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Gravity from Poincaré Gauge Theory of the Fundamental Particles. III

—— *Weak Field Approximation* ——

Kenji HAYASHI and Takeshi SHIRAFUJI*

Institute of Physics, University of Tokyo, Tokyo 153
**Physics Department, Saitama University, Urawa 338*

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We apply the weak field approximation to the most general gravitational field equations in Poincaré gauge theory. The weak gravitational field $h_{\mu\nu}$ is a multimass field obeying a fourth-order field equation. In the Newtonian approximation we show that there are two routes to arrive at the Newtonian potential. The torsion field is decomposed into six irreducibe building blocks with spin$^{\text{parity}}$, 2^+, 2^-, 1^+, 1^-, 0^+ and 0^-, each of which obeys the Klein-Gordon equation. Finally, we construct a possible candidate for the massless graviton field which obeys the linearized Einstein equation.

§ 1. Introduction

In previous papers called I and II, respectively,[1] we studied Poincaré gauge theory with linear and quadratic Lagrangian densities in the translation and Lorentz gauge field strengths with ten free parameters included in the gravity action. The most general gravitational field equations were derived by means of the action principle with independent variations of the translation and Lorentz gauge fields.

Here in this paper we shall choose the *conventional* method that the Lorentz gauge field $\boldsymbol{A}$ can be decomposed into the Ricci rotation coefficient $\boldsymbol{\Delta}$ (which is given by first derivatives of the tetrad field) and the contorsion field $\boldsymbol{K}$,

$$\boldsymbol{A}=\boldsymbol{\Delta}+\boldsymbol{K}\,, \qquad (1\cdot 1)$$

and then apply the weak field approximation to the most general gravitational field equations. (In the Riemann-Cartan space-time the geometrical method also gives rise to the conventional method.) So independent variables are the linearized gravitational field $h_{\mu\nu}$ and the contorsion field $K_{\lambda\mu\nu}$, the former of which satisfies a *fourth-order differential field equation* just because of the conventional method formulated by $(1\cdot 1)$. Therefore, $h_{\mu\nu}$ describes a multimass field which splits into a graviton field with vanishing mass, a particle with mass m_2 and spin 2 and a particle with mass m_0 and spin 0. However, the redefinition of a multimass field, which incorporates second derivative terms, will give us a fairly good candidate for a massless graviton field with spin 2. This redefined field $\phi_{\mu\nu}$

obeys the usual wave equation of second order, whose source term is exactly the symmetrized energy-momentum tensor appearing in the linearized Einstein equation.

The arrangement of the present paper is as follows. In the next section we shall apply the weak field approximation method to the most general gravitational field equations derived in a previous paper I. In § 3 the Newtonian approximation will be carried on, showing that there are two different routes in arriving at the Newtonian potential: Denoting a_i $(i=1, 2, \cdots, 6)$ as coefficients multiplied by Lagrangian densities quadratic in the Lorentz gauge field strength, one is to take all a_i as finite, and the other is to take all a_i as infinite, the latter of which is just the result of New General Relativity.[2] In § 4 the torsion field is decomposed into its irreducible building blocks, which have spin$^{\text{parity}}$ as 2^+, 2^-, 1^+, 1^-, 0^+ and 0^- and propagate in vacuum. In § 5 the massless graviton field $\phi_{\mu\nu}$ is redefined and satisfies the usual wave equation whose source term is given by the symmetrized energy-momentum tensor occuring in the linearized Einstein equation. The final section will be devoted to conclusion.

§ 2. The linearized gravitational field equations

We shall apply the most general gravitational field equations to weak field situations, where the translation gauge field $a^k{}_\mu$ and the Lorentz gauge field $A_{ij\mu}$ are both so weak that it is sufficient to keep only those terms which are linear in $a^k{}_\mu$ and $A_{ij\mu}$. In this approximation we need not distinguish Latin indices from Greek ones; so we shall use Greek indices throughout this paper with the understanding that they are raised and lowered with the Minkowski metric $\eta_{\mu\nu}$. The metric tensor $g_{\mu\nu}$ is written as

$$g_{\mu\nu}=\eta_{\mu\nu}+h_{\mu\nu} \tag{2·1}$$

with

$$h_{\mu\nu}=a_{\mu\nu}+a_{\nu\mu}\,. \tag{2·2}$$

Following the *conventional* method mentioned in the Introduction, we shall express the linearized gravitational field equations in terms of $a_{\mu\nu}$ and the contorsion field $K_{\lambda\mu\nu}$. The torsion field is given in the weak field approximation by

$$T_{\lambda\mu\nu}=A_{\lambda\mu\nu}-A_{\lambda\nu\mu}-\partial_\mu a_{\lambda\nu}+\partial_\nu a_{\lambda\mu}\,. \tag{2·3}$$

The contorsion field is related to the three irreducible parts of the torsion field, $t_{\lambda\mu\nu}$, v_μ and a_μ, by*)

$$K_{\lambda\mu\nu}=\frac{1}{2}(T_{\lambda\mu\nu}-T_{\mu\lambda\nu}-T_{\nu\lambda\mu})$$

*) We denote symmetrization and antisymmetrization of tensor indices by round brackets ( ) and square brackets [ ], respectively: For example,

$$V_{[\mu\rho\cdots\sigma\nu]}=\tfrac{1}{2}(V_{\mu\rho\cdots\sigma\nu}-V_{\nu\rho\cdots\sigma\mu})\,.$$

$$= -\frac{4}{3}t_{\nu[\lambda\mu]} - \frac{2}{3}\eta_{\nu[\lambda}v_{\mu]} + \frac{1}{2}\varepsilon_{\lambda\mu\nu\rho}a^{\rho}, \qquad (2\cdot4)$$

or conversely

$$t_{\lambda\mu\nu} = K_{\nu(\lambda\mu)} + \frac{1}{3}\eta_{\nu(\lambda}v_{\mu)} - \frac{1}{3}\eta_{\lambda\mu}v_{\nu}, \qquad (2\cdot5a)$$

$$v_{\mu} = K_{\mu\nu}^{\cdot\cdot\nu}, \qquad (2\cdot5b)$$

$$a_{\mu} = \frac{1}{3}\varepsilon_{\mu\nu\rho\sigma}K^{\nu\rho\sigma}. \qquad (2\cdot5c)$$

Applying the weak field approximation to the alternative form of the gravitational field equation (I. 5·12), we have

$$2aG_{\mu\nu}^{(1)} + 2\partial^{\rho}F'^{(1)}_{\mu\nu\rho} = T_{\mu\nu}, \qquad (2\cdot6)$$

where $F'^{(1)}_{\lambda\mu\nu}$ is the linearized expression for $F'_{\lambda\mu\nu}$ of (I. 5·15),

$$F'^{(1)}_{\lambda\mu\nu} = \left(\alpha + \frac{2a}{3}\right)(t_{\lambda\mu\nu} - t_{\lambda\nu\mu}) + \left(\beta - \frac{2a}{3}\right)(\eta_{\lambda\mu}v_{\nu} - \eta_{\lambda\nu}v_{\mu})$$
$$-\frac{1}{3}\left(\gamma + \frac{3a}{2}\right)\varepsilon_{\lambda\mu\nu\rho}a^{\rho}, \qquad (2\cdot7)$$

and $G_{\mu\nu}^{(1)}$ is the linearized Einstein tensor,

$$G_{\mu\nu}^{(1)} = -\frac{1}{2}\{\Box\bar{h}_{\mu\nu} - \partial^{\rho}(\partial_{\mu}\bar{h}_{\nu\rho} + \partial_{\nu}\bar{h}_{\mu\rho}) + \eta_{\mu\nu}\partial^{\rho}\partial^{\sigma}\bar{h}_{\rho\sigma}\} \qquad (2\cdot8)$$

with

$$\bar{h}_{\mu\nu} = h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}h, \quad h = \eta^{\mu\nu}h_{\mu\nu}. \qquad (2\cdot9)$$

Here the d'Alembertian $\Box$ is defined by $\Box = \partial^{\mu}\partial_{\mu}$.

Equation (I. 5·18) takes the following form in the weak field approximation:

$$P_{\lambda\mu\nu} = -\frac{1}{2}S_{\lambda\mu\nu} \qquad (2\cdot10)$$

with

$$P_{\lambda\mu\nu} \equiv (3a_2 + 2a_5)\partial_{[\lambda}G_{\mu]\nu}^{(1)} + (a_2 + a_5 + 4a_6)\eta_{\nu[\lambda}\partial_{\mu]}G^{(1)}$$
$$-2\partial^{\rho}J^{(1)}_{[\lambda\mu][\nu\rho]}(K) - H^{(1)}_{\lambda\mu\nu}, \qquad (2\cdot11)$$

where $J^{(1)}_{\lambda\mu\nu\rho}(K)$ and $H^{(1)}_{\lambda\mu\nu}$ are the linearized expressions for $J_{\lambda\mu\nu\rho}(K)$ of (I. 5·6) and $H_{\lambda\mu\nu}$ of (I. 3·32), respectively. Here $G^{(1)} = \eta^{\mu\nu}G^{(1)}_{\mu\nu}$. By a straightforward calculation we get

$$P_{\lambda\mu\nu}=(3a_2+2a_5)\partial_{[\lambda}G^{(1)}_{\mu]\nu}+(a_2+a_5+4a_6)\eta_{\nu[\lambda}\partial_{\mu]}G^{(1)}$$

$$-\frac{2}{3}(3a_2+4a_3)\Big\{\Box t_{\nu[\lambda\mu]}-\partial^\rho t_{\rho[\lambda\mu],\nu}$$

$$+\frac{3}{2}(\partial^\rho t_{\rho(\lambda\nu),\mu}-(\lambda\rightleftarrows\mu))+\frac{3}{4}\eta_{\nu[\lambda}\partial^\rho\partial^\sigma t_{\rho\sigma\mu]}\Big\}$$

$$+(3a_2+2a_5)\Big\{\frac{1}{3}(\partial^\rho t_{\rho[\lambda\nu],\mu}-(\lambda\rightleftarrows\mu))+(\partial^\rho t_{\rho(\lambda\nu),\mu}-(\lambda\rightleftarrows\mu))\Big\}$$

$$-\frac{2}{3}(a_4+a_5)\Big\{\eta_{\nu[\lambda}\Big(\Box v_{\mu]}-\partial_{\mu]}\partial^\rho v_\rho-\frac{3}{2}\partial^\rho\partial^\sigma t_{\rho\sigma\mu]}\Big)$$

$$+v_{[\lambda,\mu]\nu}+(\partial^\rho t_{\rho[\lambda\nu],\mu}-(\lambda\rightleftarrows\mu))\Big\}$$

$$-\frac{2}{3}(a_5+12a_6)\eta_{\nu[\lambda}\partial_{\mu]}\partial^\rho v_\rho+a_3(\varepsilon_{\lambda\mu\nu\rho}\Box a^\rho-\varepsilon_{\lambda\mu\rho\sigma}\partial_\nu\partial^\rho a^\sigma)$$

$$-\frac{1}{2}a_4(\varepsilon_{\lambda\nu\rho\sigma}\partial_\mu\partial^\rho a^\sigma-(\lambda\rightleftarrows\mu))+a_1\varepsilon_{\lambda\mu\nu\rho}\partial^\rho\partial^\sigma a_\sigma$$

$$+2\Big(\alpha+\frac{2a}{3}\Big)t_{\nu[\lambda\mu]}+2\Big(\beta-\frac{2a}{3}\Big)\eta_{\nu[\lambda}v_{\mu]}+\frac{2}{3}\Big(\gamma+\frac{3a}{3}\Big)\varepsilon_{\lambda\mu\nu\rho}a^\rho\,. \quad (2\cdot 12)$$

The last three terms come from $H^{(1)}_{\lambda\mu\nu}$ and stand for the mass terms of the torsion field.

The energy-momentum tensor $T_{\mu\nu}$ and the spin tensor $S_{\lambda\mu\nu}$ are taken to lowest order in $a_{\mu\nu}$ and $K_{\lambda\mu\nu}$; so they are independent of $a_{\mu\nu}$ and $K_{\lambda\mu\nu}$, and satisfy the ordinary conservation law,

$$\partial^\nu T_{\mu\nu}=0\,, \qquad (2\cdot 13)$$

and the Tetrode formula in special relativity,

$$2T_{[\mu\nu]}=\partial^\rho S_{\mu\nu\rho}\,. \qquad (2\cdot 14)$$

Since the translation gauge field $a_{\mu\nu}$ appears in (2·6) and (2·10) only through $G^{(1)}_{\mu\nu}$, the linearized gravitational field equations are invariant under the following gauge transformations:

$$h_{\mu\nu}\to h'_{\mu\nu}=h_{\mu\nu}+\partial_\mu\Lambda_\nu+\partial_\nu\Lambda_\mu\,, \qquad (2\cdot 15a)$$

$$a_{[\mu\nu]}\to a'_{[\mu\nu]}=a_{[\mu\nu]}+\omega_{\mu\nu}\,,\quad \omega_{\mu\nu}+\omega_{\nu\mu}=0\,, \qquad (2\cdot 15b)$$

where Λ_μ and $\omega_{\mu\nu}$ are small but otherwise arbitrary four and six functions, respectively: These are the linearized version of general coordinate and local Lorentz transformations, respectively. The transformation (2·15b) means that the antisymmetric part $a_{[\mu\nu]}$ is devoid of physical singnificance. Also, (2·15a) allows us to put the harmonic condition,

$$\partial^\nu \bar{h}_{\mu\nu}=0\,, \tag{2·16}$$

which we assume henceforth. The linearized Einstein tensor is then given by

$$G^{(1)}_{\mu\nu}=-\frac{1}{2}\Box\bar{h}_{\mu\nu}\,. \tag{2·17}$$

It is convenient to decompose (2·6) into the symmetric and antisymmetric parts;

$$2aG^{(1)}_{\mu\nu}-6\left(\alpha+\frac{2a}{3}\right)\partial^\rho t_{\rho(\mu\nu)}-2\left(\beta-\frac{2a}{3}\right)(\partial_{(\mu}v_{\nu)}-\eta_{\mu\nu}\partial^\rho v_\rho)=T_{(\mu\nu)}\,, \tag{2·18}$$

$$2\left(\alpha+\frac{2a}{3}\right)\partial^\rho t_{\rho[\mu\nu]}+2\left(\beta-\frac{2a}{3}\right)\partial_{[\mu}v_{\nu]}+\frac{2}{3}\left(\gamma+\frac{3a}{2}\right)\varepsilon_{\mu\nu\rho\sigma}\partial^\rho a^\sigma=-T_{[\mu\nu]}\,. \tag{2·19}$$

Taking trace of (2·18), we have

$$2aG^{(1)}+6\left(\beta-\frac{2a}{3}\right)\partial^\rho v_\rho=T \tag{2·20}$$

with $T=\eta^{\mu\nu}T_{\mu\nu}$. Both sides of (2·6) are divergenceless with respect to ν owing to (2·13), while the divergence of (2·10) with respect to ν yields (2·19) by virtue of (2·14). Therefore, Eqs. (2·6) and (2·10) give $(16+24)-(4+6)=30$ independent equations for $6+24=30$ independent field variables, i.e., for $\bar{h}_{\mu\nu}$ satisfying (2·16) and $K_{\lambda\mu\nu}$.

As is seen from (2·6) and (2·10), six parameters a_i ($i=1, 2, \cdots, 6$) enter the linearized gravitational field equations through the six combinations, $3a_2+4a_3$, $3a_2+2a_5$, a_4+a_5, $2a_3+a_4$, a_5+12a_6 and a_1+a_3: These combinations are not independent, however, and the condition,

$$3a_2+4a_3=3a_2+2a_5=a_4+a_5=2a_3+a_4=a_5+12a_6=a_1+a_3=0\,, \tag{2·21}$$

is satisfied when the parameters a_i are given by

$$a_2=\frac{4}{3}a_1,\quad a_3=-a_1,\quad a_4=-a_5=2a_1,\quad a_6=\frac{1}{6}a_1 \tag{2·22}$$

with a_1 arbitrary. For this choice the quadratic Lagrangian density L_F of (I. 3·23) becomes

$$\begin{aligned}L_F&=a_1\Big[(A_{ijmn}A^{ijmn})+\frac{4}{3}(B_{ijmn}B^{ijmn})-(C_{ijmn}C^{ijmn})\\&\quad+2(E_{ij}E^{ij}-I_{ij}I^{ij})+\frac{1}{6}F^2\Big]\\&=a_1(F_{ijmn}F^{mnij}-4F_{ij}F^{ji}+F^2)\,.\end{aligned} \tag{2·23}$$

Thus, the variation of the action A_F,

$$A_F \equiv \int d^4xe\,(F_{ijmn}F^{mnij} - 4F_{ij}F^{ji} + F^2), \tag{2·24}$$

identically vanishes in the *weak* field approximation: Namely, we have

$$\delta A_F \equiv 0 \tag{2·25}$$

in the weak field approximation. This property of the Lorentz gauge field strength F_{ijmn} is to be compared to the Bach-Lanczos identity[3] in the Riemann space-time,

$$\delta \int d^4xe\,[R_{\mu\nu\rho\sigma}(\{\ \})R^{\mu\nu\rho\sigma}(\{\ \}) - 4R_{\mu\nu}(\{\ \})R^{\mu\nu}(\{\ \}) + R(\{\ \})^2] \equiv 0. \tag{2·26}$$

[Note added in proof: It can be shown that the identity (2·25) does hold *exactly*. See V of this series.]

§ 3. The Newtonian approximation

Using (2·18) and (2·20) in the divergence of (2·10),

$$\partial^\rho P_{\rho(\mu\nu)} = -\frac{1}{2}\partial^\rho S_{\rho(\mu\nu)}, \tag{3·1}$$

we obtain the fourth-order field equation for $\bar{h}_{\mu\nu}$,

$$2aG^{(1)}_{\mu\nu} - \frac{2a}{(m_2)^2}\left\{\Box G^{(1)}_{\mu\nu} - \frac{1}{3}(\eta_{\mu\nu}\Box - \partial_\mu\partial_\nu)G^{(1)}\right\}$$
$$-\frac{2a}{(m_0)^2}\frac{1}{3}(\eta_{\mu\nu}\Box - \partial_\mu\partial_\nu)G^{(1)} = T^{(\mathrm{eff})}_{\mu\nu}, \tag{3·2}$$

where m_2 and m_0 are given by

$$m_2 = \left\{-\frac{2a(\alpha + 2a/3)}{\alpha(3a_2 + 2a_5)}\right\}^{1/2}, \tag{3·3a}$$

$$m_0 = \left\{\frac{2a(\beta - 2a/3)}{\beta(a_5 + 12a_6)}\right\}^{1/2}, \tag{3·3b}$$

and $T^{(\mathrm{eff})}_{\mu\nu}$ is defined by

$$T^{(\mathrm{eff})}_{\mu\nu} = T^{(\mathrm{sym})}_{\mu\nu} - \frac{a_5 + 12a_6}{9(\beta - 2a/3)}(\eta_{\mu\nu}\Box - \partial_\mu\partial_\nu)T$$
$$-\frac{3a_2 + 2a_5}{3(\alpha + 2a/3)}\left\{\Box T_{(\mu\nu)} - \frac{1}{3}(\eta_{\mu\nu}\Box - \partial_\mu\partial_\nu)T + \partial^\rho\partial^\sigma\partial_{(\mu}S_{\nu)\rho\sigma}\right\} \tag{3·4}$$

with $T^{(\mathrm{sym})}_{\mu\nu}$ the symmetrized energy-momentum tensor in special relativity,

$$T^{(\mathrm{sym})}_{\mu\nu} = T_{\mu\nu} - \frac{1}{2}\partial^\rho(S_{\mu\nu\rho} + S_{\rho\mu\nu} + S_{\rho\nu\mu}). \tag{3·5}$$

In the harmonic gauge of (2·16), the field equation for $\bar{h}_{\mu\nu}$ given by (3·2) becomes

$$a\Box\bar{h}_{\mu\nu}-\frac{a\Box}{(m_2)^2}\left\{\Box\bar{h}_{\mu\nu}-\frac{1}{3}(\eta_{\mu\nu}\Box-\partial_\mu\partial_\nu)\bar{h}\right\}$$
$$-\frac{a\Box}{3(m_0)^2}(\eta_{\mu\nu}\Box-\partial_\mu\partial_\nu)\bar{h}=-T^{(\mathrm{eff})}_{\mu\nu}, \tag{3·6}$$

which contains the fourth-order differential operators like $\Box^2$ and $\Box\partial_\mu\partial_\nu$ operated on $\bar{h}_{\mu\nu}$. Here $\bar{h}=\eta^{\mu\nu}\bar{h}_{\mu\nu}$.

Let us consider the gravitational field around a static, spinless source located at the origin, for which $T_{\mu\nu}$ is given by

$$T_{\mu\nu}=\begin{cases}M\delta^3(\boldsymbol{x}), & \mu=\nu=0,\\ 0 & , \quad \text{otherwise}\end{cases} \tag{3·7}$$

with $\boldsymbol{x}=(x^1,x^2,x^3)$ and $S_{\lambda\mu\nu}$ is vanishing. It then follows from (3·4) that

$$\left.\begin{aligned}
&T^{(\mathrm{eff})}_{00}=M\delta^3(\boldsymbol{x})-\left\{\frac{a_5+12a_6}{9(\beta-2a/3)}+\frac{2(3a_2+2a_5)}{9(\alpha+2a/3)}\right\}M\Delta\delta^3(\boldsymbol{x}),\\
&T^{(\mathrm{eff})}_{0\alpha}=0,\\
&T^{(\mathrm{eff})}_{\alpha\beta}=\left\{\frac{a_5+12a_6}{9(\beta-2a/3)}-\frac{3a_2+2a_5}{9(\alpha+2a/3)}\right\}M(\delta_{\alpha\beta}\Delta-\partial_\alpha\partial_\beta)\delta^3(\boldsymbol{x}),
\end{aligned}\right\} \tag{3·8}$$

where α and β run over 1, 2 and 3. Using (3·8) in (3·6), we get

$$\phi=-\frac{1}{2}h_{00}=-\frac{M}{16\pi ar}\left\{1+\frac{4}{3}R_2\exp(-m_2r)+\frac{1}{6}R_0\exp(-m_0r)\right\} \tag{3·9}$$

with $r=|\boldsymbol{x}|$, where R_2 and R_0 are given by

$$R_2=\frac{(3a_2+2a_5)(m_2)^2}{2a}=-\frac{\alpha+2a/3}{\alpha}, \tag{3·10a}$$

$$R_0=\frac{(a_5+12a_6)(m_0)^2}{a}=\frac{2(\beta-2a/3)}{\beta}. \tag{3·10b}$$

Since the world line of test particles is the geodesics of the metric $g_{\mu\nu}$ as has been shown in II, the ϕ of (3·9) is the gravitational potential in Poincaré gauge theory.

The second and third terms of ϕ are the Yukawa potentials with the range $1/m_2$ and $1/m_0$, respectively.*) We must choose the parameters in such a way that

*) As will be shown in §5, the weak gravitational field $h_{\mu\nu}$ is represented as a linear combination of three classes of fields; the massless graviton field of spin 2, a field with mass m_2 and spin 2 and a field with mass m_0 and spin 0. (See Eq. (5·11).) The $(1/r)$-term of ϕ comes from the massless graviton exchange, whereas the two Yukawa terms are due to exchange of massive particles of spin 2 and spin 0, respectively.

the $(1/r)$-term of ϕ coincides with the Newtonian potential, $\phi=-GM/r$, and that the Yukawa potentials are of sufficiently short range. It is convenient to treat two cases separately according as the parameters, a_i $(i=1, 2, \cdots, 6)$, are finite or not.

(i) *The case that the parameters a_i are all finite*

When m_2 and m_0 vanish, the strength of the Yukawa potentials vanishes because of the factors, $(m_2)^2$ and $(m_0)^2$, in R_2 and R_0. Therefore, the first term of ϕ, $-M/16\pi ar$, must coincide with the Newtonian potential, irrespectively of whether m_2 and m_0 are vanishing or not. We thus choose the parameter a as

$$a=1/2\kappa\,, \qquad \kappa=8\pi G\,. \tag{3·11}$$

The remaining two terms of ϕ are possible corrections to the Newtonian potential, and they should be sufficiently small compared with the first term for macroscopic distances. So we choose the remaining parameters as follows: (1) The masses, m_2 and m_0, are large enough to ensure that the exponential factors of the Yukawa potentials rapidly die out, or (2) the coefficients, R_2 and R_0, are sufficiently small. In view of (3·10a and b), we see that the Yukawa potentials are attractive or repulsive according as $-\alpha(\alpha+2a/3)$ and $\beta(\beta-2a/3)$ are positive or negative, respectively. It should be mentioned that, for example, the massive spin-2 particle with $\alpha(\alpha+2a/3)>0$ yields the repulsive force and becomes ghost.*)

The case of $a=0$ deserves special care, since it is assumed in (3·9) that the parameter a is nonvanishing. By letting the parameter a tend to zero in (3·6) and then solving the resulting equation with respect to h_{00}, we have the following gravitational potential,

$$\phi=-\frac{1}{2}h_{00}=\left(\frac{1}{\alpha}+\frac{1}{4\beta}\right)\frac{M}{18\pi r}-\left\{\frac{1}{3a_2+2a_5}+\frac{1}{4(a_5+12a_6)}\right\}\frac{Mr}{12\pi}\,, \tag{3·12}$$

which contains the *linearly rising potential* in addition to the $(1/r)$-potential:[4),5)] This linearly rising potential stands for the characteristic feature of the case of $a=0$ with a_i finite. As the basic postulate we demand that the space-time around a static localized source should be asymptotically flat. This requires that the linearly rising potential should be vanishing,

$$\frac{1}{3a_2+2a_5}+\frac{1}{4(a_5+12a_6)}=0\,. \tag{3·13}$$

Furthermore, since $h=\eta^{\mu\nu}h_{\mu\nu}$ is given by

*) The massive particles of spin 2 and spin 0 are normal particles with positive-definite energy and positive mass, when the parameters satisfy the conditions,

$3a_2+2a_5>0$, $\alpha(\alpha+2a/3)<0$ for the spin-2 particle,

$a_5+12a_6>0$, $\beta(\beta-2a/3)>0$ for the spin-0 particle.

This will be shown in a next paper (see §4 of IV).

$$h = \frac{M}{6\pi\beta r} - \frac{Mr}{4\pi(a_5 + 12a_6)}, \tag{3·14}$$

we must put

$$\frac{1}{a_5 + 12a_6} = 0 . \tag{3·15}$$

Equations (3·13) and (3·15) contradict the assumption that the parameters a_i are finite numerical constants. Consequently, we conclude that *when the parameters a_i are finite the case of $a=0$ should be disregarded.**)

(ii) *The case of $a_i = \infty$ $(i = 1, 2, \cdots, 6)$*

This case is just New General Relativity[2] as was shown in § 4 of II. Since m_2 and m_0 are vanishing with R_2 and R_0 finite, all the three terms of the gravitational potential (3·9) behave like $1/r$, giving the following gravitational potential,

$$\phi = \left(\frac{1}{\alpha} + \frac{1}{4\beta}\right)\frac{M}{18\pi r}, \tag{3·16}$$

which coincides with the Newtonian potential if we put

$$\alpha + 4\beta + 9\alpha\beta\kappa = 0. \tag{3·17}$$

This is just the Newtonian approximation condition in New General Relativity. (See also (II. 4·22).)

Since we have studied the latter case in Ref. 2), we shall assume henceforth throughout this series (unless otherwise stated explicitly) that the parameter a is given by (3·11).

§ 4. Propagation of the torsion field

Let us now derive the field equations for the torsion field by using Eqs. (2·18) and (2·19) in the linearized field equation for the Lorentz gauge field (2·10) with (2·12). For this purpose, it is more convenient to rewrite (2·10) by decomposing $P_{\lambda\mu\nu}$ into three irreducible parts, $P^{(t)}_{\lambda\mu\nu}$, $P_\mu{}^{(v)}$ and $P_\mu{}^{(a)}$, in the same manner as the spin tensor (see the Appendix of II for the decomposition scheme of the spin tensor). Denoting the irreducible parts of the spin tensor as $S^{(t)}_{\lambda\mu\nu}$, $S_\mu{}^{(v)}$ and $S_\mu{}^{(a)}$, we have

$$\frac{1}{2}(3a_2 + 4a_3)\Big\{\Box t_{\lambda\mu\nu} - (\partial^\rho t_{\rho(\lambda\nu),\mu} + \partial^\rho t_{\rho(\mu\nu),\lambda} - 2\partial^\rho t_{\rho(\lambda\mu),\nu})$$

*) This conclusion is due to our assumption that the parameters a_i are finite numerical constants independent of space-time points. If it happens that the parameters a_i are finite only in the inside of a bounded region around the source due to some, yet unknown mechanism,[5] then the gravitational potential (3·12) is compatible with asymptotic flatness of the space-time.

$$-\frac{2}{3}(\partial^\rho t_{\rho[\lambda\nu],\mu}+\partial^\rho t_{\rho[\mu\nu],\lambda})$$

$$-\frac{1}{4}(\eta_{\lambda\nu}\partial^\rho\partial^\sigma t_{\rho\sigma\mu}+\eta_{\mu\nu}\partial^\rho\partial^\sigma t_{\rho\sigma\lambda}-2\eta_{\lambda\mu}\partial^\rho\partial^\sigma t_{\rho\sigma\nu})\Big\}$$

$$-(3a_2+2a_5)\Big\{\frac{1}{4}(G^{(1)}_{\lambda\nu},{}_\mu+G^{(1)}_{\mu\nu},{}_\lambda-2G^{(1)}_{\lambda\mu},{}_\nu)$$

$$-\frac{1}{12}(\eta_{\lambda\nu}\partial_\mu G^{(1)}+\eta_{\mu\nu}\partial_\lambda G^{(1)}-2\eta_{\lambda\mu}\partial_\nu G^{(1)})$$

$$-\frac{1}{2}(\partial^\rho t_{\rho(\lambda\nu),\mu}+\partial^\rho t_{\rho(\mu\nu),\lambda}-2\partial^\rho t_{\rho(\lambda\mu),\nu})$$

$$+\frac{1}{6}(\partial^\rho t_{\rho[\lambda\nu],\mu}+\partial^\rho t_{\rho[\mu\nu],\lambda})\Big\}$$

$$-\frac{1}{18}(a_4+a_5)\Big\{\eta_{\lambda\nu}\Big(\Box v_\mu-\partial_\mu\partial_\rho v^\rho-\frac{3}{2}\partial^\rho\partial^\sigma t_{\rho\sigma\mu}\Big)$$

$$+\eta_{\mu\nu}\Big(\Box v_\lambda-\partial_\lambda\partial_\rho v^\rho-\frac{3}{2}\partial^\rho\partial^\sigma t_{\rho\sigma\lambda}\Big)$$

$$-2\eta_{\lambda\mu}\Big(\Box v_\nu-\partial_\nu\partial_\rho v^\rho-\frac{3}{2}\partial^\rho\partial^\sigma t_{\rho\sigma\nu}\Big)$$

$$-6(\partial^\rho t_{\rho[\lambda\nu],\mu}+\partial^\rho t_{\rho[\mu\nu],\lambda})$$

$$-3(v_{\lambda,\mu\nu}+v_{\mu,\lambda\nu}-2v_{\nu,\lambda\mu})\Big\}$$

$$+\frac{1}{4}(2a_3+a_4)(\varepsilon_{\lambda\nu\rho\sigma}\partial_\mu\partial^\rho a^\sigma+\varepsilon_{\mu\nu\rho\sigma}\partial_\lambda\partial^\rho a^\sigma)-\frac{3}{2}\Big(\alpha+\frac{2a}{3}\Big)t_{\lambda\mu\nu}$$

$$=-\frac{1}{2}S^{(t)}_{\lambda\mu\nu}, \tag{4·1}$$

$$\frac{2}{3}(a_4+a_5)\Big(\Box v_\mu-\partial_\mu\partial_\rho v^\rho-\frac{3}{2}\partial^\rho\partial^\sigma t_{\rho\sigma\mu}\Big)-3\Big(\beta-\frac{2a}{3}\Big)v_\mu$$

$$-(a_5+12a_6)\Big(\frac{1}{2}\partial_\mu G^{(1)}-\partial_\mu\partial_\rho v^\rho\Big)=-\frac{1}{2}S_\mu{}^{(v)}, \tag{4·2}$$

$$\frac{2}{3}(2a_3+a_4)\Big(\Box a_\lambda+\frac{2}{3}\varepsilon_{\lambda\mu\nu\rho}\partial^\rho\partial_\sigma t^{\sigma[\mu\nu]}\Big)+\frac{2}{3}(3a_1+a_3-a_4)\partial_\lambda\partial_\rho a^\rho$$

$$+\frac{4}{3}\Big(\gamma+\frac{3a}{2}\Big)a_\lambda=-\frac{1}{2}S_\lambda{}^{(a)}. \tag{4·3}$$

A. *Reduction of Eq. (4·2)*

Using (2·20) in the divergence of (4·2), we get

$$[\Box-(m_0)^2]\sigma=j^{(\sigma)} \tag{4·4}$$

with $\sigma\equiv\partial^\rho v_\rho$, and

$$m_0=\left\{\frac{2a(\beta-2a/3)}{\beta(a_5+12a_6)}\right\}^{1/2}, \tag{4·5}$$

$$j^{(\sigma)}=\frac{1}{6\beta}\Box T-\frac{a}{3\beta(a_5+12a_6)}\partial^\rho S_\rho{}^{(v)}. \tag{4·6}$$

The field σ describes a massive particle having spin$^{\text{parity}}$ $J^P=0^+$. The mass of the σ coincides with m_0 defined by (3·3b), showing that the Yukawa potential with the range $1/m_0$ in the gravitational potential (3·9) is due to σ-exchange.

By virtue of (4·4) and the divergence of (2·18), Eq. (4·2) can be rewritten as

$$[\Box-m_v{}^2]\tilde{v}_\mu=j_\mu{}^{(v)}, \tag{4·7}$$

where $\tilde{v}_\mu$, m_v and $j_\mu{}^{(v)}$ are defined by

$$\tilde{v}_\mu=v_\mu-(m_0)^{-2}\partial_\mu\sigma, \tag{4·8}$$

$$m_v=\left\{\frac{9(\alpha+2a/3)(\beta-2a/3)}{2(\alpha+\beta)(a_4+a_5)}\right\}^{1/2}, \tag{4·9}$$

$$j_\mu{}^{(v)}=-\frac{m_v{}^2}{6(\beta-2a/3)}S_\mu{}^{(v)}+\frac{1}{2(\alpha+\beta)}\partial^\rho\partial^\sigma S_{\mu\rho\sigma}$$
$$+\frac{1}{6\beta}\left(\frac{m_v}{m_0}\right)^2\partial_\mu T-(m_0)^{-2}\partial_\mu j^{(\sigma)}. \tag{4·10}$$

The field $\tilde{v}_\mu$ is *divergenceless in vacuum* by virtue of (4·4) (see also the Appendix), so it describes a massive particle with $J^P=1^-$.

B. *Reduction of Eq. (4·3)*

The divergence of (4·3) gives

$$[\Box-m_B{}^2]B=j^{(B)} \tag{4·11}$$

with $B\equiv\partial^\rho a_\rho$, and

$$m_B=\left\{-\frac{2(\gamma+3a/2)}{3(a_1+a_3)}\right\}^{1/2}, \tag{4·12}$$

$$j^{(B)}=-\frac{1}{4(a_1+a_3)}\partial^\rho S_\rho{}^{(a)}. \tag{4·13}$$

The field B describes a massive particle with $J^P=0^-$.

Using (2·19) and (4·11) in (4·3), we have

$$[\Box-m_a{}^2]\tilde{a}_\mu=j_\mu{}^{(a)} \tag{4·14}$$

with $\tilde{a}_\mu$, m_a and $j_\mu{}^{(a)}$ defined by

$$\tilde{a}_\mu = a_\mu - (m_B)^{-2}\partial_\mu B\,, \tag{4·15}$$

$$m_a = \left\{ -\frac{2(\alpha+2a/3)(\gamma+3a/2)}{(\alpha-4\gamma/9)(2a_3+a_4)} \right\}^{1/2}, \tag{4·16}$$

$$j_\mu{}^{(a)} = \frac{3m_a{}^2}{8(\gamma+3a/2)} S_\mu{}^{(a)} + \frac{1}{3(\alpha-4\gamma/9)} \varepsilon_{\mu\rho\sigma\tau}\partial^\rho T^{[\sigma\tau]} - (m_B)^{-2}\partial_\mu j^{(B)}. \tag{4·17}$$

The field $\tilde{a}_\mu$ is *divergenceless in vacuum* (see the Appendix), and therefore it describes a massive particle with $J^P=1^+$.

C. *Reduction of Eq. (4·1)*

Use of (2·18) in the divergence of (4·1) with respect to ν gives after a straightforward calculation,

$$[\Box - (m_2)^2]\chi_{\mu\nu} = j^{(\chi)}_{\mu\nu}, \tag{4·18}$$

where $\chi_{\mu\nu}$, m_2 and $j^{(\chi)}_{\mu\nu}$ are given by

$$\chi_{\mu\nu} = \partial^\rho t_{\rho(\mu\nu)} + \frac{\beta-2a/3}{3(\alpha+2a/3)}\partial_{(\mu}\tilde{v}_{\nu)}\,, \tag{4·19}$$

$$m_2 = \left\{ -\frac{2a(\alpha+2a/3)}{\alpha(3a_2+2a_5)} \right\}^{1/2}, \tag{4·20}$$

$$\begin{aligned} j^{(\chi)}_{\mu\nu} = {} & \frac{(m_2)^2}{6(\alpha+2a/3)}\left(\partial^\rho S^{(t)}_{\mu\nu\rho} - \frac{1}{3}\partial_{(\mu}S^{(v)}_{\nu)}\right) - \frac{a_5+12a_6}{18\alpha(3a_2+2a_5)}\partial_\mu\partial_\nu T \\ & - \frac{1}{6\alpha}\left\{\Box T_{(\mu\nu)} - \frac{1}{3}(\eta_{\mu\nu}\Box - \partial_\mu\partial_\nu)T + \frac{2a}{3(\alpha+2a/3)}\partial_{(\mu}\partial^\rho\partial^\sigma S_{\nu)\rho\sigma}\right\} \\ & - \frac{\beta-2a/3}{3(\alpha+2a/3)}(m_0)^{-2}\partial_\mu\partial_\nu j^{(\sigma)}. \end{aligned} \tag{4·21}$$

The field $\chi_{\mu\nu}$ is symmetric by definition, and furthermore it is *traceless and divergenceless in vacuum* (see the Appendix): Consequently, the $\chi_{\mu\nu}$ describes a massive particle with $J^P=2^+$. The mass of $\chi_{\mu\nu}$ coincides with m_2 defined by (3·3a), and so the Yukawa potential with the range $1/m_2$ in the gravitational potential (3·9) is due to χ-exchange.

The field $\chi_{\mu\nu}$ is the "traceless" and "divergenceless" part of $\partial^\rho t_{\rho(\mu\nu)}$: As for the antisymmetric field $\partial^\rho t_{\rho[\mu\nu]}$, Eq. (2·19) shows that it is represented by v_μ and a_μ, and so we need not be concerned with it.

Next, let us extract the "traceless" and "divergenceless" part of $t_{\lambda\mu\nu}$. This is performed by the following three steps: (i) Eliminate $G^{(1)}_{\mu\nu}$ and $G^{(1)}$ in (4·1) by using (2·18), (ii) represent $\partial^\rho t_{\rho(\mu\nu)}$, $\partial^\rho t_{\rho[\mu\nu]}$ and v_μ in terms of $\chi_{\mu\nu}$, $\tilde{v}_\mu$ and $\tilde{a}_\mu$, and finally, (iii) rewrite a resulting expression for (4·1) by using the Klein-Gordon equations for $\chi_{\mu\nu}$, $\tilde{v}_\mu$ and $\tilde{a}_\mu$ so that it becomes a Klein-Gordon equation. As a result, we have

$$[\Box - m_t^2]\tilde{t}_{\lambda\mu\nu} = j^{(t)}_{\lambda\mu\nu}\,, \tag{4·22}$$

where $\tilde{t}_{\lambda\mu\nu}$, m_t and $j^{(t)}_{\lambda\mu\nu}$ are defined by

$$\tilde{t}_{\lambda\mu\nu} = t_{\lambda\mu\nu} - (m_2)^{-2}(\chi_{\lambda\nu,\,\mu} + \chi_{\mu\nu,\,\lambda} - 2\chi_{\lambda\mu,\,\nu})$$
$$+ \frac{\beta - 2a/3}{\alpha + 2a/3}\left\{\frac{1}{6}(\eta_{\lambda\nu}\tilde{v}_\mu + \eta_{\mu\nu}\tilde{v}_\lambda - 2\eta_{\lambda\mu}\tilde{v}_\nu) + m_v^{-2}(\tilde{v}_{\nu,\,\lambda\mu} - \tilde{v}_{(\lambda,\,\mu)\nu})\right\}$$
$$+ \frac{\gamma + 3a/2}{3(\alpha + 2a/3)} m_a^{-2}(\varepsilon_{\lambda\nu\rho\sigma}\partial_\mu + \varepsilon_{\mu\nu\rho\sigma}\partial_\lambda)\partial^\rho\tilde{a}^\sigma\,, \tag{4·23}$$

$$m_t = \left\{\frac{3(\alpha + 2a/3)}{3a_2 + 4a_3}\right\}^{1/2}, \tag{4·24}$$

$$j^{(t)}_{\lambda\mu\nu} = \frac{1}{3a_2 + 4a_3}\Big[-S^{(t)}_{\lambda\mu\nu} + \frac{3a_2 + 2a_5}{4a}(T_{(\lambda\nu),\,\mu} + T_{(\mu\nu),\,\lambda} - 2T_{(\lambda\mu),\,\nu})$$
$$- \frac{9a_2 + 8a_3 - 2a_4}{6(\alpha + 2a/3)}(T_{[\lambda\nu],\,\mu} + T_{[\mu\nu],\,\lambda})$$
$$- \frac{2a_2 + a_5 - 4a_6}{8a}(\eta_{\lambda\nu}T_{,\,\mu} + \eta_{\mu\nu}T_{,\,\lambda} - 2\eta_{\lambda\mu}T_{,\,\nu})$$
$$- \frac{1}{12}(\eta_{\lambda\nu}S_\mu^{(v)} + \eta_{\mu\nu}S_\lambda^{(v)} - 2\eta_{\lambda\mu}S_\nu^{(v)})\Big]$$
$$+ \frac{1}{12(\alpha + 2a/3)}\partial^\rho\partial^\sigma(\eta_{\lambda\nu}S_{\mu\rho\sigma} + \eta_{\mu\nu}S_{\lambda\rho\sigma} - 2\eta_{\lambda\mu}S_{\nu\rho\sigma})$$
$$- (m_2)^{-2}(j^{(\chi)}_{\lambda\nu,\,\mu} + j^{(\chi)}_{\mu\nu,\,\lambda} - 2j^{(\chi)}_{\lambda\mu,\,\nu})$$
$$- \frac{\beta - 2a/3}{\alpha + 2a/3}\left[\frac{1}{m_v^2}(j^{(v)}_{(\lambda,\,\mu)\nu} - j^{(v)}_{\nu,\,\lambda\mu}) + \frac{1}{6(m_0)^2}(\eta_{\lambda\nu}\partial_\mu + \eta_{\mu\nu}\partial_\lambda - 2\eta_{\lambda\mu}\partial_\nu)j^{(\sigma)}\right]$$
$$+ \frac{\gamma + 3a/2}{3m_a^2(\alpha + 2a/3)}(\varepsilon_{\lambda\nu\rho\sigma}\partial_\mu\partial^\rho j^{(a)\sigma} + \varepsilon_{\mu\nu\rho\sigma}\partial_\lambda\partial^\rho j^{(a)\sigma})\,. \tag{4·25}$$

The field $\tilde{t}_{\lambda\mu\nu}$ is symmetric with respect to λ and μ, and satisfies the cyclic identity,

$$\tilde{t}_{\lambda\mu\nu} + \tilde{t}_{\mu\nu\lambda} + \tilde{t}_{\nu\lambda\mu} = 0\,. \tag{4·26}$$

As for the trace and divergence, it is *traceless and divergenceless in vacuum* (see the Appendix): Therefore, the field $\tilde{t}_{\lambda\mu\nu}$ involves five independent degrees of freedom, and describes a massive particle with $J^P = 2^-$. For a particle at rest, we have

$$\tilde{t}_{0\alpha\beta} = \tilde{t}_{00\alpha} = \tilde{t}_{000} = 0\,, \tag{4·27}$$

and the space-components $t_{\alpha\beta\gamma}$ $(\alpha, \beta, \gamma = 1, 2, 3)$ are expressed in terms of five independent quantities, for example, $\tilde{t}_{311}$, $\tilde{t}_{312}$, $\tilde{t}_{321}$, $\tilde{t}_{331}$ and $\tilde{t}_{332}$.*)

We have thus seen that the torsion field can be decomposed into the *six independent fields*, $\tilde{t}_{\lambda\mu\nu}$, $\chi_{\mu\nu}$, $\tilde{v}_\mu$, $\tilde{a}_\mu$, $\sigma=\partial^\rho v_\rho$ and $B=\partial^\rho a_\rho$, all of which obey the Klein-Gordon equations: Each of these fields is *irreducible* in the sense that it is traceless and divergenceless in vacuum, thus describing a massive particle with definite spin-parity. It is still to be clarified, however, whether these six classes of the irreducible fields are *normal* (with positive mass and positive energy) or abnormal (with imaginary mass and/or negative energy). The energy of these irreducible fields will be calculated in a next paper of this series (see § 4 of IV).

§ 5. The massless graviton field

We shall now extract from $\bar{h}_{\mu\nu}$ the massless graviton field. For this purpose, we write (2·18) in terms of the irreducible torsion fields $\chi_{\mu\nu}$ and σ,

$$2aG^{(1)}_{\mu\nu}-6\left(\alpha+\frac{2a}{3}\right)\chi_{\mu\nu}+2\left(\beta-\frac{2a}{3}\right)(\eta_{\mu\nu}-(m_0)^{-2}\partial_\mu\partial_\nu)\sigma=T_{(\mu\nu)}\,. \tag{5·1}$$

Let us define $\bar{h}^*_{\mu\nu}$ by

$$\bar{h}^*_{\mu\nu}=\bar{h}_{\mu\nu}+\frac{6(\alpha+2a/3)}{a(m_2)^2}\chi_{\mu\nu}-\frac{2(\beta-2a/3)}{a(m_0)^2}(\eta_{\mu\nu}-(m_0)^{-2}\partial_\mu\partial_\nu)\sigma\,, \tag{5·2}$$

and rewrite (5·1) by using $\bar{h}^*_{\mu\nu}$ in place of $\bar{h}_{\mu\nu}$. It follows from (2·8) and the above definition of $\bar{h}^*_{\mu\nu}$ that the fields $\chi_{\mu\nu}$ and σ appear in (5·1) in the forms, $\{\Box-(m_2)^2\}\chi_{\mu\nu}$, $\{\Box-(m_0)^2\}\sigma$ and $\partial^\rho\chi_{\rho\nu}$, showing that $\chi_{\mu\nu}$ and σ can be eliminated by means of their field equations, (4·4) and (4·18): The terms with $\partial^\rho\chi_{\rho\nu}$ are eliminated by using (A·8). We then get the following field equation for $\bar{h}^*_{\mu\nu}$:

$$\begin{aligned}
&\Box\bar{h}^*_{\mu\nu}-\partial^\rho(\partial_\mu\bar{h}^*_{\nu\rho}+\partial_\nu\bar{h}^*_{\mu\rho})+\eta_{\mu\nu}\partial^\rho\partial^\sigma\bar{h}^*_{\rho\sigma}\\
&=-\frac{1}{a}T^{(\mathrm{sym})}_{\mu\nu}-\frac{a_5+12a_6}{6a^2}(\eta_{\mu\nu}\Box-\partial_\mu\partial_\nu)T\\
&\quad+\frac{3a_2+2a_5}{2a^2}\left\{\Box T_{(\mu\nu)}-\frac{1}{3}(\eta_{\mu\nu}\Box-\partial_\mu\partial_\nu)T+\partial^\rho\partial^\sigma\partial_{(\mu}S_{\nu)\rho\sigma}\right\}\\
&\quad+\frac{2(\beta-2a/3)}{a(m_0)^2}\left\{\frac{1}{(m_0)^2}-\frac{1}{(m_2)^2}\right\}(\eta_{\mu\nu}\Box-\partial_\mu\partial_\nu)j^{(\sigma)}\,,
\end{aligned} \tag{5·3}$$

where $T^{(\mathrm{sym})}_{\mu\nu}$ is given by (3·5). The source term of (5·3) can be simplified by introducing the new field variable $\phi_{\mu\nu}$ by

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*) For the spin component J_3 along the 3rd axis, $J_3=\pm2$ for $\{\tilde{t}_{311}\pm(i/2)(\tilde{t}_{312}+\tilde{t}_{321})\}$, $J_3=\pm1$ for $(\tilde{t}_{331}\pm i\tilde{t}_{332})$ and $J_3=0$ for $(\tilde{t}_{312}-\tilde{t}_{321})$.

$$\phi_{\mu\nu}\equiv\bar{h}^{*}_{\mu\nu}-\frac{3a_2+2a_5}{2a^2}\left(T_{(\mu\nu)}-\frac{1}{6}\eta_{\mu\nu}T\right)+\frac{a_5+12a_6}{12a^2}\eta_{\mu\nu}T$$
$$-\frac{\beta-2a/3}{a(m_0)^2}\left\{\frac{1}{(m_0)^2}-\frac{1}{(m_2)^2}\right\}\eta_{\mu\nu}j^{(\sigma)}$$
$$=\bar{h}_{\mu\nu}-\frac{3\alpha(3a_2+2a_5)}{a^2}\chi^{*}_{\mu\nu}-\frac{\beta(a_5+12a_6)}{a^2}(\eta_{\mu\nu}-(m_0)^{-2}\partial_\mu\partial_\nu)\sigma$$
$$-\frac{\beta-2a/3}{a(m_0)^2}\eta_{\mu\nu}j^{(\sigma)} \tag{5·4}$$

with $\chi^{*}_{\mu\nu}$ defined by

$$\chi^{*}_{\mu\nu}\equiv\chi_{\mu\nu}+\frac{1}{6\alpha}T_{(\mu\nu)}-\frac{1}{36\alpha}\left(1+\frac{a_5+12a_6}{3a_2+2a_5}\right)\eta_{\mu\nu}T$$
$$+\frac{\beta-2a/3}{6(\alpha+2a/3)(m_0)^2}\eta_{\mu\nu}j^{(\sigma)}. \tag{5·5}$$

In fact, it can be shown that the field $\phi_{\mu\nu}$ obeys

$$\Box\phi_{\mu\nu}-\partial^\rho(\partial_\mu\phi_{\nu\rho}+\partial_\nu\phi_{\mu\rho})+\eta_{\mu\nu}\partial^\rho\partial^\sigma\phi_{\rho\sigma}=-\frac{1}{a}T^{(\mathrm{sym})}_{\mu\nu}$$
$$=-2\kappa T^{(\mathrm{sym})}_{\mu\nu}, \tag{5·6}$$

where we have used (3·11) in the last term. This equation (5·6) is just the linearized Einstein equation, and so *we can interpret the field $\phi_{\mu\nu}$ as the massless graviton field of spin* 2.

It follows from the field equation for $\chi_{\mu\nu}$ that the field $\chi^{*}_{\mu\nu}$ obeys the following Fierz-Pauli equation:

$$\Box\chi^{*}_{\mu\nu}-\partial^\rho(\partial_\mu\chi^{*}_{\nu\rho}+\partial_\nu\chi^{*}_{\mu\rho})+\eta_{\mu\nu}\partial^\rho\partial^\sigma\chi^{*}_{\rho\sigma}-(m_2)^2\chi^{*}_{\mu\nu}$$
$$=-\frac{(m_2)^2}{6(\alpha+2a/3)}\left\{T^{(\mathrm{sym})}_{\mu\nu}+\frac{2a}{3\alpha}T_{(\mu\nu)}\right\}+\frac{(m_0)^2}{36\alpha}\left\{1+\frac{a_5+12a_6}{3a_2+2a_5}\right\}\eta_{\mu\nu}T$$
$$-\frac{\beta(a_5+12a_6)}{6\alpha(3a_2+2a_5)}\eta_{\mu\nu}j^{(\sigma)}, \tag{5·7}$$

in which the source term does not involve second-order derivatives of $T_{\mu\nu}$ but only $\Box T$ through $j^{(\sigma)}$. By rewriting the Klein-Gordon equation for $\chi_{\mu\nu}$, (4·18), into a form similar to (5·7), on the other hand, we see that the source term of $\chi_{\mu\nu}$ contains $\Box T_{\mu\nu}$ as well as $\Box^2 T$. Since higher derivative terms in the source are dangerous in quantized theory, the field $\chi^{*}_{\mu\nu}$ seems to be preferable to $\chi_{\mu\nu}$ as the dynamical field variable.

The field equation (5·6) is invariant under gauge transformations,

$$\phi_{\mu\nu}\to\phi'_{\mu\nu}=\phi_{\mu\nu}+\partial_\mu\Lambda_\nu+\partial_\nu\Lambda_\mu-\eta_{\mu\nu}\partial^\rho\Lambda_\rho \tag{5·8}$$

with Λ_μ small but otherwise arbitrary four functions. Owing to this gauge freedom, we can set the divergenceless condition,

$$\partial^\nu \phi_{\mu\nu}=0. \tag{5·9}$$

Then Eq. (5·6) becomes the d'Alembert equation,

$$\Box \phi_{\mu\nu}=-2\kappa T_{\mu\nu}^{(\mathrm{sym})}. \tag{5·10}$$

The weak gravitational field $\bar{h}_{\mu\nu}$ is represented as a linear sum of the three classes of fields, $\phi_{\mu\nu}$, $\chi^*_{\mu\nu}$ and σ,

$$\bar{h}_{\mu\nu}=\phi_{\mu\nu}+\frac{3\alpha(3a_2+2a_5)}{a^2}\chi^*_{\mu\nu}+\frac{\beta(a_5+12a_6)}{a^2}(\eta_{\mu\nu}-(m_0)^{-2}\partial_\mu\partial_\nu)\sigma$$
$$+\frac{\beta-2a/3}{a(m_0)^2}\eta_{\mu\nu}j^{(\sigma)}, \tag{5·11}$$

showing that the field $\bar{h}_{\mu\nu}$ is the *multimass field*. This is also the reason why $\bar{h}_{\mu\nu}$ obeys the fourth-order field equation (3·6).

§ 6. Conclusion

We have applied the weak field approximation to the most general gravitational field equations derived in Poincaré gauge theory with the linear and quadratic Lagrangian densities. Adopting the conventional method, the linearized field equation for the linearized gravitational field $h_{\mu\nu}$ was of fourth order, showing that $h_{\mu\nu}$ is the multimass field, which is given by the graviton with $m=0$, and the particles with $m=m_2$ and $m=m_0$. This fact was also viewed from the Newtonian approximation method, where there are two Yukawa potentials in addition to the famous Newtonian potential.

On the other hand, there are the six irreducible fields in the torsion field, which have $J^P=\mathrm{spin}^{\mathrm{parity}}$ as 2^+, 2^-, 1^+, 1^-, 0^+ and 0^-, each of which satisfies the Klein-Gordon equation. Thus, the previous multimass field can be redefined as the purely genuine field $\phi_{\mu\nu}$ of spin 2 and mass 0, obeying the usual wave equation of second order, whose source term is exactly the symmetrized energy-momentum tensor appearing in the linearized Einstein field equation: So we call it the massless graviton field of spin 2. The two additional fields are called $\chi_{\mu\nu}$ and σ, which have $J^P=2^+$ and mass m_2 and $J^P=0^+$ and mass m_0, respectively. We denote in Table I the massless graviton field in terms of $\bar{h}_{\mu\nu}$, $\chi_{\mu\nu}$ and σ, and the six irreducible torsion fields in terms of the original irreducible torsion fields, $t_{\lambda\mu\nu}$, v_μ and a_μ.

These irreducible torsion fields might be normal (i.e., the energy is positive-definite and the mass is positive) or abnormal, depending on parameters a_i involved. This issue will be reported in a forthcoming paper of this series; IV. Mass and energy of particle spectrum.

Table I. Massless and Massive fields in Poincaré gauge theory. In the definition of $\phi_{\mu\nu}$ the matter terms are omitted here (see (5·4)).

Fields	(Mass)^2	J^P
$\phi_{\mu\nu}=\bar{h}_{\mu\nu}+\dfrac{6(\alpha+2a/3)}{a(m_2)^2}\chi_{\mu\nu} - \dfrac{2(\beta-2a/3)}{a(m_0)^2}\left(\eta_{\mu\nu}-\dfrac{\partial_\mu\partial_\nu}{(m_0)^2}\right)\sigma$	0	2^+
$\sigma=\partial^\mu v_\mu$	$(m_0)^2=\dfrac{2a(\beta-2a/3)}{\beta(a_5+12a_6)}$	0^+
$\tilde{v}_\mu=v_\mu-(m_0)^{-2}\partial_\mu\sigma$	$m_v{}^2=\dfrac{9(\alpha+2a/3)(\beta-2a/3)}{2(a_4+a_5)(\alpha+\beta)}$	1^-
$\chi_{\mu\nu}=\partial^\rho t_{\rho(\mu\nu)}+\dfrac{\beta-2a/3}{3(\alpha+2a/3)}\partial_{(\mu}\tilde{v}_{\nu)}$	$(m_2)^2=-\dfrac{2a(\alpha+2a/3)}{\alpha(3a_2+2a_5)}$	2^+
$B=\partial^\mu a_\mu$	$m_B{}^2=-\dfrac{2(\gamma+3a/2)}{3(a_1+a_3)}$	0^-
$\tilde{a}_\mu=a_\mu-(m_B)^{-2}\partial_\mu B$	$m_a{}^2=-\dfrac{2(\alpha+2a/3)(\gamma+3a/2)}{(2a_3+a_4)(\alpha-4\gamma/9)}$	1^+
$\tilde{t}_{\lambda\mu\nu}=t_{\lambda\mu\nu}-\dfrac{2}{(m_2)^2}(\partial_{(\lambda}\chi_{\mu)\nu}-\partial_\nu\chi_{\lambda\mu}) +\dfrac{\beta-2a/3}{\alpha+2a/3}\left[\dfrac{1}{3}(\eta_{\nu(\lambda}\tilde{v}_{\mu)}-\eta_{\lambda\mu}\tilde{v}_\nu) +\dfrac{1}{m_v{}^2}(\partial_\lambda\partial_\mu\tilde{v}_\nu-\partial_\nu\partial_{(\lambda}\tilde{v}_{\mu)})\right] +\dfrac{2(\gamma+3a/2)}{3(\alpha+2a/3)m_a{}^2}\partial_{(\lambda}\varepsilon_{\mu)\nu\rho\sigma}\partial^\rho a^\sigma$	$m_t{}^2=\dfrac{3(\alpha+2a/3)}{3a_2+4a_3}$	2^-

Appendix

——*Divergence and Trace of the Irreducible Fields, $\tilde{v}_\mu$, $\tilde{a}_\mu$, $\chi_{\mu\nu}$ and $\tilde{t}_{\lambda\mu\nu}$*——

(i) The fields $\tilde{v}_\mu$ and $\tilde{a}_\mu$:

Taking the divergence of (4·10) and (4·17), we get

$$\partial^\mu j_\mu{}^{(v)}=-[\Box-m_v{}^2]j^{(\sigma)}/(m_0)^2, \tag{A·1}$$

$$\partial^\mu j_\mu{}^{(a)}=-[\Box-m_a{}^2]j^{(B)}/m_B^2. \tag{A·2}$$

Therefore, $\partial^\mu\tilde{v}_\mu$ and $\partial^\mu\tilde{a}_\mu$ are frozen at the place of matter, given by

$$\partial^\mu\tilde{v}_\mu=-j^{(\sigma)}/(m_0)^2, \tag{A·3}$$

$$\partial^\mu\tilde{a}_\mu=-j^{(B)}/m_B^2. \tag{A·4}$$

(ii) The field $\chi_{\mu\nu}$:

It follows from (4·21) that

$$\eta^{\mu\nu}j_{\mu\nu}^{(\chi)}=-[\Box-(m_2)^2]\,\frac{(\beta-2a/3)j^{(\sigma)}}{3(\alpha+2a/3)(m_0)^2} \tag{A·5}$$

and

$$\partial^\nu j^{(\chi)}_{\mu\nu} = [\Box - (m_2)^2]\left\{\frac{1}{12(\alpha+2a/3)}\partial^\rho\partial^\sigma S_{\mu\rho\sigma} - \frac{(\beta-2a/3)\partial_\mu j^{(\sigma)}}{3(\alpha+2a/3)(m_0)^2}\right\}, \qquad (\mathrm{A}\cdot 6)$$

showing that $\eta^{\mu\nu}\chi_{\mu\nu}$ and $\partial^\nu\chi_{\mu\nu}$ cannot propagate in vacuum:

$$\eta^{\mu\nu}\chi_{\mu\nu} = -\frac{(\beta-2a/3)j^{(\sigma)}}{3(\alpha+2a/3)(m_0)^2}, \qquad (\mathrm{A}\cdot 7)$$

$$\partial^\nu\chi_{\mu\nu} = \frac{1}{12(\alpha+2a/3)}\partial^\rho\partial^\sigma S_{\mu\rho\sigma} - \frac{(\beta-2a/3)\partial_\mu j^{(\sigma)}}{3(\alpha+2a/3)(m_0)^2}. \qquad (\mathrm{A}\cdot 8)$$

(iii) The field $\tilde{t}_{\lambda\mu\nu}$:

From (4·25) we have

$$\eta^{\lambda\mu} j^{(t)}_{\lambda\mu\nu} = -2\eta^{\lambda\mu} j^{(t)}_{\nu\lambda\mu}$$
$$= \frac{\Box - m_t^2}{\alpha+2a/3}\left\{\left(\beta-\frac{2a}{3}\right)\left(j_\nu^{(v)} + \frac{\partial_\nu j^{(\sigma)}}{(m_0)^2}\right) - \frac{1}{6(m_2)^2}\partial^\rho\partial^\sigma S_{\nu\rho\sigma}\right\}, \qquad (\mathrm{A}\cdot 9)$$

$$\partial^\lambda j^{(t)}_{\lambda[\mu\nu]} = \frac{\Box - m_t^2}{12(\alpha+2a/3)}\left\{-\frac{3}{(m_2)^2}\partial^\rho\partial^\sigma\partial_{[\mu}S_{\nu]\rho\sigma} - 6T_{[\mu\nu]}\right.$$
$$\left. + \frac{18(\beta-2a/3)}{m_v^2}\partial_{[\mu}j^{(v)}_{\nu]} + \frac{4(\gamma+3a/2)}{m_a^2}\varepsilon_{\mu\nu\rho\sigma}\partial^\rho j^{(a)\sigma}\right\}, \qquad (\mathrm{A}\cdot 10)$$

$$\partial^\lambda j^{(t)}_{\lambda(\mu\nu)} = \frac{\Box - m_t^2}{12(\alpha+2a/3)}\left[\frac{1}{(m_2)^2}\left\{\partial^\rho\partial^\sigma\partial_{(\mu}S_{\nu)\rho\sigma} - 12\left(\alpha+\frac{2a}{3}\right)j^{(\chi)}_{\mu\nu}\right\}\right.$$
$$+ \frac{6(\beta-2a/3)}{m_v^2}\partial_{(\mu}j^{(v)}_{\nu)}$$
$$\left. - \frac{2(\beta-2a/3)}{(m_0)^2}\left\{\eta_{\mu\nu} + \left(\frac{2}{(m_2)^2} - \frac{3}{m_v^2}\right)\partial_\mu\partial_\nu j^{(\sigma)}\right\}\right]. \qquad (\mathrm{A}\cdot 11)$$

Since $[\Box - m_t^2]$ is factored out in these expressions, the trace and the divergence of $\tilde{t}_{\lambda\mu\nu}$, i.e., $\eta^{\lambda\mu}\tilde{t}_{\lambda\mu\nu}$, $\partial^\lambda\tilde{t}_{\lambda[\mu\nu]}$ and $\partial^\lambda\tilde{t}_{\lambda(\mu\nu)}$, are frozen at the place of matter: The expressions for them can easily be derived from (A·9)~(A·11).

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