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Gravity from Poincaré Gauge Theory of the Fundamental Particles. V

——*The Extended Bach-Lanczos Identity*——

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In the Riemann-Cartan space-time we prove the extended form of the Bach-Lanczos identity,

$$\delta\int ed^4x(F_{ijmn}F^{mnij}-4F_{ij}F^{ji}+F^2)\equiv 0\,,$$

where δ denotes that infinitesimal variations in the integral are taken with respect to both the translation gauge field and the Lorentz gauge field, and F_{ijmn} stands for the curvature tensor formed of the Lorentz gauge field; $F_{ij}=\eta^{mn}F_{imjn}$ and $F=\eta^{ij}F_{ij}$. By virtue of this identity, there are five independent invariants quadratic in the Lorentz gauge field strength, when one derives field equations from the action principle.

§ 1. Introduction

In Poincaré gauge theory we have started with the action principle, the integrand of which consists of the translation gauge field and the Lorentz gauge field. It is very convenient to use the translation gauge field strength and the Lorentz gauge field strength; the 'strength' is defined as a tensor, being linear in first derivatives in the gauge field in question. Geometrically speaking, the space-time underlying Poincaré gauge theory is the Riemann-Cartan space-time, and these two strengths are the torsion tensor and the curvature tensor.

Algebraically independent invariants quadratic in the Lorentz gauge field strength are shown to be six; this enumeration was performed by the help of the Young table method. Here we have assumed that the action principle is invariant under parity operation. This fact was derived in I of this series.[1),*)] However, are these six invariants really independent for the formation of field equations derived from the action principle? Is there some identity which holds under the integral sign? This is the problem to which we address ourselves.

The reason for this question arised from the weak field approximation we have made in III of this series.[2),**)] In fact, there are five independent invariants

*) We shall refer to this reference as I henceforth.

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quadratic in Lorentz gauge field strength. But we did not know whether this is due to the method of approximation we have made, or not.

On the other hand, it is well known that in the Riemann space-time there is an identity,[3] sometimes called the Gauss-Bonnet theorem,

$$\delta \int \sqrt{-g}\ d^4x (R_{\mu\nu\rho\sigma} R^{\rho\sigma\mu\nu} - 4 R_{\mu\nu} R^{\nu\mu} + R^2) \equiv 0\ , \qquad (1\cdot 1)$$

where δ represents the infinitesimal variation of the metric tensor under the integral sign and $R_{\mu\nu\rho\sigma}$ denotes the Riemann-Christoffel curvature tensor; $R_{\mu\nu} = g^{\rho\sigma} R_{\mu\rho\nu\sigma}$ and $R = g^{\mu\nu} R_{\mu\nu}$. Unfortunately, the above identity has often been quoted without reference.

Motivated by the use of the weak field approximation, we shall see that the following identity holds:

$$\delta \int e d^4x (F_{ijmn} F^{mnij} - 4 F_{ij} F^{ji} + F^2) \equiv 0\ , \qquad (1\cdot 2)$$

where δ this time means that one takes the infinitesimal variations of both the translation gauge field and the Lorentz gauge field in the integrand, and F_{ijmn} stands for the curvature tensor for this Lorentz gauge field; $F_{ij} = \eta^{mn} F_{imjn}$ and $F = \eta^{ij} F_{ij}$.

This extended form of the Bach-Lanczos identity holds in Poincaré gauge theory, or equivalently, in the Riemann-Cartan space-time. What differs from the previous identity of (1·1) is that there is no longer any symmetry property in the curvature tensor, $F_{ijmn} \neq F_{mnij}$, although in the Riemann space-time one sees the symmetry property of the Riemann-Christoffel curvature tensor, $R_{\mu\nu\rho\sigma} = \mathrm{R}_{\rho\sigma\mu\nu}$. The mis-extrapolation of this fact has frequently been seen in the literatures.[4),*)] Maybe some mathematicians would know what this identity of (1·2) implies, but so far as we know, there are no literatures about it,

By the help of this powerful identity denoted by (1·2), there are, therefore, five independent invariants quadratic in the Lorentz gauge field strength when field equations are derived from the action principle. Thus, six parameters multiplied by the six invariants are reduced to five parameters.

In the next section we shall explicitly show that the identity of (1·2) holds, by computing each term on the left-hand side. The final section is devoted to conclusion.

*) The misquoted identity reads

$$\int e d^4x (R_{\mu\nu ab} R^{\mu\nu ab} - 4 R_{\mu a} R^{\mu a} + R^2) = 0\ ,$$

but this 'identity' fails to vanish. There are many references where a similar mistake or mis-spelling is made.

§ 2. The identity

Let us first explain briefly how we are led to the particular quadratic form in the integrand of the identity (1·2). The most general gravitational Lagrangian density L_G derived in I is written as

$$L_G = aF + L_T + L_F . \tag{2·1}$$

The second term L_T contains three parameters, α, β and γ, being made of three invariants quadratic in the translation gauge field strength T_{ijk}. The third term L_F is constructed by adding the six invariants quadratic in the Lorentz gauge field strength F_{ijmn}:

$$L_F = a_1 A_{ijmn}A^{ijmn} + a_2 B_{ijmn}B^{ijmn} + a_3 C_{ijmn}C^{ijmn} + a_4 E_{ij}E^{ij} + a_5 I_{ij}I^{ij} + a_6 F^2 , \tag{2·2}$$

where a_1, a_2,$\cdots$ and a_6 are dimensionless parameters, and the tensors A_{ijmn}, B_{ijmn}, C_{ijmn}, E_{ij}, I_{ij} and F are the irreducible building blocks of F_{ijmn} (see (I·3·10), (I·3·11) and (I·3·14)~(I·3·17) for their definition). Thus, the gravitational Lagrangian density L_G contains *ten* parameters, a, α, β, γ, a_1, a_2, $\cdots$ and a_6. The gravitational field equations for the translation gauge field and the Lorentz gauge field are derived by the action principle. It is shown in III that in the weak field approximation the six parameters, a_1, a_2, $\cdots$ and a_6, appear in the gravitational field equations only through the combinations, a_5+12a_6, a_1+a_3, a_4+a_5, $2a_3+a_4$, $3a_2+2a_5$ and $3a_2+4a_3$. These combinations are all vanishing when a_1, a_2, $\cdots$ and a_6 take the following particular values:

$$a_1 = \frac{3}{4}a_2 = -a_3 = \frac{1}{2}a_4 = -\frac{1}{2}a_5 = 6a_6 = c \tag{2·3}$$

with c an arbitrary parameter. Let us denote by L_{GB} the quadratic Lagrangian L_F for the particular choice (2·3) with $c=1$: Namely, we define L_{GB} by*)

$$L_{GB} = A_{ijmn}A^{ijmn} + \frac{4}{3}B_{ijmn}B^{ijmn} - C_{ijmn}C^{ijmn} + 2E_{ij}E^{ij} - 2I_{ij}I^{ij} + \frac{1}{6}F^2 . \tag{2·4}$$

Using the explicit expressions for A_{ijmn}, B_{ijmn}, C_{ijmn}, E_{ij} and I_{ij} in (2·4), we see that L_{GB} is represented as

*) The suffix *GB* of L_{GB} means the abbreviation of Gauss and Bonnet.

$$L_{GB}=F_{ijmn}F^{mnij}-4F_{ij}F^{ji}+F^2 . \tag{2·5}$$

Notice the order of the indices in this expression.

Now we proceed to the proof of the identity (1·2), which is written as

$$\delta\int ed^4xL_{GB}\equiv 0 \tag{2·6}$$

with $e=\det(e^i{}_\mu)$. Here δ plays the dual role of taking the infinitesimal variations with respect to the translation gauge field $a^i{}_\mu$ and the Lorentz gauge field $A_{ij\mu}$, respectively:*) That is, the identity (2·6) means that the Euler derivatives of eL_{GB} with respect to $a^i{}_\mu$ and $A_{ij\mu}$ are both identically vanishing.

From (2·5) we obtain the following expressions for the Euler derivatives of L_{GB} with respect to $a^i{}_\mu$ and $A_{ij\mu}$:

$$\frac{1}{e}\delta(eL_{GB})/\delta a^i{}_\mu=-2F_{kmni}J^{[km][n\mu]}+e_i{}^\mu L_{GB} , \tag{2·7}$$

$$\frac{1}{e}\delta(eL_{GB})/\delta A_{ij\mu}=2\hat{D}_\nu J^{[ij][\mu\nu]} , \tag{2·8}$$

where

$$\begin{aligned}J^{[ij][mn]}&=e^n{}_\mu J^{[ij][m\mu]}=e^m{}_\mu e^n{}_\nu J^{[ij][\mu\nu]}\\&=2\Big[F^{mnij}-(\eta^{mi}F^{nj}+\eta^{nj}F^{mi}-\eta^{mj}F^{ni}-\eta^{ni}F^{mj})\\&\qquad+\frac{1}{2}(\eta^{mi}\eta^{nj}-\eta^{mj}\eta^{ni})F\Big] .\end{aligned} \tag{2·9}$$

Here $\hat{D}_\nu$ denotes the covariant derivative with respect to the Lorentz gauge field $A_{ij\nu}$ when it acts on a quantity with Latin indices, while it means the ordinary covariant derivative with respect to the Christoffel symbol when it operates on a quantity with Greek indices.**)

The identity (2·6) thus takes the form,

$$F_{kmni}J^{[km][nj]}-\frac{1}{2}\delta_i{}^jL_{GB}\equiv 0 , \tag{2·10}$$

*) The tetrad field $e^i{}_\mu$ is related to the translation gauge field $a^i{}_\mu$ by

$$e^i{}_\mu=\delta^i{}_\mu+a^i{}_\mu .$$

**) Namely, we define as

$$\hat{D}_\nu J^{[ij][\mu\nu]}=\partial_\nu J^{[ij][\mu\nu]}+A^i{}_{k\nu}J^{[kj][\mu\nu]}+A^j{}_{k\nu}J^{[ik][\mu\nu]}+\left\{{\mu \atop \rho\nu}\right\}J^{[ij][\rho\nu]}+\left\{{\nu \atop \rho\nu}\right\}J^{[ij][\mu\rho]} .$$

This covariant derivative $\hat{D}_\lambda$ satisfies the metric condition in the sense that $\hat{D}_\lambda\eta_{ij}=0$ and $\hat{D}_\lambda g_{\mu\nu}=0$: However, $\hat{D}_\lambda$ *does not commute* with the tetrad field because $\hat{D}_\lambda e^i{}_\mu=K^i{}_{j\lambda}e^j{}_\mu\neq 0$.

$$\hat{D}_\nu J^{[ij][\mu\nu]} \equiv 0 , \tag{2·11}$$

which we shall show below. The identity (2·10) is a generalization of the Bach-Lanczos identity[3] for the Riemann-Christoffel curvature tensor denoted by (I·5·14) to the curvature tensor in the Riemann-Cartan space-time.

By virtue of the relation,*)

$$\begin{aligned} -\frac{1}{4}\varepsilon_{ijab}\varepsilon_{mncd}F^{abcd} &= F_{mnij} - (\eta_{mi}F_{nj} + \eta_{nj}F_{mi} - \eta_{mj}F_{ni} - \eta_{ni}F_{mj}) \\ &\quad + \frac{1}{2}(\eta_{mi}\eta_{nj} - \eta_{mj}\eta_{ni})F , \end{aligned} \tag{2·12}$$

we can rewrite L_{GB} and $J^{[ij][mn]}$ as

$$L_{GB} = -\frac{1}{4}\varepsilon_{ijab}\varepsilon_{mncd}F^{ijmn}F^{abcd} , \tag{2·13}$$

$$J^{[ij][mn]} = -\frac{1}{2}\varepsilon^{ijab}\varepsilon^{mncd}F_{abcd} . \tag{2·14}$$

Let us calculate $F_{kmni}J^{[km][nj]}$ using (2·13) and (2·14). For $i \neq j$, we choose p and q so that $(ijpq)$ is an even permutation of (0123), then we have**)

$$\begin{aligned} F_{kmni}J^{[km][nj]} &= -\frac{1}{2}\varepsilon^{kmab}\varepsilon^{njcd}F_{kmni}F_{abcd} \\ &= -\varepsilon^{kmab}(\varepsilon^{pjiq}F_{kmpi}F_{abiq} + \varepsilon^{qjip}F_{kmqi}F_{abip}) \\ &= -\varepsilon^{kmab}(F_{kmip}F_{abiq} - F_{kmiq}F_{abip}) \\ &= 0 . \end{aligned} \tag{2·15}$$

For $i = j$, on the other hand, we choose p, q and r such that $(ipqr)$ is an even permutation of (0123); then we get***)

$$\begin{aligned} F_{kmni}J^{[km][nj]} &= -\frac{1}{2}\varepsilon^{kmab}\varepsilon^{nicd}F_{kmni}F_{abcd} \\ &= -\varepsilon^{kmab}(\varepsilon^{piqr}F_{kmpi}F_{abqr} + \varepsilon^{qirp}F_{kmqi}F_{abrp} \\ &\quad + \varepsilon^{ripq}F_{kmri}F_{abpq}) \end{aligned}$$

*) The totally antisymmetric tensors ε_{ijmn} and ε^{ijmn} are normalized as $\varepsilon_{(0)(1)(2)(3)} = -1$ and $\varepsilon^{(0)(1)(2)(3)} = +1$ with Latin indices enclosed in parentheses.

**) Summation is not carried out over the indices i, p and q in (2·15).

***) Summation is not carried out over the indices i, p, q and r in (2·16).

$$= -\varepsilon^{kmab}(F_{kmip}F_{abqr} + F_{kmiq}F_{abrp} + F_{kmir}F_{abpq})$$

$$= -\frac{1}{8}\varepsilon^{kmab}\varepsilon^{lncd}F_{kmln}F_{abcd}$$

$$= \frac{1}{2}L_{GB} . \qquad (2\cdot16)$$

Thus, the identity (2·10) has been proven.

We next calculate $\hat{D}_\nu J^{[ij][\mu\nu]}$ with the help of (2·14). The totally antisymmetric tensors ε^{ijab} and $\varepsilon^{\mu\nu\rho\sigma}$ defined by*)

$$\varepsilon^{\mu\nu\rho\sigma} = e_i{}^\mu e_j{}^\nu e_m{}^\rho e_n{}^\sigma \varepsilon^{ijmn} \qquad (2\cdot17)$$

commute with the covariant derivative $\hat{D}_\lambda$, because $\hat{D}_\lambda$ coincides with the Poincaré covariant derivative D_λ of (I·2·16) when it acts on ε^{ijab}, and because $\hat{D}_\lambda$ is just the ordinary covariant derivative with respect to the Christoffel symbol when it operates on $\varepsilon^{\mu\nu\rho\sigma}$. Accordingly, we have

$$\hat{D}_\nu J^{[ij][\mu\nu]} = -\frac{1}{2}\hat{D}_\nu(\varepsilon^{ijab}\varepsilon^{\mu\nu\rho\sigma}F_{ab\rho\sigma})$$

$$= -\frac{1}{2}\varepsilon^{ijab}\varepsilon^{\mu\nu\rho\sigma}\hat{D}_\nu F_{ab\rho\sigma}$$

$$= -\frac{1}{6}\varepsilon^{ijab}\varepsilon^{\mu\nu\rho\sigma}(\hat{D}_\nu F_{ab\rho\sigma} + \hat{D}_\rho F_{ab\sigma\nu} + \hat{D}_\sigma F_{ab\nu\rho}) \qquad (2\cdot18)$$

by cyclically interchanging the indices ν, ρ and σ at the last step. The last term in (2·18) vanishes identically because of the "Bianchi identity",

$$\hat{D}_\lambda F_{ij\mu\nu} + \hat{D}_\mu F_{ij\nu\lambda} + \hat{D}_\nu F_{ij\lambda\mu} \equiv 0, \qquad (2\cdot19)$$

which follows from the Jacobi identity,**)

$$[\hat{D}_\lambda,[\hat{D}_\mu,\hat{D}_\nu]] + [\hat{D}_\mu,[\hat{D}_\nu,\hat{D}_\lambda]] + [\hat{D}_\nu,[\hat{D}_\lambda,\hat{D}_\mu]] \equiv 0 . \qquad (2\cdot20)$$

Therefore, the identity (2·11) has also been proven.

Owing to the identity (2·6), the gravitational field equations for the translation gauge field and the Lorentz gauge field are invariant under the

*) The totally antisymmetric tensors $\varepsilon^{\mu\nu\rho\sigma}$ and $\varepsilon_{\mu\nu\rho\sigma}$ with Greek indices are normalized as $\varepsilon^{0123} = 1/e$ and $\varepsilon_{0123} = -e$.

**) Two identities follow from the Jacobi identity (2·20). The one is (2·19), and the other is the following well-known identity for the Riemann-Christoffel curvature tensor:

$$R_{\rho\sigma\mu\nu;\lambda} + R_{\rho\sigma\nu\lambda;\mu} + R_{\rho\sigma\lambda\mu;\nu} \equiv 0$$

with a semicolon denoting the ordinary covariant derivative with respect to the Christoffel symbol.

following change of the quadratic Lagrangian density L_F:

$$L_F \to L_F + cL_{GB} , \tag{2·21}$$

where c is an arbitrary dimensionless parameter.*) Thus, the six invariants quadratic in the Lorentz gauge field strength are reduced to five *independent* invariants when field equations are derived by the action principle. Hence, there are *five independent parameters* multiplied by the five independent invariants. Consequently, the most general linear and quadratic Lagrangian density L_G contains *nine* independent parameters.

§ 3. Conclusion

Long time ago Bach and Lanczos proved the identity,[3]

$$\delta \int \sqrt{-g}\, d^4x\, R^{\mu\nu\rho\sigma} R^{**}_{\mu\nu\rho\sigma} \equiv 0 , \tag{3·1}$$

where $R^{\mu\nu\rho\sigma}$ is the Riemann-Christoffel curvature tensor in the Riemann space-time, and $R^{**}_{\mu\nu\rho\sigma}$ is the dual tensor,

$$R^{**}_{\mu\nu\rho\sigma} = \frac{1}{4} \varepsilon_{\mu\nu\alpha\beta} \varepsilon_{\rho\sigma\gamma\delta} R^{\alpha\beta\gamma\delta} , \tag{3·2}$$

and finally, δ denotes the infinitesimal variation of the metric tensor. This identity can be rewritten as

$$\delta \int \sqrt{-g}\, d^4x (R_{\mu\nu\rho\sigma} R^{\rho\sigma\mu\nu} - 4R_{\mu\nu} R^{\nu\mu} + R^2) \equiv 0 \tag{3·3}$$

with $R_{\mu\nu} = g^{\rho\sigma} R_{\mu\rho\nu\sigma}$ and $R = g^{\mu\nu} R_{\mu\nu}$.

This time we have proven the extended Bach-Lanczos identity,

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*) Alternatively, the gravitational field equations are invariant under the following transformation of the parameters:

$$a_1 \to a_1 + c, \qquad a_2 \to a_2 + \frac{4}{3}c, \qquad a_3 \to a_3 - c,$$

$$a_4 \to a_4 + 2c, \qquad a_5 \to a_5 - 2c, \qquad a_6 \to a_6 + \frac{1}{6}c . \tag{2·22}$$

Therefore, among the six parameters, $a_1, a_2, \cdots$ and a_6, there are five independent parameters. In particular, one can freely choose one of $a_1, a_2, \cdots$ and a_6 as zero without loss of generality. We also notice that the combinations of the parameters appearing in the weak field approximation, namely, $a_5 + 12a_6$, $a_1 + a_3$, $a_4 + a_5$, $2a_3 + a_4$, $3a_2 + 2a_5$ and $3a_2 + 4a_3$, are invariant under (2·22).

$$\delta \int e d^4 x (F_{ijmn} F^{mnij} - 4 F_{ij} F^{ji} + F^2) \equiv 0 \,, \tag{3·4}$$

where F_{ijmn} is the curvature tensor formed from the Lorentz gauge field in the Riemann-Cartan space-time underlying Poincaré gauge theory, and δ means the infinitesimal variations with respect to the translation gauge field and the Lorentz gauge field.

For mathematicians of topological geometry it may be better to cast the above identity to the form

$$\delta \int e d^4 x F^{ijmn} F^{**}_{ijmn} \equiv 0 \,, \tag{3·5}$$

where

$$F^{**}_{ijmn} = \frac{1}{4} \varepsilon_{ijpq} \varepsilon_{mnrs} F^{pqrs} \,, \tag{3·6}$$

and δ takes the same meaning as the above identity does. This way of writing the identity (3·5) will be suitable for calling it the Gauss-Bonnet theorem of the topological invariant, $\int e d^4 x F^{ijmn} F^{**}_{ijmn}$, in the Riemann-Cartan space-time. However, we have not proven the identity (3·5) topologically but explicitly by computing both sides of the identity.

References

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4) For example, E. Sezgin and P. van Nieuwenhuizen, Phys. Rev. **D21** (1980), 3269, Eq. (6).

Note added in proof: After completing this work, we are noticed by Prof. Kimura of a recent paper by H.T. Nieh, J. Math. Phys. **21** (1980), 1439, which treats a similar problem to ours, though different in method.