case of vacuum (where $\{\mathcal{E}_{\alpha\beta},\ \mathcal{H}_{\alpha\beta}\}=\{\mathbb{E}_{\alpha\beta},\ \mathbb{H}_{\alpha\beta}\}$), are Eqs. (149)-(150) of Sec. 6 below. One may write as well dynamical equations for the electromagnetic tidal tensors, which for the case of vacuum, and an inertial frame, are Eqs. (156)-(157) of Sec. 6.1; however in the electromagnetic case the fundamental physical fields are the vectors E^{α}, B^{α}, whose covariant field equations are Eqs. (151)-(154) (the tidal field equations (156)-(157) follow trivially from these).

2.2 Gravity vs Electromagnetism

In the tidal tensor formalism, cf. Table 1, the gravitational field is described by five (independent) algebraic equations, four of which analogous to the Maxwell equations, plus an additional equation, involving the tensor $\mathbb{F}_{\alpha\beta}$, which has no parallel in electromagnetism. Conversely, in Maxwell equations there are terms with no gravitational counterpart; these correspond to the antisymmetric parts / time projections (with respect to the observer congruence) of the electromagnetic tidal tensors.

The tensor $\mathbb{F}_{\alpha\beta}$ — whereas Maxwell's equations can be fully expressed in terms of tidal tensors and sources, the same is only true, in general, for the temporal part of Einstein's equations. The Space-Space part, Eq. (21), involves the tensor $\mathbb{F}_{\alpha\beta}$, which has no electromagnetic analogue. This tensor, however is not an additional independent object, as it is completely determined via (21) given the sources and $\mathbb{E}_{\alpha\beta}$. In vacuum $\mathbb{F}_{\alpha\beta}=-\mathbb{E}_{\alpha\beta}$.

Sources — The source of the gravitational field is the rank two energy momentum tensor $T^{\alpha\beta}$, whereas the source of the electromagnetic field is the current 4-vector j^{α}. Using the projectors (2) one can split $T^{\alpha\beta}=\rho U^{\alpha}U^{\beta}+2U^{(\alpha}h^{\beta)}_{\ \mu}J^{\mu}+T^{\langle\alpha\rangle\langle\beta\rangle}$, and $j^{\alpha}=\rho_c U^{\alpha}+h^{\alpha}_{\ \mu}j^{\mu}$. Eqs. (1.4) show that the source of $E_{\alpha\beta}$ is ρ_c, and its gravitational analogue, as the source of $\mathbb{E}_{\alpha\beta}$, is $2\rho+T^{\alpha}_{\ \alpha}$ ($\rho+3p$ for a perfect fluid). The magnetic/gravitomagnetic tidal tensors are analogously sourced by the charge/mass-energy currents $j^{\langle\mu\rangle}/J^{\langle\mu\rangle}$, as shown by Eqs. (1.5). Note that, in stationary (in the observer's frame) setups, $\star F_{\alpha\beta;\gamma}U^{\gamma}$ vanishes and equations (1.5a) and (1.5b) match up to a factor of 2, identifying $j^{\langle\mu\rangle} \leftrightarrow J^{\langle\mu\rangle}$. Eq. (23) shows that ρ is the source of $\mathbb{F}_{\alpha\beta}$. Eq. (1.6), sourced by the space-space part $T^{\langle\alpha\rangle\langle\beta\rangle}$, as well as the contribution $T^{\alpha}_{\ \alpha}$ for (1.4b), manifest the well known fact that in gravity, by contrast with electromagnetism, pressure and stresses act as sources of the field.

Symmetries and time projections of tidal tensors — The gravitational and electromagnetic tidal tensors do not generically exhibit the same symmetries; moreover, the former tidal tensors are spatial, whereas the latter have a time projection (with respect to the observer measuring them), signaling fundamental differences between the two interactions. In the general case of fields that vary along the observer's worldline (that is the case of an intrinsically non-stationary field, or an observer moving in a stationary non-uniform field), $E_{\alpha\beta}$ possesses an antisymmetric part; $\mathbb{E}_{\alpha\beta}$, by contrast, is always symmetric. $E_{[\alpha\beta]}$ encodes Faraday's law of induction: as discussed above, $E_{\alpha\beta}$ is a covariant derivative of the electric field as measured in the the momentarily comoving reference frame (MCRF); thus Eq. (13a) is a covariant way of writing the Maxwell-Faraday equation $\nabla \times \vec{E} = -\partial\vec{B}/\partial t$. Therefore, the statement encoded in the equation $\mathbb{E}_{[\alpha\beta]} = 0$ is that there are no analogous induction effects in the *physical* (i.e., tidal) gravitational forces (in the language of GEM vector fields of Sec. 3, we can say that the curl of the gravitoelectric field $\vec{G}$ does not manifest itself in the tidal forces, unlike its electromagnetic counterpart; see Sec. 3.5 for explicit demonstration). To see a physical consequence, let δx^{α} in Eq. (1.1a) — the separation vector between a pair of particles with the same q/m and the same 4-velocity U^{α} — be spatial with respect to U^{α} ($\delta x^{\alpha}U_{\alpha} = 0$); and note that the spatially projected antisymmetric part of $E_{\mu\nu}$ can be written in terms of the dual spatial vector α^{μ}: $E_{[\langle\mu\rangle\langle\nu\rangle]} = \epsilon_{\mu\nu\gamma\delta}\alpha^{\gamma}U^{\delta}$. Then the spatial components (1.1a) can be written as (using $E_{\langle\mu\rangle\langle\nu\rangle} = E_{(\langle\mu\rangle\langle\nu\rangle)} + E_{[\langle\mu\rangle\langle\nu\rangle]}$):

$$\frac{D^2\delta x_{\langle\mu\rangle}}{d\tau^2} = \frac{q}{m}\left[E_{(\langle\mu\rangle\langle\nu\rangle)}\delta x^{\nu} + \epsilon_{\mu\nu\gamma\delta}\alpha^{\gamma}U^{\delta}\delta x^{\nu}\right] \quad \Leftrightarrow \quad \frac{D^2\delta\vec{x}}{d\tau^2} = \frac{q}{m}\left[\overleftrightarrow{E}\cdot\delta\vec{x} + \delta\vec{x}\times\vec{\alpha}\right], \tag{25}$$

the second equation holding in the frame $U^i = 0$, where we used the dyadic notation $\overleftrightarrow{E}$ of e.g. [102]. From the form of the second equation we see that $q\vec{\alpha}/m$ is *minus* an angular acceleration. Using relation (4), we see that $\alpha^{\mu} = -B^{\mu}_{\ \beta}U^{\beta}$; and in an inertial frame $\vec{\alpha} = \partial\vec{B}/\partial t = -\nabla\times\vec{E}$. In the gravitational case, since $\mathbb{E}_{\mu\nu} = \mathbb{E}_{(\mu\nu)} = \mathbb{E}_{\langle\mu\rangle\langle\nu\rangle}$, we have

$$\frac{D^2\delta x_{\langle\mu\rangle}}{d\tau^2} = \frac{D^2\delta x_{\mu}}{d\tau^2} = -\mathbb{E}_{(\mu\nu)}\delta x^{\nu} \quad \Leftrightarrow \quad \frac{D^2\delta\vec{x}}{d\tau^2} = -\overleftrightarrow{\mathbb{E}}\cdot\delta\vec{x}\ . \tag{26}$$

That is, given a set of neighboring charged test particles, the electromagnetic field "shears" the set via $E_{(\mu\nu)}$, and induces an accelerated rotation[4] via the laws of electromagnetic induction encoded in $E_{[\mu\nu]}$. The gravitational field, by contrast, only shears[5] the set, since $\mathbb{E}_{[\mu\nu]} = 0$.

[4]By rotation we mean here absolute rotation, i.e, measured with respect to a comoving Fermi-Walker transported frame. As one can check from the connection coefficients (39) below, in such frame ($\Omega_{\alpha\beta} = 0$) we have $d^2\delta x^{\hat{i}}/d\tau^2 = D^2\delta x^{\hat{i}}/d\tau^2$. See also in this respect [126].

[5]If the two particles were connected by a "rigid" rod then the symmetric part of the electric tidal tensor would also, in general, torque the rod; hence in such system we would have a rotation even in the gravitational case, see [25] pp. 154-155. The same is true for a quasi-rigid extended body; however, even in this case the effects due to the symmetric part are very different from the ones arising from electromagnetic induction: first, the former do not require the fields to vary along the particle's worldline, they exist even if the body is at rest in a stationary field; second, they vanish if the body is spherical, which does not happen with the torque generated by the induced electric field, see [6].

Further physical evidence for the absence of a gravitational analogue for Faraday's law of induction in the *physical* forces and torques is given in the companion paper [6]: consider a spinning spherical charged body in an electromagnetic field; and choose the MCRF; if the magnetic field is not constant in this frame, by virtue of equation $\nabla \times \vec{E} = -\partial \vec{B}/\partial t$, a torque will in general be exerted on the body by the induced electric field, changing its angular momentum and kinetic energy of rotation. By contrast, *no gravitational torque* is exerted on a spinning "spherical" body (i.e., a particle whose multipole moments in a local orthonormal frame match the ones of a spherical body in flat spacetime) placed in an *arbitrary* gravitational field; its angular momentum and kinetic energy of rotation are *constant*.

As discussed in the previous section, the symmetry of $\mathbb{E}_{\alpha\beta}$ follows from the algebraic Bianchi identity $R^{\alpha}{}_{[\beta\gamma\delta]} = 0$; this identity states that $R^{\alpha}{}_{\beta\gamma\delta}$ is the curvature tensor of a connection with vanishing torsion (the Levi-Civita connection of the space-time manifold). So one can say that the absence of electromagnetic-like induction effects is the statement that the physical gravitational forces are described by the curvature tensor of a connection without torsion.

There is also an antisymmetric contribution $\star F_{\alpha\beta;\gamma}U^{\gamma}$ to $B_{\alpha\beta}$; in vacuum, Eq. (1.5a) is a covariant form of $\nabla \times \vec{B} = \partial \vec{E}/\partial t$; hence the fact that, in vacuum, $\mathbb{H}_{[\alpha\beta]} = 0$, means that there is no gravitational analogue to the antisymmetric part $B_{[\alpha\beta]}$ (i.e., the curl of $\vec{B}$) induced by the time varying field $\vec{E}$. Some physical consequences of this fact are explored in [6]: Eq. (1.5a) implies, via (1.2a), that whenever a magnetic dipole moves in a non-homogeneous field, it measures a non-vanishing $B_{[\alpha\beta]}$ (thus also $B_{\alpha\beta} \neq 0$), and therefore (except for very special orientations of the dipole moment μ^{α}) a force will be exerted on it; in the gravitational case, by contrast, the gravitational force on a gyroscope is not constrained to be non-vanishing when it moves in a non-homogeneous field; it is found that it may actually move along geodesics, as is the case of radial motion in Schwarzschild spacetime[6], or circular geodesics in Kerr-dS spacetime.

The spatial character of the gravitational tidal tensors, contrasting with their electromagnetic counterparts, is another difference in the tensorial structure related to the laws of electromagnetic induction: as can be seen from Eqs. (10) and (12), the antisymmetric parts of $E_{\alpha\beta}$ and $B_{\alpha\beta}$ (in vacuum, for the latter) consist of time projections of these tidal tensors. Physically, these time projections are manifest for instance in the fact that the electromagnetic force on a magnetic dipole has a non-vanishing projection along the particle's 4-velocity U^{α}, which is the rate of work done on it by the induced electric field [1, 6], and is reflected in a variation of the particle's proper mass. The projection, along U^{α}, of the gravitational force (1.2b), in turn, vanishes, and the gyroscope's proper mass is constant.

2.3 The analogy for differential precession

Eqs. (1.2) in Table 1 give $B_{\alpha\beta}$ and $\mathbb{H}_{\alpha\beta}$ a physical interpretation as the tensors which, when contracted with a magnetic/gravitomagnetic dipole vector, yield the force exerted on magnetic dipoles/gyroscopes. We will now show that these tensors can also be interpreted as tensors of "relative", or "differential", precession for these test particles; i.e., tensors that, when contracted

[6] This example is particularly interesting in this discussion. In the electromagnetic analogous problem, a magnetic dipole in (initially) radial motion in the Coulomb field of a point charge experiences a force; that force, as shown in [6], comes entirely from the antisymmetric part of the magnetic tidal tensor, $B_{\alpha\beta} = B_{[\alpha\beta]}$; it is thus a natural realization of the arguments above that $\mathbb{H}_{\alpha\beta} = 0$ in the analogous gravitational problem.

with a separation vector δx^{β}, yield the angular velocity of precession of a spinning particle at given point $\mathcal{P}_2$ relative to a system of axes anchored to spinning particles, with *the same 4-velocity* (and the same gyromagnetic ratio σ, if an electromagnetic field is present), at the infinitesimally close point $\mathcal{P}_1$. This is analogous to the electric tidal tensors $E_{\alpha\beta}$ and $\mathbb{E}_{\alpha\beta}$, which, when contracted with δx^{β}, yield the relative acceleration of two infinitesimally close test particles with the same 4-velocity and the same ratio q/m in the electromagnetic case.

For clarity we will treat the gravitational and electromagnetic interactions separately. We will start with the gravitational problem. Our goal is to compute the precession of a gyroscope at some point $\mathcal{P}_2$ relative to a frame attached to guiding gyroscopes at the neighboring point $\mathcal{P}_1$. Assuming the Mathisson-Pirani [47, 50] spin condition $S^{\alpha\beta}U_{\beta}=0$, it follows from the Mathisson-Papapetrou equations [47, 49] that the spin vector of a gyroscope undergoes Fermi-Walker transport,

$$\frac{DS^{\alpha}}{d\tau}=S_{\nu}a^{\nu}U^{\alpha} \tag{27}$$

(for more details, see [6]; in the comoving frame, the spatial part reads $D\vec{S}/d\tau=0$). Thus the frame we are looking for is a tetrad Fermi-Walker transported along the worldline L of the set of gyroscopes 1 (passing trough the location $\mathcal{P}_1$). There is a locally rectangular coordinate system associated to such tetrad, the so-called[7] "Fermi coordinates"; let $\mathbf{e}_{\alpha}$ denote its basis vectors and $\Gamma^{\alpha}_{\beta\gamma}$ its Christoffel symbols, $\Gamma^{\alpha}_{\beta\gamma}\mathbf{e}_{\alpha}=\nabla_{\mathbf{e}_{\beta}}\mathbf{e}_{\gamma}$. The vectors $\mathbf{e}_{\alpha}$ are Fermi-Walker transported along L, so $\langle\nabla_{\mathbf{e}_0}\mathbf{e}_i,\mathbf{e}_j\rangle|_{\mathcal{P}_1}=0\Rightarrow\Gamma^{j}_{0i}(\mathcal{P}_1)=0$. Hence, a gyroscope at $\mathcal{P}_1$, momentarily at rest in this frame, by Eq. (27) obviously does not precess relative to it, $d\vec{S}/d\tau|_{\mathcal{P}_1}=\dot{\vec{S}}|_{\mathcal{P}_1}=0$. Here the dot denotes ordinary derivative along $\mathbf{e}_0$, $\dot{A}^{\alpha}\equiv\partial_0A^{\alpha}$. However, outside L, the basis vectors $\mathbf{e}_{\alpha}$ are no longer Fermi-Walker transported, $\langle\nabla_{\mathbf{e}_0}\mathbf{e}_i,\mathbf{e}_j\rangle|_{\mathcal{P}_2}\neq0\Rightarrow\Gamma^{j}_{0i}(\mathcal{P}_2)\neq0$. That means that gyroscope 2, at a point $\mathcal{P}_2$ (outside L), will be be seen to precess relative to the frame $\mathbf{e}_{\alpha}$: $d\vec{S}_2/d\tau_2|_{\mathcal{P}_2}\neq0$. If the gyroscope is at rest in this frame ($U_2^i=0$), we have

$$\left.\frac{dS_2^i}{d\tau_2}\right|_{\mathcal{P}_2}=-\Gamma^{i}_{0j}(\mathcal{P}_2)S_2^j\ . \tag{28}$$

The Christoffel symbol $\Gamma^i_{0j}(\mathcal{P}_2)$ can be obtained from e.g. Eqs. (20) of [96] (making $\Omega_{ij}=0$ therein, and neglecting higher order terms); it reads $\Gamma^i_{0j}(\mathcal{P}_2)=R^i{}_{jk0}(\mathcal{P}_1)\delta x^k$. From Eq. (4) above, we note that

$$R_{\langle\alpha\rangle\langle\beta\rangle\gamma\tau}=\epsilon^{\mu\nu}{}_{\sigma\delta}\epsilon_{\mu\alpha\beta\lambda}U^{\lambda}U_{\nu}R^{\sigma\delta}{}_{\gamma\tau}=\epsilon_{\mu\alpha\beta\lambda}U^{\lambda}U_{\nu}\star R^{\mu\nu}{}_{\gamma\tau}\ ,$$

which, in the Fermi frame $\mathbf{e}_{\alpha}$ (orthonormal at $\mathcal{P}_1$), reads: $R_{ij\gamma\tau}=\epsilon_{ijk}\star R^k{}_{0\gamma\tau}$. We thus have

$$\Gamma^i_{0j}(\mathcal{P}_2)=\epsilon^i{}_{jk}\star R^k{}_{0l0}\,(\mathcal{P}_1)\delta x^l=\epsilon^i{}_{jk}\mathbb{H}^k{}_l\delta x^l\ .$$

$\Gamma^i_{0j}(\mathcal{P}_2)$ is an antisymmetric matrix, which we can write as $\Gamma^i_{0j}(\mathcal{P}_2)=\epsilon^i{}_{jk}\delta\Omega^k_{\mathrm{G}}$, where

$$\delta\Omega^i_{\mathrm{G}}\equiv\mathbb{H}^i{}_l\delta x^l\ . \tag{29}$$

[7] Following Synge [95], by Fermi coordinates we mean the locally rectangular coordinate system associated to a tetrad Fermi-Walker transported along a worldline (the curve being the origin of the frame, and its tangent the time axis). Note the existence of different conventions in the literature: the so-called "Fermi normal coordinates" of e.g. [97, 53] are a special case of our definition, for the case that the worldline is geodesic. The "Fermi coordinates" of [96], in turn, are a generalization of our definition, for the case that the tetrad is not Fermi-Walker transported.

Substituting into (28),

$$\left.\frac{d\vec{S}_2}{d\tau_2}\right|_{\mathcal{P}_2} = \delta\vec{\Omega}_{\rm G}\times\vec{S}_2 = \dot{\vec{S}}_2|_{\mathcal{P}_2} \tag{30}$$

(in the last equality we noted that $U_2^\alpha = (1+\mathcal{O}(\delta x),\vec{0})$). $\delta\vec{\Omega}_{\rm G}$ is thus the angular velocity of precession of gyroscopes at $\mathcal{P}_2$ with respect to the Fermi frame $\mathbf{e}_i$, locked to the guiding gyroscopes at $\mathcal{P}_1$. Obviously, this is just *minus* the angular velocity of rotation of the basis vectors $\mathbf{e}_i$ relative to Fermi-Walker transport *at* $\mathcal{P}_2$. This result was first obtained in a recent work [4] through a different procedure; we believe the derivation above is more clear, and shows that one of the assumptions made in [4] to obtain $\delta\vec{\Omega}_{\rm G}$ — that the gyroscopes at $\mathcal{P}_1$ and $\mathcal{P}_2$ have the same acceleration — is not necessary; in order for (30) to hold, they only need to be momentarily comoving[8], in the sense that they have zero 3-velocity in the Fermi coordinate system of $\mathcal{P}_1$. It is however worth noting that if the basis worldline L through $\mathcal{P}_1$ is geodesic, then, as one may easily check, Eq. (30) still holds when the gyroscopes at $\mathcal{P}_2$ have an infinitesimal velocity $v\propto\delta x$ in such frame. Hence it applies to gyroscopes carried by *infinitesimally close* observers belonging to an arbitrary *geodesic congruence* (in a certain analogy with the geodesic deviation equation (1.1b) of Table 1). The differential dragging effect in terms of the eigenvectors of $\mathbb{H}_{\alpha\beta}$ and their associated field lines, as well as their visualization in different spacetimes, are discussed in detail in [4].

Let us turn now to the analogous electromagnetic problem. Consider, in flat spacetime, a triad of orthogonal magnetic dipoles (magnetic moment $\mu^\alpha = \sigma S^\alpha$), with the same gyromagnetic ratio σ and 4-velocity U^α (so that they all precess with the same frequency), moving along a worldline L of tangent $\mathbf{U}$ passing through $\mathcal{P}_1$. If the Mathisson-Pirani condition holds, the spin vector of a magnetic dipole moving along L evolves as (e.g. [6]):

$$\frac{DS^\mu}{d\tau} = S_\nu a^\nu U^\mu + \sigma\epsilon^\mu_{\ \ \alpha\beta\nu}U^\nu S^\alpha B^\beta\ , \tag{31}$$

where $B^\alpha\equiv\star F^{\alpha\beta}U_\beta$. The second term marks an obvious difference with the case of the gyroscope, as it means that magnetic dipoles *do precess* (with angular velocity $\Omega^\alpha = -\sigma B^\alpha$) with respect to the comoving Fermi-Walker transported frame. We shall compare this precession for dipoles at infinitesimally close points, and relate with the gravitational analogue. In the Fermi frame $\mathbf{e}_\alpha$ with origin at L, the space part of Eq. (31) reads (at $\mathcal{P}_1$),

$$\left.\frac{d\vec{S}}{d\tau}\right|_{\mathcal{P}_1} = \vec{\Omega}(\mathcal{P}_1)\times\vec{S}\ ;\qquad \vec{\Omega}(\mathcal{P}_1)\equiv-\sigma\vec{B}(\mathcal{P}_1)\ . \tag{32}$$

Now consider a magnetic dipole at the neighboring point $\mathcal{P}_2$, *at rest in the frame* $\mathbf{e}_\alpha$ (i.e. $U_2^i=0$, so that we are using the same notion of comoving as in the gravitational case); we have

$$\left.\frac{dS_2^i}{d\tau_2}\right|_{\mathcal{P}_2} = -\Gamma^i_{0j}(\mathcal{P}_2)S_2^j + (\vec{\Omega}(\mathcal{P}_2)\times\vec{S}_2)^i = (\vec{\Omega}(\mathcal{P}_2)\times\vec{S}_2)^i\ ;\qquad \vec{\Omega}(\mathcal{P}_2)\equiv-\sigma\vec{B}(\mathcal{P}_2)\ , \tag{33}$$

[8]The relative velocity of objects at different points is not a well defined notion in curved spacetime, since there is no natural way of comparing vectors at different points, see e.g. [98, 99]. The notion of relative velocity implied above is dubbed in [98] "Fermi relative velocity".

where we noted that, in flat spacetime, $\Gamma^i_{0j} = 0$ everywhere in a Fermi frame (cf. Eqs (20) of [96]). Being at rest in the frame $\mathbf{e}_\alpha$ implies, in flat spacetime[9], $\mathbf{U}_2 = \mathbf{U}$ (note that parallelism between vectors at different points is well defined herein). Thus, $dS_2^i/d\tau_2 \equiv (S_2)^i{}_{,\alpha}U_2^\alpha = dS_2^i/d\tau$, and $B^\alpha(\mathcal{P}_2) \equiv \star F^\alpha{}_\beta(\mathcal{P}_2)U_2^\beta = \star F^\alpha{}_\beta(\mathcal{P}_2)U^\beta$. Performing a Taylor expansion of $F^\alpha{}_\beta$ about $\mathcal{P}_1$ (and using *for this operation* a rectangular coordinate system, which one can always do in flat spacetime, so that $F^\alpha{}_{\beta,\gamma} = F^\alpha{}_{\beta;\gamma}$), we may write $F^\alpha{}_\beta(\mathcal{P}_2) = F^\alpha{}_\beta(\mathcal{P}_2) + F^\alpha{}_{\beta;\gamma}(\mathcal{P}_1)\delta x^\gamma + \mathcal{O}(\delta x^2)$. Therefore,

$$B^\alpha(\mathcal{P}_2) = B^\alpha(\mathcal{P}_1) + B^\alpha{}_\gamma(\mathcal{P}_1)\delta x^\gamma + \mathcal{O}(\delta x^2) \ , \tag{34}$$

where $B_{\alpha\beta} = \star F_{\alpha\gamma;\beta}U^\gamma$ is the magnetic tidal tensor as defined in Eq. (1.2a) of Table 1. Eqs. (32) and (33) are precessions measured with respect to the same frame $\mathbf{e}_\alpha$; taking the difference $\delta\vec{\Omega}_{\rm EM}$, we obtain

$$\delta\Omega^i_{\rm EM} = \Omega^i(\mathcal{P}_2) - \Omega^i(\mathcal{P}_1) = -\sigma B^i{}_\gamma \delta x^\gamma \ , \tag{35}$$

which is analogous to Eq. (29), only with $-\sigma B_{\alpha\beta}$ in the place of $\mathbb{H}_{\alpha\beta}$.

It should be noted, however, that Eq. (35) does *not*, *in general*, yield the precession of dipole 2 with respect to a frame whose axes are *anchored* to the spin vectors of guiding magnetic dipoles at $\mathcal{P}_1$ (which would be perhaps the most natural analogue of the gravitational problem considered above). Let us denote by $(\mathbf{e}_{\rm dip})_\alpha$ the basis vectors of the coordinate system adapted to such frame (originating at L, where it is rectangular, with $(\mathbf{e}_{\rm dip})_0 = \mathbf{U}$; this is a generalized version of the Fermi-coordinates of L, for the case that the spatial triad is not Fermi-Walker transported, see [96]). The spin evolution equation for dipole 2 reads, in this frame,

$$\left.\frac{d\vec{S}_2}{d\tau}\right|_{\mathcal{P}_2} = \delta\vec{\Omega}_{\rm EM} \times \vec{S}_2 + (\vec{a} \times \vec{U}_2) \times \vec{S}_2 \ , \tag{36}$$

where we used the connection coefficients given in Eqs. (20) of [96] (in particular, $\Gamma^i_{0j}(\mathcal{P}_2) = \Omega^i{}_j = -\sigma\epsilon^i{}_{kj}B^k(\mathcal{P}_1)$), and noted that dipole 2 (since it is at rest in the Fermi frame $\mathbf{e}_\alpha$) moves in the frame $(\mathbf{e}_{\rm dip})_\alpha$ with spatial velocity $U_2^i \approx -\vec{\Omega}(\mathcal{P}_1) \times \vec{\delta x}$. Thus, only when L is geodesic one has in such frame $d\vec{S}_2/d\tau|_{\mathcal{P}_2} = \delta\vec{\Omega}_{\rm EM} \times \vec{S}_2$ (as for the acceleration of dipole 2, it can be arbitrary). It should also be noted that, by contrast with the gravitational Eqs. (29)-(30), Eqs. (35)-(36), do not hold when the dipoles possess an infinitesimal relative velocity $\delta U \propto \delta x$ (even if the basis worldline L is geodesic, as an extra term $\star F^\alpha{}_\beta \delta U^\beta$ would show up in (34); $\delta\mathbf{U} = \mathbf{U}_2 - \mathbf{U}$); they must be *strictly* comoving. This is analogous to the situation with the worldline deviation equations (1.1) of Table 1, where the gravitational equation allows the particles to have an infinitesimal deviation velocity, whereas the electromagnetic one does not (cf. footnote 1).

3 Gravito-electromagnetic analogy based on inertial fields from the 1+3 splitting of spacetime

This approach has a different philosophy from the tidal tensor analogy of Sec. 2. Therein we aimed to compare physical, covariant forces of both theories; which was accomplished through

[9]This can be easily seen from the fact that the triad $\mathbf{e}_i$ coincides with the basis vectors of a momentarily comoving inertial frame; thus $U_2^i = 0$ implies that, in the inertial frame, $U_2^\alpha = (1, \vec{0}) = U^\alpha$. This can also be seen from Eqs. (22)-(23) of [98], to which we refer for a detailed discussion of the Fermi relative velocity in flat spacetime.

the tidal forces. Herein the analogy drawn is between the electromagnetic fields E^α, B^α and spatial inertial fields G^α, H^α (i.e., fields of inertial forces, or "acceleration" fields), usually dubbed "gravitoelectromagnetic" (GEM) fields, that mimic E^α and B^α in gravitational dynamics. Inertial forces are fictitious forces, attached to a specific reference frame, and in this sense one can regard this analogy as a parallelism between physical forces from one theory, and reference frame effects from the other.

The GEM 3-vector fields are best known in the context of linearized theory for stationary spacetimes, e.g. [7, 11], where they are (somewhat naively) formulated as derivatives of the temporal components of the linearized metric tensor (the GEM potentials, in analogy with the EM potentials). More general approaches are possible if one observes that these are fields associated not to the local properties of a particular spacetime, but, as stated above, to the kinematical quantities of the reference frame. In particular, the GEM fields of the usual linearized approaches are but, up to some factors, the acceleration and vorticity of the congruence of zero 3-velocity observers ($u^\alpha \simeq \delta^\alpha_0$) in the chosen background. Taking this perspective, the GEM fields may actually be cast in an *exact* form, applying to arbitrary reference frames in arbitrary fields, through a general 1+3 splitting of spacetime. In this section we present such an exact and general formulation. We take an arbitrary orthonormal reference frame, which can be thought as a continuous field of orthonormal tetrads, or, alternatively, as consisting of a congruence of observers, each of them carrying an orthonormal tetrad whose time axis is the observer's 4-velocity; the spatial triads, spanning the local rest space of the observers, are generically left arbitrary (namely their rotation with respect to Fermi-Walker transport). The inertial fields associated to this frame are, in this framework, encoded in the mixed time-space part of the connection coefficients: the acceleration a^α and vorticity ω^α of the observer congruence, plus the rotation frequency Ω^α of the spatial triads with respect to Fermi-Walker transport. The connection coefficients encode also the shear/expansion $K_{(\alpha\beta)}$ of the congruence. A "gravitoelectric" field is defined in this framework as $G^\alpha \equiv -a^\alpha$, and a gravitomagnetic field as $H^\alpha \equiv \Omega^\alpha + \omega^\alpha$; the motivation for these definitions being the geodesic equation, whose space part, in such frame, resembles the Lorentz force, with G^α in the role of an electric field, H^α in the role of a magnetic field, plus a third term with no electromagnetic analogue, involving $K_{(\alpha\beta)}$.

The treatment herein is to a large extent equivalent to the approach in [27, 28]; the main difference (apart from the differences in the formalism) is that we use a more general definition of gravitomagnetic field, allowing for an arbitrary transport law of the spatial frame (i.e., Ω^α is left arbitrary), and so the equations adjust to any frame. We also try to use a simplified formalism, as the one in [27, 28], albeit very precise and rigorous, is not easy to follow though. For that we work with orthonormal frames in most of our analysis, which are especially suited for our purposes because the connection coefficients associated to them are very simply related with the inertial fields. The price to pay is that our approach is not manifestly covariant at each step (although the end results are easily written in covariant form), by contrast with the formalism in [27, 28].

3.1 The reference frame

To an arbitrary observer moving along a worldline of tangent vector u^α, one naturally associates an adapted *local* orthonormal frame (e.g. [27]), which is a tetrad $\mathbf{e}_{\hat{\alpha}}$ whose time axis is the observer's 4-velocity, $\mathbf{e}_{\hat{0}} = \mathbf{u}$, and whose spatial triad $\mathbf{e}_{\hat{i}}$ spans the local rest space of the observer. The latter is for now undefined up to an arbitrary rotation. The evolution of the tetrad along the observer's

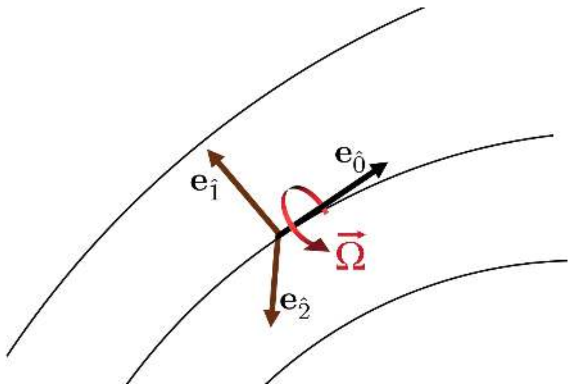

Figure 1: The reference frame: a congruence of of time-like curves — the observers' worldlines — of of tangent $\mathbf{u}$; each observer carries an orthonormal tetrad $\mathbf{e}_{\hat{\alpha}}$ such that $\mathbf{e}_{\hat{0}} = \mathbf{u}$ and the spatial triad $\mathbf{e}_{\hat{i}}$ spans the observer's local rest space. The triad $\mathbf{e}_{\hat{i}}$ rotates, relative to Fermi-Walker transport, with some prescribed angular velocity $\vec{\Omega}$.

worldline is generically described by the equation:

$$\nabla_{\mathbf{u}} \mathbf{e}_{\hat{\beta}} = \Omega^{\hat{\alpha}}_{\ \hat{\beta}} \mathbf{e}_{\hat{\alpha}}; \quad \Omega^{\alpha\beta} = 2u^{[\alpha} a^{\beta]} + \epsilon^{\alpha\beta}_{\ \ \nu\mu} \Omega^{\mu} u^{\nu} \tag{37}$$

where $\Omega^{\alpha\beta}$ is the (anti-symmetric) infinitesimal generator of Lorentz transformations, whose spatial part $\Omega_{\hat{i}\hat{j}} = \epsilon_{\hat{i}\hat{k}\hat{j}} \Omega^{\hat{k}}$ describes the arbitrary angular velocity $\vec{\Omega}$ of rotation of the spatial triad $\mathbf{e}_{\hat{i}}$ relative to a Fermi-Walker transported triad. Alternatively, from the definition of the connection coefficients,

$$\nabla_{\mathbf{e}_{\hat{\beta}}} \mathbf{e}_{\hat{\gamma}} = \Gamma^{\hat{\alpha}}_{\hat{\beta}\hat{\gamma}} \mathbf{e}_{\hat{\alpha}} \ ,$$

we can think of the components of $\Omega^{\alpha\beta}$ as some of these coefficients:

$$\Omega^{\hat{i}}_{\ \hat{0}} = \Gamma^{\hat{i}}_{\hat{0}\hat{0}} = \Gamma^{\hat{0}}_{\hat{0}\hat{i}} = a^{\hat{i}} \ ; \tag{38}$$

$$\Omega^{\hat{i}}_{\ \hat{j}} = \Gamma^{\hat{i}}_{\hat{0}\hat{j}} = \epsilon_{\hat{i}\hat{k}\hat{j}} \Omega^{\hat{k}} \ . \tag{39}$$

Unlike the situation in flat spacetime (and Lorentz coordinates), where one can take the tetrad adapted to a given inertial observer as a global frame, in the general case such tetrad is a valid frame only locally, in an infinitesimal neighborhood of the observer. In order to define a reference frame over an extended region of spacetime, one needs a congruence of observers, that is, one needs to extend u^α to a field of unit timelike vectors tangent to a congruence of time-like curves. A connecting vector X^α between two neighboring observers in the congruence satisfies

$$[\mathbf{u}, \mathbf{X}] = \mathbf{0} \Leftrightarrow u^\beta \nabla_\beta X^\alpha - X^\beta \nabla_\beta u^\alpha = 0. \tag{40}$$

The evolution of the connecting vector along the worldline of an observer in the congruence is then given by the linear equation

$$\nabla_{\mathbf{u}} X^\alpha = \left(\nabla^\beta u^\alpha \right) X_\beta. \tag{41}$$

The component of the connecting vector orthogonal to the congruence,

$$Y^\alpha = (h^u)^\alpha_{\ \beta} X^\beta = X^\alpha + \left(u_\beta X^\beta \right) u^\alpha, \tag{42}$$

satisfies

$$\nabla_{\mathbf{u}} Y^\alpha = K^{\alpha\beta} Y_\beta + \left(a_\beta Y^\beta\right) u^\alpha, \tag{43}$$

where $K^{\alpha\beta}$ denotes the spatially projected covariant derivative of u^α

$$K^{\alpha\beta} \equiv (h^u)^\alpha{}_\lambda (h^u)^\beta{}_\tau u^{\lambda;\tau} = \nabla^\beta u^\alpha + a^\alpha u^\beta \tag{44}$$

The decomposition of this tensor into its trace, symmetric trace-free and anti-symmetric parts yields the expansion

$$\theta = u^\alpha{}_{;\alpha}$$

the shear

$$\sigma_{\alpha\beta} = K_{(\alpha\beta)} - \frac{1}{3}\theta g_{\alpha\beta} - \frac{1}{3}\theta u_\alpha u_\beta \tag{45}$$

and the vorticity

$$\omega_{\alpha\beta} = K_{[\alpha\beta]} \tag{46}$$

of the congruence. It is useful to introduce the vorticity vector

$$\omega^\alpha = \frac{1}{2}\epsilon^{\alpha\beta\gamma\delta} u_{\gamma;\beta} u^\delta = -\frac{1}{2}\epsilon^{\alpha\beta\gamma\delta}\omega_{\alpha\beta} u^\delta. \tag{47}$$

According to definition above, ω^α yields *half* the curl of u^α; this is in agreement with the convention in e.g. [20, 27], but differs by a minus sign from the definition in e.g. [35, 87]. Note however that for the vorticity tensor $\omega_{\alpha\beta}$ we are using the more general definition given in [35, 87], differing from a minus sign from the one in [27] (consequently, ω^α given by Eq. (47) is *minus* the dual of $\omega_{\alpha\beta}$). The non-vanishing tetrad components of $K_{\alpha\beta}$ are

$$K_{\hat{i}\hat{j}} = \sigma_{\hat{i}\hat{j}} + \frac{1}{3}\theta\delta_{\hat{i}\hat{j}} + \omega_{\hat{i}\hat{j}}. \tag{48}$$

These components determine the following connection coefficients:

$$K_{\hat{i}\hat{j}} = \nabla_{\hat{j}} u_{\hat{i}} = \Gamma^{\hat{0}}_{\hat{j}\hat{i}} = \Gamma^{\hat{i}}_{\hat{j}\hat{0}}. \tag{49}$$

The remaining temporal connection coefficients (other than the ones given in Eqs. (38)-(39), (49) above) are trivially zero:

$$\Gamma^{\hat{0}}_{\hat{\alpha}\hat{0}} = -\mathbf{e}_{\hat{0}} \cdot \nabla_{\mathbf{e}_{\hat{\alpha}}} \mathbf{e}_{\hat{0}} = -\frac{1}{2}\nabla_{\mathbf{e}_{\hat{\alpha}}}(\mathbf{e}_{\hat{0}} \cdot \mathbf{e}_{\hat{0}}) = 0.$$

Each observer in the congruence carries its own adapted tetrad, c.f. Fig. 1, and to define the reference frame one must provide the law of evolution for the spatial triads orthogonal to u^α. A natural choice would be Fermi-Walker transport, $\vec{\Omega} = 0$ (the triad does not rotate relative to local guiding gyroscopes); another natural choice, of great usefulness in this framework, is to lock the rotation of the spatial triads to the vorticity of the congruence, $\vec{\Omega} = \vec{\omega}$. We will dub such frame "congruence adapted frame"[10]; as argued in [57, 55], where it was introduced, this is the most

[10]Note however that in some literature, e.g. [126], the term "congruence adapted" is employed with a different meaning, designating any tetrad field whose time axis is tangent to the congruence, without any restriction on the transport law for the spatial triad (namely without requiring the triads to co-rotate with the congruence). Hence "adapted" therein means what, in our convention, we would call adapted to each individual observer.

natural generalization of the non-relativistic concept of reference frame; and the corresponding transport law $\vec{\Omega} = \vec{\omega}$ has been dubbed "co-rotating Fermi-Walker transport" [27, 28]. This choice is more intuitive in the special case of a shear-free congruence, where, as we will show next, the axes of the frame thereby defined point towards *fixed* neighboring observers. Indeed, if X^α is a connecting vector between two neighboring observers of the congruence and Y^α is its component orthogonal to the congruence, we have

$$\begin{aligned} \nabla_{\mathbf{u}} Y^{\hat{i}} &= \dot{Y}^{\hat{i}} + \Gamma^{\hat{i}}_{\hat{0}\hat{0}} Y^{\hat{0}} + \Gamma^{\hat{i}}_{\hat{0}\hat{j}} Y^{\hat{j}} \\ &= \dot{Y}^{\hat{i}} + \Omega^{\hat{i}}_{\ \hat{j}} X^{\hat{j}}. \end{aligned} \tag{50}$$

where the dot denotes the ordinary derivative along $\mathbf{u}$: $\dot{A}_{\hat{\alpha}} \equiv A_{\hat{\alpha},\hat{\beta}} u^{\hat{\beta}}$. Since, from (43), $\nabla_{\mathbf{u}} Y^{\hat{i}} = K^{\hat{i}}_{\ \hat{j}} Y^{\hat{j}}$, we conclude that

$$\dot{Y}_{\hat{i}} = \left(\sigma_{\hat{i}\hat{j}} + \frac{1}{3}\theta\delta_{\hat{i}\hat{j}} + \omega_{\hat{i}\hat{j}} - \Omega_{\hat{i}\hat{j}} \right) Y^{\hat{j}}. \tag{51}$$

This tells us that for a shear-free congruence ($\sigma_{\hat{i}\hat{j}} = 0$), if we lock the rotation $\vec{\Omega}$ of the tetrad to the vorticity $\vec{\omega}$ of the congruence, $\Omega_{\hat{i}\hat{j}} = \omega_{\hat{i}\hat{j}}$, the connecting vector's direction is fixed on the tetrad (and if in addition $\theta = 0$, i.e., the congruence is rigid, the connecting vectors have constant components on the tetrad). A familiar example is the rigidly rotating frame in flat spacetime; in the non-relativistic limit, the vorticity of the congruence formed by the rigidly rotating observers is constant, and equals the angular velocity of the frame; in this case, by choosing $\vec{\Omega} = \vec{\omega}$, one is demanding that the spatial triads $\mathbf{e}_{\hat{i}}$ carried by the observers co-rotate with the angular velocity of the congruence; hence it is clear that the axes $\mathbf{e}_{\hat{i}}$ always point to the same neighboring observers. For relativistic rotation, the vorticity $\vec{\omega}$ is not constant and no longer equals the (constant) angular velocity of the rotating observers; but it is still the condition $\vec{\Omega} = \vec{\omega}$ that ensures that the tetrads are rigidly anchored to the observer congruence. Another example is the family of the so-called "static" observers in Kerr spacetime, which is very important in this context, because it is this construction which allows one to determine the rotation of the frame of the "distant stars" with respect to a local gyroscope, as we shall see in Sec. 3.3.

3.2 Geodesics. Inertial forces — "gravitoelectromagnetic fields"

The spatial part of the geodesic equation for a test particle of 4-velocity U^α, $\nabla_{\mathbf{U}} U^\alpha \equiv DU^\alpha/d\tau = 0$, reads, in the frame $e_{\hat{\alpha}}$:

$$\frac{dU^{\hat{i}}}{d\tau} + \Gamma^{\hat{i}}_{\hat{0}\hat{0}} (U^{\hat{0}})^2 + \left(\Gamma^{\hat{i}}_{\hat{0}\hat{j}} + \Gamma^{\hat{i}}_{\hat{j}\hat{0}} \right) U^{\hat{0}} U^{\hat{j}} + \Gamma^{\hat{i}}_{\hat{j}\hat{k}} U^{\hat{k}} U^{\hat{j}} = 0 \; .$$

Substituting (38), (39) and (49), we have

$$\frac{\tilde{D}\vec{U}}{d\tau} = U^{\hat{0}} \left[U^{\hat{0}} \vec{G} + \vec{U} \times \vec{H} - \sigma^{\hat{i}}_{\ \hat{j}} U^{\hat{j}} \mathbf{e}_{\hat{i}} - \frac{1}{3}\theta \vec{U} \right] \equiv \vec{F}_{\rm GEM} \tag{52}$$

where

$$\frac{\tilde{D}U^{\hat{i}}}{d\tau} = \frac{dU^{\hat{i}}}{d\tau} + \Gamma^{\hat{i}}_{\hat{j}\hat{k}} U^{\hat{k}} U^{\hat{j}} \; . \tag{53}$$

Here $\vec{G} = -\vec{a}$ is the *"gravitoelectric field"*, and $\vec{H} = \vec{\omega} + \vec{\Omega}$ is the *"gravitomagnetic field"*. These designations are due to the analogy with the roles that the electric and magnetic fields play in the electromagnetic Lorentz force, which reads in the tetrad

$$\frac{D\vec{U}}{d\tau} = \frac{q}{m}\left(U^{\hat{0}}\vec{E} + \vec{U} \times \vec{B}\right), \tag{54}$$

with $\vec{E} \equiv \vec{E}(u)$ and $\vec{B} \equiv \vec{B}(u)$ denoting the electric and magnetic fields as measured by the observers u^α. It is useful to write the GEM fields in a manifestly covariant form:

$$(G^u)^\alpha = -\nabla_{\mathbf{u}} u^\alpha \equiv -u^\alpha{}_{;\beta} u^\beta \; ; \qquad (H^u)^\alpha = \omega^\alpha + \Omega^\alpha \; . \tag{55}$$

The gravitomagnetic field $(H^u)^\alpha$ thus consists of two parts of different origins: the angular velocity Ω^α of rotation of the spatial triads relative to Fermi-Walker transport (i.e., to local guiding gyroscopes), plus the vorticity ω^α of the congruence of observers u^α. If we lock the rotation of the triads to the vorticity of the congruence, $\Omega^\alpha = \omega^\alpha$, the gravitomagnetic field becomes simply twice the vorticity: $(H^u)^\alpha = 2\omega^\alpha$.

The last two terms of (52) have no electromagnetic counterpart; they consist of the shear/expansion tensor $K_{(\alpha\beta)}$, which is sometimes called the *second fundamental form* of the distribution of hyperplanes orthogonal to $\mathbf{u}$. If this distribution is integrable (that is, if there is no vorticity) then $K_{(\alpha\beta)}$ is just the extrinsic curvature of the time slices orthogonal to $\mathbf{u}$. These terms correspond to the time derivative of the spatial metric $(h^u)_{\alpha\beta}$, that locally measures the spatial distances between neighboring observers; this can be seen noting that $K_{(\alpha\beta)} = \mathcal{L}_{\mathbf{u}}(h^u)_{\alpha\beta} = (h^u)_{\alpha\beta,0}u^0$, the last equality holding in the $u^i = 0$ frame.

3.2.1 The derivative operator $\tilde{D}/d\tau$ and inertial forces

$\vec{F}_{\rm GEM} = \tilde{D}\vec{U}/d\tau$ describes the inertial accelerations (forces) associated to an *arbitrary* orthonormal frame; we shall now justify this statement, and the splitting of the connection made in Eq. (52). We start by noticing that $\tilde{D}\vec{U}/d\tau$ is a spatial vector which is the derivative of another spatial vector (the spatial velocity $U^{\langle\alpha\rangle} = (h^u)^\alpha{}_\beta U^\beta$, or $\vec{U}$ in the $u^i = 0$ frame); mathematically, this is determined by a connection on the vector bundle of all spatial vectors. There is a Riemannian metric naturally defined on this vector bundle, the spatial metric $(h^u)_{\alpha\beta}$, and the most obvious connection preserving the spatial metric is the orthogonal projection $\nabla^\perp$ of the ordinary spacetime covariant derivative,

$$\nabla^\perp_\alpha X^\beta \equiv (h^u)^\beta{}_\gamma \nabla_\alpha X^\gamma, \tag{56}$$

which in terms of the tetrad components is written

$$\nabla^\perp_{\hat{\alpha}} X^{\hat{i}} = \nabla_{\hat{\alpha}} X^{\hat{i}} = X^{\hat{i}}{}_{,\hat{\alpha}} + \Gamma^{\hat{i}}_{\hat{\alpha}\hat{j}} X^{\hat{j}}. \tag{57}$$

We shall call $\nabla^\perp$ the *Fermi-Walker connection*, since the parallel transport that it defines along the congruence is exactly the Fermi-Walker transport; this is because the spatially projected covariant derivative of a *spatial* vector X^α equals its Fermi-Walker derivative:

$$\nabla^\perp_{\mathbf{u}} X^\alpha = (h^u)^\alpha_\beta \nabla_{\mathbf{u}} X^\beta = \nabla_{\mathbf{u}} X^\alpha - u^\alpha X_\beta \nabla_{\mathbf{u}} u^\beta \tag{58}$$

Along any curve with tangent vector $\mathbf{U}$ we have

$$\begin{aligned}\frac{D^\perp X^{\hat{i}}}{d\tau} &\equiv \nabla^\perp_{\mathbf{U}} X^{\hat{i}} = \frac{dX^{\hat{i}}}{d\tau} + \Gamma^{\hat{i}}_{\hat{0}\hat{j}} U^{\hat{0}} X^{\hat{j}} + \Gamma^{\hat{i}}_{\hat{j}\hat{k}} U^{\hat{j}} X^{\hat{k}} \\ &= \frac{dX^{\hat{i}}}{d\tau} + \Omega^{\hat{i}}_{\ \hat{j}} U^{\hat{0}} X^{\hat{j}} + \Gamma^{\hat{i}}_{\hat{j}\hat{k}} U^{\hat{j}} X^{\hat{k}} \ ,\end{aligned} \tag{59}$$

and so along the congruence,

$$\nabla^\perp_{\mathbf{u}} X^{\hat{i}} = \dot{X}^{\hat{i}} + \Omega^{\hat{i}}_{\ \hat{j}} X^{\hat{j}}. \tag{60}$$

Notice that the Fermi-Walker connection preserves the spatial metric: if $\vec{X}$ and $\vec{Y}$ are spatial vector fields then we have

$$\begin{aligned}\frac{d}{d\tau}(\vec{X}\cdot\vec{Y}) &= \frac{d}{d\tau}(\delta_{\hat{i}\hat{j}} X^{\hat{i}} Y^{\hat{j}}) = \delta_{\hat{i}\hat{j}}\left(\frac{dX^{\hat{i}}}{d\tau} Y^{\hat{j}} + X^{\hat{i}} \frac{dY^{\hat{j}}}{d\tau}\right) = \delta_{\hat{i}\hat{j}}\left(\frac{D^\perp X^{\hat{i}}}{d\tau} Y^{\hat{j}} + X^{\hat{i}} \frac{D^\perp Y^{\hat{j}}}{d\tau}\right) \\ &= \frac{D^\perp \vec{X}}{d\tau}\cdot\vec{Y} + \vec{X}\cdot\frac{D^\perp \vec{X}}{d\tau},\end{aligned}$$

where we used $\Omega^{\hat{i}}_{\ \hat{j}} = -\Omega^{\hat{j}}_{\ \hat{i}}$ and $\Gamma^{\hat{i}}_{\hat{j}\hat{k}} = -\Gamma^{\hat{k}}_{\hat{j}\hat{i}}$.

Eq. (59) yields the variation, along a curve of tangent $\mathbf{U}$, of a spatial vector X^α, with respect to a triad of spatial axes undergoing Fermi-Walker transport *along the congruence*. But our goal is to measure "accelerations" (i.e, the variation of the spatial velocity $U^{\langle\alpha\rangle} \equiv (h^u)^\alpha_{\ \beta} U^\beta$) with respect to some chosen orthonormal frame, whose triad vectors $\mathbf{e}_{\hat{i}}$ rotate along the congruence (according to the Fermi-Walker connection, cf. Eq. (37)) with an angular velocity $\vec{\Omega}$, that one may arbitrarily specify. We need thus to define a connection for which the triad vectors are constant along $\mathbf{u}$; the simplest modification of the Fermi-Walker connection which achieves this goal is given by

$$\tilde{\nabla}_\alpha X^\beta = \nabla^\perp_\alpha X^\beta + u_\alpha (h^u)^\beta_{\ \gamma} \Omega^\gamma_{\ \delta} X^\delta = (h^u)^\beta_{\ \gamma} \nabla_\alpha X^\gamma + u_\alpha (h^u)^\beta_{\ \gamma} \Omega^\gamma_{\ \delta} X^\delta, \tag{61}$$

or, in tetrad components,

$$\tilde{\nabla}_{\hat{\alpha}} X^{\hat{i}} = \nabla^\perp_{\hat{\alpha}} X^{\hat{i}} - \delta^{\hat{0}}_{\hat{\alpha}} \Omega^{\hat{i}}_{\ \hat{j}} X^{\hat{j}} = X^{\hat{i}}_{\ ,\hat{\alpha}} + \Gamma^{\hat{i}}_{\hat{\alpha}\hat{j}} X^{\hat{j}} - \delta^{\hat{0}}_{\hat{\alpha}} \Omega^{\hat{i}}_{\ \hat{j}} X^{\hat{j}}, \tag{62}$$

so that

$$\frac{\tilde{D} X^{\hat{i}}}{d\tau} = \nabla^\perp_{\mathbf{U}} X^{\hat{i}} - \Omega^{\hat{i}}_{\ \hat{j}} U^{\hat{0}} X^{\hat{j}} = \frac{dX^{\hat{i}}}{d\tau} + \Gamma^{\hat{i}}_{\hat{j}\hat{k}} U^{\hat{j}} X^{\hat{k}}.$$

Similarly to what was done for the Fermi-Walker connection, it is easy to see that $\tilde{\nabla}$ preserves the spatial metric:

$$\frac{d}{d\tau}(\vec{X}\cdot\vec{Y}) = \frac{\tilde{D}\vec{X}}{d\tau}\cdot\vec{Y} + \vec{X}\cdot\frac{\tilde{D}\vec{X}}{d\tau}.$$

This justifies the status of the operator $\tilde{D}/d\tau$ as a covariant derivative along a curve, *preserving the spatial metric* $(h^u)_{\alpha\beta}$, for spatial vectors [56]. The inertial forces ($F^\alpha_{\rm GEM}$) of a given frame are given by the derivative $\tilde{D}/d\tau$ (setting Ω^α for such frame) acting on the spatial velocity $U^{\langle\alpha\rangle}$ of a particle undergoing geodesic motion. In covariant form, we have[11]

$$F^\alpha_{\rm GEM} \equiv \frac{\tilde{D} U^{\langle\alpha\rangle}}{d\tau} = \frac{D^\perp U^{\langle\alpha\rangle}}{d\tau} + \gamma \epsilon^\alpha_{\ \beta\gamma\delta} u^\delta U^\beta \Omega^\gamma \ = -\gamma \frac{D^\perp u^\alpha}{d\tau} + \gamma \epsilon^\alpha_{\ \beta\gamma\delta} u^\delta U^\beta \Omega^\gamma. \tag{63}$$

[11] This corresponds to a generalized version of Eqs. (6.13) or (6.18) of [27], which in the scheme therein would follow from a generalized spatial "derivative" of the type (5.3), but allowing for an arbitrary rotation of the spatial triad, in the place of the 3-fold definitions (2.19), (3.11) therein.

where $\gamma \equiv -U_\alpha u^\alpha$. In the last equality, we decomposed U^α in its projections parallel and orthogonal to the congruence, $U^\alpha = \gamma u^\alpha + U^{\langle\alpha\rangle}$, and used the geodesic equation, $DU^\alpha/d\tau = 0$, to note that $D^\perp U^{\langle\alpha\rangle}/d\tau = -D^\perp u^\alpha/d\tau$. Eq. (63) manifests that $F^\alpha_{\rm GEM}$ consists of two terms of distinct origin: the first term which depends only on the variation of the observer velocity u^α along the test particle's worldline, and the second term which is *independent* of the observer congruence, and arises from the transport law for the spatial triads along u^α. These two contributions are illustrated, for simple examples in flat spacetime, in Appendix A.

Using $Du^\alpha/d\tau = u^{\alpha;\beta}U_\beta$, and decomposing $u_{\alpha;\beta}$ in the congruence kinematics, cf. Eqs. (44), (46), (47),

$$u_{\alpha;\beta} = -a(u)_\alpha u_\beta - \epsilon_{\alpha\beta\gamma\delta}\omega^\gamma u^\delta + K_{(\alpha\beta)} \ , \tag{64}$$

we get, substituting into (63),

$$F^\alpha_{\rm GEM} = \gamma\left[\gamma G^\alpha + \epsilon^\alpha_{\ \beta\gamma\delta}u^\delta U^\beta(\omega^\gamma + \Omega^\gamma) - K^{(\alpha\beta)}U_\beta\right] \ , \tag{65}$$

which is Eq. (52) in covariant form.

The derivative (53) has a geometrical interpretation that is simple when the $\Gamma^{\hat{i}}_{\hat{j}\hat{k}}$ are the Levi-Civita connection coefficients of some space Riemannian manifold. The quotient of spacetime by the congruence (the "space of observers") has a natural, well defined 3-D spatial metric[12] in two special cases: $K_{(\alpha\beta)} = 0$ and/or $\omega^\alpha = 0$; in those cases $\Gamma^{\hat{i}}_{\hat{j}\hat{k}}$ are the (Levi-Civita) connection coefficients of such metric provided that *also* $\Omega^\alpha = \omega^\alpha$ (see footnote 14 below). The sub-case $K_{(\alpha\beta)} = 0$ (rigid congruence), explored in detail in Sec. 3.2.2 below, is especially interesting, as in this case the spatial metric $(h^u)_{\alpha\beta}$ (γ_{ij}, in the notation of Sec. 3.2.2) is time-independent, measuring the *constant* distance between neighboring observers[13]. This is the natural metric of the quotient space (and is independent of the time-slice). In this case there is also a well defined 3-D curve obtained by projecting the particle's worldline $z^\alpha(\tau)$ on the space manifold. Let such 3-D curve still be parametrized by τ; then $\vec{U}$ is its tangent vector, and $\tilde{D}\vec{U}/d\tau$ is just the usual covariant derivative, *with respect the spatial metric*, of $\vec{U}$ — that is, the 3-D acceleration of the curve. Note that expression (53) is identical to the usual definition of 3-D acceleration for curved spaces (or non-rectangular coordinate systems in Euclidean space), e.g. Eq. (6.9) of [127].

It is easy to see in this sub-case ($\Omega^\alpha = \omega^\alpha$, $K_{(\alpha\beta)} = 0$) that $\vec{F}_{\rm GEM} = \tilde{D}\vec{U}/d\tau$ corresponds to the usual notion of inertial force from classical mechanics. Take a familiar example, a rigidly rotating frame in *flat* spacetime (as discussed in Appendix A); we are familiar with the inertial forces arising in such frame, from e.g. a merry go round. They are in this case a gravitoelectric field $\vec{G} = \vec{\omega} \times (\vec{r} \times \vec{\omega})$, due to the acceleration of the rigidly rotating observers, that causes a centrifugal force, plus a gravitomagnetic field $\vec{H} = 2\vec{\omega}$, half of it originating from the observers vorticity $\vec{\omega}$, and the other half from the rotation $\vec{\Omega} = \vec{\omega}$ (relative to Fermi-Walker transport) of the spatial triads they carry. $\vec{H}$ causes the Coriolis (or gravitomagnetic) force $\vec{U} \times \vec{H} = 2\vec{U} \times \vec{\omega}$. These centrifugal and Coriolis forces become, in the non-relativistic limit (so that the vorticity $\vec{\omega}$ equals the angular velocity of the rotating frame) the well known expressions from non-relativistic mechanics e.g. Eq. (4-107) of [128]. This example is also useful to understand the splitting of the connection made in Eq. (52). The fields $\vec{G}$ and $\vec{H}$ are codified in the mixed time-space components of the

[12]One always has a spatial metric $(h^u)_{\alpha\beta}$ defined at each point; but generically this is a metric defined on a fiber bundle (the bundle of spatial vectors), and does not coincide with the metric of a manifold.

[13]Obtained by lifting vectors on the quotient to vectors orthogonal to the corresponding observer's worldline.

connection coefficients $\Gamma^{\hat{\alpha}}_{\hat{\beta}\hat{\gamma}}$ of this frame, cf. Eqs. (38), (39), (49); and they vanish if one switches to an inertial frame. The purely spatial connection coefficients $\Gamma^{\hat{i}}_{\hat{j}\hat{k}}$, in turn, do not necessarily vanish in such frame; they do not encode inertial forces, but merely correct for the fact that the orientation of the spatial triads may trivially change from point to point along the directions orthogonal to the congruence (for instance the triads associated with a non-rectangular coordinate system). In a general curved spacetime, there are no global inertial frames, and therefore $\vec{F}_{\rm GEM}$ cannot be made to vanish (that is, one can always make $\vec{G}$ and $\vec{H}$ vanish by choosing inertial, vorticity free observer congruences, and Fermi-Walker transported spatial triads along them; but the shear/expansion tensor $K_{(\alpha\beta)}$ will not vanish for such congruences).

In the sub-case $\omega^{\alpha} = 0$, the congruence is hypersurface orthogonal, and there is a natural (time-dependent, in general) 3-D metric defined on those hypersurfaces; but (if $K_{(\alpha\beta)} \neq 0$) the interpretation of $\vec{F}_{\rm GEM}$ is a bit more tricky, because we cannot talk of a projected curve, as the space manifold itself changes with time. In this case, we can make a point-wise interpretation of (53): *at each point*, it is the 3-D covariant acceleration of the projection of the particle's worldline on that hypersurface *at that point*.

In the more general case ($\omega^{\alpha} \neq 0$ and $K_{(\alpha\beta)} \neq 0$) there is no natural (even time-dependent) metric associated to the quotient space, as the distance between observers depends on the time slices; the $\Gamma^{\hat{i}}_{\hat{j}\hat{k}}$ are not the Levi-Civita connection coefficients of some space Riemannian manifold (we can see from Eq. (101) below that the associated 3-D curvature tensor does not obey the first Bianchi identity). This is also the situation in general if[14] $\omega^{\alpha} \neq \Omega^{\alpha}$. Therefore, in these cases, it is not possible to interpret (53) as the acceleration of a projected curve in the space manifold (as none of them is defined in general); but it still yields what one would call the inertial forces of the given frame, which is exemplified in the case of flat spacetime in Appendix A.

Usefulness of the general equation. — An equation like (52), yielding the inertial forces in an arbitrary frame, in particular allowing for an arbitrary rotation Ω^{α} of the spatial triads along u^{α}, *independent* of ω^{α}, is of interest in many applications. Although the congruence adapted frame, $\vec{\Omega} = \vec{\omega} = \vec{H}/2$, might seem the most natural frame associated to a given family of observers, other frames are used in the literature, and the associated gravitomagnetic effects discussed therein. Our general definition of $\vec{H}$, Eq. (55), encompasses the many different gravitomagnetic fields that have been defined in the different approaches. That includes the case of the reference frames sometimes employed in the context of black hole physics and astrophysics [25, 116, 117, 118, 26, 119]: the tetrads carried by hypersurface orthogonal observers, whose spatial axis are taken to be fixed to the background symmetries; for instance, in the Kerr spacetime, the congruence are the zero angular momentum observers (ZAMOS), and the spatial triads are fixed to the Boyer-Lindquist spatial coordinate basis vectors. This tetrad field has been dubbed in some literature "locally non-rotating frames"[15] [117, 116, 118, 26] or "proper reference frames of the fiducial observers" [25]. It is regarded as important for black hole physics because it is a reference frame that is defined everywhere (unlike for instance the star fixed static observers, see Sec. 3.3 below, that do not exist

[14]The generalization of Eq. (101) for non-congruence adapted frames is $\star\tilde{R}^{j}{}_{ji} = 2\epsilon_{ikj}\omega^{j}\Omega^{k} - 2K_{(ik)}\omega^{k}$; the first term is not zero in general when $\vec{\Omega} \neq \vec{\omega}$. One could also immediately see from Eq. (92) that $\tilde{R}_{ijkl}$ does not possess the pair exchange symmetry $\{ij\} \leftrightarrow \{jk\}$ if $\Omega_{ij} \neq \omega_{ij}$.

[15]Somewhat erroneously, as the tetrads *do rotate* with respect to the local compass of inertia, since they are not Fermi-Walker transported in general [118, 119].

past the ergosphere). Eq. (52) allows us to describe the inertial forces in these frames, where $\vec{\omega} = 0$, and $\vec{H} = \vec{\Omega} = N^{-1}\tilde{\nabla} \times \vec{\beta}$; that is, all the gravitomagnetic accelerations come from Ω^α (N, $\vec{\beta}$ denote, respectively, the lapse function and the shift vector [25, 26]). Frames corresponding to a congruence with vorticity, but where the spatial triads are chosen to be Fermi-Walker transported, $\vec{\Omega} = 0$, have also been considered; in such frames $\vec{H} = \vec{\omega}$ (dubbed the "Fermi-Walker gravitomagnetic field" [27, 28]).

Finally, it is worth noting that the GEM "Lorentz" forces from the more popular linearized theory [11, 7] or post-Newtonian approaches [70, 63, 62] are special cases of Eq. (52) (e.g. linearizing it, one obtains Eq. (2.5) of [51]; further specializing to stationary fields, one obtains e.g. (6.1.26) of [7]).

3.2.2 Stationary fields — "quasi-Maxwell" formalism

If one considers a stationary spacetime, *and* a frame where it is explicitly time-independent (i.e., a congruence of observers u^α tangent to a time-like Killing vector field, which necessarily means that the congruence is rigid [113]), the last two terms of Eq. (52) vanish and the geodesic equation becomes formally similar to the Lorentz force (54):

$$\frac{\tilde{D}\vec{U}}{d\tau} = U^{\hat{0}}\left(U^{\hat{0}}\vec{G} + \vec{U} \times \vec{H}\right). \tag{66}$$

The line element of a *stationary spacetime* is generically described by:

$$ds^2 = -e^{2\Phi}(dt - \mathcal{A}_i dx^i)^2 + \gamma_{ij}dx^i dx^j \tag{67}$$

with Φ, $\vec{\mathcal{A}}$, γ_{ij} time-independent. Here $\gamma_{ij} = (h^u)_{ij}$ is a Riemannian metric, not flat in general, that measures the time-constant distance between stationary observers, as measured by the Einstein light signaling procedure [18]. This is the metric one can naturally associate with the quotient space in the case a rigid congruence; and, as discussed above, Eq. (66) is just the acceleration of the 3-D curves obtained by projecting test-particle's geodesics in the 3-D manifold of metric γ_{ij}. The GEM fields measured by the static observers (i.e. the observers of zero 3-velocity in the coordinate system of 67) are related with the metric potentials by [20]:

$$\vec{G} = -\tilde{\nabla}\Phi; \quad \vec{H} = e^{\Phi}\tilde{\nabla} \times \vec{\mathcal{A}}, \tag{68}$$

with $\tilde{\nabla}$ denoting the covariant differentiation operator *with respect to the spatial metric* γ_{ij}. The formulation (68) of GEM fields applying to stationary spacetimes is the most usual one; it was introduced in [18], and further worked out in e.g. [21, 20, 25, 23, 19], and is sometimes called the "quasi-Maxwell formalism".

3.3 Gyroscope precession

One of the main results of this approach is that, within this formalism (and if the Mathisson-Pirani spin condition is employed), the equation describing the evolution of the spin-vector of a gyroscope in a gravitational field takes a form exactly analogous to precession of a magnetic dipole under the action of a magnetic field *when expressed in a local orthonormal tetrad comoving with the test particle.*

As we have seen in Sec. 2.3, if the Mathisson-Pirani condition holds, the spin vector of a torque-free gyroscope is Fermi-Walker transported, cf. Eq. (27). Let U^α be the 4-velocity of the gyroscope; in a comoving orthonormal tetrad $e_{\hat{\alpha}}$, $U^{\hat{\alpha}} = \delta^{\hat{\alpha}}_{\hat{0}}$ and $S^{\hat{0}} = 0$; therefore, Eq. (27) reduces in such frame to:

$$\frac{DS^{\hat{i}}}{d\tau} = 0 \Leftrightarrow \frac{dS^{\hat{i}}}{d\tau} = -\Gamma^{\hat{i}}_{\hat{0}\hat{k}} S^{\hat{k}} = \left(\vec{S} \times \vec{\Omega}\right)^{\hat{i}}. \tag{69}$$

This is the natural result. If we choose a Fermi-Walker transported frame ($\vec{\Omega} = 0$), which is mathematically defined as a frame with no "absolute" spatial rotation, cf. Eq. (37), then gyroscopes, which are understood as objects that "oppose" to changes in direction (and interpreted as determining the local "compass of inertia" [7]), have their axes fixed with respect to such frame: $d\vec{S}/d\tau = 0$. Otherwise gyroscopes are seen to "precesses" with an angular velocity $-\vec{\Omega}$, that is in fact just *minus* the angular velocity of rotation *of the chosen frame* relative to a Fermi-Walker transported frame. Now, for a congruence adapted frame, $\vec{\Omega} = \vec{\omega}$, Eq. (69) becomes:

$$\frac{d\vec{S}}{d\tau} = \frac{1}{2}\vec{S} \times \vec{H}. \tag{70}$$

Thus, the "precession" of a gyroscope is given, in terms of the gravitomagnetic field $\vec{H}$, by an expression identical (up to a factor of 2) to the precession of a magnetic dipole under the action of a magnetic field $\vec{B}$, cf. Eq. (31):

$$\frac{D\vec{S}}{d\tau} = \vec{\mu} \times \vec{B}. \tag{71}$$

This holds for arbitrary fields, hence in this case the one to one correspondence with electromagnetism is more general than the one for the geodesics described above (between Eqs. (66) and (54)), which required the fields to be stationary *and* the observers to be stationary (i.e., their worldlines be tangent to a Killing vector field); herein by contrast the only conditions are the observer to be comoving with the gyroscope, and using an orthonormal tetrad. Also, the earlier result obtained for weak fields in [51] (that the analogy holds even if the fields are time dependent) is just a special case of this principle.

However important differences should be noted: whereas in the electromagnetic case it is the same field $\vec{B}$ that is at the origin of both the magnetic force $q(\vec{U} \times \vec{B})$ in Eq. (54) and the torque $\vec{\mu} \times \vec{B}$ on the magnetic dipole, in the case of the gravitomagnetic force $\vec{U} \times \vec{H}$ it has, in the general formulation, a different origin from gyroscope "precession", since the former arises not only from the rotation $\vec{\Omega}$ of the frame relative to a local Fermi-Walker transported tetrad, but also from the vorticity $\vec{\omega}$ of the congruence. In this sense, one can say that the Lense-Thirring effect detected in the LAGEOS satellite data [59] (and presently under experimental scrutiny by the ongoing LARES mission [61]), measuring $\vec{H}$ from test particle's deflection, is of a different mathematical origin from the one which was under scrutiny by the Gravity Probe B mission [60], measuring $\vec{\Omega}$ from gyroscope precession, the two being made to match by measuring both effects relative to the "frame of the distant stars" (discussed below), for which $\vec{\Omega} = \vec{\omega} = \vec{H}/2$. This type of frame (i.e. congruence adapted) is the most usual in the literature; in this case the fields causing the gravitomagnetic force and the precession of a gyroscope differ only by a relative factor of 2. But it is important to not overlook their distinct origin, as in the literature GEM fields of frames which are not congruence adapted are considered as well; for instance the "Fermi-Walker gravitomagnetic field" defined in [27], which is the $\vec{H}$ of a frame corresponding to a congruence with vorticity, but where the spatial triads are chosen to be Fermi-Walker transported: $\vec{\Omega} = 0$. Thus there is a non-vanishing $\vec{H} = \vec{\omega}$ in this frame, whereas at the same time gyroscopes do not precess relative to it.

Another obvious difference between Eqs. (71) and (70) is the presence of a covariant derivative in the former, whereas in the latter we have a simple derivative, signaling that $\vec{B}$ is a *physical* field, and $\vec{H}$ a mere artifact of the reference frame (which can be anything, depending on the frame one chooses), that can always be made to vanish (in the congruence adapted case, $\vec{H} = 2\vec{\omega}$, by choosing a vorticity-free observer congruence).

Frame dragging. — The fact that $\vec{H}$ is a reference frame artifact does not mean it is necessarily meaningless; indeed it has no *local* physical significance, but it can tell us about frame dragging, which is a *non-local* physical effect. That is the case when one chooses the so-called "frame of the distant stars", a notion that applies to asymptotically flat spacetimes. In stationary spacetimes, such frame is setup as follows: consider a rigid congruence of stationary observers such that at infinity it coincides with the asymptotic inertial rest frame of the source — the axes of the latter define the directions fixed relative to the distant stars. These observers are interpreted as being "at rest" with respect to the distant stars (and also at rest with respect to the asymptotic inertial frame of the source); since the congruence is rigid, it may be thought as a grid of points rigidly fixed to the distant stars. For this reason we dub them "static observers"[16]. This congruence fixes the time axis of the local tetrads of the frame. Now if we demand the rotation $\vec{\Omega}$ of the spatial triads (relative to Fermi-Walker transport) to equal the vorticity $\vec{\omega}$ of the congruence, we see from Eq. (51) that the connecting vectors between different observers are constant in the tetrad:

$$\dot{Y}^{\hat{i}} = 0 \ ;$$

in other words, each local spatial triad $e_{\hat{i}}$ is locked to this grid, and therefore has directions fixed to the distant stars. Hence, despite having no *local* meaning, the gravitomagnetic field $\vec{H} = 2\vec{\Omega} = 2\vec{\omega}$ describes in this case a consequence of the frame dragging effect: the fact that a torque free gyroscope at finite distance from a rotating source precesses with respect to an inertial frame at infinity. This is a physical effect, that clearly distinguishes, for instance, the Kerr from the Schwarzschild spacetimes, but is non-local (i.e., it cannot be detected in any local measurement; only by locking to the distant stars by means of telescopes). It should be noted, however, that the relative precession of two neighboring (comoving) system of gyroscopes is locally measurable, and encoded in the curvature tensor (more precisely, in the gravitomagnetic tidal tensor $\mathbb{H}_{\alpha\beta}$, as discussed in Sec. 2.3).

3.4 Field equations

The Einstein field equations and the algebraic Bianchi identity, Eqs. (14), can be generically written in this exact GEM formalism — i.e., in terms of $\vec{G}$, $\vec{\Omega}$, $\vec{\omega}$ and $K_{(\alpha\beta)}$. These equations will be compared in this section with the analogous electromagnetic situation: Maxwell's equations in an arbitrarily accelerated, rotating and shearing frame. The latter will be of use also in Sec. 6. As

[16] In the case of Kerr spacetime, these are the observers whose worldlines are tangent to the temporal Killing vector field $\xi = \partial/\partial t$, i.e., the observers of zero 3-velocity in Boyer-Lindquist coordinates. This agrees with the convention in [112, 53]. We note however that the denomination "static observers" is employed in some literature (e.g. [100, 101]) with a different meaning, where it designates hypersurface orthogonal time-like Killing vector fields (which are rigid, *vorticity-free* congruences, existing only in *static spacetimes*).

a special case, we will also consider *stationary spacetimes* (and rigid, congruence adapted frames therein), where we recover the quasi-Maxwell formalism of e.g. [20, 24, 18, 22, 23, 19]. In this case, the similarity with the electromagnetic analogue — Maxwell's equations in a rigid, but arbitrarily accelerated and rotating frame — becomes closer, as we shall see.

Before proceeding, let us write the following relations which will be useful. Let A^α be a *spatial* vector; in the tetrad we have:

$$\tilde{\nabla}_{\hat{i}} A^{\hat{j}} = A^{\hat{j}}{}_{,\hat{i}} + \Gamma^{\hat{j}}_{\hat{k}\hat{i}} A^{\hat{k}} \; ; \tag{72}$$

$$\nabla_{\mathbf{u}} A^{\hat{i}} = \dot{A}^{\hat{i}} + \Gamma^{\hat{i}}_{\hat{0}\hat{j}} A^{\hat{j}} = \dot{A}^{\hat{i}} + \Omega^{\hat{i}}{}_{\hat{j}} A^{\hat{j}} \; ; \tag{73}$$

$$A^{\beta}{}_{;\beta} = \left(A^{\hat{i}}{}_{,\hat{i}} + A^{\hat{i}} \Gamma^{\hat{j}}_{\hat{j}\hat{i}} \right) + A^{\hat{i}} \Gamma^{\hat{0}}_{\hat{0}\hat{i}} = \tilde{\nabla} \cdot \vec{A} + \vec{A} \cdot \vec{a} \; , \tag{74}$$

where we used (62) and the connection coefficients (38)-(39), and the dot denotes the ordinary derivative along the observer worldline, $\dot{A}^{\hat{\alpha}} \equiv A^{\hat{\alpha}}{}_{,\hat{\beta}} u^{\hat{\beta}}$. $\tilde{\nabla}$ is the connection defined in Eqs. (61)-(62); since in expressions (72) and (74) the derivatives are along the spatial directions, one could as well have used the Fermi-Walker connection $\nabla^\perp$, Eqs. (56)-(57), they are the same along these directions.

3.4.1 Maxwell equations for the electromagnetic fields measured by an arbitrary congruence of observers

Using decomposition (1), we write Maxwell's Eqs. (6) in terms of the electric and magnetic fields $(E^u)^\alpha = F^\alpha_{\ \beta} u^\beta$ and $(B^u)^\alpha = \star F^\alpha_{\ \beta} u^\beta$ measured by the congruence of observers of 4-velocity u^α. All the fields below are measured with respect to this congruence, so we may drop the superscripts u: $(E^u)^\alpha \equiv E^\alpha$, $(B^u)^\alpha \equiv B^\alpha$. The time projection of Eq. (6a) with respect to u^α (see point 2 of Sec. 1.1) reads:

$$E^{\beta}{}_{;\beta} = 4\pi\rho_c + E^\alpha a_\alpha + 2\omega_\alpha B^\alpha \; . \tag{75}$$

Using (74), we have in the tetrad:

$$\tilde{\nabla} \cdot \vec{E} = 4\pi\rho_c + 2\vec{\omega} \cdot \vec{B} \; . \tag{76}$$

Analogously, for the time projection of (6b) we get

$$B^{\beta}{}_{;\beta} = B^\alpha a_\alpha - 2\omega^\mu E_\mu \; , \tag{77}$$

which in the tetrad becomes

$$\tilde{\nabla} \cdot \vec{B} = -2\vec{\omega} \cdot \vec{E} \; . \tag{78}$$

The space projection of Eq. (6a) reads:

$$\epsilon^{\alpha\gamma\beta} B_{\beta;\gamma} = \nabla^\perp_{\mathbf{u}} E^\alpha - K^{(\alpha\beta)} E_\beta + \theta E^\alpha - \epsilon^\alpha_{\ \beta\gamma} \omega^\beta E^\gamma + \epsilon^\alpha_{\ \beta\gamma} B^\beta a^\gamma + 4\pi j^{\langle\alpha\rangle} \; , \tag{79}$$

where the index notation $\langle\mu\rangle$ stands for the spatially projected part of a vector, $V_{\langle\mu\rangle} \equiv h_\mu^{\ \nu} V_\nu$, and $\epsilon^{\mu\beta\sigma} \equiv \epsilon^{\mu\beta\sigma\alpha} u_\alpha$. The tetrad components of (79) in the frame defined in Sec. 3.1 read:

$$(\tilde{\nabla} \times \vec{B})^{\hat{i}} = \nabla_{\mathbf{u}} E^{\hat{i}} - K^{(\hat{i}\hat{j})} E_{\hat{j}} + \theta E^{\hat{i}} - (\vec{\omega} \times \vec{E})^{\hat{i}} + (\vec{G} \times \vec{B})^{\hat{i}} + 4\pi j^{\hat{i}} \; . \tag{80}$$

Using (73), this becomes

$$(\tilde{\nabla} \times \vec{B})^{\hat{i}} = \dot{E}^{\hat{i}} + (\vec{G} \times \vec{B})^{\hat{i}} + \left[(\vec{\Omega} - \vec{\omega}) \times \vec{E}\right]^{\hat{i}} - K^{(\hat{i}\hat{j})} E_{\hat{j}} + \theta E^{\hat{i}} + 4\pi j^{\hat{i}} \ . \tag{81}$$

The space projection of (6b) is

$$\epsilon^{\alpha\gamma\beta} E_{\beta;\gamma} = -\nabla^{\perp}_{\mathbf{u}} B^{\alpha} + K^{(\alpha\beta)} B_{\beta} - \theta B^{\alpha} + \epsilon^{\alpha}{}_{\beta\gamma} \omega^{\beta} B^{\gamma} + \epsilon^{\alpha\mu\sigma} E_{\mu} a_{\sigma} \ , \tag{82}$$

which, analogously, reads in the tetrad,

$$(\tilde{\nabla} \times \vec{E})^{\hat{i}} = -\dot{B}^{\hat{i}} + (\vec{G} \times \vec{E})^{\hat{i}} + \left[(\vec{\omega} - \vec{\Omega}) \times \vec{B}\right]^{\hat{i}} + K^{(\hat{i}\hat{j})} B_{\hat{j}} - \theta B^{\hat{i}} \ . \tag{83}$$

In the congruence adapted frame ($\vec{\omega} = \vec{\Omega} = \vec{H}/2$), Eqs. (76), (78), (81) and (83) above become,

$$\tilde{\nabla} \cdot \vec{E} = 4\pi \rho_c + \vec{H} \cdot \vec{B} \ ; \tag{84}$$

$$\tilde{\nabla} \times \vec{B} = \dot{\vec{E}} + \vec{G} \times \vec{B} + 4\pi \vec{j} - K^{(\hat{i}\hat{j})} E_{\hat{j}} \vec{e}_{\hat{i}} + \theta \vec{E} \ ; \tag{85}$$

$$\tilde{\nabla} \cdot \vec{B} = -\vec{H} \cdot \vec{E} \ ; \tag{86}$$

$$\tilde{\nabla} \times \vec{E} = -\dot{\vec{B}} + \vec{G} \times \vec{E} + K^{(\hat{i}\hat{j})} B_{\hat{j}} \vec{e}_{\hat{i}} - \theta \vec{B} \ . \tag{87}$$

In the special case of a rigid frame ($K^{(\hat{i}\hat{j})} = \theta = 0$) and time-independent fields ($\dot{\vec{E}} = \dot{\vec{B}} = 0$), this yields Eqs. (2.4a)-(3.8a) of Table 2.

3.4.2 Einstein equations

We start by computing the tetrad components of the Riemann tensor in the frame of Sec. 3.1:

$$R_{\hat{0}\hat{i}\hat{0}\hat{j}} = -\tilde{\nabla}_{\hat{i}} G_{\hat{j}} + G_{\hat{i}} G_{\hat{j}} - \dot{K}_{\hat{j}\hat{i}} + K_{\hat{l}\hat{i}} \Omega^{\hat{l}}{}_{\hat{j}} + \Omega^{\hat{l}}{}_{\hat{i}} K_{\hat{j}\hat{l}} - K^{\hat{l}}{}_{\hat{i}} K_{\hat{j}\hat{l}} \ ; \tag{88}$$

$$R_{\hat{0}\hat{i}\hat{j}\hat{k}} = \tilde{\nabla}_{\hat{k}} K_{\hat{i}\hat{j}} - \tilde{\nabla}_{\hat{j}} K_{\hat{i}\hat{k}} + 2 G_{\hat{i}} \omega_{\hat{j}\hat{k}} \ ; \tag{89}$$

$$R_{\hat{i}\hat{j}\hat{k}\hat{l}} = \tilde{R}_{\hat{i}\hat{j}\hat{k}\hat{l}} - K_{\hat{l}\hat{i}} K_{\hat{k}\hat{j}} + K_{\hat{l}\hat{j}} K_{\hat{k}\hat{i}} + 2 \omega_{\hat{i}\hat{j}} \Omega_{\hat{k}\hat{l}} \ . \tag{90}$$

In the expressions above we kept $\vec{\Omega}$ independent of $\vec{\omega}$, so that they apply to an arbitrary orthonormal tetrad field. Here

$$\tilde{R}^{\hat{i}}{}_{\hat{j}\hat{k}\hat{l}} \equiv \Gamma^{\hat{i}}_{\hat{l}\hat{j},\hat{k}} - \Gamma^{\hat{i}}_{\hat{k}\hat{j},\hat{l}} + \Gamma^{\hat{i}}_{\hat{k}\hat{m}} \Gamma^{\hat{m}}_{\hat{l}\hat{j}} - \Gamma^{\hat{i}}_{\hat{l}\hat{m}} \Gamma^{\hat{m}}_{\hat{k}\hat{j}} - C^{\hat{m}}_{\hat{k}\hat{l}} \Gamma^{\hat{i}}{}_{\hat{m}\hat{j}} \tag{91}$$

(where $C^{\hat{m}}_{\hat{k}\hat{l}} = \Gamma^{\hat{m}}_{\hat{k}\hat{l}} - \Gamma^{\hat{m}}_{\hat{l}\hat{k}}$) is the restriction to the spatial directions of the curvature of the connection $\tilde{\nabla}$, given by

$$\tilde{R}(\vec{X}, \vec{Y}) \vec{Z} = \tilde{\nabla}_{\vec{X}} \tilde{\nabla}_{\vec{Y}} \vec{Z} - \tilde{\nabla}_{\vec{Y}} \tilde{\nabla}_{\vec{X}} \vec{Z} - \tilde{\nabla}_{[\vec{X}, \vec{Y}]} \vec{Z}$$

for any spatial vector fields $\vec{X}, \vec{Y}, \vec{Z}$. It is related by

$$\tilde{R}_{\hat{i}\hat{j}\hat{k}\hat{l}} = R^{\perp}_{\hat{i}\hat{j}\hat{k}\hat{l}} - 2 \omega_{\hat{i}\hat{j}} \Omega_{\hat{k}\hat{l}} \tag{92}$$

to the curvature tensor of the distribution of hyperplanes orthogonal to the congruence, $R^{\perp}_{\hat{i}\hat{j}\hat{k}\hat{l}}$, that is, the restriction to the spatial directions of the curvature of the Fermi-Walker connection $\nabla^{\perp}$ on the vector bundle of spatial vectors, given by

$$R^{\perp}(\vec{X}, \vec{Y}) \vec{Z} = \nabla^{\perp}_{\vec{X}} \nabla^{\perp}_{\vec{Y}} \vec{Z} - \nabla^{\perp}_{\vec{Y}} \nabla^{\perp}_{\vec{X}} \vec{Z} - \nabla^{\perp}_{[\vec{X}, \vec{Y}]} \vec{Z} \ .$$

In some special regimes the interpretation of $\tilde{R}_{\hat{i}\hat{j}\hat{k}\hat{l}}$ is more straightforward, as it becomes the curvature of (the Levi-Civita connection of) a 3-D space manifold. In the quasi-Maxwell limit of Sec. 3.2.2 — that is, rigid ($K_{(\alpha\beta)} = 0$), congruence adapted ($\vec{\Omega} = \vec{\omega}$) frames — it is the curvature tensor of the spatial metric γ_{ij} (which yields the infinitesimal distances between neighboring observers of the congruence). In the case $\vec{\Omega} = 0$, $\tilde{R}_{\hat{i}\hat{j}\hat{k}\hat{l}} = R^{\perp}_{\hat{i}\hat{j}\hat{k}\hat{l}}$. The same equality holds in the case that the vorticity is zero ($\vec{\omega} = 0$), regardless of $\vec{\Omega}$; however, in this case the distribution is integrable, that is, the observers are hypersurface orthogonal, and $\tilde{R}_{\hat{i}\hat{j}\hat{k}\hat{l}}$ gives the curvature of these hypersurfaces. In the general case however $\tilde{R}_{\hat{i}\hat{j}\hat{k}\hat{l}}$ cannot be identified with the Levi-Civita connection of some space manifold, and that is manifest in the fact that does not satisfy the algebraic Bianchi identities for a 3-D curvature tensor, as we shall see below.

We shall now compute the tetrad components of the Ricci tensor, but specializing to congruence adapted frames: $\Omega_{ij} = \omega_{ij} = K_{[ij]} = -\epsilon_{ijk}H^k/2$, so that the Ricci tensor comes in terms of the three GEM fields: $\vec{G}$, $\vec{H}$ and $K_{(ij)}$. These read

$$R_{\hat{0}\hat{0}} = -\tilde{\nabla}\cdot\vec{G} + \vec{G}^2 + \frac{1}{2}\vec{H}^2 - \dot{\theta} - K^{(\hat{i}\hat{j})}K_{(\hat{i}\hat{j})} ; \tag{93}$$

$$R_{\hat{0}\hat{i}} = \tilde{\nabla}^{\hat{j}}K_{(\hat{j}\hat{i})} - \theta_{,\hat{i}} + \frac{1}{2}(\tilde{\nabla}\times\vec{H})_{\hat{i}} - (\vec{G}\times\vec{H})_{\hat{i}} ; \tag{94}$$

$$\begin{aligned} R_{\hat{i}\hat{j}} = {} & \tilde{R}_{\hat{i}\hat{j}} + \tilde{\nabla}_{\hat{i}}G_{\hat{j}} - G_{\hat{i}}G_{\hat{j}} + \dot{K}_{(\hat{i}\hat{j})} + K_{(\hat{i}\hat{j})}\theta \\ & + \frac{1}{2}\left[\dot{H}_{\hat{i}\hat{j}} + H_{\hat{i}\hat{j}}\theta + \vec{H}^2\delta_{\hat{i}\hat{j}} - H_{\hat{i}}H_{\hat{j}} + K_{(\hat{i}\hat{l})}H^{\hat{l}}{}_{\hat{j}} - H_{\hat{i}}{}^{\hat{l}}K_{(\hat{l}\hat{j})}\right] , \end{aligned} \tag{95}$$

where $H_{ij} = \epsilon_{ijk}H^k$ is the dual of $\vec{H}$, and $\tilde{R}_{\hat{i}\hat{j}} \equiv \tilde{R}^{\hat{l}}{}_{\hat{i}\hat{l}\hat{j}}$ is the Ricci tensor associated to $\tilde{R}_{\hat{i}\hat{j}\hat{k}\hat{l}}$; this tensor is *not* symmetric in the general case of a congruence possessing both vorticity and shear. Using $T^{\hat{0}\hat{0}} = \rho$ and $T^{\hat{0}\hat{i}} = J^{\hat{i}}$, the time-time, time-space, and space-space components of the Einstein field equations with sources, Eq. (14a), read, respectively:

$$\tilde{\nabla}\cdot\vec{G} = -4\pi(2\rho + T^{\alpha}{}_{\alpha}) + \vec{G}^2 + \frac{1}{2}\vec{H}^2 - \dot{\theta} - K^{(\hat{i}\hat{j})}K_{(\hat{i}\hat{j})} ; \tag{96}$$

$$\tilde{\nabla}\times\vec{H} = -16\pi\vec{J} + 2\vec{G}\times\vec{H} + 2\tilde{\nabla}\theta - 2\tilde{\nabla}_{\hat{j}}K^{(\hat{j}\hat{i})}\vec{e}_{\hat{i}} ; \tag{97}$$

$$\begin{aligned} 8\pi\left(T_{\hat{i}\hat{j}} - \frac{1}{2}\delta_{\hat{i}\hat{j}}T^{\alpha}{}_{\alpha}\right) = {} & \tilde{R}_{\hat{i}\hat{j}} + \tilde{\nabla}_{\hat{i}}G_{\hat{j}} - G_{\hat{i}}G_{\hat{j}} + \dot{K}_{(\hat{i}\hat{j})} + K_{(\hat{i}\hat{j})}\theta \\ & + \frac{1}{2}\left[\dot{H}_{\hat{i}\hat{j}} + H_{\hat{i}\hat{j}}\theta + \vec{H}^2\delta_{\hat{i}\hat{j}} - H_{\hat{i}}H_{\hat{j}} + K_{(\hat{i}\hat{l})}H^{\hat{l}}{}_{\hat{j}} - H_{\hat{i}}{}^{\hat{l}}K_{(\hat{l}\hat{j})}\right] . \end{aligned} \tag{98}$$

Eqs. (96)-(97) are the gravitational analogues of the electromagnetic equations (84) and (85), respectively; Eq. (98) has no electromagnetic counterpart.

As for the the algebraic Bianchi identities (14b), using (88)-(90), the time-time (equal to space-space, as discussed in Sec. 14), time-space and space-time components become, respectively:

$$\tilde{\nabla}\cdot\vec{H} = -\vec{G}\cdot\vec{H} ; \tag{99}$$

$$\tilde{\nabla}\times\vec{G} = -\dot{\vec{H}} - \vec{H}\theta + H_{\hat{j}}K^{(\hat{i}\hat{j})}\vec{e}_{\hat{i}} ; \tag{100}$$

$$K_{(ij)}H^j = -\star\tilde{R}^j{}_{ji} . \tag{101}$$

Eqs. (99)-(100) are the gravitational analogues of the time and space projections of the electromagnetic Bianchi identities, Eqs. (86)-(87), respectively[17]; Eq. (101) has no electromagnetic analogue. This equation states that if the observer congruence has both vorticity and shear/expansion then $\tilde{R}_{ijkl}$ does not obey the algebraic Bianchi identities for a 3D curvature tensor.

Note this remarkable aspect: all the terms in the Maxwell equations (84), (86) and (87) have a gravitational counterpart in (96), (99) and (100), respectively, substituting $\{\vec{E},\vec{B}\} \rightarrow \{\vec{G},\vec{H}\}$ (up to some numerical factors). As for (85), there are clear gravitational analogues in (97) to the terms $\vec{G}\times\vec{B}$ and the current $4\pi\vec{j}$, but not to the remaining terms. It should nevertheless be noted that, as shown in Sec. 5 below, in the post-Newtonian regime (or in the "GEM limit" of linearized theory), the term $2\tilde{\nabla}\theta$ of (97) embodies a contribution analogous to the displacement current term $\dot{\vec{E}}$ of (85). The gravitational equations in turn contain, as one might expect, terms with no parallel in electromagnetism, most of them involving the shear/expansion tensor $K_{(\alpha\beta)}$.

Finally, it is also interesting to compare Eqs. (96)-(101) with the tidal tensor version of the same equations, Eqs. (19)-(22).

3.4.3 Special cases: quasi-Maxwell regime (1+3 formalism), and hypersurface orthogonal observers (3+1 "ADM" formalism)

Eqs. (96)-(101) encompass two notable special cases: i) the "quasi-Maxwell" regime (see Sec. 3.2.2 above), corresponding to stationary fields, and a frame adapted to a *rigid* congruence of stationary observers, which is obtained by setting $K_{(\alpha\beta)}=\theta=0$, and all time derivatives also to zero; and ii) the case that the frame is adapted to an hypersurface orthogonal (i.e., vorticity free) congruence, corresponding to the "hypersurface point of view" [27, 28], leading to the well known ADM "3+1 formalism" (see e.g. [122, 121]) which is obtained by setting $\vec{H}=0$ in the equations above. Let us start by case i), also known as the "1+3 formalism" (e.g. [24]) or threading picture [27, 28] for stationary spacetimes, which is where the similarity with the electromagnetism gets closer, since, as we have seen in the previous section, most of the differing terms between the gravitational field Eqs. (96),(97), (99), (100), and their electromagnetic counterparts (84)-(87), involve $K_{(\alpha\beta)}$. The field equations in this regime are given in Table 2; and the expressions above for the Riemann and Ricci tensors match the ones given in [20]. In Table 2 we drop the hats in the indices, for the following reason: as discussed in Sec. 3.2, in this regime there is a natural 3-D metric γ_{ij} on the quotient space (measuring the *fixed* distance between neighboring observers); we thus interpret the spatial fields $\vec{G}$ and $\vec{H}$ as vector fields on this 3-D Riemannian manifold. The spatially projected covariant derivative operator $\tilde{\nabla}$ becomes the covariant derivative with respect to γ_{ij} (as $\Gamma^i_{jk} =^{(3)} \Gamma^i_{jk}$, i.e., the 4-D spatial connection coefficients equal the Levi-Civita connection coefficients for γ_{ij}), and $\tilde{R}_{ij}$ becomes its Ricci tensor, which is symmetric (contrary to the general case). The equations in this "quasi-Maxwell" regime exhibit a striking similarity with their electromagnetic counterparts, Eqs. (2.4a)-(2.8a) of Table 2, in spite of some natural differences that remain — numerical factors, the source and quadratic terms in (2.4b) with no electromagnetic counterpart. We note in particular that, by simply replacing $\{\vec{E},\vec{B}\} \rightarrow \{\vec{G},\vec{H}\}$ in (2.5a)-(2.8a), one obtains, up to some numerical factors, Eqs. (2.5b), (2.7b)-(2.8b). Of course, the electromagnetic terms involving products of GEM fields

[17] Eqs. (99)-(100) are equivalent to Eqs. (7.3) of [27]; therein they are obtained through a different procedure, not by projecting the identity $\star R^{\gamma\alpha}{}_{\gamma\beta}=0 \Leftrightarrow R_{[\alpha\beta\gamma]\delta}=0$, but instead from the splitting of the identity $d^2\mathbf{u}=0 \Leftrightarrow u_{[\alpha;\beta\gamma]}=0$. Noting that $u_{[\alpha;\beta\gamma]}=-R_{[\alpha\beta\gamma]\lambda}u^\lambda$, we see that the latter is indeed encoded in the time-time and space-time parts (with respect to u^α) of the former.

with EM fields, are mimicked in gravity by second order terms in the GEM fields. This is intrinsic to the non-linear nature of the gravitational field, and may be thought of as manifesting the fact that the gravitational field sources itself. It is interesting to note in this context that the term $2\vec{G}\times\vec{H}$ in

Table 2: The gravito-electromagnetic analogy based on inertial GEM fields.

Electromagnetism	Gravity
Lorentz Force: $\frac{D\vec{U}}{d\tau}=\frac{q}{m_0}\left(U^0\vec{E}+\vec{U}\times\vec{B}\right)$ (2.1a)	Geodesic Equation ($\vec{H}=\vec{\Omega}+\vec{\omega}$): $\frac{\tilde{D}\vec{U}}{d\tau}=U^{\hat{0}}\left[U^{\hat{0}}\vec{G}+\vec{U}\times\vec{H}-\sigma^{\hat{i}}{}_{\hat{j}}U^{\hat{j}}\mathbf{e}_{\hat{i}}-\frac{1}{3}\theta\vec{U}\right]$ (2.1b)
Precession of magnetic dipole: $\frac{D\vec{S}}{d\tau}=\vec{\mu}\times\vec{B}$ (2.2a)	Gyroscope "precession": $\frac{d\vec{S}}{d\tau}=\vec{S}\times\vec{\Omega}$ (2.2b)
Stationary fields, rigid, congruence adapted frame: $\vec{\Omega}=\vec{\omega}=\vec{H}/2$ (quasi-Maxwell formalism)	
Force on magnetic dipole: $\frac{D\vec{P}}{d\tau}=\tilde{\nabla}(\vec{B}\cdot\vec{\mu})-\frac{1}{2}\vec{\mu}(\tilde{\nabla}\cdot\vec{B})-\frac{1}{2}(\vec{\mu}\cdot\vec{H})\vec{E}$ (2.3a)	Force on gyroscope: $\frac{D\vec{P}}{d\tau}=\frac{1}{2}\left[\tilde{\nabla}(\vec{H}\cdot\vec{S})-\vec{S}(\tilde{\nabla}\cdot\vec{H})-2(\vec{S}\cdot\vec{H})\vec{G}\right]$ (2.3b)
Maxwell Source Equations: $F^{\alpha\beta}{}_{;\beta}=4\pi J^\beta$	Einstein Equations: $R_{\mu\nu}=8\pi\left(T_{\mu\nu}-\frac{1}{2}g_{\mu\nu}T^\alpha{}_\alpha\right)$
• Time Component: $\tilde{\nabla}\cdot\vec{E}=4\pi\rho_c+\vec{H}\cdot\vec{B}$ (2.4a)	• Time-Time Component: $\tilde{\nabla}\cdot\vec{G}=-4\pi(2\rho+T^\alpha{}_\alpha)+\vec{G}^2+\frac{1}{2}\vec{H}^2$ (2.4b)
• Space Components: $\tilde{\nabla}\times\vec{B}=\vec{G}\times\vec{B}+4\pi\vec{j}$ (2.5a)	• Time-Space Components: $\tilde{\nabla}\times\vec{H}=2\vec{G}\times\vec{H}-16\pi\vec{J}$ (2.5b)
No electromagnetic analogue	• Space-Space Component: $\tilde{\nabla}_iG_j-G_iG_j+\frac{1}{2}\vec{H}^2\gamma_{ij}+\tilde{R}_{ij}=8\pi\left(\frac{1}{2}\gamma_{ij}T^\alpha{}_\alpha+T_{ij}\right)$ (2.6)
Bianchi Identity: $\star F^{\alpha\beta}{}_{;\beta}=0\quad(\Leftrightarrow F_{[\alpha\beta;\gamma]}=0\ )$	Algebraic Bianchi Identity: $\star R^{\gamma\alpha}{}_{\gamma\beta}=0\quad(\Leftrightarrow R_{[\alpha\beta\gamma]\delta}=0)$
• Time Component: $\tilde{\nabla}\cdot\vec{B}=-\vec{H}\cdot\vec{E}$ (2.7a)	• Time-Time (or Space-Space) Component: $\tilde{\nabla}\cdot\vec{H}=-\vec{H}\cdot\vec{G}$ (2.7b)
• Space Components: $\tilde{\nabla}\times\vec{E}=\vec{G}\times\vec{E}$ (2.8a)	• Space-Time Components: $\tilde{\nabla}\times\vec{G}=0$ (2.8b)

Eq. (2.5b), sourcing the curl of the gravitomagnetic field, resembles the electromagnetic Poynting vector $\vec{p}_{EM}=\vec{E}\times\vec{B}/4\pi$; and the contribution $\vec{G}^2+\vec{H}^2/2$ in Eq. (2.5b), sourcing the divergence of the gravitoelectric field, resembles the electromagnetic energy density $\rho_{EM}=(\vec{E}^2+\vec{B}^2)/8\pi$. For these reasons these quantities are dubbed in e.g. [19, 23, 24] gravitational "energy density" and "energy current density", respectively. It is also interesting that, in the asymptotic limit, $\vec{p}_G\equiv-\vec{G}\times\vec{H}/4\pi$ corresponds to the time-space components of the Landau-Lifshitz [18] pseudo-tensor $t^{\mu\nu}$ [62]. One should however bear in mind that, by contrast with their electromagnetic counterparts, these quantities are artifacts of the reference frame, with no physical significance from a *local* point of view (see related discussion in Sec. 6.1).

Let us turn now to case ii); taking a vorticity-free congruence of observers (i.e., $\vec{\omega} = \vec{H} = 0$), the Einstein Eqs. (96)-(98) can be written as, respectively,

$$16\pi\rho = \tilde{R} + \theta^2 - K^{(\hat{i}\hat{j})}K_{(\hat{i}\hat{j})} \; ; \tag{102}$$

$$8\pi\vec{J} = \tilde{\nabla}\theta - \tilde{\nabla}_{\hat{j}}K^{(\hat{j}\hat{i})}\vec{e}_{\hat{i}} \; ; \tag{103}$$

$$\dot{K}_{(\hat{i}\hat{j})} = \tilde{\nabla}_{\hat{i}}G_{\hat{j}} - G_{\hat{i}}G_{\hat{j}} + \tilde{R}_{ij} + \theta K_{(\hat{i}\hat{j})} - 8\pi\left(T_{\hat{i}\hat{j}} - \frac{1}{2}\delta_{\hat{i}\hat{j}}T^{\alpha}_{\ \alpha}\right) . \tag{104}$$

In this regime, $\tilde{R}$ and $\tilde{R}_{ij}$ are, respectively, the 3-D Ricci scalar and tensor of the hypersurfaces orthogonal to u^α. Eq. (102) is known in the framework of the ADM formalism [122, 121, 123] as the "Hamiltonian constraint". Since this equation is the tetrad time-time component of Eq. (14a), it can either be directly obtained from the latter by noting that, when $\omega^\alpha = 0$, $K_{(\alpha\beta)}$ is the extrinsic curvature of the hypersurfaces orthogonal to the congruence, and employing the scalar Gauss equation (e.g. Eq. (2.95) of [121]); or from Eq. (96) above, computing $R = R_{\hat{i}\hat{j}} - R_{\hat{0}\hat{0}}$ from Eqs. (93), (94) (with $\vec{H} = \vec{0}$), substituting into (96), and then using (14a) to eliminate R. Eq. (103) follows directly from Eq. (97) by making $\vec{H} = \vec{0}$, and is known as the "momentum constraint" [121, 123]. The space-space Eq. (104) is the dynamical equation for extrinsic curvature, corresponding to writing in tetrad components e.g. Eq. (3.42) of [121]; in order to obtain one from the other, one notes that $G_\alpha = -(1/N)\tilde{\nabla}_\alpha N$, where N is the lapse factor [121]. Eqs. (102)-(103) have now little resemblance to their *physical* electromagnetic counterparts (84)-(85) (for $\vec{H} = \vec{B} = \vec{0}$); but in this framework a different (purely *formal*) analogy is sometimes drawn (e.g. [123]): a parallelism between Eqs. (102)-(103) and the two electromagnetic constraints (for Lorentz frames in flat spacetime) $\partial_i E^i = 4\pi\rho_c$ and $\partial_i B^i = 0$, and between the ADM evolution equations for $K_{(ij)}$ and for the spatial metric, written in a coordinate system adapted to the foliation (e.g. Eqs. (4.63)-(4.64) of [121]), and the dynamical equations for the curls of $\vec{B}$ and $\vec{E}$.

3.5 Relation with tidal tensor formalism

The analogy based on the gravito-electromagnetic fields $\vec{G}$ and $\vec{H}$ is intrinsically different from the gravito-electromagnetic analogy based on tidal tensors of Sec. 2; the latter stems from tensor equations, whereas the former are fields of inertial forces, i.e., artifacts of the reference frame. A relationship between the two formalisms exists nevertheless, as in an arbitrary frame one can express the gravitational tidal tensors in terms of the GEM fields, using the expressions for the tetrad components of Riemann tensor Eqs. (88)-(89). This relationship is in many ways illuminating, as we shall see; it is one of the main results in this work, due the importance of using the two formalisms together in practical applications, to be presented elsewhere (e.g. [29]). In an arbitrary frame one can express the gravitational tidal tensors in terms of the GEM fields, using the expressions for the tetrad components of Riemann tensor (88)-(90). The expressions obtained are to be compared with the analogous electromagnetic situation, i.e., the electromagnetic tidal tensors computed from the fields as measured in an arbitrarily accelerating, rotating, and shearing frame (in flat or curved spacetime).

We start by the electromagnetic tidal tensors; using the abbreviated notation $E_{\alpha\beta} \equiv (E^u)_{\alpha\beta} = F_{\alpha\gamma;\beta}u^\gamma$, $B_{\alpha\beta} \equiv (B^u)_{\alpha\beta} = \star F_{\alpha\gamma;\beta}u^\gamma$, cf. Table 1, it follows that

$$E_{\alpha\gamma} = E_{\alpha;\gamma} - F_{\alpha\beta}u^\beta_{\ ;\gamma} \ ; \qquad B_{\alpha\gamma} = B_{\alpha;\gamma} - \star F_{\alpha\beta}u^\beta_{\ ;\gamma} \ .$$

Using decompositions (1), and Eq. (73), we obtain the tetrad components ($E_{\hat{0}\hat{i}} = B_{\hat{0}\hat{i}} = 0$):

$$E_{\hat{i}\hat{j}} = \tilde{\nabla}_{\hat{j}}E_{\hat{i}} - \epsilon_{\hat{i}}^{\ \hat{l}\hat{m}}B_{\hat{m}}K_{\hat{l}\hat{j}} \ ; \tag{105}$$

$$B_{\hat{i}\hat{j}} = \tilde{\nabla}_{\hat{j}}B_{\hat{i}} + \epsilon_{\hat{i}}^{\ \hat{l}\hat{m}}E_{\hat{m}}K_{\hat{l}\hat{j}} \ ; \tag{106}$$

$$E_{\hat{i}\hat{0}} = \dot{E}_{\hat{i}} + (\vec{\Omega} \times \vec{E})_{\hat{i}} + (\vec{G} \times \vec{B})_{\hat{i}} \ ; \tag{107}$$

$$B_{\hat{i}\hat{0}} = \dot{B}_{\hat{i}} + (\vec{\Omega} \times \vec{B})_{\hat{i}} - (\vec{G} \times \vec{E})_{\hat{i}} \ , \tag{108}$$

or, using $K_{ij} = \omega_{ij} + K_{(ij)}$, and choosing a congruence adapted frame ($\vec{\omega} = \vec{\Omega} = \vec{H}/2$),

$$E_{\hat{i}\hat{j}} = \tilde{\nabla}_{\hat{j}}E_{\hat{i}} - \frac{1}{2}\left[\vec{B}\cdot\vec{H}\delta_{\hat{i}\hat{j}} - B_{\hat{j}}H_{\hat{i}}\right] - \epsilon_{\hat{i}}^{\ \hat{l}\hat{m}}B_{\hat{m}}K_{(\hat{l}\hat{j})} \ ; \tag{109}$$

$$B_{\hat{i}\hat{j}} = \tilde{\nabla}_{\hat{j}}B_{\hat{i}} + \frac{1}{2}\left[\vec{E}\cdot\vec{H}\delta_{\hat{i}\hat{j}} - E_{\hat{j}}H_{\hat{i}}\right] + \epsilon_{\hat{i}}^{\ \hat{l}\hat{m}}E_{\hat{m}}K_{(\hat{l}\hat{j})} \ ; \tag{110}$$

$$E_{\hat{i}\hat{0}} = \dot{E}_{\hat{i}} + \frac{1}{2}(\vec{H} \times \vec{E})_{\hat{i}} + (\vec{G} \times \vec{B})_{\hat{i}} \ ; \tag{111}$$

$$B_{\hat{i}\hat{0}} = \dot{B}_{\hat{i}} + \frac{1}{2}(\vec{H} \times \vec{B})_{\hat{i}} - (\vec{G} \times \vec{E})_{\hat{i}} \ . \tag{112}$$

Let us compute the gravitational tidal tensors. From the definitions of $\mathbb{E}_{\alpha\beta}$ and $\mathbb{H}_{\alpha\beta}$ in Table 1, and using the tetrad components of the Riemann tensor, Eqs. (88)-(89), we obtain ($\mathbb{E}_{\hat{0}\hat{\alpha}} = \mathbb{E}_{\hat{\alpha}\hat{0}} = \mathbb{H}_{\hat{0}\hat{\alpha}} = \mathbb{H}_{\hat{\alpha}\hat{0}} = 0$):

$$\mathbb{E}_{\hat{i}\hat{j}} = -\tilde{\nabla}_{\hat{j}}G_{\hat{i}} + G_{\hat{i}}G_{\hat{j}} - \dot{K}_{\hat{i}\hat{j}} + K_{\hat{l}\hat{j}}\Omega^{\hat{l}}_{\ \hat{i}} + \Omega^{\hat{l}}_{\ \hat{j}}K_{\hat{i}\hat{l}} - K^{\hat{l}}_{\ \hat{j}}K_{\hat{i}\hat{l}} \ ; \tag{113}$$

$$\mathbb{H}_{\hat{i}\hat{j}} = -\tilde{\nabla}_{\hat{j}}\omega_{\hat{i}} + \delta_{\hat{i}\hat{j}}\tilde{\nabla}\cdot\vec{\omega} + 2G_{\hat{j}}\omega_{\hat{i}} + \epsilon_{\hat{i}}^{\ \hat{l}\hat{m}}\tilde{\nabla}_{\hat{l}}K_{(\hat{j}\hat{m})} \ . \tag{114}$$

For a congruence adapted frame these expressions become:

$$\begin{aligned}\mathbb{E}_{\hat{i}\hat{j}} = & -\tilde{\nabla}_{\hat{j}}G_{\hat{i}} + G_{\hat{i}}G_{\hat{j}} + \frac{1}{4}\left(\vec{H}^2\gamma_{ij} - H_jH_i\right) + \frac{1}{2}\epsilon_{\hat{i}\hat{j}\hat{k}}\dot{H}^{\hat{k}} + \epsilon^{\hat{l}}_{\ \hat{j}\hat{m}}H^{\hat{m}}K_{(\hat{i}\hat{l})} \\ & -\dot{K}_{(\hat{i}\hat{j})} - \delta^{\hat{l}\hat{m}}K_{(\hat{i}\hat{l})}K_{(\hat{m}\hat{j})} \ ;\end{aligned} \tag{115}$$

$$\mathbb{H}_{\hat{i}\hat{j}} = -\frac{1}{2}\left[\tilde{\nabla}_jH_i + (\vec{G}\cdot\vec{H})\gamma_{ij} - 2G_jH_i\right] + \epsilon_{\hat{i}}^{\ \hat{l}\hat{m}}\tilde{\nabla}_{\hat{l}}K_{(\hat{j}\hat{m})} \ . \tag{116}$$

In (116) we substituted $\tilde{\nabla}\cdot H = -\vec{G}\cdot\vec{H}$ using Eq. (99). Note the formal similarities with the electromagnetic analogues (109)-(110). All the terms present in E_{ij} and B_{ij}, except for the last term of the latter, have a correspondence in their gravitational counterparts $\mathbb{E}_{ij}$, $\mathbb{H}_{ij}$, substituting $\{\vec{E},\vec{B}\} \to -\{\vec{G},\vec{H}\}$ and correcting some factors of 2. However, the gravitational tidal tensors contain additional terms, which (together with the differing numerical factors) encode the crucial differences in the tidal dynamics of the two interactions. The fourth and fifth terms in (115) have the role of canceling out the antisymmetric part of $\tilde{\nabla}_{\hat{j}}G_{\hat{i}}$, that is, canceling out the contribution of the curl of $\vec{G}$ to the gravitoelectric tidal tensor, as can be seen from Eq. (100). Note in particular the term involving $\dot{H}^i$, which has no counterpart in the electric tidal tensor (109); in Eq. (100), that term shows up "inducing" the curl of $\vec{G}$, in a role analogous to $\dot{B}^i$ in the equation (87) for $\tilde{\nabla} \times \vec{E}$, which might lead one to think about gravitational induction effects in analogy with Faraday's law of electromagnetism. The fact that it is being subtracted in (115), means, however, that the curl of $\vec{G}$ does not translate into *physical, covariant* forces. For instance, it does not induce rotation in a set of free neighboring particles (see Eq. (26) above and discussion therein), nor does it torque an extended rigid body, as shown in the companion paper [6].

There are some interesting special regimes where the relation between the tidal tensor and the inertial fields formalism becomes simpler. One is the "quasi-Maxwell" regime of Sec. 3.2.2, i.e., *stationary spacetimes*, and a *frame adapted to a rigid (i.e., shear and expansion-free) congruence of stationary observers*. The gravitational tidal tensors *as measured in such frame* can be expressed entirely in terms of the gravitoelectric $\vec{G}$ and gravitomagnetic $\vec{H}$ fields; the non-vanishing components are:

$$\mathbb{E}_{ij} = -\tilde{\nabla}_j G_i + G_i G_j + \frac{1}{4}\left(\vec{H}^2\gamma_{ij} - H_j H_i\right); \tag{117}$$

$$\mathbb{H}_{ij} = -\frac{1}{2}\left[\tilde{\nabla}_j H_i + (\vec{G}\cdot\vec{H})\gamma_{ij} - 2G_j H_i\right]. \tag{118}$$

The hats in the indices of these expressions are dropped because, since in the quasi-Maxwell regime there is a natural metric γ_{ij} associated to the quotient space (see Sec. 3.2), we express these tensors in terms of an arbitrary (possibly coordinate) basis in such space (as we did in Sec. 3.2.2 and in Table 2), instead of tetrad components.

The non-vanishing components of the electromagnetic tidal tensors are, under the same conditions,

$$E_{ij} = \tilde{\nabla}_j E_i - \frac{1}{2}\left[\vec{B}\cdot\vec{H}\gamma_{ij} - B_j H_i\right] \quad \text{(a)} \qquad E_{i0} = \frac{1}{2}(\vec{H}\times\vec{E})_i + (\vec{G}\times\vec{B})_i \quad \text{(b)} \tag{119}$$

$$B_{ij} = \tilde{\nabla}_j B_i + \frac{1}{2}\left[\vec{E}\cdot\vec{H}\gamma_{ij} - E_j H_i\right] \quad \text{(a)} \qquad B_{i0} = \frac{1}{2}(\vec{H}\times\vec{B})_i - (\vec{G}\times\vec{E})_i \quad \text{(b)} \tag{120}$$

Thus again, even in the stationary regime, the electromagnetic tidal tensors have non-vanishing time components, unlike their gravitational counterparts. The spatial parts, however, are very similar in form; note that replacing $\{\vec{E},\vec{B}\} \to -\{\vec{G},\vec{H}/2\}$ in (120), the time components vanish, and one *almost* obtains the space part (118), apart from the factor of 2 in the third term; and that a similar substitution in (119) *almost* leads to (117), apart from the term $G_i G_j$, which has no electromagnetic counterpart. The gravitational and electromagnetic tidal tensors are nevertheless very different, even in this regime; namely in their symmetries. E_{ij} is not symmetric, whereas $\mathbb{E}_{ij}$ is (the second and third terms in (117) are obviously symmetric; and that the first one also is can be seen from Eq. (2.8b) of Table 2). As for the magnetic tidal tensors, note that, by virtue of Eq. (2.5b), the last term of (118) ensures that, in vacuum, the antisymmetric part $H_{[i;j]}$ (i.e., the curl of $\vec{H}$) is subtracted from $H_{i;j}$ in (114), thus keeping $\mathbb{H}_{ij}$ symmetric, by contrast with B_{ij}. This can be seen explicitly by noting that *in vacuum* (118) can be put in the equivalent form

$$\mathbb{H}_{ij} = -\frac{1}{2}\left[H_{i;j} - H_{[i;j]} + (\vec{G}\cdot\vec{H})\gamma_{ij} - 2G_{(j}H_{i)}\right],$$

where we used $H_{[i;j]} = 2G_{[j}H_{i]}$, as follows from Eq. (2.5b).

Another interesting regime to consider is the weak field limit, where the non-linearities of the gravitational field are negligible, and compare with electromagnetism in inertial frames. From Eqs. (109)-(112), the non-vanishing components of the electromagnetic tidal tensors measured by observers at rest in an inertial frame are:

$$E_{ij} = E_{i,j} \ ; \qquad E_{i0} = \dot{E}_i \ ; \qquad B_{ij} = B_{i,j} \ ; \qquad B_{i0} = \dot{B}_i \ ,$$

i.e., they reduce to ordinary derivatives of the electric and magnetic fields. The *linearized* gravitational tidal tensors are, from Eqs. (115)-(116):

$$\mathbb{E}_{ij} \approx -G_{i,j} + \frac{1}{2}\epsilon_{ijk}\dot{H}^k - \dot{K}_{(ij)} \ ; \quad \text{(a)} \qquad \mathbb{H}_{ij} \approx -\frac{1}{2}H_{i,j} + \epsilon_i^{\ lm}K_{(jm),l} \ . \quad \text{(b)} \tag{121}$$

Thus, even in the linear regime, the gravitational tidal tensors cannot, in general, be regarded as derivatives of the gravitoelectromagnetic fields $\vec{G}$ and $\vec{H}$. Noting, from Eq. (134) below, that $K_{(ij)}$ is the time derivative of the spatial metric, we see that *only if the fields are time independent* in the chosen frame do we have $\mathbb{E}_{ij} \approx -G_{i,j}$, $\mathbb{H}_{ij} \approx -\frac{1}{2}H_{i,j}$.

3.6 Force on a gyroscope

In the framework of the inertial GEM fields, there is also an analogy relating the gravitational force on a gyroscope and the electromagnetic force on a magnetic dipole. This is different from the analogy based on tidal tensors, and not as general. We start with equations (1.2) of Table 1, which tell us that the forces are determined by the magnetic/gravitomagnetic tidal tensors as *seen by the particle*. For the spatial part of the forces, only the space components of the tidal tensors, as measured in the particle's proper frame, contribute. Comparing Eqs. (110) and (116), which yield the tidal tensors in terms of the electromagnetic/gravitoelectromagnetic fields, we see that a close formal analogy is possible only when $K_{(\alpha\beta)} = 0$ in the chosen frame. Thus, a close analogy between the forces in this formalism can hold only when the particle is at rest with respect to a congruence for which $K_{(\alpha\beta)} = 0$; that is, a rigid congruence. The rigidity requirement can be satisfied only in special spacetimes [113]; it is ensured in the "quasi-Maxwell" regime — that is, stationary spacetimes, and congruences tangent to time-like Killing vector fields therein.

Let us start by the electromagnetic problem — a magnetic dipole at rest in a rigid, but arbitrarily accelerating and rotating frame. Since the dipole is at rest in that frame, we have $\mu^\alpha = (0, \mu^i)$; hence the spatial part of the force is $F^i_{EM} = B^{ji}\mu_j$. Substituting (120a) in this expression yields the force exerted on the dipole, in terms of the electric and magnetic fields *as measured in this frame*:

$$\vec{F}_{EM} = \tilde{\nabla}(\vec{B} \cdot \vec{\mu}) + \frac{1}{2}\left[\vec{\mu}(\vec{E} \cdot \vec{H}) - (\vec{\mu} \cdot \vec{H})\vec{E}\right] \ . \tag{122}$$

Using $\vec{H} \cdot \vec{E} = -\tilde{\nabla} \cdot \vec{B}$, cf. Eq. (2.7a), we can re-write this expression as

$$\vec{F}_{EM} = \tilde{\nabla}(\vec{B} \cdot \vec{\mu}) - \frac{1}{2}\left[\vec{\mu}(\tilde{\nabla} \cdot \vec{B}) + (\vec{\mu} \cdot \vec{H})\vec{E}\right] \ . \tag{123}$$

Consider now the analogous gravitational situation: a gyroscope at rest (i.e., with zero 3-velocity, $U^i = 0$) with respect to stationary observers (arbitrarily accelerated and rotating) in a stationary gravitational field. If the Mathisson-Pirani condition is employed (see [6] for details), the force exerted on it is described by Eq. (1.2b) of Table 1; using (118) we write this force in terms of the GEM fields:

$$\vec{F}_G = \frac{1}{2}\left[\tilde{\nabla}(\vec{H}\cdot\vec{S}) + \vec{S}(\vec{G}\cdot\vec{H}) - 2(\vec{S}\cdot\vec{H})\vec{G}\right] \ . \tag{124}$$

From Eq. (2.7b) we have $\vec{G}\cdot\vec{H} = -\tilde{\nabla}\cdot\vec{H}$; substituting yields [20]:

$$\vec{F}_G = \frac{1}{2}\left[\tilde{\nabla}(\vec{H}\cdot\vec{S}) - \vec{S}(\tilde{\nabla}\cdot\vec{H}) - 2(\vec{S}\cdot\vec{H})\vec{G}\right] \ . \tag{125}$$

Note that replacing $\{\vec{\mu},\vec{E},\vec{B}\} \to \{\vec{S},\vec{G},\vec{H}/2\}$ in Eq. (122) one *almost* obtains (124), except for a factor of 2 in the last term. The last term of (124)-(125), in this framework, can be interpreted as the "weight" of the dipole's energy [20]. It plays, together with Eq. (2.5b), a crucial role in the dynamics, as it cancels out the contribution of the curl of $\vec{H}$ to the force, ensuring that it is given by a contraction of S^α with a *symmetric* tensor $\mathbb{H}_{\alpha\beta}$ (see the detailed discussion in Sec. 3.5). This contrasts with the electromagnetic case, where the curl of $\vec{B}$ is manifest in $B_{\alpha\beta}$ (which has an antisymmetric part) and in the force F^α_{EM}.

The expression (125) was first found in [20], where it was compared to the force on a magnetic dipole as measured in the *inertial* frame *momentarily* comoving with it, in which case the last two terms of (123) vanish; herein we add expression (123), which is its electromagnetic counterpart for analogous conditions (the frame where the particle is at rest can be arbitrarily accelerating and rotating), and shows that the analogy is even stronger.

4 "Ultra-stationary" spacetimes

Ultra-stationary spacetimes are stationary spacetimes admitting *rigid* geodesic time-like congruences. In the coordinate system adapted to such congruence, the metric is generically obtained by taking $\Phi = 0$ in Eq. (67), leading to:

$$ds^2 = -\left(dt - \mathcal{A}_i(x^k)dx^i\right)^2 + \gamma_{ij}(x^k)dx^i dx^j \ . \tag{126}$$

Examples of these spacetimes are the Som-Raychaudhuri metrics [89], the van Stockum interior solution [90], and the Gödel [91] spacetime; see [2] for their discussion in this context. This is an interesting class of spacetimes in the context of GEM, due to the close *similarity* with electrodynamics, which was explored in an earlier work [1] by one of the authors: 1) they are exactly mapped [69, 1], via the Klein-Gordon equation, into curved 3-spaces with a "magnetic" field; 2) their gravitomagnetic tidal tensor is linear [1] (just like in the case of electromagnetism), and, up to a factor, matches the covariant derivative of the magnetic field of the electromagnetic analogue. A link between these two properties was suggested in [1] [18]; however, the non-vanishing gravitoelectric

[18] In the earlier work Refs. [1, 2] by one of the authors (to whom the *exact* GEM fields analogy of Sec. 3 was not yet known), it was suggested that the above mapping could be interpreted as arising from the similarity of magnetic tidal forces manifest in relations (129). It seems, however, to be much more related to the analogy based on GEM "vector" fields manifest in Eqs. (127) and (128). Even though the exact correspondence (129) reinforces in some sense the analogy, tidal forces do not seem to be the underlying principle behind the mapping, since: i) there is no electromagnetic counterpart to the non-vanishing gravitoelectric tidal tensor $\mathbb{E}_{\alpha\beta}$; ii) the Klein-Gordon Eq. $\Box\Phi = m^2\Phi$ and the Hamiltonian in Sec. IV of [1] are for a (free) monopole particle, which feels no tidal forces. Thus one would expect it to reveal coordinate artifacts such as the fields $\vec{G}$, $\vec{H}$, and not physical tidal forces.

tidal tensor (while no electric field is present in the map) was a question left unanswered. Herein, putting together the knowledge from the tidal tensor with the inertial force formalisms (Secs. 2 and 3), we revisit these spacetimes and shed new light on these issues.

Eqs. (68) yield the GEM fields corresponding to the frame adapted to the rigid geodesic congruence, $u^\alpha = \delta^\alpha_0$. They tell us that the gravitoelectric field vanishes, $\vec{G} = 0$, which is consistent with the fact that no electric field arises in the mapping above; the gravitomagnetic field $\vec{H}$ is *linear* in the metric potentials:

$$\vec{H} = \tilde{\nabla} \times \vec{\mathcal{A}}. \tag{127}$$

These properties can be interpreted as follows. The fact that $\vec{G} = 0$ means that the metric is written in a frame corresponding to a congruence of freely falling observers (as their acceleration $a^\alpha = -G^\alpha$ is zero); the very special property of these spacetimes (unlike the situation in general, e.g. the Kerr or Schwarzschild spacetimes) is that such congruence is rigid, i.e. has no shear/expansion, allowing the metric to be time independent in the coordinates associated to that frame. The gravitomagnetic field, on the other hand, does not vanish in this frame, which means in this context (since the frame is congruence adapted, see Sec. 3.1 and Eq. (55)), that the congruence has vorticity. The equation of motion for a free particle in this frame, cf. Eq. (66), reduces to

$$\frac{\tilde{D}\vec{U}}{d\tau} = U^{\hat{0}}\vec{U} \times \vec{H} \ , \tag{128}$$

similar to the equation of motion of a charged particle under the action of a magnetic field; and since $\vec{H}$ is linear in the metric, the similarity with the electromagnetic analogue is indeed close.

Let us now examine the tidal effects. This type of spacetimes have a very special property: the gravitomagnetic tidal tensor measured by the observers $u^\alpha = \delta^\alpha_0$ is *linear* in the fields (and thus in the metric potentials), cf. Eq. (118), and, just like in the electromagnetic analogue, it is given by the covariant derivative of $\vec{H}$ with respect to γ_{ij}:

$$\mathbb{H}_{ij} = -\frac{1}{2}\tilde{\nabla}_j H_i = -\frac{1}{2}\tilde{\nabla}_j(\tilde{\nabla} \times \vec{\mathcal{A}})_i \tag{129}$$

($\mathbb{H}_{0j} = \mathbb{H}_{00} = \mathbb{H}_{j0}$ for these observers). This reinforces the similarity with electromagnetism. The gravitoelectric tidal tensor is, however, non-zero, as seen from Eq. (117):

$$\mathbb{E}_{ij} = \frac{1}{4}\left(\vec{H}^2\gamma_{ij} - H_j H_i\right) \ , \tag{130}$$

even though $\vec{G} = \vec{0}$. This should not be surprising, for the following reasons: i) it is always possible to make $\vec{G}$ vanish by choosing freely falling observers (this is true in an arbitrary spacetime), but that does not eliminate the tidal effects, as they arise from the curvature tensor; ii) in the present case of ultrastationary spacetimes, $\mathbb{E}_{\alpha\beta}$ is actually a non-linear tensor in $\vec{H}$, which merely reflects the fact that, except on very special circumstances, $\mathbb{E}_{\alpha\beta}$ (unlike its electromagnetic counterpart) cannot be thought of as simply a covariant derivative of some gravitoelectric field $\vec{G}$.

The tidal tensor (130) exhibits other interesting properties. It vanishes along the direction of the gravitomagnetic field H^α: let X^α be a spatial vector (with respect to u^α, $X^\alpha u_\alpha = 0$); if it is parallel to H^α, then $\mathbb{E}^\alpha_{\ \beta} X^\beta = 0$. That is, the tidal force, or the relative acceleration of two neighboring test particles of 4-velocity u^α, connected by X^α, vanishes. If X^α is orthogonal to the gravitomagnetic field, $H^\alpha X_\alpha = 0$, then it is an *eigenvector* of $\mathbb{E}^\alpha_{\ \beta}$, with eigenvalue $\vec{H}^2$. Thus, in the two dimensional subspace (on the rest space $u^i = 0$) spanned by the vectors orthogonal to H^α, the tidal force $-\mathbb{E}^{\alpha}{}_{\beta}X^\beta$ is *proportional* to the separation vectors X^α. Next we will physically interpret this for the special case of the Gödel universe.

4.1 The Gödel Universe

The Gödel universe is a solution corresponding to an *homogeneous* rotating dust with negative cosmological constant. The homogeneity implies that the dust rotates around every point. The line element can be put in the form (126), with

$$\mathcal{A}_i dx^i = e^{\sqrt{2}\omega x} dy \ , \quad \gamma_{ij} dx^i dx^j = dx^2 + \frac{1}{2} e^{2\sqrt{2}\omega x} dy^2 + dz^2 \ , \tag{131}$$

where ω is a constant. It is straightforward to show that the gravitomagnetic field $\vec{H}$ is *uniform*, and equal to $\vec{H} = 2\omega \vec{e}_z$; hence, by virtue of (129), the gravitomagnetic tidal tensor vanishes: $\mathbb{H}_{\alpha\beta} = 0$. For this reason, this universe has been interpreted in [1, 2] as being analogous to an uniform magnetic field in the curved 3-manifold with metric γ_{ij}, and the homogeneous rotation physically interpreted in analogy with a gas of charged particles subject to an uniform magnetic field — as in that case one equally has Larmor orbits around any point.

Now we will interpret its gravitoelectric tidal tensor. In the coordinate system (131) it reads, for the $u^i = 0$ observers:

$$\mathbb{E}_{ij} = \omega^2 \left(\gamma_{ij} - \delta_i^z \delta_j^z \right) \ .$$

It vanishes along z, and is isotropic in the spatial directions x, y orthogonal to $\vec{H}$. It is similar to the Newtonian tidal tensor $\partial_i \partial_j V$ of a potential $V = \omega^2 (x^2 + y^2)/2$, corresponding to a 2-D harmonic oscillator, which is the potential of the Newtonian analogue of the Gödel Universe [92]: an infinite cylinder of dust rotating *rigidly* with angular velocity ω. The potential V is such that the the gravitational attraction exactly balances the centrifugal force on each fluid element of the rigidly rotating cylinder. This causes a curious effect in the Newtonian system: the fluid is seen to be rotating about any point at rest in the frame comoving with the "original" cylinder; indeed, through an arbitrary point there passes an axis of rotation relative to which the system is indistinguishable from the "original one".

Therefore, whilst the gravitomagnetic field and tidal tensor, as well as the mapping via Klein-Gordon equation in [1], link to the magnetic analogue of the Gödel universe, the gravitoelectric tidal tensor links to the Newtonian analogue, both yielding consistent models to picture the homogeneous rotation of this universe.

5 Linear gravitoelectromagnetism

The oldest and best known gravito-electromagnetic analogies are the ones based on linearized gravity, which have been worked out by many authors throughout the years, see e.g. [7, 8, 14, 9, 10, 11, 12, 103, 16, 13, 120]. As is usually presented, one considers a metric given by small perturbations $|\varepsilon_{\alpha\beta}| \ll 1$ around Minkowski spacetime, $g_{\alpha\beta} = \eta_{\alpha\beta} + \varepsilon_{\alpha\beta}$, and from the components $\varepsilon_{\alpha\beta}$ one defines the 3-vectors $\vec{G}$ and $\vec{H}$, in terms of which one writes the gravitational equations. Let us write the line element of such metric in the general form

$$ds^2 = -(1 + 2\Phi)\, dt^2 + 2\mathcal{A}_j dt dx^j + (\delta_{ij} + 2\xi_{ij})\, dx^i dx^j \ . \tag{132}$$

If ones considers stationary perturbations, as is more usual (e.g. [7, 12, 103, 16, 13, 120]), the GEM fields are (up to numerical factors in the different definitions) $\vec{G} = -\nabla\Phi$, $\vec{H} = \nabla \times \vec{\mathcal{A}}$, where, *in this section* (and only herein!), $\nabla_i \equiv \partial/\partial x_i$ (equaling the covariant derivative operator associated to the background Euclidean metric δ_{ij}). These fields are straightforwardly related to the ones in Sec. 3: they are just, *to linear order*, minus the acceleration and twice the vorticity of the zero 3-velocity observers ($u^i = 0$) with respect to the coordinate system used in (132) (they can be called "static observers"). Thus they are simply a linear approximation to the quasi-Maxwell fields in Eqs. (68).

If the fields depend on time, different definitions of the fields exist in the literature, as a complete, one to one GEM analogy based on inertial fields, holding simultaneously for the geodesic equation and for the field equations, is not possible, as we shall see below (cf. also [8, 9, 10, 2, 51]). So if one chooses to write one of them in an electromagnetic like form, the other will contain extra terms. We stick to defining $\vec{G}$ and $\vec{H}$ by minus the acceleration and twice the vorticity of the observer congruence (i.e. the same definitions given in Sec. 3.2 for congruence adapted frames, only this time linearized), which seems to make more sense from a physical point of view, as with these definitions the fields appear in the equation of geodesics playing roles formally analogous to the electric and magnetic fields in the Lorentz force. That amounts to define:

$$\vec{G} = -\nabla\Phi - \frac{\partial\vec{\mathcal{A}}}{\partial t}\ ; \quad \vec{H} = \nabla \times \vec{\mathcal{A}}\ .$$

The space part of the linearized equation for the geodesics, in the coordinate basis $\mathbf{e}_\alpha \equiv \partial_\alpha$ associated to the coordinate system in (132), is obtained from the corresponding exact equation (52), for orthonormal tetrads, as follows[19]. One first notes that the coordinate triad of basis vectors $\mathbf{e}_i$ are connecting vectors between the $u^i = 0$ observers; thus they co-rotate with the congruence, and therefore the orthonormal tetrad which follows $\mathbf{e}_\alpha$ as close as possible is the *congruence adapted* tetrad (obtained by setting $\vec{\Omega} = \vec{\omega} = \vec{H}/2$, cf. Sec. 3.1); i.e., a tetrad such that $\mathbf{e}_{\hat{0}} \propto \mathbf{e}_0$ (for one to be dealing with the same observers) and that $\mathbf{e}_{\hat{i}}$ co-rotates with the $\mathbf{e}_i$, but without enduring the shear and expansion effects of the former (since the $\mathbf{e}_{\hat{\alpha}}$ remain orthonormal). Let $e^{\beta}_{\ \hat{\alpha}}$ denote the transformation matrix between $\mathbf{e}_\alpha$ and $\mathbf{e}_{\hat{\alpha}}$: $\mathbf{e}_{\hat{\alpha}} = e^{\beta}_{\ \hat{\alpha}}\mathbf{e}_\beta$. To linear order, $e^{\beta}_{\ \hat{\alpha}}$, and its inverse $e^{\hat{\beta}}_{\ \alpha}$, are given by:

$$\begin{aligned} &\mathbf{e}_{\hat{0}} = (1-\Phi)\mathbf{e}_0\ ; \qquad \mathbf{e}_{\hat{i}} = \mathbf{e}_i - \xi_i^{\ j}\mathbf{e}_j - \mathcal{A}_i\mathbf{e}_0\ ; \\ &\mathbf{e}_0 = (1+\Phi)\mathbf{e}_{\hat{0}}\ ; \qquad \mathbf{e}_i = \mathbf{e}_{\hat{i}} + \xi_i^{\ \hat{j}}\mathbf{e}_{\hat{j}} + \mathcal{A}_i\mathbf{e}_{\hat{0}}\ . \end{aligned} \tag{133}$$

Thus, $U^{\hat{i}} = e^{\hat{i}}_{\ \alpha}\, U^\alpha = U^i + \xi_i^{\ j}U_j$; using $U^i = dx^i/d\tau$, substituting into (52), linearizing in the perturbations and keeping lowest order terms in U^i, and noting that, to linear order,

$$K_{(ij)} \equiv u_{(i;j)} = \sigma_{ij} + \frac{1}{3}\theta\delta_{ij} \approx \frac{\partial\xi_{ij}}{\partial t}; \qquad \theta = K^i_{\ i} = \frac{\partial\xi^i_{\ i}}{\partial t}\ , \tag{134}$$

the equation for the geodesics reads:

$$\frac{d\vec{U}}{dt} = \vec{G} + \vec{U} \times \vec{H} - 2\frac{\partial\xi^i_{\ j}}{\partial t}U^j\vec{e}_i\ . \tag{135}$$

[19]One could also obtain it directly from the covariant version (63), by setting therein $\Omega^\alpha = \omega^\alpha = H^\alpha/2$, noting that, to linear order $dU^{\langle i\rangle}/dt = dU^i/dt + U^\beta d(h^u)^i_{\ \beta}/dt = dU^i/dt + 2U^j d\xi^i_{\ j}/dt$, and using (134), as done below.

That is, the extra term, compared to the Lorentz force of electromagnetism, comes *from the time derivative of the spatial metric* (which is true also in the exact case, as we have seen in Sec. 3.2). Noting that $d\vec{U}/dt \approx d^2\vec{x}/dt^2 - \vec{v}\partial\Phi/\partial t$, with $\vec{v} = d\vec{x}/dt$, we can also write this result as

$$\frac{d^2\vec{x}}{dt^2} = \vec{G} + \vec{v}\times\vec{H} - 2\frac{\partial \xi^i{}_j}{\partial t}v^j\vec{e}_i + \frac{\partial\Phi}{\partial t}\vec{v}\,. \tag{136}$$

The gravitational field equations in this regime are obtained by linearizing (96)-(101) and substituting relations (134):

$$\begin{aligned}
&\nabla\cdot\vec{G} = -4\pi(2\rho + T^\alpha_{\ \alpha}) - \frac{\partial^2 \xi^i{}_i}{\partial t^2}\ ; \qquad \text{(i)} \qquad\qquad \nabla\times\vec{G} = -\frac{\partial\vec{H}}{\partial t}\ ; \qquad \text{(ii)}\\
&\nabla\cdot\vec{H} = 0\ ; \qquad \text{(iii)} \qquad\qquad \nabla\times\vec{H} = -\,16\pi\vec{J} + 4\frac{\partial}{\partial t}\xi_j{}^{[j,k]}\vec{e}_k\ ; \qquad \text{(iv)}\\
&G_{i,j} + \frac{1}{2}\epsilon_{ijk}\frac{\partial H^k}{\partial t} + \frac{\partial^2}{\partial t^2}\xi_{ij} + 2\xi^k{}_{(j,i)k} - \nabla^2\xi_{ij} - \xi^k{}_{k,ij} = 8\pi\left(T_{ij} + \frac{1}{2}\delta_{ij}T^\alpha_{\ \alpha}\right)\,. \qquad \text{(v)}
\end{aligned} \tag{137}$$

Eqs. (137i), (137iv), and (137v), are, respectively, the time-time, time-space, and space-space components of Einstein's equations with sources (14a); Eqs. (137iii) and (137ii) are, respectively the time-time and space-time components of the identities (14b). To obtain (137v) from the exact Eq. (98), we note that $\tilde{R}_{ij}$ reads, to linear order

$$\tilde{R}_{ij} \simeq \Gamma^k_{ij,k} - \Gamma^k_{kj,i} \simeq 2\xi^k{}_{(j,i)k} - \nabla^2\xi_{ij} - \xi^k{}_{k,ij}$$

As for the time-space component of the identity (14b), i.e., Eq. (101), it yields the trivial, at linear order, equation $\star\tilde{R}^j{}_{ji} = 0$.

Eqs. (137) encompass two particularly important regimes: the "GEM limit", and gravitational radiation. Starting by the latter, in a source free region ($T^{\alpha\beta} = 0$) one can, as is well known, through gauge transformations (employing the harmonic gauge condition, and further specializing to the transverse traceless, or radiation, gauge, see e.g. [12]) make $\vec{\mathcal{A}} = \Phi = \xi^i{}_i = \xi^{ij}{}_{,j} = 0$; with this choice, the only non trivial equation left is (137v), yielding the 3-D wave equation $\partial^2\xi_{ij}/\partial t^2 = \nabla^2\xi_{ij}$.

The GEM regime is obtained making $\xi_{ij} = -\Phi\delta_{ij}$ (which effectively neglects radiation); in this case, the traceless shear of the congruence of zero 3-velocity observers ($u^i = 0$ in the coordinates system of (132)) vanishes, $\sigma_{\alpha\beta} = 0$, and we have $u_{(i;j)} = \theta\delta_{ij}/3 = -\delta_{ij}\partial\Phi/\partial t$. This is also the case for the post-Newtonian regime (e.g. [27, 63, 58, 111, 70]). Moreover, the source is assumed to be non-relativistic, so that the contribution of the pressure and stresses in Eq. (137) is negligible: $2\rho + T^\alpha_{\ \alpha} \approx \rho$. The two versions of the equation for the geodesics, (135) and (136), then read, respectively,

$$\frac{d\vec{U}}{dt} = \vec{G} + \vec{U}\times\vec{H} + 2\frac{\partial\Phi}{\partial t}\vec{v}\ ; \qquad \frac{d^2\vec{x}}{dt^2} = \vec{G} + \vec{v}\times\vec{H} + 3\frac{\partial\Phi}{\partial t}\vec{v} \tag{138}$$

and Eqs. (137) above become

$$\begin{aligned}
&\nabla\cdot\vec{G} = -4\pi\rho + 3\frac{\partial^2\Phi}{\partial t^2}\ ; \qquad \text{(i)} \qquad\qquad \nabla\times\vec{G} = -\frac{\partial\vec{H}}{\partial t}\ ; \qquad \text{(ii)}\\
&\nabla\cdot\vec{H} = 0\ ; \qquad \text{(iii)} \qquad \nabla\times\vec{H} = -\,16\pi\vec{J} + 4\frac{\partial\vec{G}}{\partial t} - 4\frac{\partial^2\vec{\mathcal{A}}}{\partial t^2}\ ; \qquad \text{(iv)}\\
&\frac{\partial}{\partial t}\mathcal{A}_{(i,j)} - \left(\frac{\partial^2\Phi}{\partial t^2} - \nabla^2\Phi\right)\delta_{ij} = -4\pi\rho\delta_{ij}\ . \qquad \text{(v)}
\end{aligned} \tag{139}$$

In some works, e.g. [11], the gravitoelectric field is given a different definition: $\vec{G}' = -\nabla\Phi - \frac{1}{4}\partial\vec{\mathcal{A}}/\partial t$. With this definition, and choosing the harmonic gauge condition, which implies $\nabla\cdot\vec{\mathcal{A}} = -4\partial\Phi/\partial t$, the non-Maxwellian term in Eq. (139i) disappears; but, on the other hand, a "non-Lorentzian" term appears in the equations for the geodesics, where in the place of $\vec{G}$ in Eqs. (135)-(136), we would have instead $\vec{G}' - \frac{3}{4}\partial\vec{\mathcal{A}}/\partial t$. As for the non-Maxwellian term in Eq. (139iv), it is neglected in the post-Newtonian regime [63, 27].

The presence of the terms $\partial\vec{H}/\partial t$ and $\partial\vec{G}/\partial t$, "inducing" curls in $\vec{G}$ and $\vec{H}$, respectively, analogous to the induction terms of electromagnetism, leads to the question of whether one can talk about gravitational induction effects in analogy with electrodynamics. Indeed, there is a debate concerning the applicability and physical content of this analogy for time-dependent fields, see e.g. [1] and references therein. Although a discussion of the approaches to this issue in the literature is outside the scope of this work, still there are some points that can be made based on the material herein. If one considers a time dependent gravitational field, such as the one generated by a moving point mass, e.g. Eq. (2.10) of [51], one finds that indeed the corresponding gravitoelectric field $\vec{G}$ is different from the one of a point mass at rest, and has a curl. That is, the acceleration $-\vec{G}$ of the congruence of observers at rest with respect to the background inertial frame (the "post-Newtonian grid", e.g. [58]), acquires a curl when the source moves with respect to that frame. From Eq. (139ii), one can think about this curl as induced by the time-varying gravitomagnetic field $\vec{H}$, see e.g. [111]. These fields are well suited to describe the *apparent* Newtonian and Coriolis-like accelerations of particles in geodesic motion, as shown by Eq. (138) above (one must just bear in mind that in the case of time-dependent fields, the motion is not determined solely by $\vec{G}$ and $\vec{H}$; there is an additional term with no analogue in the Lorentz force law, which leads to important differences). However, the latter are artifacts of the reference frame; the *physical* (i.e., tidal) forces tell a different story, as one does not obtain the correct tidal forces by differentiation of $\vec{G}$ and $\vec{H}$ (as is the case with electrodynamics). Namely, the curls of the GEM fields do not translate into these forces. The linearized gravitoelectric tidal tensor, Eq. (121a), reads in the GEM regime ($K_{(ij)} = -\delta_{ij}\partial\Phi/\partial t$),

$$\mathbb{E}_{ij} \approx -G_{i,j} + \frac{1}{2}\epsilon_{ijk}\frac{\partial H^k}{\partial t} - \frac{\partial\Phi}{\partial t}\delta_{ij} = -G_{(i,j)} - \frac{\partial\Phi}{\partial t}\delta_{ij} \;, \tag{140}$$

where we see that the curl (139ii) is subtracted from the derivative of $\vec{G}$. That is, only the symmetrized derivative $G_{(i,j)}$ describes physical, covariant forces. This is manifest in the fact that the curl of $\vec{G}$ does not induce a rotation on a set of neighboring particles (the gravitational field only shears the set, see Sec. 2.2 and Eq. (26) therein), nor does it torque a rigid test body, see [6]. Note that in electromagnetism this rotation and torque are tidal manifestations of Faraday's law of induction. Likewise, the curl of $\vec{H}$ is not manifest in the gravitomagnetic tidal effects (e.g., the force on a gyroscope); the linearized gravitomagnetic tidal tensor (121b) reads, in this regime:

$$\mathbb{H}_{ij} \approx -\frac{1}{2}\left[H_{i,j} - 2\epsilon_{ijl}\left(\frac{\partial G^l}{\partial t} - \frac{\partial^2\mathcal{A}^l}{\partial t^2}\right)\right] \;, \tag{141}$$

where again we can see that the induction contribution $4\partial\vec{G}/\partial t$ (and also the one of the term $\partial^2\vec{\mathcal{A}}/\partial t^2$) to the curl of $\vec{H}$ is subtracted from the derivative of $\vec{H}$. The physical consequences are explored in [6]: in electromagnetism, due to vacuum equation $\nabla\times\vec{B} = -\partial\vec{E}/\partial t$, there is a non-vanishing force on a magnetic dipole, $F^i_{EM} = B^{\beta i}\mu_\beta$ ($= \nabla^i(\vec{\mu}\cdot\vec{B})$ in the comoving inertial frame, cf. Eq. (122)), whenever it moves in a non-homogeneous field; this is because the electric field measured by the particle is time-varying, and so $\nabla \times \vec{B} \neq 0 \Rightarrow B_{ij} \neq 0 \Rightarrow \vec{F}_{EM} \neq 0$. That is not necessarily the case in gravity. In vacuum, from Eqs. (139iv) and (141), we have $\mathbb{H}_{ij} = -H_{(i,j)}/2$, and the gravitational force on a gyroscope, cf. Eq. (1.2b) of Table 1, is $F^i_G = \frac{1}{2}H^{(i,j)}S_j$. Thus no analogous induction effect is manifest in the force, and in fact spinning particles in non-homogeneous gravitational fields can move along geodesics, as exemplified in [6].

As for the equation of motion for the gyroscope's spin vector, from Eq. (27) we get, in terms of components in the coordinate system associated to (132),

$$\frac{dS^i}{dt} = -\Gamma^i_{0j}S^j = \frac{1}{2}(\vec{S} \times \vec{H})^i - \frac{\partial \Phi}{\partial t}S^i \ . \tag{142}$$

Comparing with the equation for the precession of a magnetic dipole (with respect to an inertial frame), $d\vec{S}/d\tau = \vec{\mu} \times \vec{B}$, there is a factor of 1/2, and an additional term. The origin of the former is explained in Sec. 3.3: it is due to the fact the the field $\vec{H}$, causing the Coriolis (or gravitomagnetic) acceleration of test particles via Eq. (135), is distinct from the field causing the gyroscope precession in (142); in general they are *independent*. $\vec{H}$ is the sum of the vorticity $\vec{\omega}$ of the observer congruence with the angular velocity of rotation $\vec{\Omega}$ of the frame's spatial triads relative to Fermi-Walker transport; and to Eq. (142) only the latter part contributes. In the case of a congruence adapted frame ($\vec{\Omega} = \vec{\omega}$), which is the problem at hand (the frame is adapted to the congruence of $u^i = 0$ observers), this originates the relative factor of 1/2. Note also that the same factor shows up also in the force on the gyroscope discussed above, but in this case by the opposite reason: to $\vec{F}_G$ the vorticity $\vec{\omega}$ is the only part of $\vec{H}$ that contributes, cf. Eq. (114). The second term in (142) merely reflects the fact that the basis vectors $\mathbf{e}_i$ expand; if using expressions (133), one transforms to the orthonormal basis $S^i = e^i_{\ \hat{i}}S^{\hat{i}}$, and substitutes in (142), that term vanishes, as expected from the exact result (70).

If the field is stationary, we have a one to one correspondence with electromagnetism *in inertial frames*. Eq. (137v) above becomes identical to (137i), and then we are left with a set of four equations — Eqs. (137i)-(137iv) with the time dependent terms dropped — similar, up to some factors, to the time-independent Maxwell equations in an inertial frame. These equations can also be obtained by linearization of Eqs. (24b)-(24b) of Table 2. The space part of the equation of the geodesics: $d^2\vec{x}/dt^2 = \vec{G} + \vec{v} \times \vec{H}$, cf. Eq. (136) above, is also similar to the Lorentz force in a Lorentz frame. The equation for the evolution of the spin vector of a gyroscope, in the coordinate basis, becomes simply $d\vec{S}/d\tau = \vec{S} \times \vec{H}/2$, which gives the precession relative to the background Minkowski frame, and is similar to the precession of a magnetic dipole in a magnetic field. The force on a gyroscope whose center of mass it *at rest* is $\vec{F}_G = \nabla(\vec{S} \cdot \vec{H})/2$, similar to the force $\vec{F}_{EM} = \nabla(\vec{\mu} \cdot \vec{B})$ on a magnetic dipole at rest in a Lorentz frame; the same for the differential precession of gyroscopes/dipoles at rest: for a spatial separation vector δx^α they read, respectively, $\delta\vec{\Omega}_G = -\nabla(\delta\vec{x} \cdot \vec{H})/2$ and $\delta\vec{\Omega}_{EM} = -\nabla(\delta\vec{x} \cdot \vec{B})$.

6 The *formal* analogy between gravitational *tidal tensors* and electromagnetic *fields*

There is a set of analogies, based on exact expressions, relating the Maxwell tensor $F^{\alpha\beta}$ and the Weyl tensor $C_{\alpha\beta\gamma\delta}$. These analogies rest on the fact that: 1) they both irreducibly decompose in an electric and a magnetic type spatial tensors; 2) these tensors obey differential equations — Maxwell's equations and the so called "higher order" gravitational field equations — which are formally analogous to a certain extent [35, 32, 36, 37]; and 3) they form invariants in a similar fashion [31, 32, 38, 39]. In this section we will briefly review these analogies and clarify their physical content in the light of the previous approaches.

The Maxwell tensor splits, with respect to a unit time-like vector u^α, into its electric and magnetic parts:

$$E^\alpha \equiv (E^u)^\alpha = F^\alpha_{\ \beta} u^\beta \ , \qquad B^\alpha \equiv (B^u)^\alpha = \star F^\alpha_{\ \beta} u^\beta \ , \tag{143}$$

i.e., the electric and magnetic fields as measured by the observers of 4-velocity u^α. These are spatial vectors: $E^\alpha u_\alpha = B^\alpha u_\alpha = 0$, thus possessing 3+3 independent components, which completely encode the 6 independent components of $F_{\mu\nu}$, as can be seen explicitly in decompositions (1). In spite of their dependence on u^α, one can use E^α and B^β to define two tensorial quantities which are u^α independent, namely

$$E^\alpha E_\alpha - B^\alpha B_\alpha = -\frac{F_{\alpha\beta}F^{\alpha\beta}}{2} \ , \qquad E^\alpha B_\alpha = -\frac{\star F_{\alpha\beta}F^{\alpha\beta}}{4} \ ; \tag{144}$$

these are the only algebraically independent invariants one can define from the Maxwell tensor.

The Weyl tensor has a formally similar decomposition: with respect to a unit time-like vector u^α, it splits irreducibly into its electric $\mathcal{E}_{\alpha\beta}$ and magnetic $\mathcal{H}_{\alpha\beta}$ parts:

$$\mathcal{E}_{\alpha\beta} \equiv (\mathcal{E}^u)_{\alpha\beta} = C_{\alpha\gamma\beta\sigma}u^\gamma u^\sigma, \qquad \mathcal{H}_{\alpha\beta} \equiv (\mathcal{H}^u)_{\alpha\beta} = \star C_{\alpha\gamma\beta\sigma}u^\gamma u^\sigma. \tag{145}$$

These two spatial tensors, both of which are symmetric and traceless (hence have 5 independent components each), completely encode the 10 independent components of the Weyl tensor, as can be seen by writing [35]

$$C_{\alpha\beta}^{\ \ \gamma\delta} = 4\left\{2u_{[\alpha}u^{[\gamma} + g^{[\gamma}_{[\alpha}\right\}\mathcal{E}^{\delta]}_{\beta]} + 2\left\{\epsilon_{\alpha\beta\mu\nu}u^{[\gamma}\mathcal{H}^{\delta]\mu}u^\nu + \epsilon^{\gamma\delta\mu\nu}u_{[\alpha}\mathcal{H}_{\beta]\mu}u_\nu\right\} \tag{146}$$

(in vacuum, this equals decomposition (15)). Again, in spite of their dependence on u^α, one can use $\mathcal{E}_{\alpha\beta}$ and $\mathcal{H}_{\alpha\beta}$ to define the two tensorial quantities which are U^α independent,

$$\mathcal{E}^{\alpha\beta}\mathcal{E}_{\alpha\beta} - \mathcal{H}^{\alpha\beta}\mathcal{H}_{\alpha\beta} = \frac{C_{\alpha\beta\mu\nu}C^{\alpha\beta\mu\nu}}{8} \ , \qquad \mathcal{E}^{\alpha\beta}\mathcal{H}_{\alpha\beta} = \frac{\star C_{\alpha\beta\mu\nu}C^{\alpha\beta\mu\nu}}{16} \ , \tag{147}$$

which are *formally* analogous to the electromagnetic scalar invariants (144). Note however that, by contrast with the latter, these are not the only independent scalar invariants one can construct from $C_{\alpha\beta\mu\nu}$; there are also two cubic invariants, see [32, 41, 29, 46, 40].

As stated above, these tensors obey also differential equations which have some formal similarities with Maxwell's; such equations, dubbed the "higher order field equations", are obtained from the Bianchi identities $R_{\sigma\tau[\mu\nu;\alpha]} = 0$. These, together with the field equations (14a), lead to:

$$C^\mu_{\ \nu\sigma\tau;\mu} = 8\pi\left(T_{\nu[\tau;\sigma]} - \frac{1}{3}g_{\nu[\tau}T_{;\sigma]}\right) \ , \tag{148}$$

Expressing $C_{\alpha\beta\delta\gamma}$ in terms of $\mathcal{E}_{\alpha\beta}$ and $\mathcal{H}_{\alpha\beta}$ using (146), and taking time and space projections of (148) using the projectors (2), we obtain, assuming a *perfect fluid*, the set of equations

$$\begin{gathered}\tilde{\nabla}^\mu\mathcal{E}_{\nu\mu} = \frac{8\pi}{3}\tilde{\nabla}_\nu\rho + 3\omega^\mu\mathcal{H}_{\nu\mu} + \epsilon_{\nu\alpha\beta}\sigma^\alpha_{\ \gamma}\mathcal{H}^{\beta\gamma} \ ; \\ \mathrm{curl}\mathcal{H}_{\mu\nu} = \nabla^\perp_{\mathbf{u}}\mathcal{E}_{\mu\nu} + \mathcal{E}_{\mu\nu}\theta - 3\sigma_{\tau\langle\mu}\mathcal{E}^{\ \tau}_{\nu\rangle} - \omega^\tau\epsilon_{\tau\rho(\mu}\mathcal{E}^{\ \rho}_{\nu)} - 2a^\rho\epsilon_{\rho\tau(\mu}\mathcal{H}^{\ \tau}_{\nu)} + 4\pi(\rho+p)\sigma_{\mu\nu} \ ;\end{gathered} \tag{149}$$

$$\tilde{\nabla}^{\mu}\mathcal{H}_{\nu\mu} = 8\pi(\rho+p)\omega_{\nu} - 3\omega^{\mu}\mathcal{E}_{\nu\mu} - \epsilon_{\nu\alpha\beta}\sigma^{\alpha}{}_{\gamma}\mathcal{E}^{\beta\gamma} \ ;$$
$$\mathrm{curl}\mathcal{E}_{\mu\nu} = -\nabla^{\perp}_{\mathbf{u}}\mathcal{H}_{\mu\nu} - \mathcal{H}_{\mu\nu}\theta + 3\sigma_{\tau\langle\mu}\mathcal{H}_{\nu\rangle}{}^{\tau} + \omega^{\tau}\epsilon_{\tau\rho(\mu}\mathcal{H}_{\nu)}{}^{\rho} - 2a^{\rho}\epsilon_{\rho\tau(\mu}\mathcal{E}_{\nu)}{}^{\tau} \ , \tag{150}$$

where, following the definitions in [35], $\epsilon_{\mu\nu\rho} \equiv \epsilon_{\mu\nu\rho\tau}u^{\tau}$, $\mathrm{curl}A_{\alpha\beta} \equiv \epsilon^{\mu\nu}{}_{(\alpha}A_{\beta)\nu;\mu}$, and the index notation $\langle\mu\nu\rangle$ stands for the spatially projected, symmetric and trace free part of a rank two tensor:

$$A_{\langle\mu\nu\rangle} \equiv h_{(\mu}{}^{\alpha}h^{\beta}_{\nu)}A_{\alpha\beta} - \frac{1}{3}h_{\mu\nu}h_{\alpha\beta}A^{\alpha\beta} \ ,$$

with $h^{\alpha}{}_{\beta} \equiv (h^{u})^{\alpha}{}_{\beta}$, cf. Eq. (2). $\nabla^{\perp}_{\mathbf{u}}$ and $\tilde{\nabla}$ (which in the equations above we could have written as well $\nabla^{\perp}$, for they are the same along the spatial directions) are the derivative operators whose action on a spatial vector is defined in Eqs. (58) and (61), respectively. For a rank two *spatial* tensor $A^{\alpha\beta}$, we have $\nabla^{\perp}_{\mathbf{u}}A^{\alpha\beta} = h^{\alpha}{}_{\mu}h^{\beta}{}_{\nu}\nabla_{\mathbf{u}}A^{\mu\nu}$ and $\tilde{\nabla}_{\alpha}A^{\alpha\beta} = h^{\alpha}{}_{\mu}h^{\beta}{}_{\nu}\nabla_{\alpha}A^{\mu\nu}$. As before, the quantities $\theta \equiv u^{\alpha}{}_{;\alpha}$, $\sigma_{\mu\nu} \equiv h^{\alpha}{}_{\mu}h^{\beta}{}_{\nu}u_{\alpha;\beta}$, $\omega^{\alpha} \equiv \epsilon^{\alpha}{}_{\beta\gamma}u_{\gamma;\beta}/2$ and a^{α} are, respectively, the expansion, shear, vorticity and acceleration of the congruence of observers with 4-velocity u^{α}.

The analogous electromagnetic equations are the ones in Sec. 3.4.1, which we can re-write as

$$\tilde{\nabla}_{\mu}E^{\mu} = 4\pi\rho_c + 2\omega_{\mu}B^{\mu} \ ; \tag{151}$$

$$\epsilon^{\alpha\gamma\beta}B_{\beta;\gamma} = \nabla^{\perp}_{\mathbf{u}}E^{\alpha} - \sigma^{\alpha}{}_{\beta}E^{\beta} + \frac{2}{3}\theta E^{\alpha} - \epsilon^{\alpha}{}_{\beta\gamma}\omega^{\beta}E^{\gamma} + \epsilon^{\alpha}{}_{\beta\gamma}B^{\beta}a^{\gamma} + 4\pi j^{\langle\alpha\rangle} \ ; \tag{152}$$

$$\tilde{\nabla}_{\mu}B^{\mu} = -2\omega_{\mu}E^{\mu} \ ; \tag{153}$$

$$\epsilon^{\alpha\gamma\beta}E_{\beta;\gamma} = -\nabla^{\perp}_{\mathbf{u}}B^{\alpha} + \sigma^{\alpha}{}_{\beta}B^{\beta} - \frac{2}{3}\theta B^{\alpha} + \epsilon^{\alpha}{}_{\beta\gamma}\omega^{\beta}B^{\gamma} + \epsilon^{\alpha\mu\sigma}E_{\mu}a_{\sigma} \ . \tag{154}$$

Eqs. (151) and (153) follow from Eqs. (75) and (77), respectively, by noting that, for an arbitrary vector A^{α},

$$A^{\beta}{}_{;\beta} = \left(\top^{\gamma}_{\beta} + h^{\gamma}_{\beta}\right)\left(\top^{\beta}_{\lambda} + h^{\beta}_{\lambda}\right)A^{\lambda}{}_{;\gamma} = \left(\top^{\gamma}_{\lambda} + h^{\gamma}_{\lambda}\right)A^{\lambda}{}_{;\gamma} = A^{\beta}a_{\beta} + \tilde{\nabla}_{\alpha}A^{\alpha}.$$

Eqs. (152) and (154) follow from Eqs. (79) and (82) by decomposing $K_{(\alpha\beta)} = \sigma_{\alpha\beta} + \theta h_{\alpha\beta}/3$.

It is worth mentioning that the *exact* wave equations for E^{α} and B^{α} in this formalism were obtained in [124], Eqs. (39)-(40) therein[20]. As for the exact wave equations for $\mathcal{E}_{\alpha\beta}$, $\mathcal{H}_{\alpha\beta}$, they have not, to our knowledge, been derived in the literature; only in some approximations, such as in e.g. [37, 88], or the linear regime of the next section.

[20]The wave equations in [124, 125] are obtained using also the Ricci identities $2\nabla_{[\gamma}\nabla_{\beta]}X_{\alpha} = R_{\delta\alpha\beta\gamma}X^{\delta}$, which couple the electromagnetic fields to the curvature tensor; this coupling is shown to lead to amplification phenomena, suggested therein as a possible explanation for the observed (and unexplained) large-scale cosmic magnetic fields.

6.1 Matte's equations vs Maxwell equations. Tidal tensor interpretation of gravitational radiation.

Table 3: Formal analogy between Maxwell's equations (differential equations for electromagnetic *fields*) and Matte's equations (differential equations for gravitational *tidal* tensors)

Electromagnetism		Linearized Gravity	
Maxwell's Equations		Matte's Equations	
$E^i{}_{,i} = 0$	(3.1a)	$\mathbb{E}^{ij}{}_{,i} = 0$	(3.1b)
$B^i{}_{,i} = 0$	(3.2a)	$\mathbb{H}^{ij}{}_{,i} = 0$	(3.2b)
$\epsilon^{ikl} E_{l,k} = -\dfrac{\partial B^i}{\partial t}$	(3.3a)	$\epsilon^{ikl} \mathbb{E}^j{}_{l,k} = -\dfrac{\partial \mathbb{H}^{ij}}{\partial t}$	(3.3b)
$\epsilon^{ikl} B_{l,k} = \dfrac{\partial E^i}{\partial t}$	(3.4a)	$\epsilon^{ikl} \mathbb{H}^j{}_{l,k} = \dfrac{\partial \mathbb{E}^{ij}}{\partial t}$	(3.4b)
Wave equations		Wave equations	
$\left(\dfrac{\partial^2}{\partial t^2} - \partial^k \partial_k\right) E^i = 0$	(3.5a)	$\left(\dfrac{\partial^2}{\partial t^2} - \partial^k \partial_k\right) \mathbb{E}_{ij} = 0$	(3.5b)
$\left(\dfrac{\partial^2}{\partial t^2} - \partial^k \partial_k\right) B^i = 0$	(3.6a)	$\left(\dfrac{\partial^2}{\partial t^2} - \partial^k \partial_k\right) \mathbb{H}_{ij} = 0$	(3.6b)

In vacuum, the Bianchi identities become:

$$R_{\sigma\tau[\mu\nu;\alpha]} = 0; \quad (a) \qquad R^{\mu}{}_{\alpha\beta\gamma;\mu} = 0 \quad (b) \tag{155}$$

(the second equation following from the first and from vacuum equation $R_{\mu\nu} = 0$). The formal analogy with Eqs. (6), for $j^\alpha = 0$, is now more clear [32]. In a nearly Lorentz frame where $u^i = 0$, and to linear order in the metric potentials, Eqs. (149)-(150), for vacuum, become Eqs. (3.1b)-(3.4b) of Table 3, which are *formally* similar to Maxwell's equations in a Lorentz frame (3.1a)-(3.4a). The analogy in Eqs. (3.1)-(3.4) was first found by Matte [31], and further studied by some other authors[21] [32, 34, 33]. Taking curls of Eqs. (3.3a)-(3.4a) we obtain the wave equations for the electromagnetic fields; and taking curls of (3.3b)-(3.4b), we obtain gravitational waves, as wave equations for gravitational tidal tensors.

Hence, to this degree of accuracy, vacuum gravitational waves can be cast as a pair of oscillatory tidal tensors $\mathbb{E}_{\alpha\beta}$, $\mathbb{H}_{\alpha\beta}$, propagating in space by mutually inducing each other, just like the pair of fields E^α, B^α, in the case of the electromagnetic waves. Also, just like E^α and B^α are equal in magnitude and mutually orthogonal for a purely radiative field, the same applies to the waves in (3.5b)-(3.6b) of Table 3. In the electromagnetic case this implies that the two invariants (144)

[21] The *exact* equations (149)-(150), in vacuum, can take the form (3.1b)-(3.4b) of Table 3, in a *local* Lorentz frame (where $a^\alpha = \omega^\alpha = \sigma_{\alpha\beta} = \theta = 0$), as done in e.g. [4]. That however holds only in a small neighborhood around a particular observer worldline; therefore, if one wants to study the propagation of radiation with the equations of this form, one must settle for the linear regime.

vanish; likewise, the gravitational invariants (147) also vanish for a solution corresponding to pure gravitational radiation according to Bel's second criterion (cf. e.g. [44] p. 53) — a definition based on "super-energy", discussed below.

An interesting aspect of this formulation of gravitational radiation, contrasting with the more usual approaches in the literature, e.g. [12, 53, 7, 3] — which consist of equations for the propagation of gauge fields (the components of the metric tensor), having no local physical significance (only their second derivatives may be related to physically measurable quantities, see in this respect [75]) — is that Eqs. (3.5b)-(3.6b) are equations for the propagation of *tensors of physical forces,* with direct translation in physical effects – the relative acceleration of two neighboring test particles via geodesic deviation equation (1.1b) of Table 1, and the force on a spinning test particle, via Mathisson-Papapetrou-Pirani Eq. (1.2b), or the relative precession of two nearby gyroscopes, via Eqs. (29)-(30). The latter effects come from the gravitomagnetic part (i.e., involving $\mathbb{H}_{\alpha\beta}$) of the radiation, little-studied in comparison to its gravitoelectric counterpart; its role, in equal footing with the gravitoelectric part, is made explicit in this formulation — they are both essential for the existence of gravitational waves, as a pair of mutually inducing fields is needed. It yields a simple and intuitive description of the interaction of a gravitational wave with a pole-dipole spinning particle, as a coupling of the wave equation (3.6b) to the spin-vector of the particle, via Eq. (1.2b). In other words: putting together Matte's equations with the physical interpretation of the electric and magnetic parts of the Riemann tensor given in Sec. 2, we obtain this suggestive interpretation of gravitational radiation, which has not (to our knowledge) been put forth in the existing literature. That might be down to two reasons: on the one hand, the works treating the higher order/Matte equations [31, 32, 75, 36, 34, 33, 37, 88], as well the as tensors $\{\mathcal{E}_{\alpha\beta}, \mathcal{H}_{\alpha\beta}\}$ in other contexts, e.g. [86, 88, 64], lack a physical interpretation of the tensors $\mathcal{H}_{\alpha\beta}/\mathbb{H}_{\alpha\beta}$ (the tensors $\mathcal{E}_{\alpha\beta}/\mathbb{E}_{\alpha\beta}$ in turn are well understood due to their role in the geodesic deviation equation), which are either portrayed as not well understood [86, 88, 64], or given inconsistent interpretations that lead to contradictions [84, 85, 35] (see [2] for more details). On the other hand, in the literature concerning the interaction of gravitational waves with spinning particles, e.g. [71, 72, 73, 74], neither gravitational waves are written in the tidal tensor form (3.5b)-(3.6b) (but instead in the more usual equations for the propagation of the metric perturbations), nor, as a matter of fact, is the force on a spinning particle explicitly related to the magnetic part of the Riemann tensor $\mathbb{H}_{\alpha\beta}$ (or to $\mathcal{H}_{\alpha\beta}$), which is likely down to the fact that in these treatments the Tulczyjew-Dixon [114] spin condition $S^{\alpha\beta}P_\beta = 0$ is employed in the force equation (159), instead of the Mathisson-Pirani [47, 50] condition $S^{\alpha\beta}U_\beta = 0$. The two conditions are equally valid (see in this respect [83]); however, it is only when one uses the latter that $\mathbb{H}_{\alpha\beta}$ appears explicitly in the equation for the dipole force (first term of Eq. (159)), which becomes in this case Eq. (1.2b) of Table 1, see [6] for more details.

It is also interesting to note that in the traditional treatments the wave equations are obtained from a linearized form of Einstein's equations (14a); whereas equations (3.5b)-(3.6b) come from a linearization of the higher order field equations (155b). (Even though the former still play a role, as in order to obtain Eqs. (3.5b)-(3.6b) from the differential Bianchi identity (155a), we have to substitute $R_{\alpha\beta}$ by the source terms using Einstein's equations (14a)).

It is important to realize that whereas in electromagnetic radiation it is the vector fields that propagate, gravitational radiation is a purely tidal effect, i.e., traveling tidal tensors not subsidiary to any associated (electromagnetic-like, or Newtonian-like) vector field; it is well known that there are no vector waves in gravity (see e.g. [66, 53, 10]; such waves would carry negative energy if they were to exist, cf. [53] p. 179). We have seen in Sec. 3.5 that, except for the very special case of the linear regime in weak, *stationary* fields (and static observers therein), the gravitational tidal tensors cannot be cast as derivatives of some vector field. In the electromagnetic case there are of course also tidal effects associated to the wave; but their dynamics follows trivially[22] from Eqs. (3.3a)-(3.4a) of Table 3; to this accuracy, the tidal tensors as measured by the background static observers are just $E_{ij} = E_{i,j}$, $B_{ij} = B_{i,j}$; hence the equations of their evolution (i.e., the "electromagnetic higher order equations") are:

$$\epsilon_i^{\ kl} E_{jl,k} = 0 \ ; \qquad \epsilon_i^{\ kl} B_{jl,k} = 0 \ ; \tag{156}$$

$$\epsilon_i^{\ kl} E_{lj,k} = \epsilon_i^{\ kl} E_{lk,j} = -\frac{\partial B_{ij}}{\partial t} \ ; \qquad \epsilon_i^{\ kl} B_{lj,k} = \epsilon_i^{\ kl} B_{lk,j} = \frac{\partial E_{ij}}{\partial t} \ . \tag{157}$$

These four equations are the *physical* analogues of the pair of gravitational Eqs. (3.3b)-(3.4b); we have two more equations in electromagnetism, since E_{ij} and B_{ij} are not symmetric. Eqs. (156), and the first equality in Eqs. (157), come from the fact that derivatives in flat spacetime commute; therefore $\epsilon_i^{\ kl} E_{jl,k} = \epsilon_i^{\ kl} E_{j,[lk]} = 0$ and $E_{lj,k} = E_{l,jk} = E_{lk,j}$. Thus, Eqs. (157), which are the only ones that contain dynamics, are obtained by simply differentiating Eqs. (3.3a)-(3.4a) with respect to x^j. The wave equations for the electromagnetic tidal tensors follow likewise from differentiating Eqs. (3.5a)-(3.6a) with respect to x^j. Note that the fact that in gravity $\mathbb{H}_{j[l,k]} \neq 0$ is again related to the fact that, even in the linear regime, the gravitational tidal tensors are not derivatives of some vector fields.

The tidal tensor interpretation of gravitational waves gives insight into some of their fundamental aspects; we will mention three. Firstly, since gravitational waves (in vacuum) are traveling tidal tensors, then, according to the discussion in Sec. 2, they couple to dipole particles (i.e., spinning particles, or "gravitomagnetic dipoles"), causing a force, and they can also cause a relative acceleration between two neighboring monopole particles; but they cannot exert any force on a monopole test particle (by contrast with their electromagnetic counterparts); that is the reason why a gravitational wave distorts a (approximately monopole) test body, but does not displace its center of mass, as is well known, e.g. [7]. Let us state this important point in terms of rigorous equations. As explained in Sec. 2, an extended test body may be represented by its electromagnetic and "gravitational" multipole moments (i.e., the moments of j_p^α and $T_p^{\alpha\beta}$, respectively). The force exerted on a charged body in an electromagnetic field is, up to quadrupole order [48, 16],

$$\frac{DP^\alpha_{\rm Dix}}{d\tau} = qF^{\alpha\beta}U_\beta + \frac{1}{2}F^{\mu\nu;\alpha}Q_{\mu\nu} + \frac{1}{3}Q_{\beta\gamma\delta}F^{\gamma\delta;\beta\alpha} + \ldots \ , \tag{158}$$

where $P^\alpha_{\rm Dix}$ is "Dixon's momentum" (see [6]), U^α is the particle's 4-velocity, q is the charge, $Q_{\alpha\beta}$ is the dipole moment 2-form, which we may write as $Q_{\alpha\beta} = 2d_{[\alpha}U_{\beta]} + \epsilon_{\alpha\beta\gamma\delta}\mu^\gamma U^\delta$, and $Q_{\alpha\beta\gamma}$ is a quadrupole moment of the charge 4-current density j^α. Using decomposition (7), we can write the second term in terms of tidal tensors ($F^{\mu\nu;\alpha}Q_{\mu\nu}/2 = E^{\beta\alpha}d_\beta + B^{\beta\alpha}\mu_\beta$), the third term in terms of derivatives of the tidal tensors, and so on. Thus we see that, in the electromagnetic case, there

[22] We thank J. Penedones for discussions on this point.

are force terms coming from the electromagnetic monopole, dipole, quadrupole, etc., moments, corresponding to the coupling to, respectively, the field, the tidal field, derivatives of tidal field, etc.

The *exact* equations of motion for a test body in a gravitational field are [47, 16], up to quadrupole order,

$$\frac{DP^{\alpha}}{d\tau}=-\frac{1}{2}R^{\alpha}_{\ \beta\mu\nu}S^{\mu\nu}U^{\beta}-\frac{1}{6}J_{\beta\gamma\delta\sigma}R^{\beta\gamma\delta\sigma;\alpha}+\dots\ , \tag{159}$$

where $S_{\alpha\beta}=\epsilon_{\alpha\beta\gamma\delta}S^{\gamma}U^{\delta}$ is the spin 2-form, which is essentially the dipole moment of the mass current 4-vector of the particle, $J^{\alpha}=-T^{\alpha\beta}U_{\beta}$, see e.g. [48, 6], and $J_{\beta\gamma\delta\sigma}$ is a quadrupole moment of the energy-momentum tensor of the test particle, $T^{\alpha\beta}$. The first term yields of course $-\mathbb{H}^{\beta\alpha}S_{\beta}$, in agreement with Eq. (1.2b) of Table 1, corresponding to the coupling of the gravitomagnetic tidal tensor with the spin vector S^{α} (the "gravitomagnetic dipole moment"), the second term yields the coupling of the derivatives of the tidal field to the quadrupole moment, and so on. Hence we see that, by contrast with electromagnetism, in the gravitational case the interaction starts only at dipole order. In other words, the lowest order (in differentiation) *physical* fields present in gravity (the real, physical content of the gravitational waves) are the tidal tensors, which do not couple to monopoles.

Secondly, this tidal tensor interpretation of gravitational waves also sheds some light on the issues of gravitational induction. As seen in Secs. 2, 3.5, and 5, electromagnetic-like induction effects are absent in the gravitational *physical* forces. However, instead of vector fields, in gravity one can talk of a different type of induction phenomena, *for tidal tensors*: a time-varying magnetic tidal tensor induces an electric tidal tensor, and vice-versa, as implied by Eqs. (3.3b)-(3.4b) of Table 3. These tidal tensors propagate by mutually inducing each other, giving rise to gravitational radiation, just like the laws of electromagnetic induction lead to electromagnetic radiation.

Finally, one can also relate the fact that what propagates in gravitational waves are the tidal tensors, without any associated electromagnetic-like vector field, with the fact that, to the gravitational waves (and the gravitational field itself), one cannot associate a *local* energy or momentum density, in the traditional sense of being manifest in the energy momentum tensor $T^{\alpha\beta}$. This becomes more clear by contrasting with the electromagnetic case. The electromagnetic field gives a contribution to the total energy momentum tensor $T^{\alpha\beta}$ equal to:

$$T^{\alpha\beta}_{EM}=\frac{1}{8\pi}\left[F^{\alpha\gamma}F^{\beta}_{\ \gamma}+\star F^{\alpha\gamma}\star F^{\beta}_{\ \gamma}\right]\ , \tag{160}$$

leading to an energy density ρ_{EM} and spatial momentum density $p^{\langle\alpha\rangle}_{EM}$ (Poynting vector), with respect to an observer of 4-velocity u^{α},

$$\rho_{EM}\ \equiv\ T^{\alpha\beta}_{EM}u_{\alpha}u_{\beta}\ =\ \frac{1}{8\pi}\left[(E^{u})^{\alpha}(E^{u})_{\alpha}+(B^{u})^{\alpha}(B^{u})_{\alpha}\right]\ ; \tag{161}$$

$$p^{\langle\alpha\rangle}_{EM}\ \equiv\ -T^{\langle\alpha\rangle\beta}_{EM}u_{\beta}=\frac{1}{4\pi}\epsilon^{\alpha}_{\ \mu\nu\sigma}u^{\sigma}(E^{u})^{\mu}(B^{u})^{\nu}\ , \tag{162}$$

or, in vector notation, $8\pi\rho_{EM}=\vec{E}(u)^2+\vec{B}(u)^2$; $\vec{p}_{EM}=\vec{E}(u)\times\vec{B}(u)/4\pi$. Now note that the significance of these expressions as energy and momentum densities can be traced back [76, 52], at the most fundamental level, to the Lorentz force (together with the Maxwell equations), the work done by it, and its spatial momentum transfer. It is thus clear that it cannot have a direct *physical* gravitational analogue, as there is no physical, covariant, gravitational counterpart to the

Lorentz force and the vector fields $\vec{E}$, $\vec{B}$. The inertial "force" (66), and its Lorentz-like form for stationary spacetimes (66) (both being but the geodesic equation in a different language, no *real* force being involved), as well as the fields $\vec{G}$ and $\vec{H}$, are mere artifacts of the reference frame, which can be made to vanish by switching to a locally inertial one, as explained in Sec. 3. It is the same for the "energy" and "momentum densities" arising from the Landau-Lifshitz pseudo-tensor $t_{\mu\nu}$, e.g. [18, 53]; the "momentum density " arising from $t_{\mu\nu}$, to post-Newtonian order, may actually be written as [62]: $\vec{p}_G \approx (-\vec{G} \times \vec{H} + 3\vec{G}\theta)/4\pi$.

6.1.1 Super-energy

On the other hand, there is a quantity built from tidal tensors (not vector fields), having thereby a local physical existence, that seems to fit well with the tidal tensor interpretation of gravitational radiation — the so-called Bel-Robinson super-energy tensor [32], which reads in vacuum:

$$T^{\alpha\beta\gamma\delta} = \frac{1}{2}\left[R^{\alpha\rho\gamma\sigma}R^{\beta}{}_{\rho}{}^{\mu}{}_{\sigma} + \star R^{\alpha\rho\gamma\sigma} \star R^{\beta}{}_{\rho}{}^{\mu}{}_{\sigma}\right] \tag{163}$$

(the more general expression in the presence of sources is given in e.g. [77]; a general superenergy tensor can also be defined, e.g. [35, 77], from the Weyl tensor; it is obtained by simply replacing $R_{\alpha\beta\gamma\delta} \to C_{\alpha\beta\gamma\delta}$ above). The *formal* analogy with the electromagnetic energy-momentum tensor, Eq. (160) above, is clear.

Given the formulation in Table 3 of gravitational radiation as a pair of propagating tidal tensors, just like the pair $\{\vec{E}, \vec{B}\}$ in the electromagnetic waves, it is natural to suppose that gravitational waves might carry some quantity formally analogous to the energy ρ_{EM} and momentum $p^{\langle\alpha\rangle}_{EM}$ densities, Eqs. (161)-(162), but with $\{\mathbb{E}_{\alpha\beta}, \mathbb{H}_{\alpha\beta}\}$ in the place of $\{E^\alpha, B^\alpha\}$. Such quantities turn out to be the well known "super-energy" W and "super-momentum" $\mathcal{P}^{\langle\alpha\rangle}$ densities, obtained from the Bel-Robinson tensor (163) in close analogy with the electromagnetic counterparts:

$$W \equiv T^{\alpha\beta\gamma\delta}u_\alpha u_\beta u_\gamma u_\delta = \frac{1}{2}\left[(\mathbb{E}^u)^{\alpha\beta}(\mathbb{E}^u)_{\alpha\beta} + (\mathbb{H}^u)^{\alpha\beta}(\mathbb{H}^u)_{\alpha\beta}\right] ; \tag{164}$$

$$\mathcal{P}^{\langle\alpha\rangle} \equiv -T^{\langle\alpha\rangle\beta\gamma\delta}u_\beta u_\gamma u_\delta = \epsilon^{\alpha}{}_{\mu\nu\sigma}u^\sigma(\mathbb{E}^u)^{\mu\lambda}(\mathbb{H}^u)^{\nu}{}_{\lambda} . \tag{165}$$

Note that expressions (164)-(165) are *exact*. W is positive definite, and is zero if and only if the curvature vanishes [32, 44]; which is in analogy with the energy scalar for the electromagnetic field. According to Bel [32, 44], gravitational radiation is characterized by a flux of super-energy $\mathcal{P}^\alpha$ (which, at least in the linear regime, is clear from the equations (3.1b)-(3.6b) above), parallel to the spatial direction of propagation of the wave; moreover, the non-vanishing of $\mathcal{P}^{\langle\alpha\rangle}$ at a given point, for *every* observer u^α (i.e., an *intrinsic* super-flux), is sufficient to ensure that gravitational radiation is present. Several criteria[23] have been proposed (see [105] for a review) to characterize radiative states, based on the electromagnetic analogy. It is indeed tempting to think of super-energy and super-momentum (or super-flux) as the form in which gravitational waves carry what then, *in the interaction with matter* — e.g., through the test body multipole moments, as described by Eq. (159) — manifests itself as ordinary energy and momentum. The fact that the gravitational super-energy tensor is not divergence-free in the presence of matter [78, 77] seems in line with this reasoning. There are however conceptual difficulties [77, 79, 104] in the physical interpretation of

[23] We thank A. García-Parrado for discussions on this issue.

super-energy (the very question of whether it has any physical reality is an open one), due to the "strange" dimensions of W, namely L^{-4}, which (taking $L = M$) can be interpreted as an energy density per unit area [42, 77, 78], or an energy density times a frequency squared [75], or an energy density squared [77]. A detailed discussion of these issues, and of the possible relation between super-energy and energy is beyond the scope of this work. Herein we would just like to point out that some interesting connections have been found in the literature[24]: in [75] it was argued that the energy available in a gravitational wave of the type $\mathbb{E}_{\alpha\beta}(0)e^{i\omega t}$ to interact with matter is $4W\omega^{-2}$ (therein it is also shown that, via W, gravitational radiation has an active attractive effect); and reinforcing the interpretation of super-energy as energy times squared frequency, it was found in [82] that W, for a massive scalar field in flat spacetime, takes a form similar to the corresponding Hamiltonian, but where the quanta are $\hbar\omega^3$. Another interesting connection, this one supporting an interpretation of super-energy as energy density per unit area, is as follows: consider a small 2-sphere (radius R), in vacuum; the quasi-local gravitational field energy of such sphere, as given by the several definitions in the literature[25] (see Refs. in [77]), is (taking into account terms up to r^2) proportional to $W|_{r=0}$ times R^5 (that is, super-energy times volume times area). In the spirit of the arguments given above — that the GEM fields are frame artifacts, the non-existence of local quantities *physically* analogous to the electromagnetic vector fields (consequence of the equivalence principle), and that gravitational waves are tidal tensors propagating by mutually inducing each other (without associated vector fields), which do not interact with the multipole structure of test bodies *in the same way* electromagnetic waves do (in particular, do not couple to monopole particles) — it seems plausible that if some fundamental quantity exists which is carried by gravitational waves, its dimensions are not the same of their electromagnetic counterparts, hence not the ones of an energy. This is consistent with the point of view in [42, 104].

6.2 The relationship with the other GEM analogies

The analogy drawn in this section is between the electromagnetic fields and the electric and magnetic parts of the Weyl tensor: $\{E^\alpha, B^\alpha\} \leftrightarrow \{\mathcal{E}_{\mu\nu}, \mathcal{H}_{\mu\nu}\}$. It is clear, from the discussion of the physical meaning of $\{\mathbb{E}_{\mu\nu}, \mathbb{H}_{\mu\nu}\}$ in Sec. 2, and from the discussion in Sec. 3 of the dynamical gravitational counterparts of $\{E^\alpha, B^\alpha\}$, that this analogy is a purely formal one. It draws a parallelism between electromagnetic fields (whose dynamical gravitational analogues are the GEM inertial fields $\{\vec{G}, \vec{H}\}$ of Sec. 3), with gravitational *tidal* fields, which, as shown in Sec. 2, are the physical analogues *not* of $\{E^\alpha, B^\alpha\}$, but instead of the electromagnetic *tidal* tensors $\{E_{\mu\nu}, B_{\mu\nu}\}$ (these, in an inertial frame, are *derivatives* of the E^α and B^α, cf. Eqs. (105)-(108)). This sheds light on some conceptual difficulties in the literature regarding the physical content of the analogy and in particular the physical interpretation of the tensor $\mathcal{H}_{\mu\nu}$, see [2] for details. It is also of crucial importance for the correct understanding of physical meaning of the curvature invariants, and their implications on the motion of test particles, which will be subject of detailed study elsewhere [29].

[24] We thank J. M. M. Senovilla for discussions on this issue.

[25] One possible definition is obtained from the Landau-Lifshitz pseudo-tensor $t_{\alpha\beta}$ as follows [80]: one sets up a system of normal coordinates at the center (where $t_{\alpha\beta} = 0$); then expands $t_{\alpha\beta}$ about it and integrates in an small 2-sphere. See also [81].

7 When can gravity be similar to electromagnetism?

A crucial point to realize is that the two exact physical gravito-electromagnetic analogies — the tidal tensor analogy of Sec. 2, and the inertial GEM fields analogy of Sec. 3 — do not rely on a close physical similarity between the interactions; the gravitational objects $\{\vec{G}, \vec{H}, \mathbb{E}_{\alpha\beta}, \mathbb{H}_{\alpha\beta}\}$, despite playing *analogous dynamical roles* to the ones played by the objects $\{\vec{E}, \vec{B}, E_{\alpha\beta}, B_{\alpha\beta}\}$ in electromagnetism, are themselves in general very different from the latter, even for seemingly analogous setups (e.g. the EM field of spinning charge, and the gravitational field of a spinning mass). In this sense, these analogies have a different status compared to the popular GEM analogy based on linearized theory, which, *in order to hold*, indeed requires a close similarity between the two interactions, to which the former two are not bound.

What the tidal tensor formalism of Sec. 2, together with the inertial fields formalism of Sec. 3, provide, is a "set of tools" to determine under which precise conditions a similarity between the gravitational and electromagnetic interactions may be expected.

The key differences between electromagnetic and gravitational tidal tensors are: a) they do not exhibit, generically, the same symmetries; b) gravitational tidal tensors are spatial whereas the electromagnetic ones are not; c) electromagnetic tidal tensors are linear, whereas the gravitational ones are not.

The electromagnetic tidal tensors, for a given observer, only have the same symmetries and time-projections as the gravitational ones when the Maxwell tensor is covariantly constant along the observer's worldline; that is implied by Eqs. (1.8) and (1.5) of Table 1. This restricts the eligible setups to intrinsically stationary fields (i.e., whose time-dependence, if it exists, can be gauged away by a change of frame), and to a special class of observers therein; for electromagnetic fields in flat spacetime, those observers must be *static* in the inertial frame where the fields are explicitly time-independent. This is an important point that is worth discussing in some detail. Consider the two basic analogous fields, the Coulomb field of a point charge, and the Schwarzschild gravitational field. Consider also in the latter observers $\mathcal{O}$ in circular motion: 4-velocity $U^\alpha = (U^0, 0, 0, U^\phi)$, angular velocity $\Omega = U^\phi/U^0$. The worldlines of these observers are tangent to Killing vector fields: $U^\alpha \parallel \xi^\alpha$; $\mathcal{L}_\xi g_{\alpha\beta} = 0$. One can say (e.g. [53, 68]) that they see a constant spacetime geometry; for this reason they are called "stationary observers". Now consider observers in circular motion around a Coulomb charge. Despite moving along worldlines tangent to vector fields which are symmetries of the electromagnetic field: $U^\alpha \parallel \xi^\alpha$; $\mathcal{L}_\xi F_{\alpha\beta} = 0$, the observers U^α do *not* see a covariantly constant field: $F_{\alpha\beta;\gamma}U^\gamma \neq 0$, which by virtue of Eqs. (1.5a), (1.8a), implies that the electromagnetic tidal tensors have an antisymmetric part (in particular the spatial part $B_{[ij]} \neq 0$), and thus means that they cannot be similar to their gravitational counterparts. This is a natural consequence of Maxwell's equations, and can be easily understood as follows. The magnetic tidal tensor measured by $\mathcal{O}$ is a covariant derivative of the magnetic field as measured in the inertial frame momentarily comoving with it: $B_{\alpha\beta} \equiv \star F_{\alpha\gamma;\beta}U^\gamma = B_{\alpha;\beta}|_{U=const} = (B_{MCRF})_{\alpha;\beta}$. Now, $B_{[ij]} \neq 0$ means that $\vec{B}_{MCRF}$ has a curl; which is to be expected, since in the MCRF the electric field is time-dependent (constant in magnitude but *varying* in direction), which, by virtue of Maxwell's equation $\nabla \times \vec{B} = \partial\vec{E}/\partial t = \gamma\vec{E} \times \vec{\Omega}$ (holding in the MCRF, and for which (1.5a) is a covariant form) induces a curl in $\vec{B}$.

Even if one considers static observers in stationary fields, so that the gravitational and electromagnetic tidal tensors have the same symmetries, still one may not see a close similarity between the interactions. The electromagnetic tidal tensors are linear in the electromagnetic fields, and the latter themselves *linear* in the electromagnetic 4-potential $A^{\alpha} = (\phi, \vec{A})$, whereas the gravitational tidal tensors are *non-linear* in the GEM fields, as shown by Eqs. (113)-(114), the gravitomagnetic field $\vec{H}$ being itself *non-linear* in the metric potentials Φ, $\vec{\mathcal{A}}$. This means that one can expect a similarity between tidal tensors in two limiting cases — linearized theory, and the ultra-stationary spacetimes considered in Sec. 4, where $\Phi = \vec{G} = 0$, and, therefore, cf. Eqs. (114) and (68), the exact gravitomagnetic tidal tensor is linear (both in the metric and in the GEM fields): $\mathbb{H}_{\hat{i}\hat{j}} = -\tilde{\nabla}_{\hat{j}} H_{\hat{i}}/2 = -\tilde{\nabla}_{\hat{j}}(\tilde{\nabla} \times \vec{\mathcal{A}})_{\hat{i}}/2$. We have seen in Sec. 4 that there is indeed an exact mapping (via the Klein-Gordon equation) between the dynamics in these spacetimes and an electromagnetic setup.

In what concerns concrete effects, the precise conditions (namely regarding the time dependence of the fields) for occurrence of a gravito-electromagnetic similarity are specific to the type of effect. For the covariant effects (implying physical gravitational forces, i.e., tidal forces) such as the force on a spinning particle or the worldline deviation of two neighboring particles, it is the tidal tensors *as measured by the test particles* (4-velocity U^{α}) that determine the effects, cf. Eqs. (1.1)-(1.2); which means that *it is along the particle's worldline that the constancy of the fields is required.* This basically implies that the similarity only occurs at the instant when the particles are at rest in stationary fields, so it does not hold in a truly dynamical situation. In the case of the correspondence between the Lorentz force, Eq. (54), and the geodesic equation formulated as an inertial force (which is a reference frame effect), we see from Eq. (52) that the requirement is that the frame is rigid, i.e. $\sigma_{\alpha\beta} = \theta = 0$; as explained in Sec. 3.2, this amounts to saying that the spatial part of the metric (in the coordinates associated to such frame) must be *time-independent.* This can also be stated in the following manner, generalizing to the exact case the conclusion obtained in [51] in the context of the post-Newtonian approximation: in the case of the GEM analogy for the geodesic equation, the *stationarity of the fields is required in the observer's frame* (not in test particle's frame! The test particles can move along arbitrary worldlines). As for the gyroscope "precession" (70) and the correspondence with the precession of a magnetic dipole (71), there is no restriction on the time dependence of the fields.

8 Conclusion

In this work we collected and further developed different gravito-electromagnetic analogies existing in the literature, and clarified the connection between them. We completed the approach based on tidal tensors expressing the full gravitational field equations (cast herein as the Einstein field equations, plus the algebraic Bianchi identities) in tidal tensor formalism, which is achieved through suitable projector techniques. Also, we added to the list of analogies manifest in this formalism the one concerning "differential" precession of spinning particles. As for the approach based on inertial forces — the so called gravitoelectromagnetic (GEM) fields — we derived an equation (65) for the inertial force acting on a test particle that applies to any frame, which was achieved by defining a suitable connection ($\tilde{\nabla}$) on the bundle of vectors orthogonal to an arbitrary congruence of time-like curves. It manifests that the gravitomagnetic field, in its most general formulation, consists of a combination of two distinct effects: the the vorticity of the observer congruence, plus the angular velocity of rotation along the congruence (relative to Fermi-Walker transport) of the triad of spatial axes that each observer "carries". Such formulation encompasses the different gravitomagnetic fields given in the literature, in particular the concurrent gravitomagnetic field measured by the Killing (or "static") observers in a stationary spacetime, studied in e.g. [20, 23, 24], and the gravitomagnetic field arising in the frames associated to the zero angular momentum observers in e.g. [25, 26]. We wrote the full gravitational and electromagnetic field equations in this formalism, obtaining a close analogy in the so-called "quasi-Maxwell" regime (Table 2). We also added to the list of exact analogies the one between the electromagnetic force on a magnetic dipole and the gravitational force on a gyroscope (Sec. 3.6). An important result in this paper — the relationship between the two approaches (i.e., between the two types of objects: tidal tensors and EM/GEM vector fields) — was established in Sec. 3.5; the weak field limit, and the relationship with the better known linearized theory approaches, was discussed in Sec. 5. For completeness, the formal analogies between the electric and magnetic parts of the Weyl tensor and the electromagnetic fields were briefly reviewed, including the issue of gravitational radiation and super-energy, and the physical interpretation of gravitational radiation was discussed based on what was learned from the approaches herein.

A more detailed detailed summary of the material in this paper is given in the introduction; herein we conclude by briefly summarizing the main outcome of each approach, and their applicability. The analogies split in two classes: physical and purely formal. In the second category falls the analogy between the electric and magnetic parts of the Weyl and Maxwell tensors, discussed in Sec 6. The physical analogies divide in two classes: exact analogies, and the best known post-Newtonian and linearized theory approaches. Exact physical analogies are the analogy between the electromagnetic fields and the inertial fields of Sec. 3, and the tidal tensor analogy of Sec. 2.

These analogies are useful from a practical point of view, as they provide a familiar formalism and insight from electromagnetic phenomena to describe otherwise more complicated gravitational problems. Indeed, there is a number of fundamental equations, summarized in Table 4, which can be obtained from the electromagnetic counterparts by simple application of the analogy. But the existence of these analogies, especially the exact, physical ones, is also interesting from the theoretical point of view, unveiling intriguing similarities — both in the tidal tensor, and in the inertial field formalism, manifest in Tables 1 and 2, respectively — and enlightening differences (namely the ones manifest in the symmetries of the tidal tensors). The deep connection between gravitation and electrodynamics is still very much an open question; the similarities and fundamental differences unveiled in the analogies and the formalism herein may be a small step in the direction of that understanding.

The tidal tensor formalism is primarily suited for a transparent comparison between the two interactions, since it is based on mathematical objects describing covariant *physical* forces common to both theories. The most natural application of the analogy is the dynamics of spinning multipole test particles, which is studied in the companion paper [6] to quadrupole order. As a formalism, consisting of mathematical objects with direct physical interpretation, encoding the gravitational *physical*, covariant forces, it can be useful in many applications, namely gravitational radiation (as discussed in Sec. 6.1), and whenever one wishes to study the physical aspects of spacetime curvature.

The analogy based on inertial GEM fields from the 1+3 formalism, Sec. 3, is a very powerful formalism, with vast applications; especially in the case of stationary spacetimes, where for arbitrarily strong fields the equation for geodesics is cast in a form similar to Lorentz force; many other effects related to frame dragging can be treated exactly with the GEM fields: gyroscope "precession" [6, 27, 28, 58, 20], the Sagnac effect [107], the Faraday rotation [24], the force on a gyroscope (Sec. 3.6 and [20]; note however that it is not as general as the the tidal tensor formulation of the same force); and other applications, such as the matching of stationary solutions [22], or describing the hidden momentum of spinning particles [6]. The general formulation of GEM fields in Sec. 3, applying to arbitrary fields and frames, extends the realm of applicability of this formalism.

Table 4: What can be computed by direct application of the GEM analogies

Result	**Approach**
• Geodesic deviation equation (1.1b) of Table 1: -Replacing $\{q, E_{\alpha\beta}\} \to \{m, -\mathbb{E}_{\alpha\beta}\}$ in (1.1a). • Force on a gyroscope (1.1b): -Replacing $\{\mu^\alpha, B_{\alpha\beta}\} \to \{S^\alpha, -\mathbb{H}_{\alpha\beta}\}$ in (1.1a). • Differential precession of gyroscopes (1.3b): -Replacing $\{\sigma, B_{\alpha\beta}\} \to \{1, -\mathbb{H}_{\alpha\beta}\}$ in (1.3a). • Gravitational field equations (1.4b)-(1.5b), (1.7b)-(1.8b): -Replacing $\{E_{\alpha\beta}, B_{\alpha\beta}\} \to \{\mathbb{E}_{\alpha\beta}, \mathbb{H}_{\alpha\beta}\}$ in Eqs. (9)-(12), *and* $\rho_c \to 2\rho + T^{\alpha}_{\ \alpha}$ in (9), $j^\alpha \to 2J^\alpha$ in (12).	**Tidal tensor analogy** (Exact, general results)
• Geodesic Equation (66) (*stationary* fields) -Replacing $\{\vec{E}, \vec{B}\} \to \{\vec{G}, \vec{H}\}$ in (54), multiplying by γ. • Gyroscope "precession" Eq. (70) (*arbitrary* fields): -Replacing $\{\vec{\mu}, \vec{B}\} \to \{\vec{S}, \vec{H}/2\}$ in (71). • Force on gyroscope Eq. (124) (*stationary* fields, particle's worldline tangent to time-like Killing vector): -Replacing $\{\vec{\mu}, \vec{E}, \vec{B}\} \to \{\vec{S}, \vec{G}, \vec{H}/2\}$ in (122), factor of 2 in the last term.	**Inertial "GEM fields" analogy** (Exact results, require special frames)
• Higher order field equations (3.1b)-(3.4b): -Replacing $\{\vec{E}, \vec{B}\} \to \{\mathbb{E}_{ij}, \mathbb{H}_{ij}\}$ in Eqs. (3.1a)-(3.4a). • Equations of gravitational waves (3.5b)-(3.6b): -Replacing $\{\vec{E}, \vec{B}\} \to \{\mathbb{E}_{ij}, \mathbb{H}_{ij}\}$ in Eqs. (3.5a)-(3.6a).	**Weyl-Maxwell tensors analogy** (Results for linearized theory)

The well known analogies between electromagnetism and post-Newtonian and linearized gravity, follow as a limiting case of the exact approach in Sec. 3. In the case of the tidal effects, they can be seen also as limiting case of the tidal tensor analogy of Sec. 2 (in the sense that for weak, *time-independent fields,* the tidal tensors are derivatives of the GEM fields). Acknowledging this fact, and understanding the conditions under which linear GEM is obtained from the rigorous, exact approaches, is important for a correct interpretation of the physical meaning of the quantities involved, which is not clear in the usual derivations in the literature (this is especially the case for many works on linear GEM), and thus prone to misconceptions (see in this respect [2, 1]. On the other hand, linear GEM is the most important in the context of experimental physics, as it pertains all gravitomagnetic effects detected to date [59, 103, 60, 111, 110, 109], and the ones we hope to detect in the near future [61].

As for the analogy between the electric and magnetic parts of the Weyl and Maxwell tensors, its most important application is gravitational radiation, where it provides equations for the propagation of *tensors of physical forces* (not components of the metric tensor, as in the more usual approaches, which are pure gauge fields), with direct translation in physical effects (via the tidal tensor formalism of Sec. 2). This analogy has also been used to address the fundamental questions of the content of gravitational waves, and the "energy" of the gravitational field. Namely, to define *covariant*, *local* quantities alternative to the gravitational energy and momentum given by the Landau-Lifshitz pseudo-tensor (which can only have a meaning in a global sense, and in asymptotically flat spacetimes): the super-energy and super-momentum encoded in the Bel tensor. The motivation for the definition of this tensor is the analogy with electromagnetism; and the existing criteria for radiative states [105], states of intrinsic radiation [32, 104] or pure radiation ([108], see also [44] p. 53), are also solely driven by it. The analogy is also useful for the understanding of the quadratic invariants of the curvature tensor; indeed, it will be shown elsewhere [29] that using the two approaches together — the formal analogy between the Weyl and Maxwell tensors to gain insight into the invariant structure, and the tidal tensor analogy as a physical guiding principle — one can explain, in the astrophysical applications of current experimental interest (as mentioned above), the significance of the curvature invariants and the implications on the motion of test particles.

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A Inertial Forces — simple examples in flat spacetime

In Sec. 3.2 we have seen that the inertial forces felt in a given frame arise from two *independent* contributions of different origin: the kinematics of the observer congruence (that is, from the derivatives of the temporal basis vector of the frame, $\mathbf{e}_{\hat{0}} = \mathbf{u}$, where $\mathbf{u}$ is the observers' 4-velocity), and the transport law for the spatial triads $\mathbf{e}_{\hat{i}}$ along the congruence. In order to illustrate these concepts with simple examples, we shall consider, in flat spacetime, the straightline geodesic motion of a free test particle (4-velocity $\mathbf{U}$), from the point of view of three distinct frames: a) a frame whose time axis is the 4-velocity of a congruence of observers at rest, but whose spatial triads rotate uniformly with angular velocity $\vec{\Omega}$; b) a frame composed of a congruence of rigidly rotating observers (vorticity $\vec{\omega}$), but carrying Fermi-Walker transported spatial triads ($\vec{\Omega} = 0$); c) a rigidly rotating frame, that is, a frame composed of a congruence of rigidly rotating observers, carrying spatial triads co-rotating with the congruence $\vec{\Omega} = \vec{\omega}$ (i.e., "adapted" to the congruence, see Sec. 3.1). This

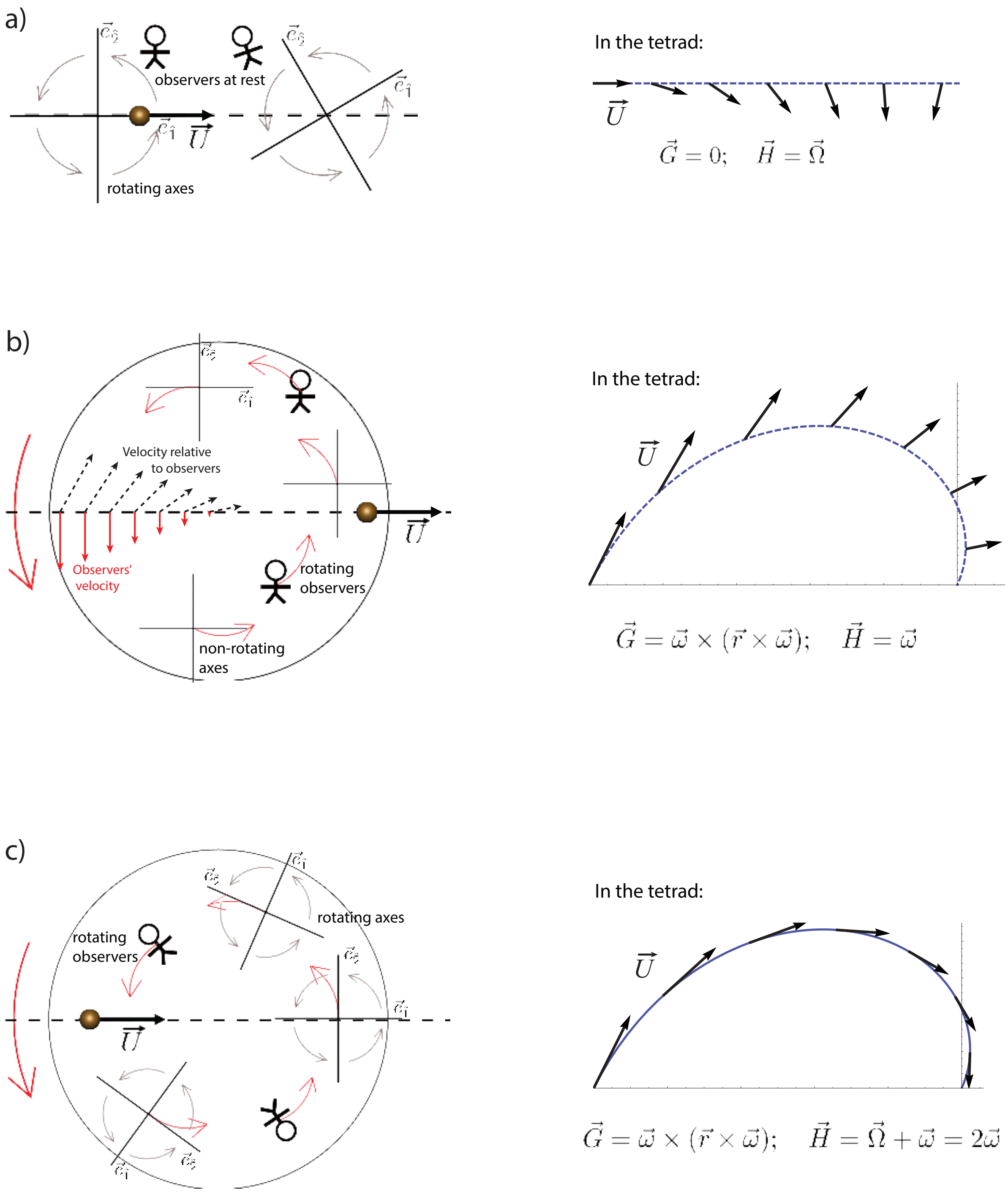

Figure 2: A test particle in uniform motion in flat spacetime from the point of view of three different frames: a) a frame composed of observers at rest, but carrying spatial triads that rotate with uniform angular velocity $\vec{\Omega}$; b) a frame consisting of a congruence of rigidly rotating observers (vorticity $\vec{\omega}$), but each of them carrying a non-rotating spatial triad (i.e., that undergoes Fermi-Walker transport); c) a rigidly rotating frame (a frame adapted to a congruence of rigidly rotating observers); the spatial triads co-rotate with the congruence, $\vec{\Omega} = \vec{\omega}$. *Note:* by observer's rotation we mean their circular motion around the center; and by axes rotation we mean their rotation (relative to FW transport) about the local tetrad's origin.

is depicted in Fig. 2.

In the first case there we have a vanishing gravitoelectric field $\vec{G} = 0$, and a gravitomagnetic field $\vec{H} = \vec{\Omega}$ arising solely from the rotation (with respect to Fermi-Walker transport) of the spatial triads; thus the only inertial force present is the gravitomagnetic force $\vec{F}_{\rm GEM} = \gamma\vec{U} \times \vec{\Omega}$, cf. Eq. (52), with $\gamma \equiv -U^{\alpha}u_{\alpha}$. In the frame b), there is a gravitoelectric field $\vec{G} = \vec{\omega} \times (\vec{r} \times \vec{\omega})$ due the observers acceleration, and also a gravitomagnetic field $\vec{H} = \vec{\omega}$, which originates solely from the vorticity of the observer congruence. That is, there is a gravitomagnetic force $\gamma\vec{U} \times \vec{\omega}$ which reflects the fact that the relative velocity $v^{\alpha} = U^{\alpha}/\gamma - u^{\alpha}$ (or $\vec{v} = \vec{U}/\gamma$, in the observer frame $\vec{u} = 0$) between the test particle and the observer it is passing by changes in time. The total inertial forces are in this frame

$$\vec{F}_{\rm GEM} = \gamma\left[\gamma\vec{\omega} \times (\vec{r} \times \vec{\omega}) + \vec{U} \times \vec{\omega}\right].$$

In the frame c), which is the relativistic version of the classical *rigid* rotating frame, one has the effects of a) and b) combined: a gravitoelectric field $\vec{G} = \vec{\omega} \times (\vec{r} \times \vec{\omega})$, plus a gravitomagnetic field $\vec{H} = \vec{\omega} + \vec{\Omega} = 2\vec{\omega}$, the latter leading to the gravitomagnetic force $2\gamma\vec{U} \times \vec{\omega}$, which is the relativistic version of the well known Coriolis acceleration, e.g. [128]. The total inertial force is in this frame

$$\vec{F}_{\rm GEM} = \gamma\left[\gamma\vec{\omega} \times (\vec{r} \times \vec{\omega}) + 2\vec{U} \times \vec{\omega}\right]$$

which is the relativistic generalization of the inertial force in e.g. Eq. (4-107) of [128]. Moreover, in this case, as discussed in Secs. 3.2.1 and 3.2.2, the $\Gamma^{\hat{i}}_{\hat{j}\hat{k}}$ in Eq. (53) are the connection coefficients of the (Levi-Civita) 3-D covariant derivative with respect to the metric γ_{ij} (defined by Eq. (67)) defined on the space manifold associated to the quotient of the spacetime by the congruence; $\vec{U}$ is the vector *tangent to the 3-D curve* (see Fig. 2c) obtained by projecting the particle's worldline on the space manifold, and $\vec{F}_{\rm GEM} = \tilde{D}\vec{U}/d\tau$ is simply the covariant 3-D acceleration of that curve.

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