T. HEAD AND N. KUROKI
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GREATEST REGULAR IMAGES OF TENSOR PRODUCTS OF COMMUTATIVE SEMIGROUPS

BY TOM HEAD AND NOBUAKI KUROKI

Abstract

Let A be a commutative semigroup which has either a greatest regular image or a greatest group image. Then for any commutative semigroup B, $A\otimes B$ has a greatest image of the same type and it is describable by standard constructions based on A and B. If a commutative semigroup A has a greatest group-with-zero image then $A\otimes B$ has such an image if and only if B is archimedean, in which case this image is again describable by standard constructions based on A and B. A handy elementary tool is the fact that the Grothendieck group of a commutative semigroup A may be regarded as the direct limit of the directed system of groups provided by $Z\otimes A$ where Z is the additive group of integers.

By a type $\mathfrak{T}$ of commutative semigroups we will mean a class of commutative semigroups that is closed under isomorphisms. We will deal with three types: the type of regular semigroups and two of its subtypes: groups and groups-with-zero. We say that a semigroup S has *a greatest image of type* $\mathfrak{T}$ if there is a homomorphism α of S onto a semigroup T in $\mathfrak{T}$ which is greatest in the sense that for every homomorphism β of S onto a semigroup U in $\mathfrak{T}$ we have $\beta=\gamma\alpha$ for some homomorphism γ of T onto U. The purpose of the present article is to show that the possession of a greatest image by a commutative semigroup A may lead to the possession of a greatest image of the same type by tensor products of the form $A\otimes B$. The study of tensor products of semigroups was initiated independently by three authors in [3], [4] and [6]. Our work here may be regarded as a synthesis of [6] with the recent investigation of greatest regular images in [9].

All semigroups considered will be commutative. Upper case letters will always denote commutative semigroups and Z will denote the additive group of integers. By a *map* we mean a semigroup homomorphism.

1. The main results. For an arbitrary commutative semigroup A, Hewitt and Zuckerman [10] (or see [1, § 4.3]) described the construction of a regular

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semigroup, which we shall call $Reg(A)$, and a map $\alpha: A \to Reg(A)$ of A *into* $Reg(A)$ which has the property that each map of A *into* each regular semigroup factors uniquely through α.

THEOREM 1. *If A has a greatest regular image then so does $A \otimes B$ and $Reg(A) \otimes Reg(B)$ is such an image.*

The proofs of our theorems will be given in § 4.

The Grothendieck group of a commutative semigroup A will be denoted $Gro(A)$. There is a canonical map $\beta: A \to Gro(A)$ of A *into* $Gro(A)$ which has the property that each map of A *into* each group factors uniquely through β.

THEOREM 2. *If A has a greatest group image then so does $A \otimes B$ and $Gro(A) \otimes Gro(B)$ is such image.*

By a group-with-zero we mean a semigroup which consists of a subgroup and one additional element which acts as an annihilator (zero). For each group G, G^0 will denote the semigroup consisting of G and the additional annihilator, 0.

THEOREM 3. *If A has a greatest group-with-zero image then $A \otimes B$ has such an image if and only if B is Archimedean. If A has G^0 as a greatest group-with-zero image and if B is Archimedean, then $(G \otimes Gro(B))^0$ is a greatest group-with-zero image of $A \otimes B$.*

The next two paragraphs prepare the way for the proofs of these theorems.

2. On Grothendieck groups. Since groups are regular, $\beta: A \to Gro(A)$ always factors uniquely through $\alpha: A \to Reg(A)$. Thus $\beta = \lambda\alpha$ for a unique map $\lambda: Reg(A) \to Gro(A)$. In [6] we noted that λ is surjective and that the congruence induced in $Reg(A)$ is the finest one which identifies the idempotents of A. This description of λ would be adequate for our purposes here, but we would like to emphasize that there is an alternate way of describing this construction of $Gro(A)$ from $Reg(A)$ that fits it into a broader algebraic context: Since $Reg(A)$ is a semilattice of groups, its structure is presentable by means of groups and group homomorphisms [1, p. 128]. This presentation constitutes a directed system of groups and maps and consequently a direct limit is associated. It is easy to see that this limit is essentially $Gro(A)$. Thus λ may be regarded as the formation of a direct limit and we may write $Gro(A) = \mathrm{Lim}\, Reg(A)$.

In [6] an isomorphism $Reg(A) \cong Z \otimes A$ was given for which the composite map $A \to Reg(A) \cong Z \otimes A$ carries each $a \in A$ into $1 \otimes a$. By means of this isomorphism we will replace $Reg(A)$ by $Z \otimes A$ and write $\alpha: A \to Z \otimes A$ where $\alpha(a) = 1 \otimes a$. This allows also: $Gro(A) = \mathrm{Lim}\, Z \otimes A$. That this view of Grothendieck groups is sometimes convenient can be seen from the following codified proof of a result of R. Fulp [3, Prop. 17]: $Gro(A \otimes B) = \mathrm{Lim}\, Z \otimes (A \otimes B) \cong \mathrm{Lim}\,(Z \otimes A) \otimes (Z \otimes B) \cong (\mathrm{Lim}\, Z \otimes A) \otimes (\mathrm{Lim}\, Z \otimes B) = Gro(A) \otimes Gro(B)$.

3. A fundamental diagram. The Proofs of Theorems 1 and 2 will be read from the commutative diagram, Figure 1. For the moment, ignore map 10 of the diagram. In view of § 2, the reader's first guess as to what map is intended at each of the remaining numbers in the diagram is almost certain to be correct and we will describe explicitly only four of them: Map 1

$$\begin{array}{ccccc}
A\otimes B & \xrightarrow{3} & Z\otimes(A\otimes B) & \xrightarrow{7} & Gro(A\otimes B) \\
\downarrow 1 & 2 & \updownarrow 4 & 11 & \downarrow 10 \\
(Z\otimes A)\otimes B & \longleftrightarrow & (Z\otimes A)\otimes(Z\otimes B) & \longrightarrow & Gro(A)\otimes Gro(B) \\
\downarrow 5 & 6 & & 8 & \uparrow 9 \\
Gro(A)\otimes B & \longleftrightarrow & (Gro(A)\otimes Z)\otimes B & \longleftrightarrow & Gro(A)\otimes(Z\otimes B)
\end{array}$$

Figure 1.

carries $a\otimes b$ into $(1\otimes a)\otimes b$. Map 3 carries $a\otimes b$ into $1\otimes(a\otimes b)$. Map 7 is λ: $Reg(A\otimes B)=Z\otimes(A\otimes B)\rightarrow Gro(A\otimes B)$. Map 11 is the tensor product of two λ-type maps. Notice that all the maps, except possibly 1 and 3, are surjective and that all the even numbered maps are isomorphisms. Now the composite map 4–11 has a group as its range and we define map 10 to be the resulting induced map. Map 10 is an isomorphism as can be verified by constructing an inverse: Map 4–7 provides a biadditive function $(Z\otimes A)\times(Z\otimes A)\rightarrow Gro(A\otimes B)$ which can be verified to induce a function $Gro(A)\times Gro(B)\rightarrow Gro(A\otimes B)$ which is also biadditive. The latter function then induces the desired inverse for map 10.

We have seen that the diagram encompasses (at 10) the result of Fulp referred to in § 2. In § 4 this result will be seen to be closely related to Theorem 2. Also included in the diagram is the following result which is closely related to Theorem 1 and subsumes Proposition 6 of [3]: *If A is regular then so is $A\otimes B$.* This can be read from the diagram as follows: If A is regular then $A\rightarrow Z\otimes A$ is an isomorphism and consequently so is map 1. By commutativity, map 3 is an isomorphism and $A\otimes B$ is regular.

4. The proofs. *Proof of Theorem* 1: From § 2 and [9, Lemma 2] we know that X has a greatest regular image if and only if $X\rightarrow Z\otimes X$ is surjective and when this map is surjective $Z\otimes X$ is a greatest regular image of X. Suppose that A has a greatest regular image. Then $A\rightarrow Z\otimes A$ is surjective and so is map 1 of Figure 1. By commutativity map 3 is surjective and $Z\otimes(A\otimes B)\cong(Z\otimes A)\otimes(Z\otimes B)=Reg(A)\otimes Reg(B)$ is a greatest regular image of $A\otimes B$.

Proof of Theorem 2: From § 2 and [9, Lemma 1] (or [12]) we know that X has a greatest group image if and only if the composite map $X\rightarrow Z\otimes X\rightarrow Gro(X)$ is surjective and when this map is surjective $Gro(X)$ is a greatest group image of X. Suppose that A has a greatest group image. Then $A\rightarrow Z\otimes A\rightarrow Gro(A)$ is surjective and so is the composite map 1-5 of Figure 1. Thus map 1-5-6-8-9 is surjective. By commutativity, map 3-7-10 is surjective. Then map 3-7 is surjective and $Gro(A\otimes B)\cong Gro(A)\otimes Gro(B)$ is a greatest group image of $A\otimes B$.

Proof of Theorem 3: From [9, Theorem 3] we know that X has a greatest group-with-zero image if and only if X has precisely two Archimedean components and the upper component contains an idempotent. From the proof of this same theorem we know that when X possesses such a greatest image, G^0 is such an image where G is the maximal subgroup (=minimal ideal) of the upper component of X. Suppose that A has a greatest group-with-zero image, that U is the upper component of A, and that G is the maximal subgroup of U. If B is Archimedean then $A\otimes B$ must again have two components ([3, Proposition 4] or [4, Proposition 1.1] or [6, Theorem 1]) and the mapping $U\otimes B\rightarrow A\otimes B$ induced by the inclusion $U\subseteq A$ is an isomorphism of $U\otimes B$ onto the upper component of $A\otimes B$ ([5, Theorem 2.3] or [8, Proposition 1]). From [7, § 1 (ii)] we know that the maximal subgroup of $U\otimes B$ is isomorphic with $G\otimes Gro(B)$. Thus $A\otimes B$ has a greatest group-with-zero image and it is isomorphic with $(G\otimes Gro(B))^0$. If B is not Archimedean then by [2] (or [11] or a direct calculation) $A\otimes B$ must have more than two Archimedean components and consequently cannot have a greatest group-with-zero image.

5. Problems. It may be possible to strengthen Theorem 3 into: $A\otimes B$ has a greatest group-with-zero image if and only if either, (1) one of the factors is Archimedean with an idempotent and the other has precisely two Archimedean components or, (2) one of the factors is Archimedean and the other has precisely two Archimedean components with an idempotent in the upper component. Indeed if conjecture 7 of [7] is correct then this statement follows. If both conjecture 7 and conjecture 2 of [7] are correct then the following strengthening of Theorem 1 will follow via [9, Theorem 2]: $A\otimes B$ has a greatest regular image if and only if either A or B does.

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UNIVERSITY OF ALASKA
COLLEGE, ALASKA

AND

UNIVERSITY OF TEXAS AT EL PASO
EL PASO, TEXAS

NIHON UNIVERSITY
NARASHINO, JAPAN