Open access • Journal Article • DOI:10.1063/1.529381

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Published on: 01 Apr 1991 - Journal of Mathematical Physics (American Institute of Physics)

Topics: De Sitter space, De Sitter universe, Graviton, Propagator and Scalar (mathematics)

Archives

CERN-TH.4605
December 1986

GRAVITON FLUCTUATIONS IN DE SITTER SPACE+)

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A B S T R A C T

We calculate the two-point correlation function of metric fluctuations in de Sitter space. The results are expressed in a gauge-invariant and O(4,1)-invariant form in terms of elementary functions of z(x,x') (a biscalar variable simply related to the invariant distance between x and x'). The Feynman functions for the transverse, trace-free and scalar metric fluctuations grow without bounds, like ln z and z ln z respectively, for large z. We interpret these results as an evidence for the quantum instability of de Sitter space.

+) Work supported in part by the US Department of Energy, Contract DE-AC03-76SF00515 and by the National Science Foundation under Grant No. PHY82-17853 supplemented by funds from the National Aeronautics and Space Administration.

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1. INTRODUCTION

de Sitter space-time is the maximally symmetric solution of Einstein's equations with a positive cosmological constant. It is simply characterized geometrically as the four-dimensional manifold with constant

$$-(X^0)^2 + (X^1)^2 + (X^2)^2 + (X^3)^2 + (X^4)^2 \equiv \frac{1}{H^2}$$

embedded in five-dimensional Minkowski space-time with metric, diag (-++++). In the last few years there has been considerable interest in understanding quantum field theory in this background space for two principal reasons:

i) The problem of vacuum energy has emerged as a fundamental issue in attempts to unify realistic quantum field theories of elementary particles with gravitation.

ii) Inflationary models of the very early Universe suggest that the Universe actually passed through a de Sitter-like phase in its early evolution [1].

In this context an instability of de Sitter space is directly relevant to the early history of the Universe and has consequences for present-day observations, as well as possibly pointing a way out of the vacuum-energy ('cosmological constant') problem. This line of thought has been pursued in a series of recent papers [2-5].

In this paper we present the strongest evidence to date that de Sitter space is indeed unstable, by considering the gravitational two-point functions in this background.

Because of the maximal O(4,1) symmetry of de Sitter space we are able to obtain exact results for the two-point functions. The essential technical tool is the decomposition of the graviton propagator for the metric fluctuations into maximally symmetric bitensors multiplied by scalar functions of the invariant distance between x and x'. We present the necessary preliminaries to this tensor decomposition in Section 2. Possessed with this information, the problem of determining all the graviton two-point functions is reduced to a (nearly) straightforward problem of solving linear differential equations for the various scalar amplitudes. These equations have solutions in terms of elementary functions of z(x,x') -- a quantity whose precise definition is given by Eq. (19) of the next section. The full solution for the graviton propagator which emerges is presented then in Sections 3 and 4.

In Section 5 we discuss the divergent behaviour of this quantity as x and x' are separated by a large time-like (or space-like) interval. The growth of the propagator for large separation depends critically on the Feynman boundary conditions ($i\varepsilon$ prescription) imposed. If instead, we take the retarded (or advanced) propagator corresponding to classical time evolution forward (or backward) in time we find *no* unbounded growth in physical gauge-invariant quantities. Thus, our results are fully consistent with previous work on the classical stability of de Sitter space (at the linearized level) [5].

Since the Feynman propagator does grow without bounds, certain gauge-invariant quantities can be constructed from the metric fluctuations which diverge. This can be understood as the instability of the de Sitter invariant Bunch-Davies state to small external perturbations of energy-momentum. Since metric fluctuations themselves can be the source of perturbations in the full non-linear theory we conclude that de Sitter space-time is unstable at the quantum level, and that the maximal O(4,1) symmetry must be broken. This conclusion and the direction of further progress on the important problem of vacuum energy are discussed in the final section as well.

2. TENSOR DECOMPOSITION IN MAXIMALLY SYMMETRIC SPACES

We regard the full metric tensor $\hat{g}_{ab}$ to be the sum of g_{ab}, the metric of de Sitter space in some coordinates (x^a) and h_{ab}, a small perturbation from the de Sitter background:

$$\hat{g}_{ab}(x) = g_{ab}(x) + h_{ab}(x) \tag{1}$$

The general linear fluctuation h_{ab} can be decomposed in the following way:

$$h_{ab} = h^{\perp}_{ab} + D_a A^{\perp}_b + D_b A^{\perp}_a + \left(D_a D_b - \tfrac{1}{4} g_{ab} \Box\right) B + \tfrac{1}{4} g_{ab} h . \tag{2}$$

Here D_a is the covariant derivative with respect to the classical background de Sitter metric g_{ab} and $\Box \equiv g^{ab} D_a D_b$. The vector field $A^{\perp}_a$ is transverse in the sense that

$$D^a A^{\perp}_a = 0 \tag{3}$$

and the tensor field $h^{\perp}_{ab}$ is both transverse and traceless,

$$D^a h^{\perp}_{ab} = 0 \tag{4a}$$

$$g^{ab} h^{\perp}_{ab} \equiv h^{\perp}{}_a{}^a = 0 \tag{4b}$$

Thus

$$h = h_a{}^a \tag{5}$$

is the trace of the metric perturbation h_{ab}.

A coordinate transformation of the full metric $\hat{g}_{ab}$ is indistinguishable (to linear order) from a gauge transformation on h_{ab}:

$$h_{ab} \longrightarrow h_{ab} + D_a \xi_b + D_b \xi_a \quad , \tag{6}$$

where $\xi_a(x)$ is an arbitrary vector function of the coordinates. If we also decompose ξ_a into transverse and longitudinal parts,

$$\xi_a = \xi^{\perp}_a + D_a \xi \quad , \quad D^a \xi^{\perp}_a = 0 \quad , \quad D^a \xi_a = \Box \xi \tag{7}$$

Then, expressions (2) and (6) imply the linearized transformation properties of the various tensors:

$$h^{\perp}_{ab} \longrightarrow h^{\perp}_{ab} \tag{8a}$$

$$A_a^\perp \rightarrow A_a^\perp + \xi_a^\perp \tag{8b}$$

$$B \rightarrow B + 2\xi \tag{8c}$$

$$h \rightarrow h + 2\Box\xi \tag{8d}$$

and

$$h - \Box B \rightarrow h - \Box B \quad . \tag{8e}$$

It is obvious that $h_{ab}^\perp$ and $h - \Box B$ are *independent* of the gauge function ξ_a.*) They contain the full coordinate invariant information about the linearized perturbations. On the other hand, $A_a^\perp$ and any other linear combination of h and B are gauge dependent and must drop out of all physical quantities. Therefore, it is natural to consider the graviton propagator only over the transverse, traceless subspace spanned by $h_{ab}^\perp$, obeying eqs. (4), and the scalar subspace spanned by $h - \Box B$.

The variation of any coordinate invariant action such as the Einstein-Hilbert action $(1/16\pi G) \int d^4x \sqrt{-g}\,(R-2\Lambda)$ gives, to linear order in h_{ab}, differential operators which depend only on $h_{ab}^\perp$ and $h - \Box B$:

$$-16\pi G \int \frac{\delta^2 S_E}{\delta g_{c'd'}(x')\,\delta g^{ab}(x)}\, h_{c'd'}(x')\, d^4x' = \delta\left(R_{ab} - \frac{R}{2} g_{ab} + \Lambda g_{ab}\right)$$

$$= -\Box h_{ab}^\perp + 2 R^c{}_{ab}{}^d\, h_{cd}^\perp + \frac{1}{2}\left[g_{ab}\Box - D_a D_b + g_{ab}\Lambda\right](h - \Box B) \tag{9}$$

In deriving Eq. (9) one has only to assume that the background metric g_{ab} is a solution of the vacuum Einstein equations with cosmological constant:

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*) Equations (8) are valid provided the identification of the various components in Eq. (2) is unique. If this is not the case special care is required (cf. Section 4).

$$R_{ab} - \frac{R}{2} g_{ab} + g_{ab} \Lambda = 0 \tag{10}$$

No assumption of de Sitter (or any other) symmetry of the background has been made, as yet. However, when the background is maximally symmetric the general decomposition (2) becomes most useful. In that case the quadratic part of the Einstein action corresponding to Eq. (9) is simply

$$\int d^4x \sqrt{-g} \left\{ \frac{1}{2} h^{ab\perp} \left(-\Box + \frac{R}{6}\right) h^{\perp}_{ab} + \frac{3}{16} (h - \Box B)\left(\Box + \frac{R}{3}\right)(h - \Box B) \right\} \tag{11}$$

Define the spin-2 part of the metric two-point function (graviton propagator) to be the inverse of $(-\Box + R/6)$ over transverse, trace-free tensors;

$$\left(-\Box + \frac{R}{6}\right) G^{(2)}_{abc'd'}(x,x') = P^{(2)}_{abc'd'}(x,x') \tag{12}$$

$$D^a G^{(2)}_{abc'd'} = D^{c'} G^{(2)}_{abc'd'} = 0 \tag{13}$$

$$G^{a\ (2)}_{\ a c'd'} = G^{(2)\ \ c'}_{ab\ \ c'} = 0 \tag{14}$$

with $P^{(2)}_{abc'd'}$ being the projector onto such tensors:

$$h^{\perp}_{ab}(x) = \int d^4x' \sqrt{-g'}\ P^{(2)}_{abc'd'}(x,x')\, h^{c'd'} \ . \tag{15}$$

In a similar manner we would like to define the spin-0 part of the metric two-point function to be the inverse of $(\Box + R/3)$ over the scalar functions, $h - \Box B$. However, in Section 4 we shall see that this operator is singular, i.e. it has scalar zero modes in de Sitter space. So the inverse can be defined only in the subspace of scalar functions with the zero modes excluded. This is discussed in full in Appendix A.

To continue with the tensor decomposition of $G^{(2)}_{abc'd'}$ we now make use of the properties of the maximally symmetric de Sitter space in an explicit way, by following the formalism of Allen and Jacobson [6]. They observe that in the maximally symmetric case all tensor functions of two arguments x and x' can be expressed in terms of a few basic bitensors whose properties can be catalogued once and for all. Let $\mu(x,x')$ denote the geodesic distance between x and x'. Then these basic tensors are the unit tangents $n_a(x,x')$ and $n_{a'}(x,x')$ to the geodesic at x and x':

$$n_a(x,x') = D_a \mu(x,x') \quad , \quad n_{a'}(x,x') = D_{a'} \mu(x,x') \tag{16}$$

together with $g^a_{b'}(x,x')$ the parallel propagator between x and x' *) and the metric tensor itself at x and at x'.

We follow the usual convention of indicating tensor indices at x by unprimed sub(super)scripts and tensor indices at x' by primed sub(super)scripts. Indices at x are raised (lowered) by $g^{ab}(g_{ab})$, while those at x' are raised (lowered) by $g^{a'b'}(g_{a'b'})$. The definitions (16) imply that $g^{b'}_a n_{b'} = -n_a$, $g_{ab} = g^{c'}_a g_{c'b}$ and $g_{a'b'} = g^c_{a'} g_{cb'}$ (arguments understood). Allen and Jacobson also show that

$$D_a n_b = A\,(g_{ab} - n_a n_b) \tag{17a}$$

$$D_a n_{b'} = C\,(g_{ab'} + n_a n_{b'}) \tag{17b}$$

$$D_a g_{bc'} = -(A+C)\,(g_{ab} n_{c'} + g_{ac'} n_b) \tag{17c}$$

with

$$A = H \cot(H\mu) \tag{18a}$$

$$C = -H \csc(H\mu) \tag{18b}$$

in de Sitter space. Other useful formulae for our subsequent work can be found in their Appendix C or derived directly from Eqs. (17) and (18).

Once it is recognized that the propagator $G^{(2)}{}_{abc'd'}$ in a de Sitter invariant state is itself a maximally symmetric bitensor, it follows that it can be decomposed into combinations of the basic tensors g_{ab}, $g^a_{b'}$, $g_{a'b'}$, n_a, and $n_{a'}$, the conditions

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*) $v^a = g^a_{b'} v^{b'}$ is the vector $v^{b'}$ at x', parallely transported to x.

(12) to (14) being imposed, and the problem of determining the spin-2 propagator being reduced to solving a set of purely scalar differential equations in the single variable μ. For our subsequent work it will prove useful to introduce the change of variable

$$z(x,x') \equiv \cos^2\left(\frac{\mu H}{2}\right) = \frac{1+\cos(\mu H)}{2} \tag{19}$$

in place of the invariant distance $\mu(x,x')$. Notice that the light cone of $x(\mu = 0)$ is at $z = 1$. Euclidean space-like x and x' are described by real μ or $0 < z < 1$, while time-like x and x' are characterized by continuation to imaginary μ or $z > 1$. Non-Euclidean space-like x and x' are described by complex μ such that $z < 0$.

3. THE SPIN-2 PROPAGATOR FUNCTION

Cataloguing all of the possible bitensors with two indices a, b at x and c', d' at x', which are symmetric under $a \rightleftarrows b$, $c' \rightleftarrows d'$ and $(a,b) \rightleftarrows (c',d')$, shows that there are exactly five such quantities which can be constructed from the basic tensors n_a, $n_{a'}$, $g_a^{\ c'}$, g_{ab}, $g^{c'd'}$. They are:

$$\tau^{(1)}{}_{ab}{}^{c'd'} = g_{ab}\, g^{c'd'} \tag{20a}$$

$$\tau^{(2)}{}_{ab}{}^{c'd'} = g_{ab}\, n^{c'} n^{d'} + n_a n_b\, g^{c'd'} \tag{20b}$$

$$\tau^{(3)}{}_{ab}{}^{c'd'} = g_a^{\ c'} g_b^{\ d'} + g_a^{\ d'} g_b^{\ c'} \tag{20c}$$

$$\tau^{(4)}{}_{ab}{}^{c'd'} = g_a^{\ c'} n_b n^{d'} + g_a^{\ d'} n_b n^{c'} + g_b^{\ c'} n_a n^{d'} + g_b^{\ d'} n_a n^{c'} \equiv 4\, g_{(a}^{\ \ (c'} n_{b)} n^{d')} \tag{20d}$$

$$\tau^{(5)}{}_{ab}{}^{c'd'} = n_a n_b n^{c'} n^{d'} \tag{20e}$$

Since it is a maximally symmetric bitensor with the same structure, the graviton propagator must be of the form:

$$G^{(2)}_{ab}{}^{c'd'}(x,x') = \sum_{i=1}^{5} G_i(\mu)\, \tau^{(i)}_{ab}{}^{c'd'} \quad , \tag{21}$$

where the G_i are *scalar* functions of the invariant distance $\mu(x,x')$, or $z(x,x')$.

The transversality conditions (13) are now easily implemented by computing, for example,

$$0 = D^a G^{(2)}_{ab}{}^{c'd'} = \sum_{i=1}^{5} \left\{ \frac{dG_i}{d\mu}\, n^a \tau^{(i)}_{ab}{}^{c'd'} + G_i D^a \tau^{(i)}_{ab}{}^{c'd'} \right\} \tag{22}$$

with

$$n^a \tau^{(1)}_{ab}{}^{c'd'} = n_b g^{c'd'} \tag{23a}$$

$$n^a \tau^{(2)}_{ab}{}^{c'd'} = n_b n^{c'} n^{d'} + n_b g^{c'd'} \tag{23b}$$

$$n^a \tau^{(3)}_{ab}{}^{c'd'} = -2\, g_b{}^{(c'} n^{d')} \tag{23c}$$

$$n^a \tau^{(4)}_{ab}{}^{c'd'} = -2\, n_b n^{c'} n^{d'} + 2\, g_b{}^{(c'} n^{d')} \tag{23d}$$

$$n^a \tau^{(5)}_{ab}{}^{c'd'} = n_b n^{c'} n^{d'} \tag{23e}$$

and

$$D^a \tau^{(1)}_{ab}{}^{c'd'} = 0 \tag{24a}$$

$$D^a \tau^{(2)}_{ab}{}^{c'd'} = 3A\, n_b g^{c'd'} + 2C\, g_b{}^{(c'} n^{d')} + 2C\, n_b n^{c'} n^{d'} \tag{24b}$$

$$D^a \tau^{(3)}_{ab}{}^{c'd'} = -(A+C)\left[8\, g_b{}^{(c'} n^{d')} + 2\, n_b g^{c'd'} \right] \tag{24c}$$

$$D^a \tau^{(4)}_{ab}{}^{c'd'} = -4(A+2C)\, n_b n^{c'} n^{d'} + 2C\, n_b g^{c'd'} + 8A\, g_b{}^{(c'} n^{d')} \tag{24d}$$

$$D^a \tau^{(5)}_{ab}{}^{c'd'} = 3A\, n_b n^{c'} n^{d'} . \tag{24e}$$

Thus there are three distinct tensors appearing on the right-hand side of Eq. (22), namely $n_b g^{c'd'}$, $g_b{}^{(c'}n^{d')}$, and $n_b n^{c'} n^{d'}$. Equating the coefficients of each of these tensors to zero gives three conditions on the five G_i:

$$\frac{d}{d\mu}\left(G_1+G_2\right) + 3AG_2 - 2(A+C)G_3 + 2CG_4 = 0 \tag{25a}$$

$$\frac{d}{d\mu}\left(G_2-2G_4+G_5\right) + 2CG_2 - 4(A+2C)G_4 + 3AG_5 = 0 \tag{25b}$$

$$\frac{d}{d\mu}\left(-G_3+G_4\right) + CG_2 - 4(A+C)G_3 - 4G_4 = 0 \;. \tag{25c}$$

On the other hand, the trace conditions (14) imply the two relations:

$$4G_1 + G_2 + 2G_3 = 0 \tag{26a}$$

$$4G_2 - 4G_4 + G_5 = 0 \tag{26b}$$

Actually, these five conditions on the G_i are not all linearly independent. If we define the linear combinations,

$$f \equiv G_4 - G_3 \tag{27}$$

$$g \equiv 3G_2 - 2G_3 \tag{28}$$

Then, Eqs. (25) and (26) yield

$$\frac{dg}{d\mu} + 4Ag + 8Cf = 0 \tag{29}$$

$$\frac{df}{d\mu} + 4Af + 2Cg = 5CG_2 \;. \tag{30}$$

We need one more equation for g. Then, Eq. (29) will determine f, Eq. (30) fixes G_2, expressions (28) and (27) determine G_3 and G_4, and G_1 and G_5 can be obtained from relations (26).

The last differential equation comes from the equation of motion (12), which in turn requires knowledge of the transverse traceless projector $P^{(2)c'd'}_{ab}$. The calculation of this tensor and the associated scalar functions Δ_0 and Δ_1 are given in the appendices. One also needs to compute

$$\Box G^{(a)\,c'd'}_{ab} = \sum_{i=1}^{5} \left\{ \left(\frac{d^2}{d\mu^2} G_i + 3A \frac{d}{d\mu} G_i \right) \tau^{(i)\,c'd'}_{ab} + G_i \, \Box \, \tau^{(i)\,c'd'}_{ab} \right\} , \tag{31}$$

which is a straightforward though tedious application of the methods of Ref. [6]. We pass over these details and merely point out that the algebra simplifies enormously if one concentrates only on the scalar product of Eq. (12) with $n^a n^b n_{c'} n_{d'}$, which turns out to contain the only new piece of information about the function $g(\mu)$. The main steps of the computation are outlined in Appendix B. The resulting differential equation for g turns out to be

$$\frac{1}{H^2}\left\{ \frac{d^2}{d\mu^2} + 7A \frac{d}{d\mu} - 10H^2 \right\} g = \left\{ z(1-z)\frac{d^2}{dz^2} + 4(1-2z)\frac{d}{dz} - 10 \right\} g$$

$$= \frac{5H^2}{24\pi^2} \frac{1}{(1-z)^2} \quad ; \quad z \neq 1 \tag{32}$$

which is just a hypergeometric equation with a particular form for the inhomogeneous term.

Let us introduce F(a,b;c;z), the standard notation for the hypergeometric function obeying

$$\mathcal{D}(a,b;c)\, F(a,b;c;z) \equiv$$

$$\left\{ z(1-z)\frac{d^2}{dz^2} + \left[c - (a+b+1)z\right]\frac{d}{dz} - ab \right\} F = 0 . \tag{33}$$

Then the homogeneous solutions to Eq. (32) are F(2,5;4;z) and F(2,5;4;1-z). But

$$F(2,5;4;z) = \frac{1}{2}\frac{1}{(1-z)^3} + \frac{1}{2}\frac{1}{(1-z)^2} \tag{34}$$

is too singular at z = 1 to give the correct flat space limit to the graviton propagator. Similarly, F(2,5;4;1-z) has the wrong singularity structure as z → 1 (namely, no singularity at all), as well as an unwanted singular behaviour as $z \to 0$. Thus both homogeneous solutions to Eq. (32) must be rejected for the boundary conditions corresponding to the Bunch-Davies de Sitter invariant state*).

To find a particular solution to the *inhomogeneous* equation (32) we follow the methods of Ref. [6] and write $(1-z)^{-2} = 2F(2,5;4;z) - F(3,4;4;z)$. The first term has the same values of a, b, and c as the hypergeometric differential operator on the left-hand side of Eq. (32). Thus, the relation

$$\mathcal{D}\left[\left(\frac{\partial}{\partial a} - \frac{\partial}{\partial b}\right) F\right] = (b-a)\, F \tag{35}$$

allows us to find a solution for this first inhomogeneous term. The second term $F(3,4;4;z)$ fits the formula

$$\mathcal{D}(a,b;c)\, F(a+1, b-1; c; z) = 2\, F(a+1, b-1; c; z) \tag{36}$$

and so an inhomogeneous solution can also be found. Working out $[(\partial/\partial a) - (\partial/\partial b)]F$ for $a = 2$, $b = 5$ from the Taylor expansion of $F(a,b;c;z)$ and taking into account the normalization factors gives finally

$$g = -\frac{5H^2}{48\pi^2}\left\{ \frac{1}{1-z} + \frac{2}{3z^3}(1+z)\ln(1-z) + \frac{2}{3z^2} + \frac{1}{z} \right\}, \tag{37}$$

which has the appropriate $(1-z)^{-1}$ behaviour at $z = 1$ to give the correct $H^2 \to 0$ limit and *no* singularity at $z = 0$. Now, using expressions (26) to (30) we obtain the full (spin-2 part of the) graviton propagator to be Eq. (21) with all the G_i of the form

$$G_i = \frac{H^2}{16\pi^2 N_i}\left\{ \frac{1}{1-z} + \frac{2}{3z^3}\left(1+z+p_i z^2 + q_i z^3\right)\ln(1-z) + \frac{2}{3}\frac{1}{z^2} + \frac{1}{z} + r_i \right\}. \tag{38}$$

*) An admixture of $F(2,5;4;1-z)$ in the solution gives the Green functions for a member of the one-parameter family of de Sitter invariant states discussed in Refs. [2] and [7].

The normalization constants N_i and the additional constants (p,q,r) are catalogued in Table 1 for each i.

This evaluation of $G_{ab}^{(2)c'd'}$ is strictly valid for $0 < z < 1$, i.e. x and x' Euclidean space-like.

The Feynman propagator function for time-like x and x' is determined from Eq. (38) by interpreting z as a complex variable and taking the boundary values of the complex functions G_i as z approaches the real axis from *above*, the cut running from $z = 1$ to $z = \infty$. For non-Euclidean space-like x and x', the propagator is just the continuation of Eq. (38) for $z < 0$ which is obtained trivially since there is no singularity at $z = 0$.

The discontinuity across the cut $G(z+i0) - G(z-i0)$ is the commutator function (which vanishes for space-like separations $z < 1$) and the retarded Green functions are

$$G^{ret}(z) = -i\,\theta(t-t')\left[G(z+io) - G(z-io)\right], \tag{39}$$

where the index i has been omitted for clarity. Since

$$\frac{1}{1-(z\pm io)} = \pm i\pi\,\delta(1-z) + P\left(\frac{1}{1-z}\right) \tag{40a}$$

and

$$\ln\left[1-(z\pm io)\right] = \mp i\pi + \ln(z-1) \quad , \; z > 1 \tag{40b}$$

$$G_i^{ret} = \frac{H^2}{8\pi N_i}\,\theta(t-t')\left\{\delta(1-z) - \frac{2}{3z^3}\left(1+z+p_i z^2+q_i z^3\right)\theta(z-1)\right\} \tag{41}$$

It is obvious from Eqs. (38) to (41) that although the *real* parts of the Feynman functions diverge like log z for $z \gg 1$ the imaginary part which contributes to the *classical* retarded (or advanced) Green functions does not diverge at all. Instead, as $z \gg 1$,

$$G_i^{ret} \xrightarrow[t \gg t']{} -\frac{H^2}{12\pi}\frac{q_i}{N_i} = \text{const}. \tag{42}$$

Thus, the transverse trace-free perturbations of the de Sitter background do not go unstable -- classically. If we had included an arbitrary amount of the homogeneous solution to Eq. (32) in the propagator function, then the commutator and classical Green functions would have been non-zero on and within the light cone of the *antipodal* point to x ($z < 0$). However, within the light cone of x, i.e. for $z > 1$, expressions (41) and (42) remain unchanged and our conclusions are not altered by the addition of such a term.

The logarithmic divergence of the spin-2 Feynman propagator at large distances agrees qualitatively with the results of Refs. [3] and [8], although more precise comparison is difficult because they use a non-covariant gauge.

4. THE SPIN-0 PROPAGATOR FUNCTION

In order to assemble the full metric two-point function we need also to consider the spin-0 gauge-invariant piece of the metric fluctuations:

$$h - \Box B = \left[g^{ab} - \frac{4}{3}\left(\Box + \frac{R}{3}\right)^{-1}\left(D^a D^b - \frac{1}{4} g^{ab}\Box\right)\right] h_{ab} , \tag{43}$$

where $-(\Box + R/3)^{-1}$ denotes the inverse of $-\Box - R/3$ over the non-zero mode subspace of the differential operator, i.e. it is just $\Delta_1(x,x')$ defined in Appendix A.

Equation (43) defines a projection of symmetric tensor functions onto scalar functions, which we shall denote by $h - \Box B = S^{ab}h_{ab}$. Since only the combination $h - \Box B$ is gauge invariant there is no unique spin-0 projection operator analagous to Eq. (15) for spin-2. If we choose the simplest such projector (operating to the left)

$$P^{(0)}_{abc'd'} = \frac{1}{4}\, S_{ab}\, g_{c'd'} \quad , \tag{44}$$

then the spin-0 part of the metric two-point function obeys:

$$\begin{aligned} &\frac{1}{2}\left[g_{ab}\Box - D_a D_b + \frac{R}{4} g_{ab} \right] S^{e''f''} G_{e''f''c'd'} = \\ &\quad \frac{3}{8}\left[S_{ab}\left(\Box + \frac{R}{3}\right) S^{e''f''} \right] G^{(0)}_{e''f''c'd'} = P^{(0)}_{abc'd'} \quad , \end{aligned} \tag{45}$$

which is the analogue of Eq. (12). A solution of the form

$$G^{(0)}_{abc'd'} = g_{ab}\, G^{(0)}\, g_{c'd'} \tag{46}$$

satisfies Eq. (45) provided

$$G^{(0)}(x,x') = \frac{1}{6}\left(\Box + \frac{R}{3}\right)^{-1} = -\frac{1}{6}\,\Delta_1(x,x') \; . \tag{47}$$

The function Δ_1 which appears in the spin-0 two-point function (47) is given by Eq. (A.15) of Appendix A. Its corresponding retarded function is

$$\Delta_1^{ret} = \frac{H^2}{8\pi}\,\theta(t-t')\left\{ \delta(1-z) + 6(2z-1)\,\theta(z-1) \right\} \quad , \tag{48}$$

which behaves like $\sim z$ for large time-like (x,x'). This is just a reflection of the fact that the operator $-(\Box + R/3)$ is that of a scalar field with *negative* mass squared, $-R/3 = -4H^2$. We will now argue that this divergence in the classical retarded spin-0 function must be a gauge artifact.

The subtlety arises because the decomposition (2) and transformation rules (8) are unique only when $D_a D_b \xi$ is *linearly independent* (as a tensor) from $g_{ab}\xi$. However when the gauge function is longitudinal, $\xi_a = D_a \xi$ and ξ obeys

$$D_a D_b \xi = \frac{1}{4} g_{ab} \Box \xi \ , \tag{49}$$

then h still transforms as expression (8d) but the transformation rule for B becomes ill defined: it is possible to add an arbitrary amount of ξ to B without affecting the metric perturbation, when ξ satisfies Eq. (49). Thus $h - \Box B$ is not gauge invariant under such gauge transformations and there remains a residual gauge freedom in our spin-0 two-point functions under transformations obeying Eq. (49).

To gain a geometric understanding of these transformations, operate on Eq. (49) with D^b and use the commutator rule for covariant derivatives $[D^b, D_a]D_b\xi = R_{ac}D^c\xi = \frac{1}{4} RD_a\xi$ in de Sitter space to find that Eq. (49) implies $D_a[(\Box + R/3)\xi] = 0$. This equation has solutions for ξ = const. and

$$\left(\Box + \frac{R}{3}\right)\xi = 0 \quad . \tag{50}$$

The constant solutions result in no change whatever in h_{ab} and may be disregarded. The solutions of Eq. (50) are most clearly visualized on the Euclidean four sphere. There, the D'Alembert operator has eigenfunctions which are the spherical harmonics on S_4. The eigenvalues of $\Box$ are $-n(n+3)H^2$ with degeneracy $2n + 3$. Thus, Eq. (50) is exactly the condition for the $n = 1$ harmonics, of which there are five. These five conformal Killing fields ξ^i, $i = 1, \ldots, 5$, of S_4 correspond to the five translations of the four sphere in the imbedding five-dimensional flat space. The changes of the metric under such transformations are clearly without geometric significance for the four sphere. The (normalized) sum

$$\frac{8\pi^2}{15 H^4} \sum_{i=1}^{5} \xi^i(x)\, \xi^i(x') = \cos\left(H\mu(x,x')\right) = 2z - 1 \tag{51}$$

is exactly the quantity appearing in Δ_1^{ret} for $z > 1$.

Since everything which has just been said applies equally well to the Lorentzian signature de Sitter space-time, the divergence of Δ_i^{ret} for large time-like (x,x') should not give rise to any physical instability of de Sitter space. To check this directly we must consider gauge-invariant quantities constructed from the metric perturbation that arises from a physical source (see also Ref. [8]). This we do in the next section.

5. THE QUANTUM INSTABILITY OF DE SITTER SPACE

The most direct application of our calculation of the metric two-point function in de Sitter space is to the evaluation of quantities of the form

$$\delta\langle \hat{g}_{ab}(x)\rangle = h_{ab}(x) = \int d^4x' \sqrt{-g'}\; G_{abc'd'}(x,x')\; T^{c'd'}(x') \quad , \tag{52}$$

that is, $G_{abc'd'}(x,x')$ gives the response of the metric expectation value in the de Sitter invariant Bunch-Davies state to small (external) perturbations in the energy-momentum tensor of matter. Here

$$G_{abc'd'} = G^{(2)}_{abc'd'} + G^{(0)}_{abc'd'} \tag{53}$$

contains the contributions from both the spin-2 and spin-0 parts of the metric two-point function calculated in Sections 3 and 4, respectively. The information about the quantum state of the gravitational field is contained in the Feynman boundary conditions ($i\varepsilon$ prescription) used in the evaluation of the various scalar functions G_i and $G^{(0)}$.

If instead we are interested in the *classical* response of the metric to stress-energy sources then the retarded (or advanced) Green functions must be used. The fact that the Feynman functions contain a log (1-z) term, whereas the retarded functions do not, leads to very different results for the two, when inserted into Eq. (52). We have shown already in expression (42) that the spin-2 parts of the two-point function do not lead to any divergences in the classical perturbations away from de Sitter space (although the perturbations do not diminish as $t \to \infty$ either). This is consistent with the work of previous authors [5]. What about the spin-0 part?

The considerations of the last section led us to believe that the divergence in the classical spin-0 retarded Green function must be a gauge artifact. If $G^{(0)ret}_{abc'd'} = -(1/6)g_{ab}\Delta_1^{ret}g_{c'd'}$ is substituted into Eq. (52) the apparently growing piece is just proportional to

$$\sum_{i=1}^{5} \xi^i(x) \int d^4x' \sqrt{-g'}\; \xi^i(x')\, T_{c'}{}^{c'}(x') \;.$$

However, using Eqs. (49) and (50) one has

$$\int d^4x' \sqrt{-g'}\; \xi^i(x')\, g_{c'd'} T^{c'd'}(x') = -\frac{1}{H^2}\int d^4x' \sqrt{-g'}\; T^{c'd'}(x')\, D_{c'} D_{d'} \xi^i(x') \qquad (54)$$

which vanishes after a by part integration, since the energy-momentum tensor is covariantly conserved, $D_{c'}T^{c'd'}(x') = 0$. No physical quantity can be constructed which diverges, classically.

The situation changes dramatically when the Feynman propagator functions are inserted into Eq. (52). These contain log (1-z) pieces in the spin-2 part, and (2z-1) ln (1-z) pieces in the spin-0 part. Explicitly, the divergent piece in $h_{ab}(x)$ is:

$$\sum_{i=1}^{5} \frac{H^2}{24\pi^2 N_i} q_i \int d^4x' \sqrt{-g'}\; \ln(1-z)\; \tau^{(i)}_{abc'd'}(x,x')\, T^{c'd'}(x')$$
$$+ \frac{H^2}{16\pi^2}\int d^4x' \sqrt{-g'}\; (2z-1)\ln(1-z)\; T_{c'}{}^{c'}(x') \;. \qquad (55)$$

It is straightforward to show that the spin-2 part leads to a log z divergence, while the scalar part leads to a linear-z growth in the metric perturbation. Indeed, using the property $(2z-1)g_{ab} = -(1/H^2)D_aD_b(2z-1)$ and doing integrations by part, the second term of Eq. (55) takes the form

$$-\frac{1}{8\pi^2}\int d^4x'\,\sqrt{-g'}\;z\,n_{c'}n_{d'}T^{c'd'} + \text{less singular terms}\;, \tag{56}$$

which indicates that the dominant divergence comes from the spin-0 part. Thus the de Sitter invariant quantum state usually discussed in the literature is unstable to arbitrarily small perturbations and the de Sitter symmetry is broken.

The response of the metric to the external perturbation in Eq. (52) is only one use to which the graviton two-point function can be put. Any quantum amplitude involving gravitons in internal lines (off-mass shell) will feature the same divergent behaviour evidenced by expressions (55) and (56). A simple example is the one-graviton exchange graph of Fig. 1 with an expression of the form

$$\int d^4x\,\sqrt{-g}\int d^4x'\,\sqrt{-g'}\;T^{ab}(x)\,G_{abc'd'}(x,x')\,T^{c'd'}(x')\;.$$

In addition we can use metric fluctuations as a self-generated quantum source, in the full non-linear theory, which will play the same role as the external perturbing source T_{ab} introduced in Eq. (52). Thus, fluctuations in the gravitational field itself will destabilize de Sitter space, and the spectrum of these fluctuations can be used to understand the direction in which the full quantum dynamics will cause the system to evolve. These considerations will be taken up in a future publication.

Acknowledgements

We acknowledge important discussions with B. Allen, J. Iliopoulos, T. Jacobson and T. Tomaras. We would also like to thank the Institute for Theoretical Physics in Santa Barbara for its hospitality when this work was initiated.

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Table 1

Numerical values of the constants N_i, p_i, q_i, r_i, (i = 1, ..., 5) appearing in the scalar coefficients G_i which determine the spin-2 part of the graviton propagator [see Eqs (21) and (38)].

i	N_i	p_i	q_i	r_i
1	12	0	-12	-8/3
2	-2	0	-2	-4/9
3	12	0	18	4
4	-3	5/2	-9/2	-1
5	3/2	-5	3	2/3

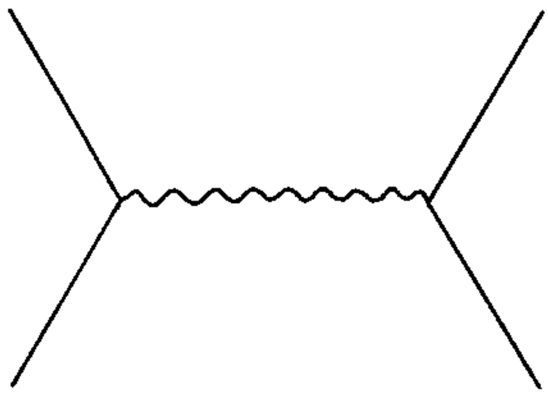

Fig. 1: One-graviton exchange graph.

Appendix A

THE FUNCTIONS Δ_0 AND Δ_1

The differential operator $-\Box + M^2$ has a unique inverse $\Delta(x,x';M^2)$ in de Sitter space provided:

i) Δ is a function only of $z(x,x')$, i.e. it is maximally symmetric;

ii) its only singularity is $(1-z)^{-1}$ as $z \to 1$ with a possible branch cut extending to $z = \infty$;

iii) $M^2 \geq (9/4)H^2$.

Condition (i) fixes $\Delta(z;M^2)$ to be a solution of the ordinary differential equation:

$$\mathcal{D}(a,b;c)\,\Delta(z;M^2) = 0 \tag{A.1}$$

for

$$a = \frac{3}{2} + i\sqrt{\frac{M^2}{H^2} - \frac{9}{4}}$$

$$b = \frac{3}{2} - i\sqrt{\frac{M^2}{H^2} - \frac{9}{4}} \qquad z \neq 1\,. \tag{A.2}$$

$$c = 2$$

The two such solutions are $F(a,b;c;z)$ and $F(a,b;c;1-z)$. Condition (ii) eliminates the latter and fixes the normalization to be

$$\Delta(z;M^2) = \frac{\Gamma(a)\,\Gamma(b)}{16\pi^2} H^2\, F(a,b;c;z)\,. \tag{A.3}$$

If condition (iii) is not satisfied the analyticity requirements on Δ imposed in (ii) may not be satisfied. In particular, if $M^2 = -n(n+3)H^2$ for $n = 0, 1, \ldots,$ then Eq. (A.3) diverges for all z because of the Γ functions in the normalization. This can be understood most directly by recognizing that (ii) is the same condition imposed on Δ by the demand that it should be the boundary value of a function defined on the Euclidean section (S_4). There $\Box$ is an elliptic differential operator with eigenfunctions

$$\Box Y_n^i = -n(n+3)H^2 Y_n^i \quad ; \quad i = 0, 1, \ldots, 2n+3 \tag{A.4}$$

that are just the spherical harmonics on S_4. Then,

$$\Delta(z; M^2) = \sum_{n=0}^{\infty} \sum_{i=0}^{2n+3} \frac{Y_n^i(x)\, Y_n^{i\,*}(x')}{n(n+3)H^2 + M^2} \tag{A.5}$$

satisfies (i) and (ii) and is equal to Eq. (A.3) for $M^2 \geq 9/4\ H^2$.

For the exceptional values $M^2 = -n(n+3)H^2$, the unique inverse obeying (i) and (ii) does not exist and we must *either* abandon the maximal symmetry implied by (i) or the normal causal behaviour implied by (ii), *or* define the inverse of $-\Box + M^2$ over the orthogonal complement to its zero mode subspace, i.e. we simply omit the divergent n in the sum of Eq. (A.5).

When $M^2 = 0$ the resulting $\Delta_0(z)$ then satisfies

$$-\Box \Delta_0 = \delta^{(4)}(x, x') - Y_0 Y_0^* \,. \tag{A.6}$$

Since

$$\int_{S_4} d^4x \sqrt{-g}\ Y_n^i(x)\, Y_{n'}^{i'\,*}(x') = \delta_{nn'}\, \delta^{ii'} \tag{A.7}$$

and Y_0 is a constant, we have

$$\Box \Delta_0(z) = H^2 \mathcal{D}(a_0, b_0; c)\, \Delta_0 = \frac{1}{Vol(S_4)} = \frac{3H^4}{8\pi^2} \quad ; \quad z \neq 1 \,. \tag{A.8}$$

The homogeneous solutions do not obey (ii) but a particular inhomogeneous solution can easily be found by taking the limit of Eq. (A.3) with the n = 0 term of Eq. (A.5) subtracted, or by direct integration of Eq. (A.8). The result is

$$\Delta_0(z) = \frac{H^2}{16\pi^2}\left\{\frac{1}{1-z} - 2\ln(1-z)\right\} + k_0 \quad . \tag{A.9}$$

In the same way, when $M^2 = -4H^2 = -R/3$, as in Section 4, we define $\Delta_1(z)$ to obey

$$-\left(\square + \frac{R}{3}\right)\Delta_1 = \delta^{(4)}(x,x') - \sum_{i=1}^{5} Y_1^i(x)\, Y_1^{i\,*}(x') \quad . \tag{A.10}$$

The sum on the right-hand side is a function only of z since it is the character of the n = 1 representation of O(5) and must be an invariant. Since

$$y_1(z) \equiv \sum_{i=1}^{5} Y_1^i(x)\, Y_1^{i\,*}(x')$$

obeys

$$\left(\square + 4H^2\right) y_1 = 0 \tag{A.11}$$

and

$$\int_{S_4} d^4x \sqrt{-g}\; y_1\big(z(x,x')\big)\Big|_{x=x'} = \mathrm{Vol}(S_4)\, y_1(1) = 5 \tag{A.12}$$

we have

$$y_1(z) = \frac{15H^4}{8\pi^2}\cos(H\mu) = \frac{15H^4}{8\pi^2}(2z-1) \; . \tag{A.13}$$

Then,

$$\mathcal{D}(4,-1;2)\,\Delta_1(z) = \frac{15H^2}{8\pi^2}(2z-1) \quad , z \neq 1 \tag{A.14}$$

determines

$$\Delta_1(z) = \frac{H^2}{16\pi^2}\left\{ \frac{1}{1-z} + 6(1-2z)\ln(1-z) - 6 \right\} + K_1 y_1 \quad . \tag{A.15}$$

Appendix B

THE SPIN-2 PROJECTOR

The tensor decomposition of Section 2 may be inverted for the various fields by operating on Eq. (2) with D^b and D^a successively. We find

$$B = \frac{4}{3}\left(\Box + \frac{R}{3}\right)^{-1} \Box^{-1}\left(D^a D^b - \frac{1}{4} g^{ab}\Box\right) h_{ab} \tag{B.1}$$

$$A_a^{\perp} = - Q_a{}^{c'} D^{d'} h_{c'd'} \tag{B.2}$$

$$h_{ab}^{\perp} = h_{ab} - 2 D_{(a} D^{c'} Q_{b)}{}^{d'} h_{c'd'} - \frac{1}{4} g_{ab} h - \\ - \frac{4}{3}\left(D_a D_b - \frac{1}{4} g_{ab}\Box\right)\left(\Box + \frac{R}{3}\right)^{-1}\Box^{-1}\left(D^{c'}D^{d'} - \frac{1}{4} g^{c'd'}\Box'\right) h_{c'd'} \;, \tag{B.3}$$

where $Q_a^{c'}(x,x')$ is the transverse vector or spin-1 projector which is the inverse of $-(\Box + R/4)$ on transverse vectors [given explicitly by Eq. (B.12) below]. The integral operators $\Box^{-1}$ and $(\Box + R/3)^{-1}$ denote the functions $-\Delta_0$ and $-\Delta_1$ found in Appendix A.

Actually, $\Box + R/4$ has zero modes when operating on transverse vectors so the inverse $Q_b^{c'}$ must again be defined over the complement to the zero-mode subspace. The zero modes are the ten Killing fields of de Sitter space ξ_a^i satisfying Killing's equation

$$D_a \xi_b^i + D_b \xi_a^i = 0 \qquad ; \; i = 1, \ldots, 10 \tag{B.4}$$

The trace of this equation tells us that the ξ_a^i are transverse and D^b on it tells us that

$$\Box \xi_a^i + \left[D^b, D_a\right] \xi_b^i = \left(\Box + \frac{R}{4}\right)\xi_a^i = 0 \;, \tag{B.5}$$

so they are indeed zero modes of $\Box + R/4$. Let

$$K_b{}^{c'} \equiv \sum_{i=1}^{10} \xi_b^i(x)\, \xi^{c'i}(x') \tag{B.6a}$$

$$= \tilde{\alpha}(z)\, g_b{}^{c'} + \tilde{\beta}(z)\, n_b n^{c'} , \tag{B.6b}$$

where the last relation follows because K is an invariant tensor. $\tilde{\alpha}$ abd $\tilde{\beta}$ are regular at z = 0, 1 and can be determined from

$$\left(\Box + \frac{R}{4}\right) K_b{}^{c'} = 0 = D^b K_b{}^{c'} . \tag{B.7}$$

These conditions imply that $\tilde{\gamma} \equiv \tilde{\alpha} - \tilde{\beta}$ is a constant and

$$\tilde{\alpha} = (2z-1)\,\tilde{\gamma} \tag{B.8a}$$

$$\tilde{\beta} = 2(z-1)\,\tilde{\gamma} . \tag{B.8b}$$

The constant $\tilde{\gamma}$ is determined by the normalization condition on the ξ^i_a:

$$\int_{S_4} d^4x \sqrt{-g}\; K_b{}^{c'} g^b{}_{c'} \Big|_{x=x'} = 10 = \left(4\tilde{\alpha} + \tilde{\beta}\right)\Big|_{z=1} \mathrm{Vol}(S_4) . \tag{B.9}$$

Thus

$$\tilde{\gamma} = \frac{15\, H^4}{16\, \pi^2} . \tag{B.10}$$

If there were no zero modes the inverse of $-(\Box + R/4)$ would be proportional simply to the transverse projector

$$P_b{}^{c'} = g_b{}^{c'} - D_b \Box^{-1} D^{c'} . \tag{B.11}$$

As it is, we define $Q_b^{c'}$ by the equation

$$-\left(\Box + \frac{R}{4}\right) Q_b{}^{c'} = P_b{}^{c'} - K_b{}^{c'} \tag{B.12}$$

with the zero modes (B.6) subtracted. Let

$$Q_b{}^{c'} = \alpha\, g_b{}^{c'} + \beta\, n_b n^{c'} \tag{B.13}$$

be decomposed in terms of its invariant tensors and define

$$\gamma = \alpha - \beta \quad . \tag{B.14}$$

This leads to the equation for γ,

$$\left(\frac{d^2}{d\mu^2} + 5A \frac{d}{d\mu} \right) \gamma = - \frac{d^2}{d\mu^2} \Delta_0 + \tilde{\gamma} \quad . \tag{B.15}$$

Since $(d^2/d\mu^2)\Delta_0 = (3H^4/32\pi^2)(1-z)^{-2}$ in terms of z, Eq. (B.15) becomes

$$\mathcal{D}(5,0;3)\,\gamma = - \frac{3H^2}{32\pi^2} \left[\frac{1}{(1-z)^2} - 10 \right] \quad . \tag{B.16}$$

The unique solution obeying conditions (i) and (ii) of Appendix A is

$$\gamma(z) = \frac{3H^2}{32\pi^2} \left\{ \frac{1}{1-z} - 2 \ln(1-z) \right\} + \text{const.} \tag{B.17}$$

Having constructed $Q_b^{c'}$ explicitly we can return to Eq. (B.3) for the spin-2 projector. $P^{(2)}_{abc'd'}$ as determined by Eq. (B.3) is a complicated term:

$$P^{(2)}_{abc'd'}(x,x') = \delta^{(c'd')}_{(ab)}(x,x') - \frac{1}{4} g_{ab} g^{c'd'} \delta(x,x') - $$
$$- 2 D_{(a} D^{(c'} Q_{b)}{}^{d')} - \frac{4}{3} \left(D_a D_b - \frac{1}{4} g_{ab} \Box \right) \left(\Box + \frac{R}{3} \right)^{-1} \Box^{-1} \left(D^{c'} D^{d'} - \frac{1}{4} g^{c'd'} \Box' \right) . \tag{B.18}$$

However, its contraction with $n^a n^b n^{c'} n^{d'}$ is much simpler and is all we actually require. Using the identities

$$n^a D_a n_b = n^a D_a n^{c'} = n^a D_a g_b{}^{c'} = 0 \tag{B.19}$$

and

$$(n^a D_a)^p f(\mu) = \frac{d^p}{d\mu^p} f(\mu) \tag{B.20}$$

for any function of μ, we have

$$n^a n^b n^{c'} n^{d'} P^{(2)}_{abc'd'} = \frac{8H^2}{3} \frac{d^2}{d\mu^2}\gamma + \frac{4}{9}\left\{ \frac{d^4}{d\mu^4}\left(\Delta_0 - \Delta_1\right) + \frac{1}{2}\frac{d^2}{d\mu^2}\Box\Delta_1 - \frac{1}{16}\Box^2\Delta_1 \right\}, \quad (x \neq x') . \tag{B.21}$$

Then the equations of motion for Δ_0 and Δ_1 and the explicit form for γ in Eq. (B.17) finally gives

$$\frac{1}{H^2} n^a n^b n^{c'} n^{d'} P^{(2)}_{abc'd'} = \frac{5H^2}{24\pi^2} \frac{1}{(1-z)^2} \tag{B.22}$$

upon converting to the variable z. This is exactly the right-hand side of Eq. (32) of Section 3 of the text.