Progress of Theoretical Physics, Vol. 60, No. 1, July 1978

Gravothermal Catastrophe and Negative Specific Heat of Self-Gravitating Systems

Izumi HACHISU and Daiichiro SUGIMOTO

Department of Earth Science and Astronomy
College of General Education, University of Tokyo
Komaba, Meguro-ku, Tokyo 153

(Received February 10, 1978)

Thermodynamics of self-gravitating gas system, which is enclosed by an adiabatic spherical wall, is discussed. When the temperature distribution is isothermal, the system is in thermodynamic equilibrium in the sense that the first order variation of the total entropy of the system vanishes. However, the second order variation of the total entropy may be positive, when the effect of gravity exceeds a certain limit. Then, the system may evolve to make its entropy increase. This is the gravothermal catastrophe, which was pointed out first by Antonov in 1962, but for which some questions were raised concerning its reality. In the present paper, this catastrophe is analysed by extending functional space of variation to include non-isothermal perturbations. It results in two merits: It is most convenient to make a close relation with usual concepts in the thermodynamics of irreversible process, and the present formulation does not contain any singular quantities which brought a confusion in the interpretation of the real physical processes.

§ 1. Introduction

In many of the astrophysical systems, the self-gravitation plays an essential role. The gravitational energy of the *self*-gravitating system is proportional to the square of the mass of the system, while the energy of other interactions and the gravitational energy in the external gravitational field are proportional to the mass of the macroscopic system. Because of this nature, the self-gravitation does not fit to the framework of the usual thermodynamics, in which internal energy is considered to be an extensive variable (i.e., the variable proportional to mass). This makes a great difference in thermodynamic behavior of the self-gravitating system from usual thermodynamic system.

For example, we consider a system which is enclosed by a solid adiabatic wall and leave it long time. Then, we ask what is the state of the system within the wall. In usual thermodynamic system without self-gravitation, it is in thermodynamic equilibrium with isothermal temperature distribution. In the case of the self-gravitating system, the temperature distribution is not always isothermal and the system is not always in thermodynamic equilibrium either, though the situation depends on the conditions. Antonov[1] and Lynden-Bell and Wood[2] investigated such a system. They considered gas particles contained within a solid adiabatic wall. They defined the entropy of the system as

$$S = -k \int f \ln f \, d\boldsymbol{p} d\boldsymbol{x} , \tag{1}$$

where $f(\boldsymbol{p}, \boldsymbol{x})$ is the distribution function in μ-space.

The entropy is larger when the distribution function is locally Maxwellian and when the temperature distribution is isothermal. However, for such state under the hydrostatic equilibrium, the entropy is not always maximum. They thought that the system will evolve to increase its entropy. In some cases, this increase in entropy is accompanied with the gradual gravitational contraction of the central part of the system so that it was called by Lynden-Bell and Wood[2] *gravothermal catastrophe.* Because such a nature of the self-gravitating system is out of common sense in thermodynamics, some questions are raised concerning the reality of the gravothermal catastrophe.[3)~5)] Antonov's discussion[1] was highly mathematical. Discussion by Lynden-Bell and Wood[2] is based on the theory of linear series of equilibria and are somewhat vague. Moreover, they considered only a perturbation with which the temperature distribution remains isothermal. In order to determine the critical point, i.e., the point of marginal stability, such a treatment is enough, because the growth time is infinitely long as compared with the time scale of heat transport.*)

However, we are interested to see how the gravothermal catastrophe develops to produce temperature gradient. For this purpose, it is better to include non-isothermal perturbations. Moreover, we can avoid the mathematical difficulty, which was raised by Taff and Van Horn,[5] by extending the functional space of variation to include non-isothermal perturbations. Such a treatment is difficult, however, if we handle the distribution function $f(\boldsymbol{p}, \boldsymbol{x})$ and the self-consistent gravitational potential. A better approach is made if we treat the system along the line of theory of stellar structure combined with thermodynamics of irreversible process. As we shall see below, such approach leads to much simpler and clearer results consistent with intuition in physics.

We discuss the self-gravitating system of *gas* in local thermodynamic equilibrium. In many of the astrophysical systems, a gas particle may be replaced with a star as in the stellar dynamics. In many examples of such systems, the mean free path of a star is not short enough, which complicates the problem. [This is the reason why people are forced to use the distribution function $f(\boldsymbol{p}, \boldsymbol{x})$.] In order to clarify the physics involved in the gravothermal system, it is better to separate the problems into one associated with the effects of gravity and the other associated with the long mean free path.

Thus we investigate the gas system, i.e., the collision frequent system. In §§ 2 and 3 we discuss the physical nature of the self-gravitating system in relation with the negative specific heat of such system. In § 4 we reformulate the criterion for the gravothermal catastrophe though the same criterion as Antonov's is obtained

*) The authors thank Professor S. Kato for pointing this out.

for the most unstable mode. In § 5 we divide such systems into the thermal system in usual sense and gravothermal system for the sake of convenience in further discussions. The development of gravothermal catastrophe will be studied by Nakada[6] in the regime of linear theory in a separate paper. Nonlinear regime of the catastrophe will be studied in a separate paper.[7] Our formulation has a further merit to allow easy inclusion of the effect of rotation, which will be discussed in the third paper.[8]

§ 2. Thermodynamics of self-gravitating gas systems

2.1. *Gravitational equilibrium*

We consider the gas contained within a spherical adiabatic wall of radius R. We assume that the time scale to establish the hydrostatic equilibrium, τ_d, which is comparable with the time scale for propagation of sound wave through the system, is much shorter than the time scale for heat transport through the system. Then, we may assume that the hydrostatic equilibrium is always established. It is described by

$$dP/dr = -GM_r\rho/r^2\,, \quad dM_r/dr = 4\pi r^2\rho\,, \tag{2}$$

where r is the radial distance of a spherical shell from the center, M_r is the mass contained within the sphere of radius r, ρ is the mass density and P is the pressure due to the momentum transfer of particles. The boundary conditions are

$$M_r = 0 \text{ at } r = 0; \quad M_r = M \text{ at } r = R\,. \tag{3}$$

For the equation of state and the specific entropy we use expressions for ideal gas, i.e.,

$$P = \frac{k}{m}\rho T\,, \tag{4}$$

$$s = \frac{k}{m}\ln\frac{ge^{5/2}(2\pi mkT)^{3/2}}{(\rho/m)h^3}\,, \tag{5}$$

where m is the mass of the particle and other symbols have their usual meanings. Here, it should be remarked that the thermodynamic behavior depends on the forms of its equation of state and of entropy.[6),9)]

In what follows, we shall use the same non-dimensional variables as in the standard theory of stellar structure:[10]

$$\phi \equiv M_r/M\,, \quad x \equiv r/R\,,$$

$$p = P(GM^2/4\pi R^4)^{-1}\,, \quad \theta = T(GMm/kR)^{-1}\,. \tag{6}$$

Then Eq. (2) for hydrostatic equilibrium is rewritten as

$$dp/d\phi = -\phi/x^4, \quad dx/d\phi = 1/x^2\psi, \tag{7}$$

where the nondimensional density is expressed as

$$\psi \equiv p/\theta = \rho(M/4\pi R^3)^{-1} \tag{8}$$

because of Eq. (4). We shall also use the non-dimensional specific entropy defined as

$$\begin{aligned}\sigma &\equiv \ln \theta^{3/2}/\psi \\ &= \frac{m}{k}s - \ln \frac{2g(2\pi e)^{5/2}G^{3/2}M^{1/2}m^4R^{3/2}}{h^3}.\end{aligned} \tag{9}$$

When a relation between p and θ or a distribution of $\theta(\phi)$ or $\sigma(\phi)$ is given, Eq. (7) may be solved under the boundary conditions,

$$x=0 \text{ for } \phi=0, \quad \text{and} \quad x=1 \text{ for } \phi=1. \tag{10}$$

2.2. *Entropy of the system and thermal equilibrium*

The entropy of the system is given

$$S = \int_0^M s\, dM_r, \tag{11}$$

which is equal to the entropy computed by Eq. (1) for the locally Maxwellian distribution. As shown in Fig. 1, we think that the heat $\delta Q_{1\to2}$ is removed from the region 1 of mass between ϕ_1 and $\phi_1 + \Delta\phi_1$ and is put into the region 2 between ϕ_2 and $\phi_2 + \Delta\phi_2$. Then the first order change in entropy is given by

$$\delta S = \left(\frac{1}{T_2} - \frac{1}{T_1}\right)\delta Q_{1\to2}. \tag{12}$$

The condition for thermal equilibrium $\delta S = 0$ requires $T_2 = T_1$. This means that the temperature distribution should be isothermal for thermal equilibrium. This is the well-known result. The temperature of this isothermal gas sphere will be denoted by T_0.

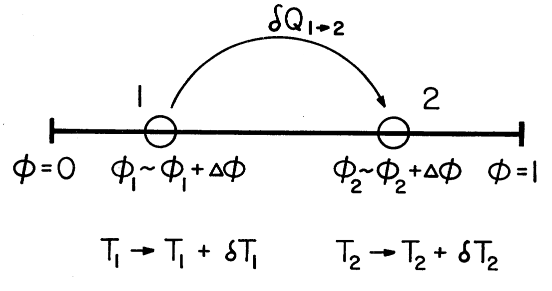

Fig. 1. When heat of $\delta Q_{1\to2}$ is shifted, the hydrostatic readjustment takes place in order to recover the gravitational equilibrium.

2.3. *Stability of thermal equilibrium*

When the heat is shifted from a region, the pressure thereof changes. Then, the readjustment of hydrostatic equilibrium takes place. If the shift of heat is made quasistatically as compared with τ_d, this hydrostatic readjustment produces no entropy but the temperature thereof changes by adiabatic compression or expansion. Then, the second order variation in the entropy is written as

$$\delta^2 S=\left(\frac{1}{T_0+\delta T_2/2}-\frac{1}{T_0+\delta T_1/2}\right)\delta Q_{1\to 2}=\frac{1}{2T_0}\left(\frac{\delta T_1}{T_0}-\frac{\delta T_2}{T_0}\right)\delta Q_{1\to 2}\,. \tag{13}$$

Here, the factor of 1/2 takes into account that a partial change in temperature is proportional to the amount of heat transferred until that time. The total change in temperature δT is found after the total heat of $\delta Q_{1\to 2}$ has been transferred.

In usual thermodynamic system, $\delta T_1<0$ and $\delta T_2>0$ for $\delta Q_{1\to 2}>0$ which lead to $\delta^2 S<0$ implying that the system was at the maximum of entropy. However, in the self-gravitating system, δT_1 is not always negative. If the effect of compression following the removal of heat overweights, δT_1 may be positive. Moreover, δT_2 may be negative due to the reverse of the situation above. When this is the case, $\delta^2 S$ may become positive, implying that there are neighboring states with higher entropy and that the system will evolve unstably to a state of higher entropy.

§ 3. **Tensor specific heats**

3.1. *Hydrostatic readjustment*

From the discussion in the preceding section, it is clear that the most important factor for the instability is the *negative* virtual specific heat. This is *virtual,* because the effect of self-gravitation is taken into computation through the hydrostatic readjustment. Because the readjustment influences the whole system, the virtual specific heat has to be a tensor.

For the first order redistribution of heat, the hydrostatic readjustment takes place according to the first order variation in Eqs. (7), (8) and (9), i.e.,

$$d(\delta \ln p)/d\phi=(\phi/px^4)(\delta \ln p+4\delta \ln x)\,, \tag{14}$$

$$d(\delta \ln \mathrm{x})/d\phi=-(1/x^3\psi)(3\delta \ln x+\delta \ln \psi)\,, \tag{15}$$

$$\delta \ln p=\delta \ln \psi+\delta \ln \theta\,, \tag{16}$$

$$\delta\sigma=(3/2)\,\delta \ln \theta-\delta \ln \psi\,. \tag{17}$$

Eliminating $\delta \ln \theta$ and $\delta \ln \psi$ from Eqs. (15) $\sim$ (17) and then using Eq. (14) together with Eq. (7), we obtain

$$\frac{d}{d\phi}\left[\frac{5p^2x^7}{8\phi\theta_0}\frac{d}{d\theta}(\delta \ln p)\right]-\frac{3}{8}\delta \ln p=\delta\sigma\,. \tag{18}$$

Here and hereafter, we assume that the unperturbed state is isothermal and denote its temperature by T_0 or θ_0.

The response of the hydrostatic readjustment to the displacement of heat

$$\delta q \equiv T_0 \delta s = \frac{GM}{R} \theta_0 \delta \sigma \,, \tag{19}$$

is expressed as

$$\delta \ln p(\phi) = \int_0^1 G(\phi, \phi') \delta\sigma(\phi') d\phi' \,, \tag{20}$$

$$\delta \ln x(\phi) = \int_0^1 H(\phi, \phi') \delta\sigma(\phi') d\phi' \,, \tag{21}$$

$$\delta \ln \theta(\phi) = \int_0^1 F(\phi, \phi') \delta\sigma(\phi') d\phi' \,. \tag{22}$$

According to Eq. (18), the Green's function satisfies

$$\begin{aligned} &L(\phi) G(\phi, \phi') = \delta(\phi - \phi') \,, \\ &L(\phi) \equiv \frac{d}{d\phi}\left[\frac{5}{8}\frac{p^2 x^7}{\phi\theta_0}\frac{d}{d\phi}\right] - \frac{3}{8} \,. \end{aligned} \tag{23}$$

Since this is a Strum-Liouville equation, we have $G(\phi, \phi') = G(\phi', \phi)$. From Eqs. (14), (16) and (17), we obtain

$$H(\phi, \phi') = \frac{1}{4}\left[\frac{px^4}{\phi}\frac{dG(\phi, \phi')}{d\phi} - G(\phi, \phi')\right], \tag{24}$$

$$F(\phi, \phi') = \frac{2}{5}[G(\phi, \phi') + \delta(\phi - \phi')]. \tag{25}$$

The boundary conditions for Eq. (23) are

$$\delta \ln x = 0 \ \text{ for } \ \phi = 1 \,, \tag{26a}$$

$$3\delta \ln x + \delta \ln \psi = 0 \ \text{ for } \ \phi = 0 \,, \tag{26b}$$

which are rewritten as

$$H(\phi, \phi') = 0 \ \text{ for } \ \phi = 1 \,, \tag{27a}$$

$$5H(\phi, \phi') + G(\phi, \phi') - \frac{2}{3}\delta(\phi - \phi') = 0 \quad \text{for } \ \phi \to 0 \,, \tag{27b}$$

respectively.

We shall call $(m/k) F(\phi, \phi')$ to be the *inverse tensor specific heat* at constant volume (constant radius R), because

$$\delta T(\phi) = \frac{m}{k}\int_0^1 F(\phi, \phi') \delta q(\phi') d\phi' \,, \tag{28}$$

holds for the isothermal system according to Eqs. (22), (19) and (6). As seen in Eq. (25), it consists of two terms: One from a scalar $\delta(\phi-\phi')$, which represents usual thermodynamics, and the other from a tensor $G(\phi, \phi')$ which represents the gravothermal effect.

3.2. *Numerical solutions*

Before proceeding to Eq. (23), we must solve Eq. (7) for the unperturbed state of the isothermal gas sphere. It is better to follow the well-known standard procedure [see, e.g., Lynden-Bell and Wood[2)]] than to integrate Eq. (7) directly. Then, we obtain a linear series of isothermal gas spheres, which is characterized by the density contrast between the center (denoted by the subscript c) and the outer boundary at $x=1$ (denoted by the subscript 1), i.e.,

$$D \equiv \rho_c/\rho_1 = \psi_c/\psi_1 . \tag{29}$$

In Fig. 2 shown against D are the non-dimensional thermal, gravitational and total energies,

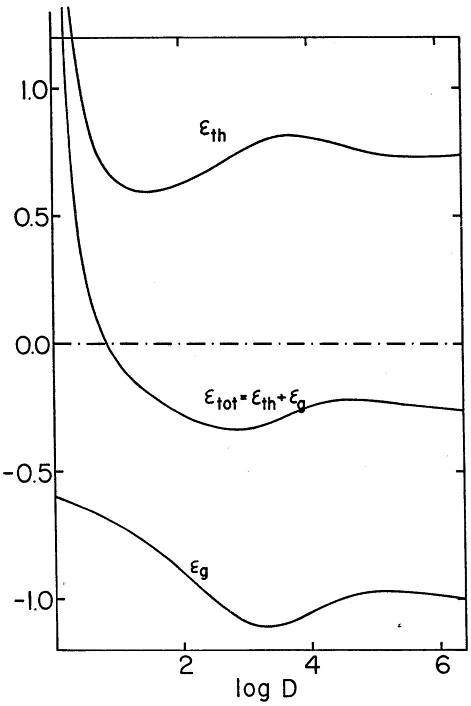

Fig. 2. Isothermal gas spheres in gravitational equilibria. The non-dimensional thermal $\varepsilon_{th}=(3/2)\theta_0$, gravitational ε_g and the total ε_{tot} energies are shown against the central condensation D.

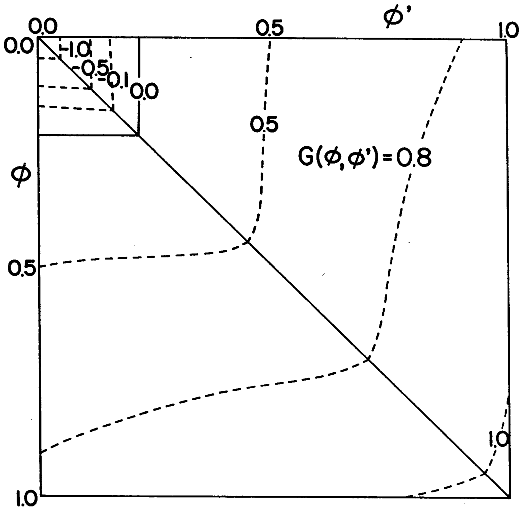

Fig. 3. Contour map of the Green's function $G(\phi, \phi')$ for the logarithmic variation in the pressure, which is proportional to the off-diagonal part of the inverse tensor specific heat $F(\phi, \phi')$. The case for a thermal system, i.e., for a relatively small density contrast, $D=2.5$.

$$\varepsilon_{\rm th}=\frac{3}{2}\theta_0\,,\quad \varepsilon_{\rm g}=-\int_0^1\frac{\phi}{x}d\phi\,,\quad \varepsilon_{\rm tot}=\varepsilon_{\rm th}+\varepsilon_{\rm g}\,, \tag{30}$$

respectively, which are energies in units of GM^2/R.

Using such unperturbed states, Eq. (23) was solved for $G(\phi,\phi')$. In practice, Eq. (23) was integrated both from $\phi=1$ and from $\phi=0$ with arbitrarily chosen normalization factors. These two integrations were fitted at $\phi=\phi'$ to determine the values of the normalization factors. The countour maps of $G(\phi,\phi')$ are shown in Figs. 3～5 for different values of the density contrast. We see that the inverse tensor specific heat is negative in the central region while it is positive in the outer region. The value of $G(\phi,\phi')$ tends to negative infinity for $\phi,\phi'\to 0$.

When D is relatively small (Fig. 3), the region of the negative inverse specific heat is small. For a larger value of D, the region of the negative inverse specific heat is wider (Figs. 4 and 5) and, moreover, the absolute value of the inverse negative specific heat is larger than the positive inverse specific heat in the outer region. Beyond the critical density contrast, i.e., $D\gtrsim D_{\rm crit}=709$ (see the next section), the contour map of $G(\phi,\phi')$ does not change much, as seen in comparison of Fig. 4 with Fig. 5. Such a tendency is also seen in Fig. 2. The ratio of the thermal energy to the absolute value of the gravitational energy is large for $D\ll D_{\rm crit}$, while it tends to 3/4 for $D\to\infty$.

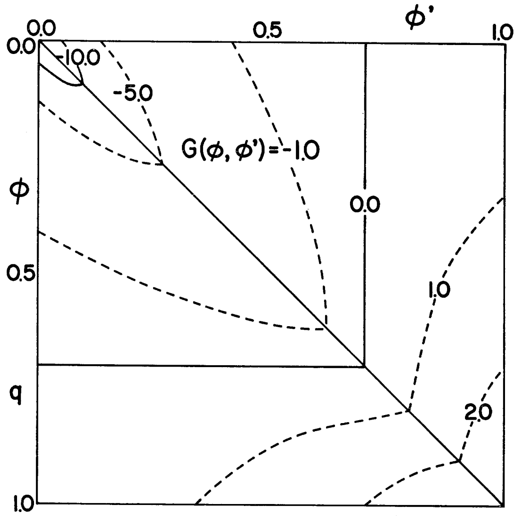

Fig. 4. The same as Fig. 3, but for a marginal system with $D=D_{\rm crit}=709$.

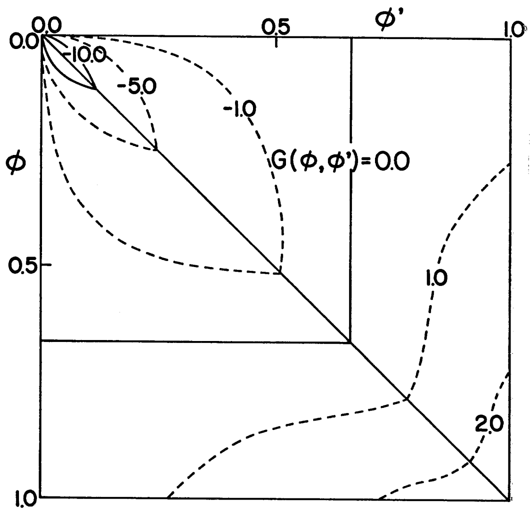

Fig. 5. The same as Fig. 3, but for a strongly gravothermal system with $D=3.16\times10^7$.

§ 4. Gravothermal catastrophe

We shall consider the heat is redistributed in the amount of

$$\delta q(\phi) = \frac{kT_0}{m}\delta\sigma = \frac{GM}{R}\theta_0(\delta\sigma)_0 f(\phi), \tag{31}$$

where $f(\phi) \equiv \delta\sigma(\phi)/(\delta\sigma)_0$. According to Eqs. (13) and (28), the entropy of the system changes by

$$\delta^2 S = -\frac{k}{2m}(\delta\sigma)_0^2 \int_0^1 d\phi \int_0^1 d\phi' F(\phi,\phi') f(\phi) f(\phi'). \tag{32}$$

The condition that the system is enclosed by an adiabatic wall is given by

$$\int_0^1 f(\phi)\, d\phi = 0, \tag{33}$$

and we use the normalization

$$\int_0^1 [f(\phi)]^2 d\phi = 1. \tag{34}$$

We shall seek $f(\phi)$, which makes $(m/k)\delta^2 S$ extremum under the conditions of Eqs. (33) and (34). It will be done by means of the Lagrangian indefinite multiplier, which will be denoted as μ and λ, i.e.,

$$\Delta\Big[-\frac{1}{2}\int_0^1 d\phi' \int_0^1 d\phi\, F(\phi,\phi') f(\phi) f(\phi') + 2\mu \int_0^1 f(\phi)\, d\phi - \lambda\Big\{\int_0^1 [f(\phi)]^2 d\phi - 1\Big\}\Big](\delta\sigma_0)^2 = 0. \tag{35}$$

Taking the variation, we obtain

$$\int_0^1 d\phi \Big[\int_0^1 F(\phi,\phi') f(\phi')\, d\phi' - 2\mu + 2\lambda f(\phi)\Big]\Delta f(\phi) = 0, \tag{36}$$

where we have used the symmetry of $F(\phi,\phi') = F(\phi',\phi)$. Since $\Delta f(\phi)$ may be chosen independently of the quantity in the brackets, the latter quantity should vanish, i.e.,

$$\int_0^1 F(\phi,\phi') f(\phi')\, d\phi' + 2\lambda f(\phi) - 2\mu = 0. \tag{37}$$

Using Eq. (25), we have equivalently

$$\int_0^1 G(\phi,\phi') f(\phi')\, d\phi' + (5\lambda+1) f(\phi) - 5\mu = 0. \tag{38}$$

Putting Eq. (37) into Eq. (32), integrating it, and then using Eqs. (33) and (34), we obtain

$$\delta^2 S = (k/m)(\delta\sigma)_0^2 \lambda. \tag{39}$$

Using Eqs. (37) and (38), variations for other quantities are easily computed from their corresponding equations as

$$\delta \ln \theta = -[2\lambda f(\phi) - 2\mu](\delta\sigma)_0, \tag{40}$$

$$\delta \ln p = -[(5\lambda+1)f(\phi) - 5\mu](\delta\sigma)_0, \tag{41}$$

$$\delta \ln x = -\left\{(5\lambda+1)\left[\frac{px^4}{\phi}\frac{df(\phi)}{d\phi} - f(\phi)\right] + 5\mu\right\}\frac{(\delta\sigma)_0}{4}, \tag{42}$$

$$\delta \ln \phi = -[(3\lambda+1)f(\phi) - 3\mu](\delta\sigma)_0. \tag{43}$$

According to Eq. (39), the system is *gravothermally unstable* when λ is positive. For the region, from which heat is removed $[f(\phi)(\delta\sigma)_0<0]$, the positive λ helps contractions as seen in Eqs. (41) and (43), and helps raising temperature as seen in Eq. (40). In this sense, the positive λ, i.e., the gravothermal instability is one of the consequences of the negative specific heat.

Equation (38) is solved for λ in the following way. We put

$$\tilde{f} \equiv f/5\mu, \quad \Lambda \equiv 5\lambda+1 \tag{44}$$

into Eq. (38), operate $L(\phi)$ of Eq. (23), and integrate the resultant equation over ϕ'. Then we obtain

$$\Lambda L(\phi)\tilde{f}(\phi) + \tilde{f}(\phi) + \frac{3}{8} = 0. \tag{45}$$

Integrating it over $0\leq\phi\leq1$ again, we obtain a boundary condition

$$\Lambda\frac{p^2x^7}{\phi\theta_0}\frac{df(\phi)}{d\phi} = -\frac{3}{5} \quad \text{for } \phi=1, \tag{46}$$

in place of the condition (33), because the quantity in the left-hand side vanishes at the center. Using Eqs. (42) and (43), the boundary conditions (26) are rewritten as

$$\Lambda\left[\frac{px^4}{\phi}\frac{d\tilde{f}(\phi)}{d\phi} - \tilde{f}(\phi)\right] + 1 = 0 \quad \text{for } \phi=1, \tag{47a}$$

$$15\Lambda\frac{px^4}{\phi}\frac{d\tilde{f}(\phi)}{d\phi} + (8-3\Lambda)\tilde{f}(\phi) + 3 = 0 \quad \text{for } \phi=0. \tag{47b}$$

When a value of Λ is assumed, we can start numerical integration with $\tilde{f}(\phi)$ and $d\tilde{f}(\phi)/d\phi$ which satisfy Eqs. (46) and (47a). When we reach $\phi=0$, it is tested whether the boundary condition (47b) is satisfied. Appropriate values of Λ are chosen so that this condition is satisfied. The value of μ can be determined from Eqs. (34) and (44) as

$$\int_0^1 [\tilde{f}(\phi)]^2 d\phi = 25\mu^2. \tag{48}$$

The sign of μ has no physical significance, because both signs of f and μ may change simultaneously according to Eq. (44).

Numerical results, which were obtained for different values of the density contrast D, are shown in Figs. 6 and 7. Here shown are not only the fundamental mode with one node of $f(\phi)$ but also the higher harmonics with more nodes. For a given value of D, the fundamental mode is most unstable if any. The system is unstable (stable) if the density contrast D is greater (smaller) than the critical value, $D_{\rm crit}=709$. This result is, of course, the same as those by Antonov,[1] and Lynden-Bell and Wood.[2]

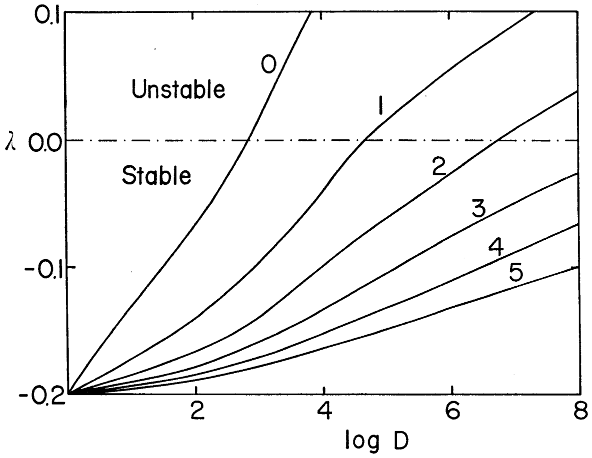

Fig. 6. Eigenvalue λ is shown against the density contrast for the fundamental mode (0) as well as higher harmonics (1, 2, ⋯). The condition for the gravothermal catastrophe is $\lambda > 0$.

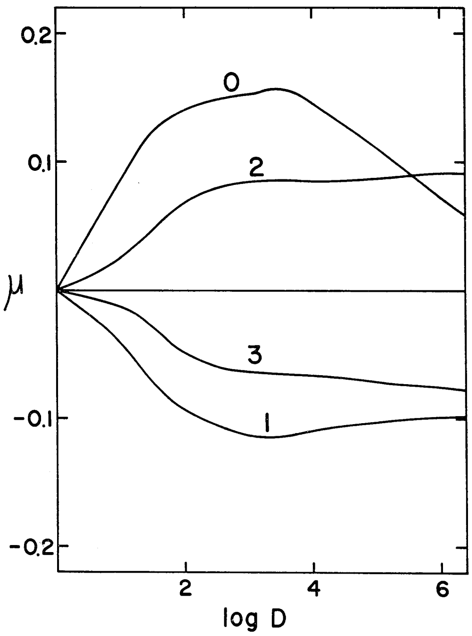

Fig. 7. Eigenvalue μ is shown against the density contrast for the fundamental mode (0) as well as higher harmonics (1, 2, 3). The sign of μ in this figure corresponds to the case of removing heat from the central region, i.e., $f(0) < 0$.

§ 5. Thermal system and gravothermal system

When the density contrast D is much smaller than $D_{\rm crit}$, the thermal energy is much greater than the absolute value of the gravitational energy (see Fig. 2), and the system behaves as a well-known thermal system. When D tends to unity, λ tends to $-1/5$ as seen in Fig. 6. In this limit, $\delta \ln p$ tends to vanish, as seen from Eq. (41) and Fig. 7, and as anticipated from the pressure balance in the system.

We shall define such a system to be *thermal system* when D is smaller than $D_{\rm crit}$. Eigenfunctions of physical quantities are shown in Fig. 8 for the

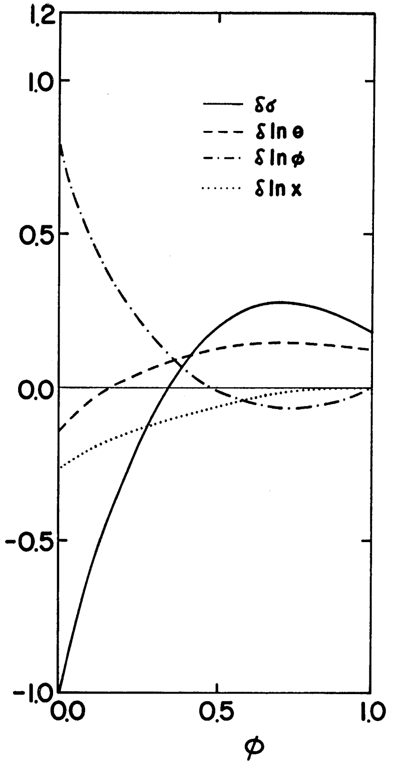

Fig. 8. Eigenfunctions of the fundamental mode are shown in arbitrary units against the Lagrangian mass coordinate for the case of a thermal system with $D=24.2$ and $\lambda=-0.11$.

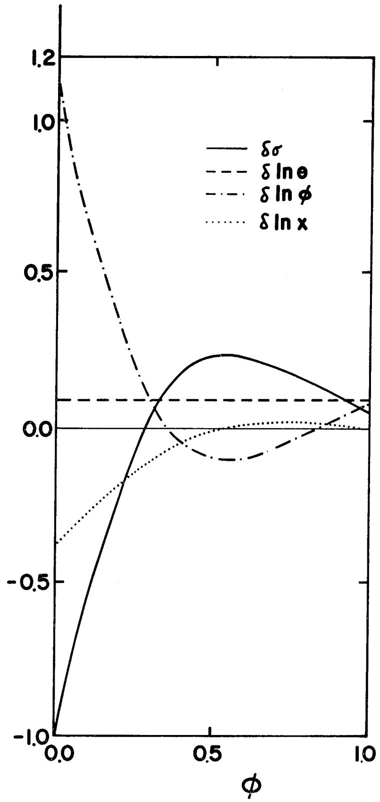

Fig. 9. The same as Fig. 8, but for the marginal stability with $D=709$ and $\lambda=0$.

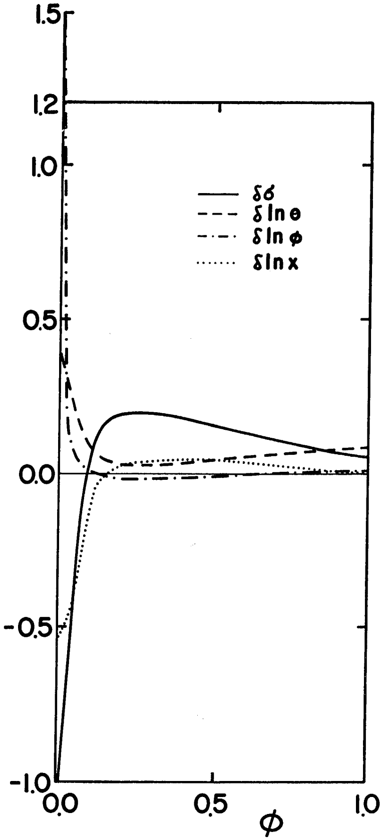

Fig. 10. The same as Fig. 8, but for the case of the gravothermal system with $D=1.41\times10^6$ and $\lambda=0.2$.

fundamental mode of such a system. When the heat (entropy) is removed from the central region, this region contracts only moderately and the temperature thereof decreases as seen in the case of a purely thermal system.

In Fig. 9 the eigenfunctions are shown for the system of the critical density contrast. We notice that $\delta \ln \theta$ is constant in space, as anticipated from Eq. (40). No additional heat flow arises any more, and the system is in marginal stability. As discussed in § 1, this is the reason why Antonov[1] and others were able to discuss correctly the point of marginal stability within the framework of isothermal perturbations.

When D is greater than $D_{\rm crit}$, it is appropriate to define the system to be *gravothermal system*. An example of such a system is shown in Fig. 10. The central region contracts strongly and the temperature thereof increases, when heat is removed from the central regions. Then, the additional heat flows outward from the central region leading to the growth of instability. As a result of the additional heat flow, the entropy of the system increases further, because the heat flow itself is an irreversible process.

We have shown that the gravothermal catastrophe is a real process when $D > D_{\rm crit}$. Our formulation is made using familiar concepts in thermodynamics. Criticism by Taff and Van Horn[5] has no relevance with our formulation, because no singular quantity appears in our formulation. In our formulation, it is easy to estimate the gravothermal effect in other situations. Here, we shall give two examples: 1) If the effect of repulsive force is included in the equation of state, the gravothermal catastrophe is suppressed, because the contraction is suppressed in the central region of the system.[6),9)] 2) There is no gravothermal catastrophe in an infinite plane-symmetric configuration; the pressure of a layer is determined only by the column mass so that the specific heat is positive for all layers.[11)]

Acknowledgements

The authors would like to thank Mr. K. Tanikawa, Dr. Y. Nakada and Dr. K. Nomoto for discussions. Numerical computations were done at the Computer Center of the University of Tokyo.

References

1) V. A. Antonov, Vest. leningr. gos. univ. **7** (1962), 135.
2) D. Lynden-Bell and R. Wood, Month. Notices Roy. Astron. Soc. **138** (1968), 495.
3) R. H. Miller, Astrophys. J. **180** (1973), 759.
4) J. R. Ipser, Astrophys. J. **193** (1974), 463.
5) L. G. Taff and H. M. Van Horn, Astrophys. J. **197** (1975), L23.
6) Y. Nakada, Pub. Astron. Soc. Japan **30** (1978), 57.
7) I. Hachisu, Y. Nakada, K. Nomoto and D. Sugimoto, Prog. Theor. Phys. **60** (1978), No. 2.
8) I. Hachisu, in preparation.
9) E. B. Aronson and C. J. Hansen, Astrophys. J. **177** (1972), 145.
10) M. Schwarzschild, *Structure and Evolution of the Stars* (Princeton University Press, 1958).
11) T. Hara, S. Ikeuchi and D. Sugimoto, Prog. Theor. Phys. **56** (1976), 818.