Greedy algorithms for high-dimensional eigenvalue problems

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Motivation

High-dimensional problems are ubiquitous: quantum mechanics, kinetic models, molecular dynamics, uncertainty quantification, finance, multiscale models etc.

How to compute $u(x_1, \cdots, x_d)$ with d potentially large?

The bottom line of deterministic approaches is to represent solutions as linear combinations of tensor products of small-dimensional functions (parallelepipedic domains):

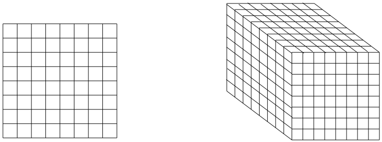
$$
\begin{aligned}
u(x_1, \cdots, x_d) &= \sum_{k \geq 1} r_k^1(x_1) r_k^2(x_2) \cdots r_k^d(x_d) \\
&= \sum_{k \geq 1} \left(r_k^1 \otimes r_k^2 \otimes \cdots \otimes r_k^d \right)(x_1, x_2, \cdots, x_d).
\end{aligned}
$$

Curse of dimensionality

Classical approach: Galerkin method using standard finite element discretization with N degrees of freedom per variate.

$$u(x_1, \cdots, x_d) \approx \sum_{(i_1,\cdots,i_d)\in\{1,\cdots,N\}^d} \lambda_{i_1,\cdots,i_d} \phi^1_{i_1} \otimes \cdots \otimes \phi^d_{i_d}(x_1, \cdots, x_d),$$

where the basis functions $\left(\phi_i^j\right)_{1\leq i\leq N,\ 1\leq j\leq d}$ are chosen a priori and the real numbers $(\lambda_{i_1,\cdots,i_d})_{1\leq i_1,\cdots,i_d\leq N}$ are to be computed.



$DIM = N^d$

This is the so-called curse of dimensionality ([Bellman, 1957])

Greedy algorithms

Progressive Generalized Decomposition: Here, we consider an approach proposed by:

- Ladevèze *et al.* to do time-space variable separation;
- Chinesta *et al.* to solve high-dimensional Fokker-Planck equations in the context of kinetic models for polymers;
- Nouy *et al* in the context of uncertainty quantification.

They are related to the so-called greedy algorithms introduced in nonlinear approximation theory: ([Temlyakov, 2008], Cohen, Dahmen, DeVore, Maday...)

The idea is to look iteratively for the "best tensor product". At the n^{th} iteration of the algorithm, an approximation u_n of the function u is given by:

$$u(x_1, \cdots, x_d) \approx u_n(x_1, \dots, x_d) = \sum_{k=1}^{n} r_k^1 \otimes r_k^2 \otimes \cdots \otimes r_k^d(x_1, \cdots, x_d).$$

$$u_n(x_1, \cdots, x_d) = u_{n-1}(x_1, \cdots, x_d) + r_n^1 \otimes r_n^2 \otimes \cdots \otimes r_n^d(x_1, \cdots, x_d).$$

$$\boxed{DIM = n \times Nd}$$

Existing results on greedy algorithms

Theoretical results for convex unconstrained minimization problems: [Le Bris, Lelièvre, Maday, 2008], [Cancès, VE, Lelièvre, 2011], [Nouy, Falco, 2012]

A greedy algorithm has been proposed in ([Chinesta, Ammar, 2010]) for eigenvalue problems, but no analysis.

Here, we propose two new greedy algorithms for eigenvalue problems and provide some theoretical convergence results for these.

Outline

1. Algorithms and theoretical convergence results

2. Numerical examples

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2. Numerical examples

Prototypical example

$\Omega = (-L_1, L_1) \times \cdots \times (-L_d, L_d)$ where for all $1 \leq i \leq d$, $\mathcal{X}_i = (-L_i, L_i)$ is a bounded open interval of $\mathbb{R}$.
We wish to compute the lowest eigenvalue μ and an associated eigenvector $u(x_1, \cdots, x_d)$ of the Schrödinger operator $-\frac{1}{2}\Delta + \Phi$ on $L^2(\Omega)$:

$$-\frac{1}{2}\Delta u + \Phi u = \mu u,$$

where $\Phi(x_1, \cdots, x_d) \in L^q(\Omega)$ with $q = 2$ if $d \leq 3$, $q > 2$ for $d = 4$ and $q = d/2$ for $d \geq 5$.

Weak formulation of the eigenvalue problem:

$$H := L^2(\Omega), \quad V := H^1_0(\Omega),$$

$$\forall v, w \in H, \quad \langle v, w \rangle_H = \int_\Omega vw,$$

$$\forall v, w \in V, \quad a(v, w) := \frac{1}{2}\int_\Omega \nabla v \cdot \nabla w + \int_\Omega \Phi vw,$$

Prototypical example

The function $u(x_1, \cdots, x_d)$ and the eigenvalue μ are then solutions of:

$$\forall v \in V, \quad a(u,v) = \mu \langle u, v \rangle_H, \quad \mu = \min_{v \in V, v \neq 0} \frac{a(v,v)}{\|v\|_H^2}, \quad u = \operatorname*{argmin}_{v \in V, v \neq 0} \frac{a(v,v)}{\|v\|_H^2}.$$

At each iteration of the algorithm, only low-dimensional functions are computed, for instance pure tensor product functions

$$\Sigma := \left\{ r^1 \otimes r^2 \otimes \cdots \otimes r^d, \ r^1 \in H_0^1(\mathcal{X}_1), \ \cdots, \ r^d \in H_0^1(\mathcal{X}_d) \right\}. \tag{1}$$

General setting and main assumptions

Let V, H be separable Hilbert spaces such that

(AV) $V \subset H$ is dense and the injection $V \hookrightarrow H$ is compact (i.e. the weak convergence in V implies the strong convergence in H).

Let $\langle \cdot, \cdot \rangle_H$ denote the scalar product on H.

Let $a : V \times V \to \mathbb{R}$ be a **continuous symmetric bilinear form** such that

(AA) $\exists \eta \geq 0$, such that the bilinear form $\langle \cdot, \cdot \rangle_a$ defined by

$$\forall v, w \in V, \ \langle v, w \rangle_a := a(v, w) + \eta \langle v, w \rangle_H$$

defines a scalar product on V whose associated norm $\| \cdot \|_a$ is equivalent to the original norm on V.

Let $\Sigma \subset V$ satisfying

(A1) Σ is a non-empty cone of V i.e. $0 \in \Sigma$ and $\forall (z, c) \in \Sigma \times \mathbb{R}, \ cz \in \Sigma$;

(A2) Σ is weakly closed in V;

(A3) $\mathrm{Span}\,(\Sigma)$ is dense in V.

Eigenvalue problem in the general framework

All the previous assumptions are satisfied in our prototypical example!

We wish to compute the lowest eigenvalue μ of the bilinear form $a(\cdot,\cdot)$ and an associated H-normalized eigenvector $u \in V$, which satisfy

$$\mu = \min_{v\in V, v\neq 0} \frac{a(v,v)}{\|v\|_H^2}, \quad u = \underset{v\in V, v\neq 0}{\mathrm{argmin}} \frac{a(v,v)}{\|v\|_H^2}.$$

In particular, we have

$$\forall v \in V,\ a(u,v) = \mu\langle u, v\rangle_H.$$

The greedy algorithm computes iteratively a sequence $(z_n)_{n\in\mathbb{N}} \subset \Sigma$ and the approximation u_n of u given at the n^{th} iteration of the algorithms satisfies

$$u_n \in \mathrm{Span}\left\{z_0, z_1, \cdots, z_n\right\}.$$

Three greedy algorithms

- Rayleigh Greedy algorithm ([Cancès, VE, Lelièvre, 2013]);
- Residual Greedy algorithm ([Cancès, VE, Lelièvre, 2013]);
- Explicit Greedy algorithm ([Chinesta, Ammar, 2010]);

All these algorithms begin with some initial guess $u_0 \in V$.

The initial guess $u_0 \in V$ is defined as follows:

Initialization $n = 0$: find $z_0 \in \Sigma$ such that

$$z_0 \in \underset{z \in \Sigma, \ z \neq 0}{\operatorname{argmin}} \frac{a(z,z)}{\|z\|_H^2}; \tag{2}$$

set $u_0 := \frac{z_0}{\|z_0\|_H}$ and $\lambda_0 := a(u_0, u_0)$.

Pure Rayleigh Greedy algorithm

Rayleigh quotient: $\forall v \in V, \ \mathcal{J}(v) := \begin{cases} \frac{a(v,v)}{\|v\|_H^2} \text{ if } v \neq 0, \\ +\infty \text{ if } v = 0. \end{cases}$

The Rayleigh Greedy Algorithm reads:

Iteration $n \geq 1$: find $z_n \in \Sigma$ such that

$$z_n \in \operatorname*{argmin}_{z \in \Sigma} \mathcal{J}\left(u_{n-1} + z\right). \qquad (3)$$

Set $u_n = \frac{u_{n-1}+z_n}{\|u_{n-1}+z_n\|_H}$, $\lambda_n := a(u_n, u_n)$ and $n = n+1$.

Residual Greedy algorithm

The Residual Greedy Algorithm reads:

Iteration $n \geq 1$: find $z_n \in \Sigma$ such that

$$z_n \in \underset{z \in \Sigma}{\text{argmin}} \frac{1}{2} \|u_{n-1} + z\|_a^2 - (\lambda_{n-1} + \eta)\langle u_{n-1}, z\rangle_H. \quad (4)$$

Set $u_n = \frac{u_{n-1}+z_n}{\|u_{n-1}+z_n\|_H}$, $\lambda_n := a(u_n, u_n)$ and $n = n+1$.

Why is it called Residual? (4) is equivalent to

$$z_n \in \underset{z \in \Sigma}{\text{argmin}} \frac{1}{2} \|R_{n-1} - z\|_a^2,$$

where R_{n-1} is the element of V such that

$$\forall v \in V, \quad \langle R_{n-1}, v\rangle_a = \lambda_{n-1}\langle u_{n-1}, v\rangle_H - a(u_{n-1}, v).$$

Euler equations for the Residual algorithm on a very simple case

In the previous prototypical example, with $d = 2$ and $\Phi = 0$ (In this case, we can take $\eta = 0$).

$$\Sigma = \left\{ r^1 \otimes r^2, r^1 \in H_0^1(\mathcal{X}_1),\ r^2 \in H_0^1(\mathcal{X}_2) \right\},$$

If $z_n = r_n^1 \otimes r_n^2 \in \Sigma$, the Euler equations associated to the previous mininmization problem read

$$\left\{ \begin{array}{l} \left[\int_{\mathcal{X}_1} |r_n^1|^2\right] (-\Delta_{x_2} r_n^2(x_2)) + \left[\int_{\mathcal{X}_1} |\nabla_{x_1} r_n^1|^2\right] r_n^2(x_2) \\ \quad = \int_{\mathcal{X}_1} \left[-\Delta_{x_1,x_2} u_{n-1}(x_1,x_2) - \lambda_{n-1} u_{n-1}(x_1,x_2)\right] r_n^1(x_1)\, dx_1, \\ \\ \left[\int_{\mathcal{X}_2} |r_n^2|^2\right] (-\Delta_{x_1} r_n^1(x_1)) + \left[\int_{\mathcal{X}_2} |\nabla_{x_2} r_n^2|^2\right] r_n^1(x_1) \\ \quad = \int_{\mathcal{X}_2} \left[-\Delta_{x_1,x_2} u_{n-1}(x_1,x_2) - \lambda_{n-1} u_{n-1}(x_1,x_2)\right] r_n^2(x_2)\, dx_2, \end{array} \right.$$

These equations leads to a system of coupled nonlinear equations, which are solved through an alternating direction method (fixed-point procedure).

Convergence results in infinite dimension

Theorem (Cancès, VE, Lelièvre, 2013)

Provided that (AA), (AV), (A1), (A2) and (A3) are satisfied, the iterations of the Rayleigh (up to a slight modification) and Residual Greedy algorithms are well-defined, in the sense that there always exists at least one solution to (2), (3) and (4).
Besides, the sequence $(\lambda_n)_{n\in\mathbb{N}}$ *converges to* λ*, an eigenvalue of the bilinear form* $a(\cdot,\cdot)$*, and if* F_λ *denotes the set of H-normalized eigenfunctions of* $a(\cdot,\cdot)$ *associated with the eigenvalue* λ*,*

$$d(u_n, F_\lambda) := \inf_{w\in F_\lambda} \|u_n - w\|_a \xrightarrow[n\to\infty]{} 0.$$

If the eigenvalue λ *is simple, the sequence* $(u_n)_{n\in\mathbb{N}}$ *strongly converges in V towards an element* $w_\lambda \in F_\lambda$ *such that* $\|w_\lambda\|_H = 1$*.*

Unfortunately, λ may not be the smallest eigenvalue of a: this depends strongly on the choice of the initial guess u_0. But this seems to be a pathological case, and in all the numerical results we have performed so far, the limit was the smallest eigenvalue of the bilinear form $a(\cdot,\cdot)$.

Convergence results in finite dimension

Lojasiewicz inequality: [Lojasiewicz, 1965], [Levitt, 2012]

Lemma

Let us assume that the dimension of V is finite and let $\mathcal{D} := \{v \in V,\ 1/2 < \|v\|_H < 3/2\}$. Besides, let F_λ be the set of H-normalized eigenvectors of $a(\cdot,\cdot)$ associated to λ. Then, $\mathcal{J} : \mathcal{D} \to \mathbb{R}$ is analytic, and there exists $K > 0$, $\theta \in (0, 1/2]$ and $\varepsilon > 0$ such that

$$\forall v \in \mathcal{D},\ d(v, F_\lambda) := \inf_{w \in F_\lambda} \|v - w\|_a \leq \varepsilon,\ |\mathcal{J}(v) - \lambda|^{1-\theta} \leq K \|\nabla \mathcal{J}(v)\|_a. \quad (5)$$

Theorem (Cancès, VE, Lelièvre, 2013)

Let us assume (AA), (AV), (A1), (A2), (A3) and that the dimension of V is finite. Then, for the Rayleigh and the Residual algorithm, the whole sequence $(u_n)_{n\in\mathbb{N}}$ strongly converges in V towards an element w_λ of F_λ. Besides, if θ denotes the same real number appearing in (5), the following convergence rates hold:

- *if $\theta = 1/2$, there exists $C > 0$ and $0 < \sigma < 1$ such that for n large enough,*

$$\|u_n - w_\lambda\|_a \leq C\sigma^n; \quad (6)$$

- *if $\theta \neq 1/2$, there exits $C > 0$ such that*

$$\|u_n - w_\lambda\|_a \leq Cn^{-\frac{\theta}{1-2\theta}}. \quad (7)$$

Explicit Greedy algorithm

The Explicit Greedy algorithm ([Chinesta, Ammar, 2010]) is only defined for sets Σ which are embedded manifolds.

For $z_n \in \Sigma$, we denote by $T_\Sigma(z_n)$ the tangent space in V to Σ at the point z_n.

Iteration $n \geq 1$**:** for $n \geq 1$, find $z_n \in \Sigma$ such that

$$\forall \delta z_n \in T_\Sigma(z_n), \quad a(u_{n-1} + z_n, \delta z_n) - \lambda_{n-1} \langle u_{n-1} + z_n, \delta z_n \rangle_H = 0. \tag{8}$$

Set $u_n = \frac{u_{n-1}+z_n}{\|u_{n-1}+z_n\|_H}$, $\lambda_n := a(u_n, u_n)$ and $n = n+1$.

This leads to a system of coupled nonlinear equations similar to the "Euler equations" associated to the minimization problems of the other algorithms, which can also be solved through a fixed-point procedure.

No mathematical results on this method, the existence of a solution to (8) is not guaranteed in general even if the algorithm seems to work in practice.

Tangent space to rank-1 tensor product functions

Rank-1 tensor product functions

$$\Sigma := \left\{ r^1 \otimes r^2 \otimes \cdots \otimes r^d,\ r^1 \in H_0^1(\mathcal{X}_1),\ \cdots,\ r^d \in H_0^1(\mathcal{X}_d) \right\}, \tag{9}$$

$$z_n = r_n^1 \otimes r_n^2 \otimes \cdots \otimes r_n^d,$$

$$T_\Sigma(z_n) := \left\{ \delta z_n \left(s^1, s^2, \dots, s^d \right),\ s^1 \in H_0^1(\mathcal{X}_1),\ \cdots,\ s^d \in H_0^1(\mathcal{X}_d) \right\}, \tag{10}$$

where

$$\begin{aligned} \delta z_n \left(s^1, s^2, \dots, s^d \right) \quad = \quad & s^1 \otimes r_n^2 \otimes \cdots \otimes r_n^d \\ & + r_n^1 \otimes s^2 \otimes \cdots \otimes r_n^d \\ & + \cdots \\ & + r_n^1 \otimes r_n^2 \otimes \cdots \otimes s^d. \end{aligned}$$

Orthogonal versions of the algorithms

[Le Bris, Lelièvre, Maday, 2009], [Nouy, Falco, 2011]

The so-called Orthogonal versions of these greedy algorithm read:

Iteration $n \geq 1$**:** find $z_n \in \Sigma$ as in the first step of the algorithms (Rayleigh, Residual, Explicit).

Find $(c_1^n, \cdots, c_n^n) \in \mathbb{R}^n$ such that

$$(c_1^n, \cdots, c_n^n) \in \underset{(c_1, \cdots, c_n) \in \mathbb{R}^n}{\operatorname{argmin}} \mathcal{J}\left(\sum_{k=1}^n c_k z_k \right);$$

Set $u_n = \frac{\sum_{k=1}^n c_k^n z_k}{\left\| \sum_{k=1}^n c_k^n z_k \right\|_H}$. If $\langle u_n, u_{n-1} \rangle_H \leq 0$, set $u_n = -u_n$ and set $n = n+1$.

The first theorem (in infinite dimension) still hold for the orthogonal versions of the Rayleigh and Residual algorithm.

Practical implementation

When Σ is the set of rank-1 tensor product functions (9), an alternating direction fixed-point procedure is used to solve the *Euler equations* associated to the minimization problems to compute the functions $(r_n^1, \cdots, r_n^d)$ at each iteration $n \in \mathbb{N}^*$.

- Residual and Explicit algorithms: only requires the inversion of *one-variable* linear problems.
- Rayleigh algorithm: requires the full diagonalization of *one-variable* bilinear forms.
- Need for an evaluation of the constant η for the Residual algorithm.

Outline

Toy numerical tests with matrices

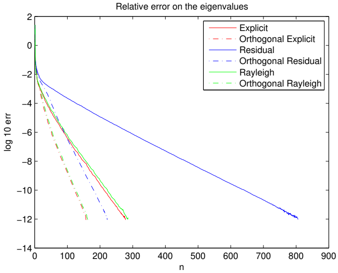
$H = V := \mathbb{R}^{N_1\times N_2}$, $\Sigma := \left\{ r^1 (r^2)^T,\ r^1 \in \mathbb{R}^{N_1},\ r^2 \in \mathbb{R}^{N_2} \right\}$.

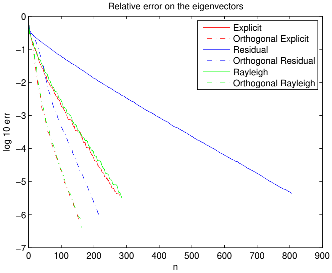
For all $M_1, M_2 \in V$,

$$a(M_1, M_2) := \mathrm{Tr}\left[M_1^T \left(P^1 M_2 P^2 + Q^1 M_2 Q^2 \right) \right],$$

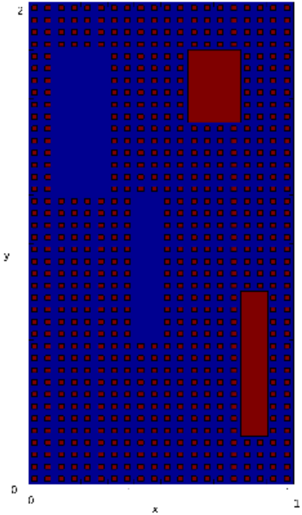
with $P^1, Q^1 \in \mathbb{R}^{N_1\times N_1}$ and $P^2, Q^2 \in \mathbb{R}^{N_2\times N_2}$ symmetric matrices.

Computing the smallest eigenvalue of $a(\cdot,\cdot)$ is equivalent to computing the smallest eigenvalue of the symmetric tensor $A = (A_{ijkl})_{1\leq i,k\leq N_1,\ 1\leq j,l\leq N_2} \in \mathbb{R}^{(N_1\times N_2)\times(N_1\times N_2)}$, where $A_{ijkl} = P^1_{ik} P^2_{jl} + Q^1_{ik} Q^2_{jl}$.

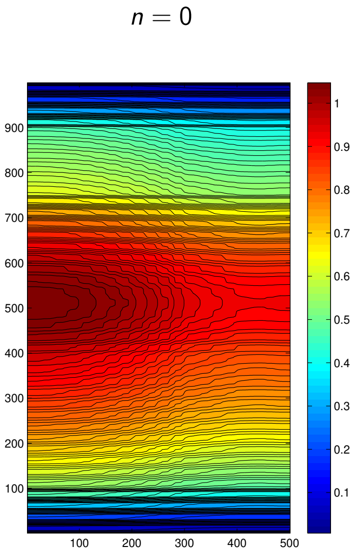


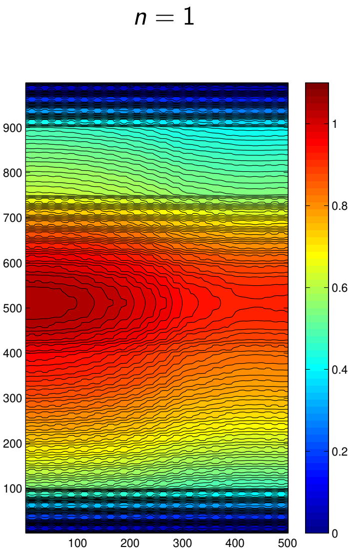


First buckling mode of a microstructured plate



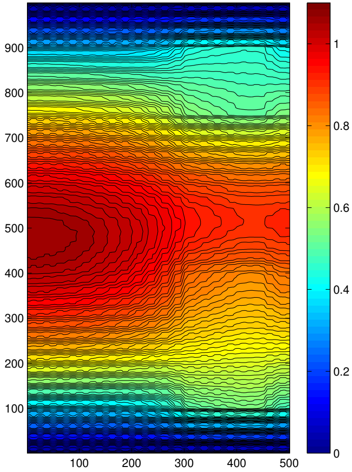
u_n: outer-plane component of the displacement field



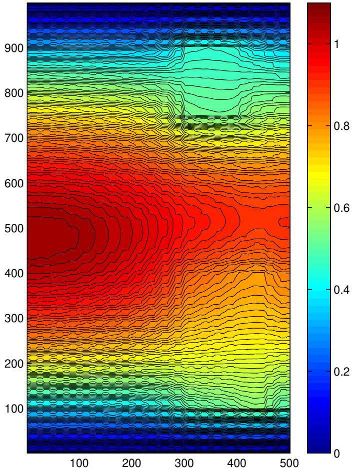


u_n: outer-plane component of the displacement field

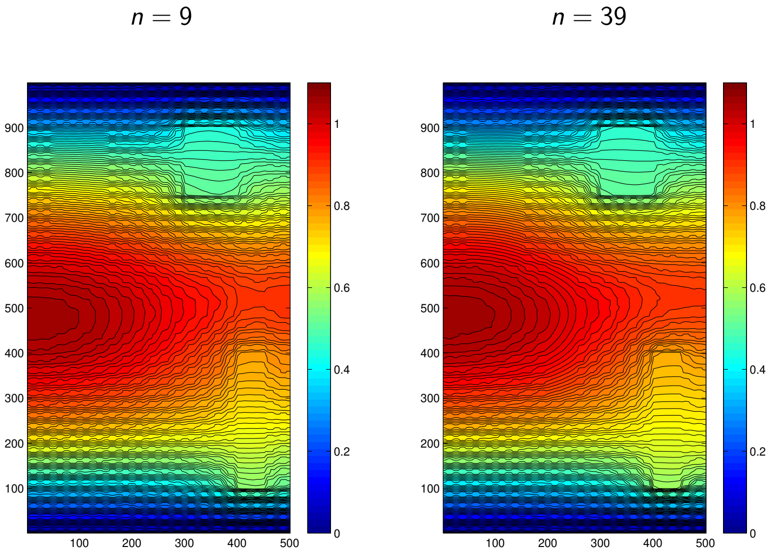
$n = 2$



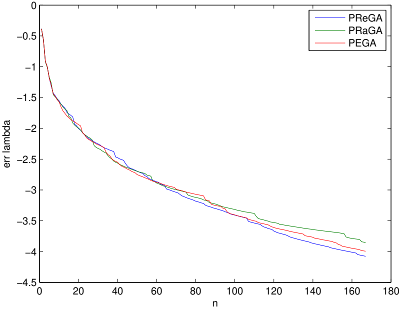
$n = 4$

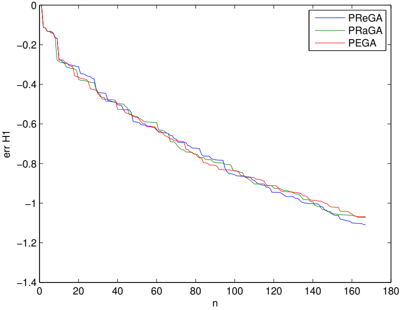


u_n: outer-plane component of the displacement field

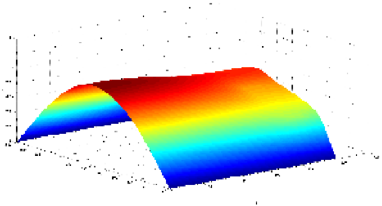


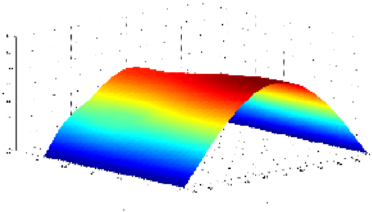
Numerical results





Numerical results





Conclusions

- Electronic structure calculations: theoretical and practical issues
- Parametric eigenvalue problems: the eigenvalue is itself a high-dimensional function!
- Nonlinear eigenvalue problems: ex:Gross-Pitaevskii model

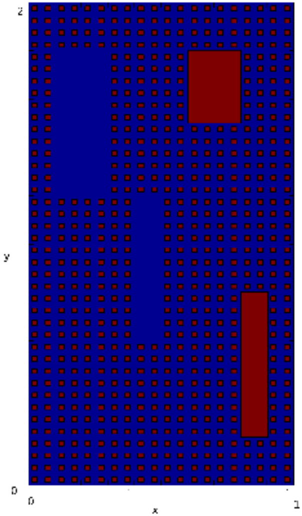
$$-\Delta u + u^3 = \mu u.$$

References

- R.E. Bellman. Dynamic Programming. Princeton University Press, 1957.
- A. Ammar, B. Mokdad, F. Chinesta, and R. Keunings. A new family of solvers for some classes of multidimensional partial differential equations encountered in kinetic theory modeling of complex fluids. J. Non-Newtonian Fluid Mech., 139:153-176, 2006.
- E. Cancès, VE, T. Lelièvre Convergence of a greedy algorithm for high-dimensional convex nonlinear problems, M3AS, 2433-2467, 2011.
- E. Cancès, VE, T. Lelièvre Greedy algorithms for high-dimensional linear eigenvalue problems, http://arxiv.org/abs/1304.2631.
- C. Le Bris, T. Lelièvre and Y. Maday. Results and questions on a nonlinear approximation approach for solving high-dimensional partial differential equations, Constructive Approximation 30(3):621-651, 2009.
- A. Nouy, A priori model reduction through Proper Generalized Decomposition for solving time-dependent partial differential equations, CMAME, 2010.
- A. Nouy, A. Falco, Proper Generalized Decomposition for Nonlinear Convex Problems in Tensor Banach Spaces, Numerische Mathematik, 2011.
- V.N. Temlyakov. Greedy approximation. Acta Numerica, 17:235-409, 2008.
- A. Ammar and F. Chinesta, Circumventing Curse of Dimensionality in the Solution of Highly Multidimensional Models Encountered in Quantum Mechanics Using Meshfree Finite Sums Decompositions, Lecture notes in Computational Science and Engineering, 65, 1-17, 2010.
- S. Lojasiewicz, Ensembles semi-analytiques, Institut des Hautes Etudes Scientifiques, 1965.
- A. Levitt, Convergence of gradient-based algorithms for the Hartree-Fock equations, accepted for publication in ESAIM : M2AN.

Thank you for your attention!

Computation of the first buckling mode of a microstructured plate



Strain tensors of the plate

Let $\Omega_x := (0,1)$, $\Omega_y := (0,2)$, $E : \Omega_x \times \Omega_y \to \mathbb{R}$ (Young modulus) and $\nu > 0$ (Poisson coefficient), $F < 0$, h thickness of the plate, $(u_x, u_y, v) : \Omega_x \times \Omega_y \to \mathbb{R}^3$ displacement field of the plate, $u = (u_x, u_y)$.

Space of cinematically admissible displacement fields:

$$V^u := \left\{ \overline{u} = (\overline{u}_x, \overline{u}_y) \in \left(H^1(\Omega_x \times \Omega_y)\right)^2, \ \overline{u}_x(x,0) = \overline{u}_y(x,0) = 0 \text{ for almost all } x \in\right.$$

$$V^v := \left\{ \overline{v} \in H^2(\Omega_x \times \Omega_y), \ \overline{v}(x,0) = \overline{v}(x,2) = \frac{\partial \overline{v}}{\partial y}(x,0) = \frac{\partial \overline{v}}{\partial y}(x,2) = 0 \text{ for almost}\right.$$

Membrane strain

$$\underline{\underline{\epsilon}}_u := \begin{bmatrix} \frac{\partial u_x}{\partial x} & \frac{1}{2}\left(\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}\right) \\ \frac{1}{2}\left(\frac{\partial u_x}{\partial y} + \frac{\partial u_x}{\partial y}\right) & \frac{\partial u_y}{\partial y} \end{bmatrix} \quad \underline{\underline{\epsilon}}_v := \begin{bmatrix} \frac{1}{2}\left(\frac{\partial v}{\partial x}\right)^2 & \frac{1}{2}\left(\frac{\partial v}{\partial x}\frac{\partial v}{\partial y}\right) \\ \frac{1}{2}\left(\frac{\partial v}{\partial x}\frac{\partial v}{\partial y}\right) & \frac{1}{2}\left(\frac{\partial v}{\partial y}\right)^2 \end{bmatrix}$$

$$\underline{\underline{\epsilon}} := \underline{\underline{\epsilon}}_u + \underline{\underline{\epsilon}}_v$$

Curvature strain

$$\underline{\underline{\chi}} := \begin{bmatrix} \frac{\partial^2 v}{\partial x^2} & \frac{\partial^2 v}{\partial x \partial y} \\ \frac{\partial^2 v}{\partial x \partial y} & \frac{\partial^2 v}{\partial y^2} \end{bmatrix}$$

Potential energy of the plate

$$W(u,v) := \int_{\Omega_x\times\Omega_y} \frac{E(x,y)h}{2(1-\nu^2)} \left[\nu\left(\mathrm{Tr}\underline{\underline{\epsilon}}\right)^2 + (1-\nu)\underline{\underline{\epsilon}} : \underline{\underline{\epsilon}}\right] dx\,dy$$

(membrane energy)

$$+ \int_{\Omega_x\times\Omega_y} \frac{E(x,y)h^3}{24(1-\nu^2)} \left[\nu\left(\mathrm{Tr}\underline{\underline{\chi}}\right)^2 + (1-\nu)\underline{\underline{\chi}} : \underline{\underline{\chi}}\right] dx\,dy$$

(bending energy)

$$- \int_{\Omega_x} F u_y(\cdot, 2)\, dx$$ (external forces)

Stationary equilibrium of the plate

Stationary equilibrium of the plate: $\left(u^0, v^0\right) \in V^u \times V^v$ such that

$$W'\left(u^0, v^0\right) = 0.$$

$\left(u^0, v^0\right) \in V^u \times V^v$ such that $v^0 = 0$ and

$$u^0 \in \operatorname*{argmin}_{u \in V^u} \mathcal{E}(u),$$

with

$$\mathcal{E}(u) := \int_{\Omega_x \times \Omega_y} \frac{E(x,y)h}{2(1-\nu^2)} \left[\nu \left(\mathrm{Tr}\underline{\underline{\epsilon}}_u\right)^2 + (1-\nu)\underline{\underline{\epsilon}}_u : \underline{\underline{\epsilon}}_u\right] dx\, dy$$

Buckling modes of the plate

There is buckling if and only if the smallest eigenvalue of the Hessian $dW^0 := W''(u^0, v^0)$ is negative. An associated eigenvector is the first buckling mode of the plate. Since $v^0 = 0$,

$$dW^0\left((u^1, v^1), (u^2, v^2)\right) = dW_u^0(u^1, u^2) + dW_v^0(v^1, v^2),$$

with

$$dW_u^0(u^1, u^2) := \int_{\Omega_x \times \Omega_y} \frac{E(x,y)h}{(1-\nu^2)} \left[\nu \mathrm{Tr}\underline{\underline{\epsilon}}_{u^1} \mathrm{Tr}\underline{\underline{\epsilon}}_{u^2} + (1-\nu)\underline{\underline{\epsilon}}_{u^1} : \underline{\underline{\epsilon}}_{u^2}\right] dx\, dy$$

$$\begin{aligned} dW_v^0(v^1, v^2) \quad := \quad & \int_{\Omega_x \times \Omega_y} \frac{E(x,y)h^3}{12(1-\nu^2)} \left[\nu \mathrm{Tr}\underline{\underline{\chi}}_{v^1} \mathrm{Tr}\underline{\underline{\chi}}_{v^2} + (1-\nu)\underline{\underline{\chi}}_{v^1} : \underline{\underline{\chi}}_{v^2}\right] dx\, dy \\ & + \int_{\Omega_x \times \Omega_y} \frac{E(x,y)h}{(1-\nu^2)} \left[\nu \mathrm{Tr}\underline{\underline{\epsilon}}_{u^0} \mathrm{Tr}\underline{\underline{e}}_{v^1,v^2} + (1-\nu)\underline{\underline{\epsilon}}_{u^0} : \underline{\underline{e}}_{v^1,v^2}\right] dx\, dy \end{aligned}$$

$$\underline{\underline{e}}_{v^1,v^2} := \begin{bmatrix} \frac{\partial v^1}{\partial x}\frac{\partial v^2}{\partial x} & \frac{1}{2}\left(\frac{\partial v^1}{\partial x}\frac{\partial v^2}{\partial y} + \frac{\partial v^1}{\partial y}\frac{\partial v^2}{\partial x}\right) \\ \frac{1}{2}\left(\frac{\partial v^1}{\partial x}\frac{\partial v^2}{\partial y} + \frac{\partial v^1}{\partial y}\frac{\partial v^2}{\partial x}\right) & \frac{\partial v^1}{\partial y}\frac{\partial v^2}{\partial y} \end{bmatrix}$$

Buckling mode of the microstructured plate

$$V^u := \left\{ \overline{u} = (\overline{u}_x, \overline{u}_y) \in \left(H^1(\Omega_x \times \Omega_y)\right)^2, \ \overline{u}_x = \overline{u}_y = 0 \text{ on } \Gamma_b \right\},$$
$$V^v := \left\{ \overline{v} \in H^2(\Omega_x \times \Omega_y), \ \overline{v} = \nabla \overline{v} \cdot \mathbf{n} = 0 \text{ on } \Gamma_b \cup \Gamma_t \right\}.$$

$$dW^0\left((u^1, v^1), (u^2, v^2)\right) = dW_u^0(u^1, u^2) + dW_v^0(v^1, v^2),$$

To determine whether there is buckling, we only need to compute the smallest eigenvalue of the bilinear form $a_v := dW_v^0 : V^v \times V^v \to \mathbb{R}$.

Continuous setting: $\Sigma := \left\{ r \otimes s, r \in V_x^v, \ s \in V_y^v \right\}$ **with**

$$V_x^v := \left\{ r \in H^2(\Omega_x), \ r(0) = r'(0) = r(2) = r'(2) = 0 \right\} \quad \textbf{and } V_y^v := H^2(\Omega_y).$$

Discrete setting: cubic splines $\otimes$ cubic splines.
The resolution of the full discretized problem via classical galerkin methods would require the computation of the lowest eigenvalue of **one** $10^6 \times 10^6$ matrix!
With the greedy algorithm, we only need the diagonalization (Rayleigh) or the inversion (Residual and Explicit) of **several** matrices whose maximum size is 2000×2000.