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Gravity's Rainbow induces topology change

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Abstract In this work, we explore the possibility that quantum fluctuations induce a topology change, in the context of Gravity's Rainbow. A semiclassical approach is adopted, where the graviton one-loop contribution to a classical energy in a background spacetime is computed through a variational approach with Gaussian trial wave functionals. The energy density of the graviton one-loop contribution, or equivalently the background spacetime, is then let to evolve, and consequently the classical energy is determined. More specifically, the background metric is fixed to be Minkowskian in the equation governing the quantum fluctuations, which behaves essentially as a backreaction equation, and the quantum fluctuations are let to evolve; the classical energy, which depends on the evolved metric functions, is then evaluated. Analyzing this procedure, a natural ultraviolet cutoff is obtained, which forbids the presence of an interior spacetime region, and this may result in a multiply connected spacetime. Thus, in the context of Gravity's Rainbow, this process may be interpreted as a change in topology, and in principle it results in the presence of a planckian wormhole.

1 Introduction

It was John A. Wheeler [1,2] who first conjectured that spacetime could be subjected to a topological fluctuation at the Planck scale, meaning that spacetime undergoes a deep and rapid transformation in its structure. The changing spacetime is best known as *spacetime foam*, which can be taken as a model for the quantum gravitational vacuum. Wheeler also considered wormhole-type solutions as objects of the spacetime quantum foam connecting different regions of spacetime at the Planck scale [2,3]. These Wheeler wormholes were obtained from the coupled equations of electromagnetism and general relativity and were denoted "geons", i.e., gravitational-electromagnetic entities. However, these solutions were singular and were not traversable [4]. In fact, the geon concept possesses interesting properties, such as the absence of charges or currents and the gravitational mass originates solely from the energy stored in the electromagnetic field, i.e., there are no material masses present. These characteristics gave rise to the terms "charge without charge" and "mass without mass", respectively.

Paging through history, one finds that these entities were further explored by several authors in different contexts. Indeed, Ernst analyzed idealized spherical "geons" using a simple adaptation of the Ritz variational principle [5], and furthermore explored toroidal geons, where the electromagnetic vector potential is vanishingly small except within a toroidal region of space [6]. In fact, the electromagnetic field physically consists of light waves circling the torus in either direction, so that the torus of electromagnetic field energy was denoted a *toroidal geon*. It was shown that toroidal geons of large major radius to minor radius ratio may be studied using an approximation of linear geons, where the electromagnetic field energy is confined to an infinitely long circular cylinder rather than to a torus. Indeed, the electromagnetic field potentials of a toroidal geon or of a linear geon possess the same general nature as the electromagnetic field potentials encountered in the solution of classical toroidal and cylindrical wave guide problems. Thus, these results provided the foundation material for a proposed later treatment of toroidal geons.

Later, Brill and Hartle [7] extended the previous analysis to the case where gravitational waves are the source of the geon's mass energy, where the background spherically symmetric metric describes the large-scale features of the geon. It was shown that the waves superimposed on this background have an amplitude small enough so that their dynamics can be analyzed in the linear approximation. However, their wavelength is so short, and their time dependence so rapid that their energy is appreciable and produces the strongly curved back-

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ground metric in which they move. It was also found that the large-scale features of the spherical gravitational geons are identical to those of the spherical electromagnetic geons analyzed previously. In fact, later work by Anderson and Brill [8] showed that the geon solution is a self-consistent solution to Einstein's equations and that, to leading order, the equations describing the geometry of the gravitational geon are identical to those derived by Wheeler for the electromagnetic geon.

Komar [9] showed that there exist solutions of the vacuum Einstein field equations with the property that exterior to the Schwarzschild radius, the solution appears to be that of a static spherically symmetric particle of mass m, whereas interior to the Schwarzschild radius the topology remains Euclidean and the solutions have the property of a bundle of gravitational radiation so intense that the mutual gravitational attraction of the various parts of the bundle prevent the radiation from spreading beyond the Schwarzschild radius. Komar also argued that no singularity can ever be observed exterior to the Schwarzschild radius. However, it was shown that the Komar bootstrap gravitational geon solution does in fact display a singular behavior along portions of an axis in the regions in which the solution deviates from the standard Schwarzschild solution [10].

An interesting geon solution was explored by Kaup [11] in the context of the Klein–Gordon Einstein equations (Klein–Gordon geons), which reveal that these geons have properties that are different from the other gravitating systems studied previously. Indeed, the equilibrium states of these geons seem analogous to other gravitating systems, but it was shown that adiabatic perturbations are forbidden, when the stability is considered from a thermodynamical viewpoint. The reason for this is that the equations of state for the thermodynamical variables are not algebraic equations, but instead they are differential equations. Consequently, the usual concept of an equation of state breaks down when Klein–Gordon geons are considered. When the question of stability is reconsidered in terms of infinitesimal perturbations of the basic fields, it was then found that Klein–Gordon geons will not undergo spherically symmetric gravitational collapse. Thus, the Klein–Gordon geons considered by Kaup are counterexamples to the conjecture that gravitational collapse is inevitable.

In fact, much work was done over the decades, but due to the extremely ambitious program and the lack of experimental evidence the geon concept soon died out. However, it is interesting to note that Misner inspired by Wheeler's geon representation, found wormhole solutions to the source-free Einstein equations [12], and with the introduction of multi-connected topologies in physics, the question of causality inevitably arose. Thus, Wheeler and Fuller examined this situation in the Schwarzschild solution and found that causality is preserved [13], as the Schwarzschild throat pinches off in a finite time, preventing the traversal of a signal from one region to another through the wormhole. Nevertheless, Graves and Brill [14], considering the Reissner–Nordström metric, also found wormhole-type solutions possessing an electric flux flowing through the wormhole. They found that the region of minimum radius, the "throat", contracted, reaching a minimum and re-expanded after a finite proper time, rather than pinching off as in the Schwarzschild case. The throat, "cushioned" by the pressure of the electric field through the throat, pulsated periodically in time.

In the context of the quantum gravitational vacuum, some authors have investigated the effects of such a foamy space on the cosmological constant, for instance, one example is the celebrated Coleman mechanism, where wormhole contributions suppress the cosmological constant, explaining its small observed value [15]. Nevertheless, how to realize such a foam-like space and also whether this represents the real quantum gravitational vacuum is still unknown. However, it is interesting to observe that Ellis et al. considered a foam-like structure built in terms of D-branes to discuss phenomenological aspects [16–19]. Wheeler when discussing the quantum fluctuations in the spacetime metric [2] considered that a typical fluctuation in a typical gravitational potential is of the order $\Delta g \sim (hG/c^3)^{1/2}/L$ which become appreciable for small length scales L. A fundamental question is whether a change in topology may be induced by large metric fluctuations. In fact, Wheeler has argued in favor of a topology change and recently researchers in quantum gravity have accepted that the notion of spacetime foam leads to topology-changing quantum amplitudes and to interference effects between spacetimes of different topologies [20].

Indeed, a classical spacetime can be modeled by a single Lorentzian manifold, which is sliced into a set of spatial hypersurfaces by a natural definition of a time parameter (see also Ref. [21] relating neutron star interiors and topology change). We can mention some results about topological constraints on the *classical* evolution of general relativistic spacetimes. These were summarized in two points by Visser [20]:

1. In causally well-behaved classical spacetimes the topology of space does not change as a function of time.
2. In causally ill-behaved classical spacetimes the topology of space can sometimes change.

From the *quantum* point of view we can separate the problem of topology change generated by a canonical quantization approach and a functional integral quantization approach. The Hawking topology change theorem is thus enough to show that the topology of space cannot change in canonically quantized gravity [22]. In the Feynman functional integral quantization of gravitation things are different. Indeed, in this formalism, it is possible to adopt an approach to spacetime foam where we know that fluctuations of topology become an important phenomenon at least at the Planck scale [23].

As discussed in Ref. [24], the Casimir energy approach involving quasi-local energy difference calculations may reflect or measure the occurrence of a topology change, and in particular, the Casimir energy was used as an indicator of topology change between wormholes and dark energy stars [25]. More specifically, the quantity

$$E_{\rm ADM}^{\rm DS} - E_{\rm ADM}^{\rm Wormhole} = \left(E_{\rm ADM}^{\rm DS} - E_{\rm ADM}^{\rm Flat}\right) - \left(E_{\rm ADM}^{\rm Wormhole} - E_{\rm ADM}^{\rm Flat}\right) \gtreqless 0, \qquad (1)$$

was used to understand which configuration is preferred with respect to the Arnowitt–Deser–Misner (ADM) energy. It was found that the classical term was not able to predict the appearance of a wormhole or the permanence of a dark energy star. Therefore one was forced to compute quantum effects. The implicit subtraction procedure of Eq. (1) can be extended in such a way that we can include quantum effects: this is the Casimir energy or in other terms, the zero point energy (ZPE). It is interesting to note that the same Casimir energy indicator described in Eq. (1) has been used in Refs. [26–34] to build a model of spacetime foam based on wormholes of different nature, namely Schwarzschild, Schwarzschild–de Sitter, Schwarzschild–anti-de Sitter and Reissner–Nordström-like wormholes. In particular one finds that if the whole universe is filled with Schwarzschild-like wormholes, one finds an agreement with the Coleman mechanism on the behavior of the cosmological constant [26].

In the present paper, we are interested in the possibility that quantum fluctuations induce a topology change, in the context of Gravity's Rainbow [35–37]. The latter is a distortion of the spacetime metric at energies comparable to the Planck energy, and a general formalism, denoted as deformed or doubly special relativity, was developed in order to preserve the relativity of inertial frames, maintain the Planck energy invariant and impose the requirement that in the limit $E/E_P \to 0$, the speed of a massless particle tends to a universal and invariant constant, c. Here, we adopt a semiclassical approach, where the graviton one-loop contribution to a classical energy in a background spacetime is computed through a variational approach with Gaussian trial wave functionals. In fact, it has been shown explicitly that the finite one-loop energy may be considered as a self-consistent source for a traversable wormhole [38]. In addition to this, a renormalization procedure was introduced and a zeta function regularization was involved to handle the divergences. The latter approach was also explored [39] in the context of phantom energy traversable wormholes [40–42]. It was shown that the latter semiclassical approach prohibits solutions with a constant equation of state parameter, which further motivates the imposition of a radial dependent parameter, $\omega(r)$, and only permits solutions with a steep positive slope proportional to the radial derivative of the equation of state parameter, evaluated at the throat [39]. Using the semiclassical approach outlined above, exact wormhole solutions in the context of noncommutative geometry were also analyzed, and their physical properties and characteristics were explored [43]. Indeed, wormhole geometries have been obtained in a wide variety of contexts, namely, in modified theories of gravity [44–51], electromagnetic signatures of accretion disks around wormhole spacetimes [52,53], etc. (we refer the reader to [54] for a review). The semiclassical procedure followed in this work relies heavily on the formalism outlined in Ref. [38]. Rather than reproduce the formalism, we shall refer the reader to Ref. [38] for details, when necessary.

In this work, we explore an alternative approach to the semiclassical approach outlined above. Note that the traditional manner is to fix a background metric and obtain self-consistent solutions. Here, we let the quantum fluctuations evolve, and the classical energy, which depends on the evolved metric functions, is then evaluated. A natural ultraviolet (UV) cutoff is obtained which forbids an interior spacetime region, and which may result in a multiply connected spacetime. Thus, in the context of Gravity's Rainbow, this process may be interpreted as a change in topology, and consequently results in the presence of a planckian wormhole.

This paper is organized in the following manner: In Sect. 2, the semiclassical approach is briefly outlined, and the graviton one-loop contribution to a classical energy is computed through a variational approach with Gaussian trial wave functionals. In Sect. 3, the self-sustained equation is interpreted in a novel way, where the quantum fluctuations are let to evolve and the classical energy is then computed, consequently one arrives at solutions which may be interpreted as a change in topology. Finally, in Sect. 4, we conclude.

2 The classical term and the one-loop energy in Gravity's Rainbow

2.1 Effective field equations in a spherically symmetric background

In this paper, using a semiclassical approach, we explore the possibility to directly compute a topology change and in particular the birth of a traversable wormhole. The starting point is represented by the semiclassical gravitational Einstein field equation given by

$$G_{\mu\nu} = \kappa \left\langle T_{\mu\nu}\right\rangle^{\rm ren}, \qquad (2)$$

where $\left\langle T_{\mu\nu}\right\rangle^{\rm ren}$ is the renormalized expectation value of the stress-energy tensor operator of the quantized field, $G_{\mu\nu}$ is the Einstein tensor and $\kappa = 8\pi G$.

The semiclassical procedure followed in this work relies heavily on the formalism outlined in Ref. [38], where the graviton one-loop contribution to a classical energy in a traversable wormhole background was computed, through a variational approach with Gaussian trial wave functionals [38,55]. (Note that our approach is very close to the gravitational *geon* considered by Anderson and Brill [8], where the relevant difference lies in the averaging procedure). More specifically, the metric may be separated into a background component, $\bar{g}_{\mu\nu}$ and a perturbation $h_{\mu\nu}$, i.e., $g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu}$. The Einstein tensor may also be split into a part describing the curvature due to the background geometry and that due to the perturbation, i.e., $G_{\mu\nu}(g_{\alpha\beta}) = G_{\mu\nu}(\bar{g}_{\alpha\beta}) + \Delta G_{\mu\nu}(\bar{g}_{\alpha\beta}, h_{\alpha\beta})$, where $\Delta G_{\mu\nu}(\bar{g}_{\alpha\beta}, h_{\alpha\beta})$ may be considered a perturbation series in terms of $h_{\mu\nu}$. If the matter field source is absent, one may define an effective stress-energy tensor for the fluctuations as

$$\langle T_{\mu\nu}\rangle^{\text{ren}} = -\frac{1}{\kappa}\langle \Delta G_{\mu\nu}\left(\bar{g}_{\alpha\beta}, h_{\alpha\beta}\right)\rangle^{\text{ren}} . \tag{3}$$

From this point of view, the equation governing quantum fluctuations behaves as a backreaction equation. If we fix our attention to the energy component of the Einstein field equations, we need to introduce a time-like unit vector u^μ such that $u^\mu u_\mu = -1$. Then the semiclassical Einstein equations (2) projected on the constant time hypersurface Σ are given by

$$\begin{aligned}\frac{1}{2\kappa}\int_\Sigma \mathrm{d}^3x\sqrt{{}^3\bar{g}}G_{\mu\nu}\left(\bar{g}_{\alpha\beta}\right)u^\mu u^\nu &= -\int_\Sigma \mathrm{d}^3x\mathcal{H}^{(0)}\\ = -\frac{1}{2\kappa}\int_\Sigma \mathrm{d}^3x\sqrt{{}^3\bar{g}}&\langle\Delta G_{\mu\nu}\left(\bar{g}_{\alpha\beta}, h_{\alpha\beta}\right)u^\mu u^\nu\rangle^{\text{ren}},\end{aligned} \tag{4}$$

where we have integrated the projected Einstein field equations over Σ and where

$$\mathcal{H}^{(0)} = 2\kappa G_{ijkl}\,\pi^{ij}\pi^{kl} - \frac{\sqrt{\bar{g}}}{2\kappa}R \tag{5}$$

is the background field super-hamiltonian, G_{ijkl} is the DeWitt super-metric [56], and R is the curvature scalar.

In a series of papers [35–37,57–59], a distortion of the gravitational metric known as Gravity's Rainbow was introduced as a tool to keep the UV divergences under control. Briefly, the situation is the following: one introduces two arbitrary functions $g_1(E/E_P)$ and $g_2(E/E_P)$, denoted as *Rainbow's functions*, with the only assumption that

$$\lim_{E/E_P\to 0} g_1(E/E_P) = 1 \quad \text{and}$$
$$\lim_{E/E_P\to 0} g_2(E/E_P) = 1. \tag{6}$$

On a general spherical symmetric metric, such functions come into play in the following manner:

$$\begin{aligned}\mathrm{d}s^2 = -N^2(r)\frac{\mathrm{d}t^2}{g_1^2(E/E_P)} &+ \frac{\mathrm{d}r^2}{\left[1-\frac{b(r)}{r}\right]g_2^2(E/E_P)}\\ &+ \frac{r^2}{g_2^2(E/E_P)}\left(\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\phi^2\right),\end{aligned} \tag{7}$$

where $N(r)$ is the lapse function and $b(r)$ denotes the shape function.

It is clear that the classical energy on the l.h.s. of Eq. (4) is modified to the following expression [59]:

$$\begin{aligned}H_\Sigma^{(0)} &= \int_\Sigma \mathrm{d}^3x\,\mathcal{H}^{(0)} = -\frac{1}{16\pi G}\int_\Sigma \mathrm{d}^3x\,\sqrt{g}\,R\\ &= -\frac{1}{2G}\int_{r_0}^\infty \frac{\mathrm{d}r\,r^2}{\sqrt{1-b(r)/r}}\frac{b'(r)}{r^2 g_2(E)}.\end{aligned} \tag{8}$$

For simplicity, we consider $N(r) = 1$ throughout this work. Note that to be a wormhole solution the following conditions need to be satisfied at the throat $b(r_0) = r_0$ and $b'(r_0) < 1$; the latter condition is a consequence of the flaring-out condition of the throat, i.e., $(b-b'r)/b^2 > 0$ [60]; asymptotic flatness imposes $b(r)/r \to 0$ as $r \to +\infty$.

2.2 The one-loop energy in Gravity's Rainbow

Note that the r.h.s. of Eq. (4) is represented by the fluctuations of the Einstein tensor, which in this context, are the fluctuations of the hamiltonian which are evaluated through a variational approach with Gaussian trial wave functionals. The divergences are treated with the help of the Rainbow's functions avoiding therefore the use of a regularization and renormalization procedure. We find that the total regularized one-loop energy is given by

$$E^{TT} = -\frac{1}{2}\sum_\tau \frac{g_1(E)}{g_2^2(E)}\left[\sqrt{E_1^2(\tau)} + \sqrt{E_2^2(\tau)}\right], \tag{9}$$

where $E_i(\tau)$ are the eigenvalues of

$$\left(\tilde{\triangle}_L^m \tilde{h}^\perp\right)_{ij} = \frac{E^2}{g_2^2(E)}\tilde{h}_{ij}^\perp, \tag{10}$$

with the condition that $E_i^2(\tau) > 0$, $h^\perp$ is the traceless-transverse component of the perturbation, and $\hat{\triangle}_L^m$ is defined by

$$\left(\hat{\triangle}_L^m h^\perp\right)_{ij} = \left(\triangle_L h^\perp\right)_{ij} - 4R_i^k h_{kj}^\perp + {}^3R h_{ij}^\perp. \tag{11}$$

The operator $\triangle_L$ is the Lichnerowicz operator, which is given by

$$(\triangle_L h)_{ij} = \triangle h_{ij} - 2R_{ikjl}h^{kl} + R_{ik}h^k_j + R_{jk}h^k_i, \tag{12}$$

with $\triangle = -\nabla^a\nabla_a$. We refer the reader to Ref. [38] for further details.

With the help of the Regge–Wheeler representation [61], the eigenvalue equation (10) can be reduced to

$$\left[-\frac{d^2}{dx^2} + \frac{l\,(l+1)}{r^2} + m_i^2\,(r)\right] f_i\,(x) = \frac{E_{i,l}^2}{g_2^2\,(E)} f_i\,(x)\ , \tag{13}$$

with $i = 1, 2$. In Eq. (13) we have used reduced fields of the form $f_i\,(x) = F_i\,(x)\,/r$ and defined two r-dependent effective masses $m_1^2\,(r)$ and $m_2^2\,(r)$

$$\begin{cases} m_1^2\,(r) = \frac{6}{r^2}\left[1 - \frac{b(r)}{r}\right] - \frac{3b'(r)}{2r^2} + \frac{3b(r)}{2r^3} \\ m_2^2\,(r) = \frac{6}{r^2}\left[1 - \frac{b(r)}{r}\right] - \frac{b'(r)}{2r^2} - \frac{3b(r)}{2r^3} \end{cases} \quad (r \equiv r\,(x))\,, \tag{14}$$

where we have implicitly defined the variable x with the help of the following relationship: $dx = dr/\sqrt{1 - b(r)/r}$.

In order to use the WKB approximation, from Eq. (13) we can extract two r-dependent radial wave numbers

$$k_i^2\left(r, l, \omega_{i,nl}\right) = \frac{E_{i,nl}^2}{g_2^2\,(E)} - \frac{l\,(l+1)}{r^2} - m_i^2\,(r) \quad i = 1, 2\,. \tag{15}$$

It is useful to use the WKB method implemented by 't Hooft in the brick wall problem [62], by counting the number of modes with frequency less than ω_i, $i = 1, 2$. This is given approximately by

$$\tilde{g}\,(E_i) = \int_0^{l_{\max}} \nu_i\,(l, E_i)\,(2l+1)\,dl, \tag{16}$$

where $\nu_i\,(l, E_i), i = 1, 2$ is the number of nodes in the mode with (l, E_i), such that $(r \equiv r\,(x))$

$$\nu_i\,(l, E_i) = \frac{1}{\pi}\int_{-\infty}^{+\infty} dx \sqrt{k_i^2\,(r, l, E_i)}. \tag{17}$$

The integration with respect to x and l is taken over those values which satisfy $k_i^2\,(r, l, E_i) \geq 0$, $i = 1, 2$. With the help of Eqs. (16) and (17), the self-sustained equation becomes

$$H_\Sigma^{(0)} = -\frac{1}{\pi}\sum_{i=1}^{2}\int_0^{+\infty} E_i \frac{g_1\,(E)}{g_2^2\,(E)}\frac{d\tilde{g}\,(E_i)}{dE_i} dE_i. \tag{18}$$

The explicit evaluation of the density of states yields

$$\begin{aligned}\frac{d\tilde{g}(E_i)}{dE_i} &= \int \frac{\partial \nu(l, E_i)}{\partial E_i}(2l+1)dl \\ &= \frac{4}{3\pi}\int_{-\infty}^{+\infty} dx r^2 \frac{d}{dE_i}\left(\frac{E_i^2}{g_2^2\,(E)} - m_i^2\,(r)\right)^{\frac{3}{2}}.\end{aligned} \tag{19}$$

Substituting Eq. (19) into Eq. (18) and taking into account the energy density, we obtain

$$\frac{1}{2G}\frac{b'(r)}{r^2 g_2\,(E)} = \frac{2}{3\pi^2}\,(I_1 + I_2)\,, \tag{20}$$

where the integrals I_1 and I_2 are, respectively, given by

$$\begin{aligned} I_1 &= \int_{E^*}^{\infty} E \frac{g_1\,(E)}{g_2^2\,(E)}\frac{d}{dE}\left[\frac{E^2}{g_2^2\,(E)} - m_1^2\,(r)\right]^{\frac{3}{2}} dE \\ &= 3\int_{E^*}^{\infty} E^2 \frac{g_1\,(E)}{g_2^3\,(E)}\sqrt{\frac{E^2}{g_2^2\,(E)} - m_1^2\,(r)}\frac{d}{dE}\left(\frac{E}{g_2\,(E)}\right) dE, \end{aligned} \tag{21}$$

and

$$\begin{aligned} I_2 &= \int_{E^*}^{\infty} E \frac{g_1\,(E)}{g_2^2\,(E)}\frac{d}{dE}\left[\frac{E^2}{g_2^2\,(E)} - m_2^2\,(r)\right]^{\frac{3}{2}} dE \\ &= 3\int_{E^*}^{\infty} E^2 \frac{g_1\,(E)}{g_2^3\,(E)}\sqrt{\frac{E^2}{g_2^2\,(E)} - m_2^2\,(r)}\frac{d}{dE}\left(\frac{E}{g_2\,(E)}\right) dE\,, \end{aligned} \tag{22}$$

where E^* is the value which annihilates the argument of the root. In I_1 and I_2 we have assumed that the effective mass does not depend on the energy E. The purpose of this paper is to show that the self-sustained equation (20) is also a source of a topology change.

3 Topology change

Now, Eq. (20) can be read off in a twofold way. The traditional manner is to fix the same background on both sides and verify the existence of a consistent solution on the remaining parameters, e.g., the wormhole throat, establishing the existence of a self-sustained traversable wormhole [38,39]. However, one may also adopt an alternative approach, where one fixes the background on the r.h.s. of Eq. (20) and consequently let the quantum fluctuations evolve, and then one verifies what kind of solutions we can extract from the l.h.s. in a recursive way. To fix ideas, Eq. (20) should be read off in the following manner:

$$\frac{1}{2G}\frac{\left(b'(r)\right)^{(n)}}{r^2 g_2(E)}=\frac{2}{3\pi^2}\left[I_1\left(b^{(n-1)}(r)\right)+I_2\left(b^{(n-1)}(r)\right)\right],\tag{23}$$

where n is the order of the approximation. In this way, if we discover that the l.h.s. has solutions which topologically differ from the fixed background of the r.h.s., we can conclude that a topology change has been induced from quantum fluctuations of the graviton for any spherically symmetric background on the r.h.s. of Eq. (20). Of course, it is not a trivial task to realize multiple topology changes, even if Eq. (23) is interpreted in this way.

The simplest way to see if Eq. (23) allows a topology change is to fix the Minkowski background on the r.h.s. This means that $b(r) = 0\ \forall r$ and for $n = 1$, so that the r.h.s. of Eq. (23) reduces to

$$\frac{1}{2G}\frac{b'(r)}{r^2 g_2(E)}=\frac{4}{\pi^2}\int_{E^*}^{\infty}E^2\frac{g_1(E)}{g_2^3(E)}\sqrt{\frac{E^2}{g_2^2(E)}-\frac{6}{r^2}}\frac{\mathrm{d}}{\mathrm{d}E}\times\left(\frac{E}{g_2(E)}\right)\mathrm{d}E.\tag{24}$$

Then all one has to do is determine the output of the l.h.s. of Eq. (24), namely the classical energy in Gravity's Rainbow. Nevertheless the result is strongly dependent on the choices that we impose on the Rainbow's functions. We will discuss two specific examples which, of course, do not exhaust all the possible choices. Nevertheless, these specific cases show that in principle if we adopt the alternative approach outlined above, one arrives at a solution that can be interpreted as a change in topology. Note that Eq. (23) is strictly related to the "Ricci flow" which is a good tool to detect and compute a topology change [63,64]. A Ricci flow is defined as follows:

$$\frac{\partial g_{\mu\nu}\left(x^\rho,\lambda\right)}{\partial\lambda}=-2R_{\mu\nu}\left(x^\rho,\lambda\right),\tag{25}$$

where $R_{\mu\nu}$ is the Ricci curvature; λ is an evolution parameter which has the dimension of (length)^2 and x^ρ are the local coordinates on a manifold M. The indices μ,ν,ρ run from 1 to $n = \dim M$. Nothing forbids to consider only the spatial part of the metric appearing in Eq. (25) assuming the following form:

$$\mathrm{d}l^2=\frac{\mathrm{d}r^2}{1-\frac{b(r,\lambda)}{r}}+r^2\left(\mathrm{d}\theta^2+\sin^2\theta\mathrm{d}\phi^2\right).\tag{26}$$

In this way the whole topology change information is encoded in the shape function $b(r,\lambda)$. In order to make a connection between the Ricci flow and Eq. (23), it is convenient to consider the following setting:

$$\lambda\to E/E_P\qquad R_{\mu\nu}\left(x^\rho,\lambda\right)\to CGR_{\mu\nu}\left(x^\rho,E/E_P\right),\tag{27}$$

where C is a dimensionless constant and G is the Newton constant. Thus, by contracting Eq. (25) with $g^{ij}\left(x^k, E/E_P\right)$, one gets

$$E_P\frac{g^{ij}\left(x^k,E/E_P\right)\partial g_{ij}\left(x^k,E/E_P\right)}{\partial E}=-2R\left(x^k,E/E_P\right)=-4\frac{b'(r,E/E_P)}{r^2}.\tag{28}$$

Multiplying both sides of the previous equation by $1/\left(8G^2 g_2(E/E_P)\right)$, we find

$$\frac{E_P}{8G^2 g_2(E/E_P)}\frac{g^{ij}\left(x^k,E/E_P\right)\partial g_{ij}\left(x^k,E/E_P\right)}{\partial E}=-\frac{Cb'(r,E/E_P)}{2Gg_2(E/E_P)r^2}.\tag{29}$$

Comparing the r.h.s. of Eq. (29) with the l.h.s. of Eq. (23) we can see that the one-loop contribution of the graviton in the r.h.s. of Eq. (23) can be connected to the contracted Ricci flow. However, a detailed analysis on the connection between the Ricci flow and the iterative procedure exposed in Eq. (23) is beyond the scope of this paper. To see if a topology change appears, in the next sections we will discuss specific examples.

3.1 Specific example I: $g_1(E/E_P)=\exp(-\alpha\frac{E^2}{E_P^2})$, $g_2(E/E_P)=1$

Following Ref. [57,58], we consider the following choice for the Rainbow's functions:

$$g_1(E/E_P)=\exp(-\alpha\frac{E^2}{E_P^2}),\qquad g_2(E/E_P)=1;\quad \alpha>0\in\mathbb{R}.\tag{30}$$

then the integrals I_1 and I_2 take the following form:

$$\begin{aligned}I_1=I_2&=3\int_{E^*}^{\infty}\exp\left(-\alpha\frac{E^2}{E_P^2}\right)E^2\sqrt{E^2-\frac{6}{r^2}}\mathrm{d}E\\&=\frac{3}{2}E_P^4\int_{\sqrt{6}/r}^{\infty}\exp(-\alpha x)\sqrt{x}\sqrt{x-\frac{6}{(rE_P)^2}}\mathrm{d}x\\&=\frac{3E_P^4}{2\sqrt{\pi}}\left(\frac{6}{\alpha(rE_P)^2}\right)\Gamma\left(\frac{3}{2}\right)\exp\left(-\frac{3\alpha}{(rE_P)^2}\right)K_1\\&\quad\times\left(\frac{3\alpha}{(rE_P)^2}\right),\end{aligned}\tag{31}$$

where we have used the change of variables $E = \sqrt{x}$ and the following relationship:

$$\int_u^\infty (x-u)^{\mu-1} x^{\mu-1} \exp(-\beta x)\,\mathrm{d}x = \frac{1}{\sqrt{\pi}}\left(\frac{u}{\beta}\right)^{\mu-1/2} \Gamma(\mu)\exp\left(-\frac{\beta u}{2}\right) K_{\mu-1/2}\left(\frac{\beta u}{2}\right) \begin{matrix}\mathrm{Re}\,\mu > 0\\ \mathrm{Re}\,\beta u > 0\end{matrix}. \tag{32}$$

Equation (24) can be rearranged in the following way:

$$\frac{1}{2G}\frac{b'(r)}{r^2} = \frac{E_P^4}{\pi^2}\left[\frac{6}{\alpha (rE_P)^2}\exp\left(-\frac{3\alpha}{(rE_P)^2}\right) K_1\left(\frac{3\alpha}{(rE_P)^2}\right)\right], \tag{33}$$

where $K_1(x)$ is a modified Bessel function of order 1. Note that it is extremely difficult to extract any useful information from this relationship, so that in the following we consider two regimes, namely the cis-planckian regime, where $rE_P \gg 1$ and the trans-planckian regime, where $rE_P \ll 1$.

In the cis-planckian regime, expanding the right hand side of Eq. (33), we find that the leading term is given by

$$\frac{1}{G}\frac{b'(r)}{r^2} \simeq \frac{E_P^4}{\pi^2}\left[\frac{4}{\alpha^2} - \frac{12}{\alpha (rE_P)^2} + O\left((rE_P)^{-4}\right)\right], \tag{34}$$

which can be rearranged to give

$$b'(r) = \frac{1}{\pi^2}\left[\frac{4r^2}{\alpha^2 G} - \frac{12}{\alpha}\right], \tag{35}$$

where we have used the definition $G = E_P^{-2} = l_P^2$. Restricting our attention to the dominant term only, we find that

$$b(r) = r_t + \frac{4E_P^2}{3\pi^2\alpha^2}\left(r^3 - r_t^3\right) - \frac{12}{\alpha}(r - r_t), \tag{36}$$

which does not represent an asymptotically flat wormhole geometry, as the condition $b(r)/r \to 0$ when $r \to +\infty$, is not satisfied. On the other hand, in the trans-planckian regime, i.e., $rE_P \ll 1$, we obtain the following approximation:

$$\frac{b'(r)}{r^2} \simeq \frac{E_P^2}{2\sqrt{\alpha^3\pi^3}}\left[\exp\left(-\alpha\frac{6}{(rE_P)^2}\right)\frac{\sqrt{6}}{rE_P} + O(rE_P)\right]. \tag{37}$$

Note that in this regime, the asymptotic expansion is dominated by the Gaussian exponential so that the quantum correction vanishes. Thus, the only solution is $b'(r) = 0$ and consequently we have a constant shape function, namely, $b(r) = r_t$. In the next two examples, we will consider different forms of the Rainbow's functions, in order to verify if a topology change may occur.

3.2 Specific example II: $g_1(E/E_P) = g_2(E/E_P) = g(E/E_P)$

Reintroducing the Planck scale E_P explicitly, we consider the specific choice

$$g_1(E/E_P) = g_2(E/E_P) = g(E/E_P), \tag{38}$$

then the integrals I_1 and I_2 take the following form:

$$I_1 = I_2 = 3\int_{E^*}^\infty \left(\frac{E}{g_2(E/E_P)}\right)^2 \sqrt{\left(\frac{E}{g_2(E/E_P)}\right)^2 - \frac{6}{r^2}} \times \frac{\mathrm{d}}{\mathrm{d}E}\left(\frac{E}{g_2(E/E_P)}\right)\mathrm{d}E = 3E_P^4\int_{\sqrt{6}/r}^{x_\infty} x^2\sqrt{x^2 - \frac{6}{r^2}}\,\mathrm{d}x, \tag{39}$$

and consequently Eq. (24) simplifies to

$$\frac{b(r)}{g(E/E_P)} = \frac{8GE_P^4}{\pi^2}\int\left[\int_{\sqrt{6}/r'}^{x_\infty} x^2\sqrt{x^2 - \frac{6}{r'^2}}\,\mathrm{d}x\right] r'^2\mathrm{d}r' + C, \tag{40}$$

where C is an arbitrary constant fixed by boundary conditions and where we have defined the following parameters:

$$x = \frac{E/E_P}{g(E/E_P)} \quad \text{and} \quad x_\infty = \lim_{E/E_P\to\infty}\frac{E/E_P}{g(E/E_P)}. \tag{41}$$

The integration inside the square brackets, in Eq. (40), is straightforward to calculate and one finally arrives at $\left(G = E_P^{-2}\right)$

$$\frac{b(r)}{g(E/E_P)} = \frac{2E_P^2}{\pi^2}\int\left[x_\infty\sqrt{\left(x_\infty^2 - \frac{6}{(E_P r')^2}\right)^3} + \frac{6x_\infty}{(E_P r')^2}\sqrt{x_\infty^2 - \frac{6}{(E_P r')^2}} - \frac{18}{(E_P r')^4}\ln\left(x_\infty + \sqrt{x_\infty^2 - \frac{6}{(E_P r')^2}}\right) + \frac{18}{(E_P r')^4}\ln\left(\sqrt{\frac{6}{(E_P r')^2}}\right)\right] r'^2\mathrm{d}r' + C. \tag{42}$$

Now, we have to fix the form of $g(E/E_P)$. As a working hypothesis, consider

$$g(E/E_P) = 1 + (E/E_P)^{\alpha} \tag{43}$$

with $\alpha > 0$, where the factor x_∞ is given by

$$x_\infty = \lim_{E/E_P \to \infty} \frac{E/E_P}{g\left(E/E_P\right)} = \begin{cases} +\infty & \alpha < 1 \\ 1 & \alpha = 1 \\ 0 & \alpha > 1 \end{cases}. \tag{44}$$

Due to the divergence present in $\alpha < 1$, and as $\alpha > 1$ leads to an imaginary result, the only possible choice is given by $\alpha = 1$. Note that the case under consideration has been extensively studied in inflationary cosmology [65–67]. By using the explicit form of $g(E/E_P)$, Eq. (42) reduces to

$$\frac{b(r)}{1 + E/E_P} = \frac{2E_P^2}{\pi^2} \int h\left(r'\right) r'^2 \mathrm{d}r' + C, \tag{45}$$

where

$$h\left(r'\right) = \frac{1}{\left(E_P r'\right)^3} \left\{ \left[\left(E_P r'\right)^2 - 6\right]^{\frac{3}{2}} + 6\left[\left(E_P r'\right)^2 - 6\right]^{\frac{1}{2}} - \frac{18}{E_P r'} \ln\left[\frac{E_P r'}{\sqrt{6}} + \left(\frac{\left(E_P r'\right)^2}{6} - 1\right)^{\frac{1}{2}}\right]\right\}. \tag{46}$$

It is immediate to observe that there is a natural UV cutoff which forbids the use of an interior region $[0, \sqrt{6}/E_P]$. Indeed, below the point $r = \sqrt{6}/E_P$, the integrand becomes imaginary. The integration in the interval $\left[0, \sqrt{6}/E_P\right]$ can be thought of as a trans-planckian contribution to the shape function which is also suppressed by a factor E/E_P. On the other hand in the cis-planckian region, $g(E/E_P) \simeq 1$ and

$$b(r) = \frac{2E_P^2}{\pi^2} \int h\left(r'\right) r'^2 \mathrm{d}r' + C. \tag{47}$$

Thus, the natural UV cutoff which forbids the use of an interior region, may be interpreted as a topology change. Now we have to verify if this change has produced a wormhole solution. The throat condition is satisfied by definition if one imposes the condition that $b(r_0) = r_0 = \sqrt{6}/E_P = C$. When $r' \gg \sqrt{6}/E_P$, we can write

$$h\left(r'\right) \simeq 1 - \frac{3}{\left(E_P r'\right)^2} - \frac{9}{2\left(E_P r'\right)^4}, \tag{48}$$

and one finds

$$\begin{aligned} b(r) &= \frac{2E_P^2}{\pi^2} \int_{r_0}^{r} \left(1 - \frac{3}{\left(E_P r'\right)^2} - \frac{9}{2\left(E_P r'\right)^4}\right) r'^2 \mathrm{d}r' + r_0 \\ &= \frac{2E_P^2}{\pi^2} \left[\frac{r^3 - r_0^3}{3} - \frac{3}{E_P^2}\left(r - r_0\right) + \frac{9}{2E_P^4}\left(\frac{1}{r} - \frac{1}{r_0}\right)\right] \\ &\quad + r_0. \end{aligned} \tag{49}$$

To be a wormhole solution, the flaring-out condition at the throat, i.e., $b'(r_0) < 1$, needs to be obeyed. The radial derivative of the shape function is given by $b'(r) = \frac{12}{\pi^2}\left[\left(\frac{r}{r_0}\right)^2 - \frac{1}{2} - \frac{1}{8}\left(\frac{r_0}{r}\right)^2\right]$, which reduces to $b'(r_0) = \frac{9}{2\pi^2}$ at the throat, so that the flaring-out condition is satisfied. Note that the resulting space is not asymptotically flat, but rather asymptotically de Sitter; therefore the condition $b(r)/r \to 0$ when $r \to \infty$ is not satisfied. However, in principle, one may match this interior solution to an exterior vacuum much in the spirit of Refs. [68–73].

3.3 Specific example III: $g_2\left(E/E_P\right) = 1 + E/E_P$ and $g_1\left(E/E_P\right) = g\left(E/E_P\right)\left(1 + E/E_P\right)^6$

A second interesting example can be proposed starting from Minkowski space with the choice

$$g_2\left(E/E_P\right) = 1 + E/E_P\,, \tag{50}$$

which is necessary in order to avoid divergent or imaginary values of the integral. Thus, we obtain $I_1 = I_2 = I$, and the integral I is given by

$$\begin{aligned} I &= E_P^2 \int_{E^*}^{\infty} E^2 \frac{g_1\left(E/E_P\right)}{\left(1 + E/E_P\right)^3} \sqrt{\left(\frac{E/E_P}{1 + E/E_P}\right)^2 - \frac{6}{\left(E_P r\right)^2}} \\ &\quad \times \frac{\mathrm{d}}{\mathrm{d}E}\left(\frac{E/E_P}{1 + E/E_P}\right) \mathrm{d}E \\ &= E_P^4 \int_{\frac{E_P\sqrt{6}}{rE_P - \sqrt{6}}}^{\infty} g\left(\frac{E}{E_P}\right)\left(1 + \frac{E}{E_P}\right)\left(\frac{E}{E_P}\right)^2 \\ &\quad \times \sqrt{\left(\frac{E/E_P}{1 + E/E_P}\right)^2 - \frac{6}{\left(E_P r\right)^2}}\, \mathrm{d}\left(\frac{E}{E_P}\right), \end{aligned} \tag{51}$$

where we have set

$$g_1\left(\frac{E}{E_P}\right) = g\left(\frac{E}{E_P}\right)\left(1 + \frac{E}{E_P}\right)^6, \tag{52}$$

and inserted explicitly the value of E^*. The term rE_P becomes relevant when $r \sim E_P^{-1}$ and integrating over r one gets

$$\frac{b(r)}{1+E/E_P}=\frac{2E_P^2}{\pi^2}\int\left[\int\limits_{\frac{E_P\sqrt{6}}{r'E_P-\sqrt{6}}}^{\infty}g\left(\frac{E}{E_P}\right)\left(1+\frac{E}{E_P}\right)\left(\frac{E}{E_P}\right)^2\right.$$

$$\left.\times\sqrt{\left(\frac{E/E_P}{1+E/E_P}\right)^2-\frac{6}{(E_Pr')^2}}\mathrm{d}\left(\frac{E}{E_P}\right)\right]$$

$$\times r'^2\mathrm{d}r'+C. \tag{53}$$

In this case, we also have a short distance cut off located at $r'=\sqrt{6}/E_P$, below which the energy E becomes negative and the argument under square root becomes imaginary. Thus, by repeating the same steps we did for the example I, we find that the integration over the first interval can be interpreted as a trans-planckian region and therefore also suppressed by a factor E/E_P. Thus, we are left with

$$b(r)=\frac{2E_P^2}{\pi^2}\int\left[\int\limits_{\frac{E_P\sqrt{6}}{r'E_P-\sqrt{6}}}^{\infty}g\left(\frac{E}{E_P}\right)\left(1+\frac{E}{E_P}\right)\left(\frac{E}{E_P}\right)^2\right.$$

$$\left.\times\sqrt{\left(\frac{E/E_P}{1+E/E_P}\right)^2-\frac{6}{(E_Pr')^2}}\mathrm{d}\left(\frac{E}{E_P}\right)\right]$$

$$\times r'^2\mathrm{d}r'+C. \tag{54}$$

Assuming that the throat is located at $r_0=\sqrt{6}/E_P$, the condition $b(r_0)=r_0=C$ is automatically satisfied. It is important to observe that the factor $6/\left(E_Pr'\right)^2$ is highly suppressed in the region $\left[\sqrt{6}/E_P,r\right]$. Therefore Eq. (54) can be approximated to give

$$b(r)=\frac{2E_P^2}{\pi^2}\int_{r_0}^{r}\left[\int\limits_{\frac{\sqrt{6}}{r'E_P}}^{\infty}g\left(\frac{E}{E_P}\right)\left(\frac{E}{E_P}\right)^3\mathrm{d}\left(\frac{E}{E_P}\right)\right]r'^2\mathrm{d}r'$$

$$+r_0. \tag{55}$$

For simplicity, fixing $g\left(E/E_P\right)=\exp\left(-\alpha E/E_P\right)\left(1+\beta E/E_P\right)$ with $\beta\in\mathbb{R}$, one obtains

$$b(r)=\frac{2E_P^2}{\pi^2}\int\limits_{r_0}^{r}\left[-\frac{\mathrm{d}^3}{\mathrm{d}\alpha^3}\int\limits_{\frac{\sqrt{6}}{r'E_P}}^{\infty}\exp\left(-\alpha E/E_P\right)\mathrm{d}\left(\frac{E}{E_P}\right)\right.$$

$$\left.+\beta\frac{\mathrm{d}^4}{\mathrm{d}\alpha^4}\int\limits_{\frac{\sqrt{6}}{r'E_P}}^{\infty}\exp\left(-\alpha E/E_P\right)\mathrm{d}\left(\frac{E}{E_P}\right)\right]r'^2\mathrm{d}r'+r_0$$

$$=\frac{12E_P^2}{\pi^2\alpha}\int\limits_{r_0}^{r}\exp\left(-\frac{\alpha\sqrt{6}}{E_Pr'}\right)h(\alpha,\beta,r')\mathrm{d}r'+r_0, \tag{56}$$

where

$$h(\alpha,\beta,r')=\frac{1}{\alpha^3}+\frac{\sqrt{6}}{E_Pr'\alpha^2}+\frac{3}{E_P^2r'^2\alpha}+\frac{\sqrt{6}}{E_P^3r'^3}$$
$$+\beta\left(\frac{6}{E_P^4r'^4}+\frac{4\sqrt{6}}{r'^3\alpha E_P^3}+\frac{12}{r'^2\alpha^2E_P^2}+\frac{4\sqrt{6}}{r'\alpha^3E_P}+\frac{4}{\alpha^4}\right). \tag{57}$$

After the integration over r', for large values of r, one obtains

$$b(r)=r_0+\frac{2E_P^2}{\pi^2}\left[\frac{2}{\alpha^5}(4\beta+\alpha)r^3-\frac{2\sqrt{6}}{e^{\alpha}E_P^3\alpha^5}\right.$$
$$\times\left[\left(\alpha^3\left(3+e^{\alpha}\right)+12\left(2+2\alpha+\alpha^2\right)\right)\beta\right.$$
$$\left.+\left(\alpha^3\left(3+e^{\alpha}\alpha\right)+6\alpha\left(1+\alpha\right)\right)\right]$$
$$\left.+\frac{9}{E_P^4r}+\frac{18\sqrt{6}}{5r^2E_P^5}(\beta-\alpha)\right]+O\left(r^{-3}\right). \tag{58}$$

Contrary to the previous case, when we set $\beta=-\alpha/4$, the de Sitter behavior is eliminated. Plugging such a choice of parameters into the shape function, one finds the following asymptotically flat solution:

$$b(r)=r_0+\frac{3\sqrt{6}}{\pi^2E_Pe^{\alpha}\alpha}\left\{1-\exp\left[\left(1-\frac{\sqrt{6}}{E_Pr}\right)\alpha\right]\right\}. \tag{59}$$

The resulting space is asymptotically flat, i.e., $b(r)/r\to0$ when $r\to\infty$ is satisfied. The radial derivative of the shape function is given by $b'(r)=-\frac{3r_0^2}{\pi e^{\alpha}r^2}\exp\left[\left(1-\frac{r_0}{r}\right)\alpha\right]$. Thus, the latter evaluated at the throat reduces $b'(r_0)=-3/(\pi e^{\alpha})$, which satisfies the flaring-out condition at the throat, i.e., $b'(r_0)<1$ for all values of α. One can observe that the topology change could have been approached also distorting the one-loop graviton by means of a noncommutative geometry like in Ref. [74–76], where the classical Liouville measure has been modified into [74]

$$\mathrm{d}n_i=\frac{\mathrm{d}^3\mathbf{x}\mathrm{d}^3\mathbf{k}}{(2\pi)^3}\exp\left(-\frac{\theta}{4}\left(\omega_{i,nl}^2-m_i^2(r)\right)\right),\quad i=1,2. \tag{60}$$

$m_i^2(r)$ are the effective masses described in (14) and θ is the noncommutative parameter. What has been obtained is that the usage of the distorted Liouville measure (60) produces a wormhole which is traversable in principle but not in practice, but from the topology change point of view it is immediate to see that a noncommutative approach is less flexible with respect to Gravity's Rainbow because of the θ parameter. On this ground one could be interested to see how different distortions can produce a topology change like in a Hořava–Lifshitz (HL) theory. Needless to say that in a HL theory, the first step is to see if traversable wormholes are allowed in the more general setting of a HL theory without detailed balanced condition. In this framework the large number of coupling constant allows a certain degree of flexibility, which is actually under investigation [77].

4 Summary and discussion

In this paper we have studied the problem of topology change induced by vacuum fluctuations on a fixed background. The calculation has been realized computing the graviton one-loop contribution in a semiclassical approach. The method is based on a variational approach with Gaussian trial wave functionals, which is closely related to the Feynman path integral approach. From this point of view, our result is not in contrast with the Hawking conjecture forbidding a topology change. It is interesting to note that the usual UV divergences are kept under control using a distortion of the usual gravitational background metric known as Gravity's Rainbow which is activated at the Planck scale, where it is supposed that the structure of spacetime begins to become foamy. In practice, the energy density of the graviton one-loop contribution, or equivalently the background spacetime, is left to evolve, and consequently the classical energy is determined. More specifically, the background metric is fixed to be Minkowskian in the equation governing the quantum fluctuations, which behaves essentially as a backreaction equation, and the quantum fluctuations are left to evolve. The classical energy, which depends on the evolved metric functions, is then evaluated. Analyzing this procedure, a natural UV cutoff is obtained, which forbids the presence of an interior spacetime region, and this may result in a multiply connected spacetime. Note that one could interpret the UV cutoff as a failure of the WKB approximation because the interior region becomes imaginary. However, this appears only in examples II and III, where the Rainbow's functions do not quickly eliminate the divergent behavior as the example I shows. It is important also to remark that the validity range of the WKB approximation is in the high energy sector which is well satisfied by the trans-planckian regime which we are probing. However, let us assume that one must include the interior region producing an imaginary contribution. From the imposition that $b\,(r)$ must be real, this imaginary contribution is automatically cut off. Moreover, let us suppose that the Gravity's Rainbow argument does not apply to this case, then to have finite results one could impose a UV cutoff by hand in Eq. (24) to have finite results. This UV cutoff simply becomes

$$r \geq \frac{\sqrt{6}}{\Lambda_{UV}}, \tag{61}$$

which means that one cannot impose the Minkowski space as a reference space and therefore there is no a topology change. However, Gravity's Rainbow allows one to consider $\Lambda_{UV} \to \infty$. This means that the previous inequality simply reduces to $r \geq 0$, namely Minkowski space. Thus, in the context of Gravity's Rainbow, this process may be interpreted as a change in topology, and consequently results in the presence of a planckian wormhole.

Note that in principle, one can adopt other backgrounds including a positive cosmological constant, i.e., a de Sitter spacetime, or a negative cosmological constant, namely an anti-de Sitter spacetime concluding that a hole can be produced by Zero Point Energy quantum fluctuations. Finally using the Casimir energy indicator [24], one can conclude that the presence of holes in spacetime seems to be favored leading therefore to a multiply connected spacetime. It is also interesting to note that one could compute the transition from Minkowski to a de Sitter spacetime or anti-de Sitter spacetime. It is important to remark that once the topology has been changed nothing can be said on the classical/quantum stability of the new spacetime because the whole calculation has been performed without a time evolution. It would also be interesting to consider solutions with charge. Indeed in [78,79], the Wheeler–DeWitt equation was considered as a device for finding eigenvalues of a Sturm–Liouville problem. In particular, the Maxwell charge was interpreted as an eigenvalue of the Wheeler–De Witt equation generated by the gravitational field fluctuations. More specifically, it was shown that electric/magnetic charges could be generated by quantum fluctuations of the pure gravitational field. It would also be interesting to consider the presence of electric/magnetic charges in the solutions outlined in this paper, and work along these lines is currently under way.

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