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Gravity from Poincaré Gauge Theory of the Fundamental Particles. II

——*Equations of Motion for Test Bodies and Various Limits*——

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We study the equations of motion for test bodies and various limits in Poincaré gauge theory with linear and quadratic Lagrangians. The classical equations of motion are derived both for spin-1/2 particles and for macroscopic test bodies. It is also shown that various limits can be taken, including General Relativity and New General Relativity.

§ 1. Introduction

In a previous paper[1),*)] the most general gravitational field equations have been derived in Poincaré gauge theory with Lagrangians linear and quadratic in the translation and the Lorentz gauge field strengths. The underlying space-time manifold possesses a locally Lorentzian metric $\boldsymbol{g}$ and a linear affine connection $\boldsymbol{\Gamma}$, which are defined by the translation and the Lorentz gauge fields, respectively. The space-time manifold of Poincaré gauge theory is then the Riemann-Cartan space-time characterized by curvature and torsion.

It has been verified experimentally that the world line of freely falling test bodies is independent of their composition and structure.[2)] The red-shift experiment indicates that this unique world line should be identified with the geodesics of the metric.[3)] In Poincaré gauge theory, gravity is described at the level of the microscopic scale, where the fundamental particles of spin 1/2 are described by spinor wave functions obeying the Dirac equation. The classical equations of motion for test bodies, therefore, should be derived either from the gravitational field equations or from the microscopic equation of motion, i.e., the Dirac equation, by taking the classical limit where the quantum mechanical uncertainty in location and the intrinsic spin of spin-1/2 particles can be neglected.

It is the first aim of this paper to derive the classical equations of motion for spin-1/2 particles and for macroscopic bodies like stars and planets.

The most general gravitational action derived in I contains ten parameters, a, α, β, γ, a_1, a_2, $\cdots$, a_6. It then seems quite reasonable to expect that many of the

*) We shall refer to this reference as I henceforth.

gravitational theories so far proposed, including General Relativity and New General Relativity,[4] can be obtained as special limiting cases. It is thus the second purpose of this paper to investigate various limiting cases of the most general framework derived in I.

In § 2 the equations of motion for classical particles of spin 1/2 are derived by applying the semiclassical approximation to the Dirac equation. In § 3 the equation of motion for macroscopic bodies is obtained from the gravitational field equations. Section 4 is devoted to discussion of various limiting cases. The main results are summarized in § 5.

Throughout this paper we use the same notations and conventions as those in I. For example, the equation (2·1) of I is referred to as Eq. (I·2·1).

§ 2. Equations of motion for spin-1/2 particles

Let us take a spin-1/2 particle of mass m as a test body, and consider its freely falling motion in the semiclassical approximation,[5] assuming that it is described by a spinor wave function of the form,

$$\psi=\exp(iS/\hbar)\left(\psi_0+\frac{\hbar}{i}\psi_1+\cdots\right), \qquad (S\gg\hbar) \tag{2·1}$$

where $\hbar$ is the reduced Planck constant, and S is a scalar function which takes values much larger than $\hbar$. Using (2·1) in the Dirac equation,*)

$$\left[i\hbar e_k{}^{\nu}\gamma^k\left(D_\nu+\frac{1}{2}v_\nu\right)-m\right]\psi=0\,, \tag{2·2}$$

and putting each order of $(\hbar/i)$ equal to zero, we get, up to first order,

$$(\hbar/i)^0:\quad [e_k{}^{\nu}(\partial_\nu S)\gamma^k+m]\psi_0=0\,, \tag{2·3}$$

$$(\hbar/i)^1:\quad ie_k{}^{\nu}\gamma^k\left(D_\nu+\frac{1}{2}v_\nu\right)\psi_0-[e_k{}^{\nu}(\partial_\nu S)\gamma^k+m]\psi_1=0\,. \tag{2·4}$$

Multiplying (2·3) and (2·4) by $[e_j{}^{\mu}(\partial_\mu S)\gamma^j-m]$, we obtain

$$g^{\mu\nu}(\partial_\mu S)(\partial_\nu S)+m^2=0\,, \tag{2·5}$$

$$\left\{2g^{\mu\nu}(\partial_\mu S)\left(D_\nu+\frac{1}{2}v_\nu\right)-e_j{}^{\mu}[D_\mu(e_k{}^{\nu}\partial_\nu S)]\gamma^j\gamma^k\right\}\psi_0=0\,, \tag{2·6}$$

where (2·5) is used in (2·6). Equations (2·3) and (2·6) are compatible with each other, because S satisfies (2·5): In fact, we can show the following commutation relation:

*) We use the unit $c=1$.

$$\left[e_i{}^\rho(\partial_\rho S)\gamma^i,\, 2g^{\mu\nu}(\partial_\mu S)\left(D_\nu+\frac{1}{2}v_\nu\right)-e_j{}^\mu\{D_\mu(e_k{}^\nu\partial_\nu S)\}\gamma^j\gamma^k\right] \tag{2·7}$$

$$=-e_i{}^\rho\partial_\rho\{g^{\mu\nu}(\partial_\mu S)\,(\partial_\nu S)\}\gamma^i$$

$$=0\,.$$

Equation (2·5) is just the Hamilton-Jacobi equation[6] describing freely falling particles in the background metric $\boldsymbol{g}=\{g_{\mu\nu}\}$. The complete solution $S(x;\alpha_1,\alpha_2,\alpha_3)$ with three parameters defines a world line $x^\mu(\sigma)$ by the condition

$$\partial S/\partial\alpha_i=0\,. \qquad (i=1,2,3)\,. \tag{2·8}$$

As is well known, this world line is the geodesics of the metric $\boldsymbol{g}$:*)

$$\frac{dx^\mu}{d\sigma}=g^{\mu\nu}\partial_\nu S\,, \tag{2·9a}$$

$$\frac{d^2x^\mu}{d\sigma^2}+\left\{\begin{matrix}\mu\\ \lambda\nu\end{matrix}\right\}\frac{dx^\lambda}{d\sigma}\frac{dx^\nu}{d\sigma}=0\,, \tag{2·9b}$$

where (2·9a) defines normalization of an affine parameter σ; for a massive particle, σ is related to the proper time τ along the world line by $\tau=m\sigma$.

We write the wave function (2·1) for $S=S(x;\alpha_1,\alpha_2,\alpha_3)$ as $\psi(x,\alpha_1,\alpha_2,\alpha_3)$. This wave function is broadly spread, however, and we cannot associate any world line with it. Therefore, let us form a wave packet by superposing $\psi(x;\alpha_1,\alpha_2,\alpha_3)$ over a small interval in α-space,

$$\psi(x;\bar{\alpha})=\int d^3\alpha w(\alpha)\psi(x;\alpha) \tag{2·10}$$

with $w(\alpha)$ a weight function sharply peaked at $\alpha_i=\bar{\alpha}_i$. Since the exponential factor $\exp(iS/\hbar)$ is rapidly oscillating with respect to α, cancellation takes place almost everywhere in space-time. The only exception is along the world line $x^\mu(\sigma)$ defined by (2·8) with $\alpha_i=\bar{\alpha}_i$: Namely, the wave packet (2·10) propagates along the world line $x^\mu(\sigma)$ satisfying (2·9a) and (2·9b) with $S=S(x;\bar{\alpha}_1,\bar{\alpha}_2,\bar{\alpha}_3)$. According to the Einstein-de Broglie relation between wave and particle, the four-momentum $p^\mu(\sigma)$ of the particle is given by

$$p^\mu(\sigma)=g^{\mu\nu}\partial_\nu S(x(\sigma);\bar{\alpha}_1,\bar{\alpha}_2,\bar{\alpha}_3)=\frac{dx^\mu}{d\sigma}\,. \tag{2·11}$$

Thus, when the quantum uncertainty in location is negligible, a freely falling particle of spin 1/2 travels along the geodesics of the metric $\boldsymbol{g}$, and *the equation of orbit is in accord with the equivalence principle.*

In the semiclassical approximation, a spin-1/2 particle is characterized by the

*) We mean by 'geodesics' the shortest (or longest) possible path between two points, with length being measured by the metric.

two classical quantities, the world line and the spin polarization.*) We shall now turn to the equation of spin for the spin-1/2 particle described by the wave packet (2·10) propagating along the world line $x^\mu(\sigma)$. Since $w(\alpha)$ has a sharp peak at $\alpha_i = \bar{\alpha}_i$, we can rewrite (2·10) as

$$\psi(x;\bar{\alpha}) = \psi_0(x(\sigma)) \int d^3\alpha w(\alpha) \exp(iS/\hbar), \tag{2·12}$$

where $\psi_0(x(\sigma))$ obeys (2·3) and (2·4) with $S = S(x; \bar{\alpha}_1, \bar{\alpha}_2, \bar{\alpha}_3)$. The spin state is then described by the spinor wave function $\psi_0(x(\sigma))$, or more strictly speaking, by appropriately renormalized version of it.

It can be shown by using (2·3) and (2·4) that the current,

$$j^\mu(x) = e_k{}^\mu \bar{\psi}_0(x) \gamma^k \psi_0(x), \tag{2·13}$$

is conserved,

$$\nabla_\mu j^\mu = 0, \tag{2·14}$$

and that $j_\mu(x)$ is proportional to $\partial_\mu S(x)$,

$$j_\mu(x) = \rho(x) \partial_\mu S(x), \tag{2·15}$$

where $\rho(x)$ is a positive function, the property of which can be obtained from the conservation law (2·14). Here ∇_μ denotes the familiar covariant derivative with respect to the Christoffel symbol. Let us define the renormalized spinor wave function $\psi_0'(x)$ by

$$\psi_0' = \rho^{-1/2} \psi_0, \tag{2·16}$$

then it follows from (2·6) that ψ_0' satisfies

$$\left[g^{\mu\nu} (\partial_\mu S) \nabla_\nu + \frac{3i}{4} \varepsilon_{ijmn} e^{m\mu} (\partial_\mu S) a^n S^{ij} \right] \psi_0' = 0, \tag{2·17}$$

where ∇_ν denotes the covariant derivative with respect to the Ricci rotation coefficients, and a^n is the axial-vector part of the torsion tensor. [Namely, ∇_ν denotes the covariant derivative in the Riemann space-time. When it acts on Greek indices it is the covariant derivative with respect to the Christoffel symbol, whereas it means the covariant derivative with respect to the Ricci rotation coefficients when it operates on Latin and spinor indices.] Here ε_{ijmn} is the usual totally antisymmetric tensor in the Minkowski space-time with $\varepsilon^{0123} = +1 = -\varepsilon_{0123}$.

We shall take $\psi_0'(x(\sigma))$, which is obtained from $\psi_0(x(\sigma))$ by (2·16), as the wave function of spin for the spin-1/2 particle described by the wave packet

*) Although the magnitude of spin vanishes as $\hbar \to 0$, the direction of spin, i.e., the spin polarization, has a well-defined classical limit.[7]

(2·12), because $\psi_0'(x(\sigma))$ is normalized as

$$e_k{}^\mu \overline{\psi}_0{}' \gamma^k \psi_0'(x(\sigma)) = p^\mu(\sigma). \tag{2·18}$$

It then follows from (2·3) and (2·17) that $\psi_0'(x(\sigma))$ satisfies

$$[e_k{}^\mu p_\mu \gamma^k + m]\psi_0'(x(\sigma)) = 0 \tag{2·19}$$

and

$$\left[\frac{\nabla}{d\sigma} + \frac{3i}{4}\varepsilon_{ijmn} p^m a^n S^{ij}\right]\psi_0'(x(\sigma)) = 0\,, \tag{2·20}$$

where $\nabla/d\sigma$ is the covariant differentiation along the world line, $\nabla/d\sigma = (dx^\mu/d\sigma)\nabla_\mu$. Equation (2·20) defines the temporal change of the spin wave function as a spin-1/2 particle moves along the world line $x^\mu(\sigma)$.

Following the ordinary procedure to describe spin precession of the electron and the muon in a slowly varying magnetic field,[7] let us introduce the spin polarization vector $S^\mu(\sigma)$ by

$$S^\mu(\sigma) = -\frac{1}{2} e_k{}^\mu \overline{\psi}_0{}' \gamma^5 \gamma^k \psi_0'(x(\sigma))\,, \tag{2·21}$$

which has only three independent components, because (2·19) implies

$$p^\mu(\sigma) S_\mu(\sigma) = 0\,. \tag{2·22}$$

For a massless particle, i.e., for a neutrino and an anti-neutrino, the spin wave function $\psi_0'(x(\sigma))$ is a two-component spinor satisfying

$$\gamma^5 \psi_0'(x(\sigma)) = \begin{cases} -\psi_0'(x(\sigma)) & \text{for neutrino,} \\ +\psi_0'(x(\sigma)) & \text{for anti-neutrino,} \end{cases} \tag{2·23}$$

and, therefore, we have

$$S^\mu(\sigma) = \begin{cases} -\dfrac{1}{2} p^\mu(\sigma) & \text{for neutrino,} \\ +\dfrac{1}{2} p^\mu(\sigma) & \text{for anti-neutrino,} \end{cases} \tag{2·24}$$

where (2·18) is used. Equation (2·24) shows that massless spin-1/2 particles have definite helicity: $-1/2$ for neutrino and $+1/2$ for anti-neutrino. This property of massless particles can also be seen directly from (2·3).

By virtue of (2·20), the spin polarization vector obeys the following equation of spin precession:*)

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*) The totally antisymmetric tensor $\varepsilon^{\mu\nu\rho\sigma}$ is defined by

$$\varepsilon^{\mu\nu\rho\sigma} = e_i{}^\mu e_j{}^\nu e_m{}^\rho e_n{}^\sigma \varepsilon^{ijmn}.$$

$$\frac{\nabla}{d\sigma}S^{\mu}=-\frac{3}{2}\varepsilon^{\mu\nu\rho\sigma}p_{\nu}a_{\rho}S_{\sigma}\,. \tag{2·25}$$

This equation implies that the motion of spin polarization violates the equivalence principle.

§ 3. Equation of motion for macroscopic bodies

Let us define $P^{k\nu}=e^{k}{}_{\mu}P^{\mu\nu}$ and $Q^{ij\nu}$ by

$$eP^{k\nu}=ee^{k}{}_{\mu}P^{\mu\nu}=-\delta eL_{G}/\delta e_{k\nu}\,, \tag{3·1}$$

$$eQ^{ij\nu}=-\delta eL_{G}/\delta A_{ij\nu}\,, \tag{3·2}$$

then from the invariance of the gravitational action under the group of general coordinate transformations follows the identity,[8]

$$e\,(P^{k\nu}e_{k\nu,\mu}+Q^{ij\nu}A_{ij\nu,\mu})-(eP_{\mu}^{\cdot\nu}+eA_{ij\mu}Q^{ij\nu})_{,\nu}=0\,. \tag{3·3}$$

Using the gravitational field equations which read

$$P^{\mu\nu}=T^{\mu\nu},\qquad Q^{ij\nu}=-\frac{1}{2}S^{ij\nu}, \tag{3·4}$$

and the formula,

$$(eP_{\mu}^{\cdot\nu})_{,\nu}-eP^{k\nu}e_{k\nu,\mu}=e\nabla_{\nu}P_{\mu}^{\cdot\nu}+e\Delta_{\lambda\nu\mu}P^{\lambda\nu} \tag{3·5}$$

in (3·3), we obtain *the response equation to gravitation*,

$$\nabla_{\nu}T_{\mu}^{\cdot\nu}+\Delta_{\lambda\nu\mu}T^{\lambda\nu}-A_{ij[\mu,\nu]}S^{ij\nu}-\frac{1}{2e}A_{ij\mu}(eS^{ij\nu})_{,\nu}=0\,, \tag{3·6}$$

where $\Delta_{\lambda\nu\mu}$ is defined by*)

$$\Delta_{\lambda\nu\mu}=\frac{1}{2}(C_{\lambda\nu\mu}-C_{\nu\lambda\mu}-C_{\mu\lambda\nu}) \tag{3·7}$$

with

$$C_{\lambda\mu\nu}=e^{k}{}_{\lambda}(\partial_{\mu}e_{k\nu}-\partial_{\nu}e_{k\mu})\,. \tag{3·8}$$

By virtue of the formula,**)

$$T^{[ij]}=\frac{1}{2e}D_{\nu}(eS^{ij\nu})\,, \tag{3·9}$$

which follows from the local Lorentz invariance of the matter action, Eq. (3·6)

*) $\Delta_{ij\nu}=e_{i}{}^{\lambda}e_{j}{}^{\mu}\Delta_{\lambda\mu\nu}$ is the Ricci rotation coefficients.

**) The covariant derivative D_{ν} acts only on Latin indices.

can be rewritten as

$$\nabla_\nu T_{\mu}^{\cdot\nu} + \frac{1}{2e} \Delta_{ij\mu} D_\nu (e S^{ij\nu}) - A_{ij[\mu'\nu]} S^{ij\nu} - \frac{1}{2e} A_{ij\mu} (e S^{ij\nu})_{,\nu} = 0 \,. \qquad (3\cdot 10)$$

Now we apply this response equation to the motion of a neutral macroscopic body such as a test particle employed in terrestrial experiments or such as an astronomical object like a star or a planet. For such a macroscopic body we can assume that the constituent particles of the body can be treated as if they were spinless. Accordingly, we can ignore in Eqs. (3·9) and (3·10) those terms which involve spin tensor $S^{ij\nu}$. The energy-momentum tensor of a macroscopic body is then symmetric and satisfies the conservation law,

$$\nabla_\nu T^{\mu\nu} = 0. \qquad (3\cdot 11)$$

Thus *the motion of a macroscopic body obeys the same law as in General Relativity.* In particular, the world line of a test particle is independent of its composition and structure, and the unique world line coincides with the geodesics of the metric $\boldsymbol{g} = \{g_{\mu\nu}\}$.

§ 4. Various limiting cases

We shall now turn to the second topic of the present paper. The most general gravitational Lagrangian L_G derived in I is represented as

$$L_G = aF + L_T + L_F \,, \qquad (4\cdot 1)$$

where L_T is formed of three quadratic forms of the translation gauge field strength with three parameters, α, β and γ, and L_F consists of six quadratic forms of the Lorentz gauge field strength with six parameters, $a_1, \cdots, a_6$. The most general gravitational field equations read

$$2aF_{ji} + 2F^{kmn}_{\cdot\cdot\cdot i} J_{[km][nj]} + 2D^k F_{ijk} + 2v^k F_{ijk} + 2H_{ij} - \eta_{ij} L_G = T_{ij} \qquad (4\cdot 2)$$

for the translation gauge field and

$$2D_m J^{[ij][km]} - (T^{k}_{\cdot mn} - 2\delta^k{}_m v_n) J^{[ij][mn]} + H^{ijk} = +(1/2) S^{ijk} \qquad (4\cdot 3)$$

for the Lorentz gauge field. These field equations can be rewritten into alternative forms,

$$2aG_{ij}(\{\ \}) + (3a_2 + 2a_5)\Big[R_{im}(\{\ \}) R_j^{\ m}(\{\ \})$$
$$+ R^{mn}(\{\ \})\Big(R_{imjn}(\{\ \}) - \frac{1}{2}\eta_{ij} R_{mn}(\{\ \})\Big)\Big]$$
$$-(2a_2 + a_5 - 4a_6) R(\{\ \})\Big(R_{ij}(\{\ \}) - \frac{1}{4}\eta_{ij} R(\{\ \})\Big) +$$

$$+2R^{kmn}_{\cdot\cdot\cdot i}(\{\ \})J_{[km][nj]}(K)+2F^{kmn}_{\cdot\cdot\cdot i}(K)\left(J_{[km][nj]}(\{\ \})+J_{[km][nj]}(K)\right)$$
$$+2D^kF'_{ijk}+2v^kF'_{ijk}+2H'_{ij}-\eta_{ij}(L'_T+L'_F)$$
$$=T_{ij} \tag{4·4}$$

and

$$(3a_2+2a_5)\nabla_{[i}G_{j]k}(\{\ \})+(a_2+a_5+4a_6)\eta_{k[i}\partial_{j]}G(\{\ \})$$
$$-2(D^m-\nabla^m)J_{[ij][km]}(\{\ \})-2D^mJ_{[ij][km]}(K)$$
$$+(T_k^{\cdot mn}-2\delta_k^{\ m}v^n)\left(J_{[ij][mn]}(\{\ \})+J_{[ij][mn]}(K)\right)-H_{ijk}$$
$$=-\frac{1}{2}S_{ijk}, \tag{4·5}$$

respectively, by means of the decomposition of the Lorentz gauge field into the Ricci rotation coefficients and the contorsion tensor. (See §§ 3 and 5 of I, especially (I·3·24), (I·3·30), (I·5·12) and (I·5·18).)

Let us consider three classes of limiting cases: (A) $a_i=0$ $(i=1, 2, \cdots, 6)$, (B) $a_i=\infty$ $(i=1, 2, \cdots, 6)$ and (C) $\alpha=\infty$, $\beta=\infty$, $\gamma=\infty$. The meaning of (A) and (B) can be interpreted as follows: Suppose that the parameters, $a_i\,(i=1, 2, \cdots, 6)$, are all of the same order of magnitude, represented as

$$a_i=\frac{1}{g^2}\bar{a}_i, \qquad (i=1, 2, \cdots, 6) \tag{4·6}$$

where g is a parameter characterizing the magnitude of a_i's, and $\bar{a}_i$'s are of the order of unity. The parameter g can be regarded as standing for the coupling strength of the Lorentz gauge field.*) The cases (A) and (B) are then the strong $(g\to\infty)$ and the weak $(g\to 0)$ coupling limits, respectively.

A1. *The case of* $a_1=a_2=\cdots=a_6=0$ *and* $\alpha=\beta=\gamma=0$

This case has been extensively studied in the past, and is usually called the Einstein-Cartan theory.[9] The gravitational Lagrangian is given by

$$L_G=aF, \tag{4·7}$$

and the torsion field is frozen at the place of matter (see **A2** below for further details). The parameter a is determined from the Newtonian limit as $a=1/2\kappa$.

*) In the abelian gauge theory, the covariant derivative is usually defined by $(\partial_\mu-ieA_\mu)$ with e the coupling constant, and the Lagrangian of the gauge field is given by $L=-(1/4)F_{\mu\nu}F^{\mu\nu}$ with $F_{\mu\nu}=\partial_\mu A_\nu-\partial_\nu A_\mu$. If we redefine the gauge field by $A'_\mu=eA_\mu$, however, the covariant derivative becomes $(\partial_\mu-iA'_\mu)$, and the Lagrangian is expressed as $L=-(1/4e^2)F'_{\mu\nu}F'^{\mu\nu}$ with $F'_{\mu\nu}=\partial_\mu A'_\nu-\partial_\nu A'_\mu$. In view of (I·2·16), we see that the Lorentz gauge field $A_{ij\mu}$ corresponds not to A_μ but to A'_μ. Therefore, we can regard g of (4·6) as a parameter characterizing the coupling strength of the Lorentz gauge field.

A2. *The case of* $a_1=a_2=\cdots=a_6=0$

This is the simplest generalization of the case **A1**.[8] The gravitational Lagrangian is

$$\begin{aligned} L_G &= aF+L_T \\ &= aF+\alpha(t_{ijk}t^{ijk})+\beta(v_iv^i)+\gamma(a_ia^i), \end{aligned} \tag{4·8}$$

and the field equations (4·2) and (4·3) read

$$2aG_{ij}(\{\ \})+2D^kF'_{ijk}+2v^kF'_{ijk}+2H'_{ij}-\eta_{ij}L'_T=T_{ij}, \tag{4·9}$$

$$H^{ijk}=\frac{1}{2}S^{ijk}, \tag{4·10}$$

where we have used the alternative form (4·4) to write (4·9). Equation (4·10) can be solved to give

$$\left(\alpha+\frac{2a}{3}\right)t_{ijk}=\frac{1}{3}S^{(t)}_{ijk}, \tag{4·11a}$$

$$\left(\beta-\frac{2a}{3}\right)v_i=\frac{1}{6}S_i^{(v)}, \tag{4·11b}$$

$$\left(\gamma+\frac{3a}{2}\right)a_i=-\frac{3}{8}S_i^{(a)}, \tag{4·11c}$$

where $S^{(t)}_{ijk}$, $S_i^{(v)}$ and $S_i^{(a)}$ are the irreducible parts of the spin tensor (see the Appendix for their explicit forms). Thus, the torsion field is still frozen at the place of matter. By virtue of the relation (see (I·5·17a)),

$$F'_{ijk}=\frac{1}{2}(H_{ijk}-H_{ikj}-H_{jki}), \tag{4·12}$$

F'_{ijk} is given by

$$F'_{ijk}=\frac{1}{4}(S_{ijk}+S_{kij}+S_{kji}). \tag{4·13}$$

It then follows from (4·9) that

$$2aG_{ij}(\{\ \})=T^{(\mathrm{sym})}_{ij}-2H'_{ij}+\eta_{ij}L'_T \tag{4·14}$$

with $T^{(\mathrm{sym})}_{ij}$ defined by

$$T^{(\mathrm{sym})}_{ij}=T_{ij}-\frac{1}{2e}D_\mu(eS_{ij}^{\cdot\cdot\mu}+eS^{\mu}_{\cdot ij}+eS^{\mu}_{\cdot ji}), \tag{4·15}$$

where D_μ acts only on Latin indices. Owing to the local Lorentz invariance of the matter action, $T^{(\mathrm{sym})}_{ij}$ is symmetric with respect to i and j. The parameter a is chosen as $a=1/2\kappa$ in order to ensure the correct Newtonian limit. The right-hand side of (4·14) contains *quadratic terms of the spin tensor,* which show that there exists contact spin-spin interaction among the fundamental particles of spin 1/2.[8),9)] These contact spin-spin interaction terms are absent from the Einstein equation of General Relativity, and are the characteristic feature of Poincaré gauge theory with $a_i=0$ $(i=1, 2, \cdots, 6)$.

B. *The case of* $a_i=\infty$ $(i=1, 2, \cdots, 6)$

Consider the weak coupling limit, $g\to 0$, of Eqs. (4·2) and (4·3). In this limit (4·3) becomes

$$2D_m\bar{J}^{[ij][km]}-(T^{k}_{\cdot mn}-2\delta_m{}^k v_n)\bar{J}^{[ij][mn]}=0 \tag{4·16}$$

with $\bar{J}^{[ij][mn]}$ defined from $J^{[ij][mn]}$ by replacing a_i's by $\bar{a}_i$'s, and so it has a trivial solution,

$$F_{ijmn}=0\,. \tag{4·17}$$

The underlying space-time of this solution is the Weitzenböck space-time characterized by the torsion tensor alone. In the gauge with

$$A_{ij\mu}=0\,, \tag{4·18}$$

the tetrad field of (I·2·11) forms a quadruplet of the parallel vector fields, which defines absolute parallelism of vectors and spinors. Using (4·17) in (4·2) gives

$$2D^kF_{ijk}+2v^kF_{ijk}+2H_{ij}-\eta_{ij}L_T=T_{ij}\,, \tag{4·19}$$

which is just the gravitational field equation of New General Relativity.[4)] Consequently, *Poincaré gauge theory reduces to New General Relativity in the weak coupling limit,* $g\to 0$*, i.e.,* $a_i\to\infty$.

Equation (4·19) contains only three parameters, α, β and γ: These parameters are connected with the parameters, c_1, c_2 and c_3 of Ref. 4) by

$$c_1=\alpha+\frac{1}{3\kappa}\,,\qquad c_2=\beta-\frac{1}{3\kappa}\,,\qquad c_3=\gamma+\frac{3}{4\kappa}\,. \tag{4·20}$$

It has been shown[4)] that the correct Newtonian limit is ensured if the parameters satisfy

$$4\bar{c}_1+\bar{c}_2+9\bar{c}_1\bar{c}_2=0 \tag{4·21}$$

with $\bar{c}_1=kc_1$ and $\bar{c}_2=kc_2$. In terms of α, β and γ, this condition reads

$$\alpha+4\beta+9\alpha\beta\kappa=0. \tag{4·22}$$

The parameters, c_1 and c_2, have been estimated from the solar system experiments as $\kappa c_1=0.001\pm 0.001$ and $\kappa c_2=-0.005\pm 0.005$. Suggested by this estimation, we have proposed to adopt the particular choice of parameters,

$$c_1 = c_2 = 0\,, \quad \text{i.e.,} \quad \alpha = -1/3\kappa = -\beta\,. \tag{4·23}$$

C. *The case of* $\alpha=\infty$, $\beta=\infty$, $\gamma=\infty$

When α, β and γ become infinitely large, it follows from (4·5) that the torsion field T_{ijk} becomes infinitesimally small with H_{ijk} kept finite: In the limit $\alpha=\infty$, $\beta=\infty$ and $\gamma=\infty$, the underlying space-time is the Riemann space-time with vanishing torsion, and H_{ijk} is given by

$$H_{ijk} = (3a_2+2a_5)\nabla_{[i}G_{j]k}(\{\ \}) + (a_2+a_5+4a_6)\eta_{k[i}\partial_{j]}G(\{\ \}) + \frac{1}{2}S_{ijk}\,. \tag{4·24}$$

Use of (4·12) and (4·24) in (4·4) gives

$$\begin{aligned}
&2aG_{ij}(\{\ \}) + (3a_2+2a_5)\left\{\nabla_k\nabla^k G_{ij}(\{\ \}) - \frac{1}{3}(\eta_{ij}\nabla_k\nabla^k - \nabla_i\nabla_j)G(\{\ \})\right\} \\
&-\frac{1}{3}(a_5+12a_6)(\eta_{ij}\nabla_k\nabla^k - \nabla_i\nabla_j)G(\{\ \}) + 2(3a_2+2a_5)R^{mn}(\{\ \})R_{imjn}(\{\ \}) \\
&-(2a_2+a_5-4a_6)R(\{\ \})R_{ij}(\{\ \}) \\
&-\frac{1}{4}\eta_{ij}\{2(3a_2+2a_5)R_{mn}(\{\ \})R^{mn}(\{\ \}) - (2a_2+a_5-4a_6)R(\{\ \})^2\} \\
&= T_{ij} - \frac{1}{2}\nabla^k(S_{ijk}+S_{kij}+S_{kji})\,.
\end{aligned} \tag{4·25}$$

This is just the gravitational field equation in General Relativity with the quadratic Lagrangian[10),11)]

$$L_{GR} = aR(\{\ \}) - bR_{mn}(\{\ \})R^{mn}(\{\ \}) + cR(\{\ \})^2, \tag{4·26}$$

where

$$b = -\frac{1}{2}(3a_2+2a_5)\,, \qquad c = -\frac{1}{4}(2a_2+a_5-4a_6)\,, \tag{4·27}$$

and a is determined to be $a=1/2\kappa$ from the Newtonian limit. Thus *Poincaré gauge theory reduces to General Relativity with the quadratic Lagrangian in the limit,* $\alpha=\infty$, $\beta=\infty$ *and* $\gamma=\infty$.

In the weak field approximation, the weak gravitational field $h_{\mu\nu}=g_{\mu\nu}-\eta_{\mu\nu}$ contains three classes of particles:[11)] The familiar massless graviton of spin 2, a massive particle of spin 2 and mass m_2,

$$m_2 = [-\kappa(3a_2+2a_5)]^{-1/2}, \tag{4·28}$$

and finally a massive particle of spin 0 and mass m_0,

$$m_0 = [\kappa(a_5+12a_6)]^{-1/2}. \tag{4·29}$$

§ 5. Conclusion

We have studied two problems in Poincaré gauge theory with linear and quadratic Lagrangians.

(i) Classical equations of motion for spin-1/2 particles and macroscopic bodies

For spin-1/2 particles we have applied the semiclassical approximation to the Dirac equation, and obtained the following results:

(1) When the wave-length is so short that the quantum mechanical uncertainty in location is negligible, the world line of spin-1/2 particles is the geodesics of the metric $\boldsymbol{g}$ in accord with the equivalence principle.

(2) The spin polarization vector obeys the equation of spin precession which violates the equivalence principle.

For macroscopic bodies we have shown with the help of the gravitational field equations that the energy-momentum tensor $T^{\mu\nu}$ is symmetric with respect to μ and ν and that $T^{\mu\nu}$ satisfies the conservation law

$$\nabla_\nu T^{\mu\nu}=0 \tag{5·1}$$

with ∇_ν the familiar covariant derivative with respect to the Christoffel symbol. Thus, the motion of macroscopic bodies obeys the equivalence principle.

(ii) Various limits

The present approach based on the Poincaré gauge invariance is so general that various limits can be taken. If the six parameters, $a_i\,(i=1, \cdots, 6)$, are all vanishing, then it is reduced to Hayashi and Bregman's theory;[8] in this case if the parameters, α, β and γ, are further vanishing, then it finally goes to Einstein and Cartar's theory.[9] On the other hand, if the limit, $\alpha=\infty$, $\beta=\infty$ and $\gamma=\infty$, is taken, then the present theory goes to General Relativity with the quadratic Lagrangian.[10),11)] Finally, if the six parameters, $a_i\,(i=1, \cdots, 6)$, are all infinite,

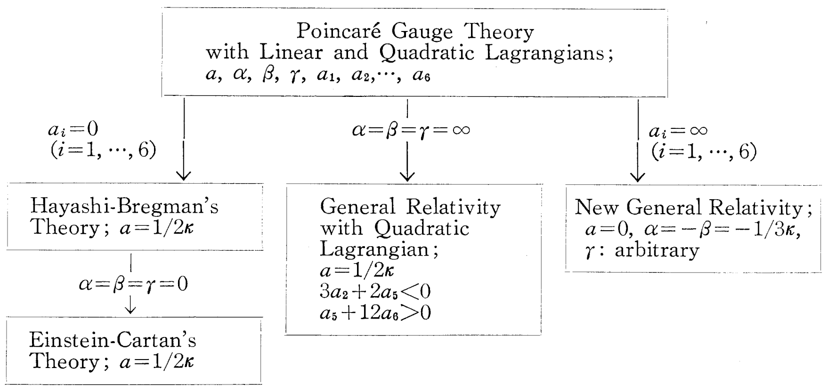

Fig. 1. Reduction of Poincaré gauge theory with the ten parameters: The gravity Lagrangian is given by $L_G=a$ (linear inv. in the Lorentz gauge field strength) $+(\alpha,\beta,\gamma)$ (inv. quadratic in the translation gauge field strength) $+(a_1,\cdots,a_6)$ (inv. quadratic in the Lorentz gauge field strength).

then the present theory is reduced to New General Relativity.[4] See Fig. 1 for this reduction.

Appendix

——*Irreducible Decomposition of the Spin Tensor*——

The spin tensor is decomposed in the same manner as in the contorsion tensor K_{ijk}, which is represented in terms of the irreducible parts, t_{ijk}, v_i and a_i, by

$$K_{ijk}=-\frac{4}{3}t_{k[ij]}-\frac{2}{3}\eta_{k[i}v_{j]}+\frac{1}{2}\varepsilon_{ijkm}a^m, \tag{A·1}$$

or conversely

$$t_{ijk}=K_{k(ij)}+\frac{1}{3}\eta_{k(i}v_{j)}-\frac{1}{3}\eta_{ij}v_k\,, \tag{A·2a}$$

$$v_i=\eta^{jk}K_{ijk}\,, \tag{A·2b}$$

$$a_i=\frac{1}{3}\varepsilon_{ijkm}K^{jkm}. \tag{A·2c}$$

Decomposition scheme of S_{ijk} is obtained by replacing K, t, v and a by S, $S^{(t)}$, $S^{(v)}$ and $S^{(a)}$, respectively.

Hayashi and Bregman[8] have adopted another decomposition scheme of S_{ijk}, according to which S_{ijk} is decomposed in the same manner as the torsion tensor T_{kij}. If we denote the irreducible parts due to the latter scheme by S^T_{ijk}, $S_i{}^V$ and $S_i{}^A$, then they are connected with $S^{(t)}_{ijk}$, $S_i{}^{(v)}$ and $S_i{}^{(a)}$ due to the former scheme by

$$S^T_{ijk}=-S^{(t)}_{ijk}\,, \tag{A·3a}$$

$$S_i{}^V=-S_i{}^{(v)}, \tag{A·3b}$$

$$S_i{}^A=\frac{1}{2}S_i{}^{(a)}. \tag{A·3c}$$

In this paper we take $S^{(t)}_{ijk}$, $S_i{}^{(v)}$ and $S_i{}^{(a)}$ due to the former scheme as the irreducible parts of the spin tensor.

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