# Green's function retrieval from the CCF of random waves and energy conservation for an obstacle of arbitrary shape: noise source distribution on a large surrounding shell

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Accepted 2013 January 25. Received 2013 January 25; in original form 2012 October 10

# SUMMARY

For imaging the earth structure, the cross-correlation function (CCF) of random waves as ambient noise or coda waves has been widely used for the estimation of the Green's function. We precisely study the condition for the Green's function retrieval in relation to the energy conservation for a single obstacle of arbitrary shape. When an obstacle is placed in a 2-D homogeneous medium, the Green's function is written by a double series expansion using Hankel functions of the first kind which represent outgoing waves. When two receivers and the scattering obstacle are illuminated by uncorrelated noise sources randomly and uniformly distributed on a closed circle of a large radius surrounding them, the lag-time derivative of the CCF of random waves at the two receivers can be written by a convolution of the antisymmetrized Green's function and the autocorrelation function of the noise source time function. We explicitly derive the constraint for the Hankel function expansion coefficients as the sufficient condition for the Green's function retrieval. We show that the constraint is equal to the generalized optical theorem derived from the energy conservation principle. Physical meaning of the generalized optical theorem becomes clear when the Hankel function expansion coefficients are transformed into scattering amplitudes in the framework of the conventional scattering theory. In the 3-D case, the Green's function is written by a double series expansion using spherical Hankel functions of the first kind and spherical harmonic functions. When two receivers and the scattering obstacle are illuminated by noise sources randomly and uniformly distributed on a closed spherical shell of a large radius surrounding them, we explicitly derive the constraint for the spherical Hankel function expansion coefficients for the Green's function retrieval and the energy conservation. We note that the derivation of the constraint does not assume that two receivers are in the far field of the scattering obstacle.

**Key words:** Interferometry; Theoretical seismology; Wave scattering and diffraction; Wave propagation; Acoustic properties.

# **1 INTRODUCTION**

For imagining a medium containing heterogeneities, the Green's function retrieval from the cross-correlation function (CCF) of random waves has been widely used in many fields including seismology (e.g. Campillo & Paul 2003). This method has a root in the microtremor survey method pioneered by Aki (1957). In most theoretical studies, the medium heterogeneity and receivers are illuminated by random waves in the equipartition state (e.g. Lobkis & Weaver 2001; Sánchez-Sesma *et al.* 2006), random waves generated by uncorrelated noise sources widely distributed in space (e.g. Roux *et al.* 2005; Snieder 2006; Sato 2010), and random waves generated by uncorrelated noise sources distributed on a closed surface surrounding them (e.g. Wapenaar & Fokkema 2006; Weaver *et al.* 2009; Sato 2009a,b; Saito 2012). The condition for the Green's function retrieval is found to be the same as the generalized optical theorem on the basis of the energy principle and the scattering theory (e.g. Snieder *et al.* 2009; Wapenaar *et al.* 2010; Lu *et al.* 2011). For the case of multiple scattering by multiple scatterers, the generalized optical theorem is found to be the condition for the Green's function retrieval: Snieder *&* Fleury (2010) proved the case of a source distribution on a closed surface, and Margerin & Sato (2011a,b) proved the case of a wide distribution of sources.

Sánchez-Sesma & Campillo (2006) confirmed that the exact solution of the Green's function satisfying the given boundary condition of a cylindrical obstacle in the 2-D homogeneous medium can be retrieved from the CCF of random waves in the equipartition state, where they

wrote the Green's function satisfying the source–receiver reciprocity by a double series expansion using Hankel functions for outgoing waves. Wapenaar *et al.* (2010) studied the Green's function retrieval when random waves are radiated from noise sources distributing on a close surface surrounding receivers and the scattering obstacle. They derived the generalized optical theorem for the Green's function retrieval, where they made a symmetric use of spherically outgoing waves in the far field in the scattering part of the Green's function for paths between the scatterer and each of receivers. Lu *et al.* (2011) derived the generalized optical theorem from the dyadic expression of the elastic Green's function in the far field. Historical review of the optical theorem in physics is given by Newton (1976).

Referring those studies, this paper precisely examines the condition for the Green's function retrieval in relation to the energy conservation for an obstacle of arbitrary shape. We first study the 2-D case. When a single obstacle of arbitrary shape is placed in a homogeneous medium, the Green's function is written by a double series expansion using Hankel functions of the first kind which represent outgoing waves from the scattering obstacle. In the case that two receivers and the scattering obstacle are illuminated by random waves radiated from uncorrelated noise sources distributed on a closed circle of large radius surrounding them, we explicitly derive the constraint for the Hankel function expansion coefficients as the condition of the Green's function retrieval from the CCF of random waves. It should be noted that our derivation does not assume that receivers are in the far field of the scattering obstacle. We show that the constraint is equal to the generalized optical theorem derived from the energy conservation principle. Writing the Green's function in the standard form of the scattering theory using the scattering amplitude, we clarify the meaning of the generalized optical theorem. Then, we study the 3-D case.

# 2 WAVE EQUATION AND GREEN'S FUNCTION

Real scalar wavefield  $u(\mathbf{x}, t)$  is governed by the wave equation for an external source N:

$$\left[\Delta - \frac{1}{V_0^2} \partial_t^2\right] u\left(\mathbf{x}, t\right) + L_j(\mathbf{x}) u\left(\mathbf{x}, t\right) = N(\mathbf{x}, t),\tag{1}$$

where  $V_0$  is the wave velocity and real  $L_j$  represents a scattering obstacle *j* located around the origin. We may imagine a localized velocity inhomogeneity, a rigid body, or a cavity for  $L_j$ . We let *L* the dimension of the obstacle *j*. The total energy is conserved in this system. The wave-field *u* is calculated by the convolution of *N* and the Green's function *G*. The Green's function *G* at **x** and time *t* for a delta function source at **x**<sub>B</sub> in space and time 0 is governed by

$$\left[\Delta - \frac{1}{V_0^2}\partial_t^2\right]G(\mathbf{x}, \mathbf{x}_{\mathrm{B}}, t) + L_j(\mathbf{x})G(\mathbf{x}, \mathbf{x}_{\mathrm{B}}, t) = \delta(\mathbf{x} - \mathbf{x}_{\mathrm{B}})\delta(t).$$
(2)

In the angular frequency domain, the Green's function  $\widehat{G}$  obeys

$$[\Delta + k_0^2]\widehat{G}(\mathbf{x}, \mathbf{x}_{\mathrm{B}}, \omega) + L_j(\mathbf{x})\widehat{G}(\mathbf{x}, \mathbf{x}_{\mathrm{B}}, \omega) = \delta(\mathbf{x} - \mathbf{x}_{\mathrm{B}}),$$
(3)

where the Fourier transform is defined by  $\widehat{G}(\omega) = \int_{\infty}^{\infty} G(t) e^{i\omega t} dt$  and  $k_0 = \omega/V_0$ .

# 3 2-D CASE

## 3.1 Series expansion of the Green's function using Hankel functions

Outside of the obstacle *j*, waves in the homogenous medium are written by outgoing wave modes  $H_m^{(1)}(r)e^{im\varphi}$  and incoming wave modes  $H_m^{(2)}(r)e^{im\varphi}$ ; however, the Green's function satisfying the radiation condition in the far field is written by using only the outgoing waves. The Green's function for a homogeneous medium is  $\hat{G}_0(\mathbf{x}_A, \mathbf{x}_B, \omega) = \frac{-i}{4}H_0^{(1)}(k_0r_{AB})$ , where  $r_{AB}$  is a distance between A and B (e.g. Snieder 2004, p. 301). Scattered waves at receiver A for source B are represented by a superposition of outgoing waves  $\sum_{l=-\infty}^{\infty} a_l H_l^{(1)}(k_0r_{Aj})e^{il\theta_{Aj}}$ , where  $r_{Aj}$  is a distance between A and the obstacle *j* and  $\theta_{Aj}$  is the angle of A measured from the *x*-axis at *j*. On the other hand, scattered waves at receiver B for source A are written by a superposition of outgoing waves  $\sum_{l=-\infty}^{\infty} b_l H_l^{(1)}(k_0r_{Bj})e^{il\theta_{Bj}}$ , where  $r_{Bj}$  is a distance between B and the obstacle *j* and  $\theta_{Aj}$  is the angle of M measured from the *x*-axis at *j*. On the other hand, scattered waves at receiver B for source A are written by a superposition of outgoing waves  $\sum_{l=-\infty}^{\infty} b_l H_l^{(1)}(k_0r_{Bj})e^{il\theta_{Bj}}$ , where  $r_{Bj}$  is a distance between B and the obstacle *j* and  $\theta_{Bj}$  is the angle of B measured from the *x*-axis at *j*. We may write the Green's function as a sum of the direct term  $\widehat{G}_0$  and the scattered wave term  $\widehat{G}_s$  by using the double series expansion of Hankel functions of the first kind and Fourier series of angles:

$$\widehat{G}(\mathbf{x}_{A}, \mathbf{x}_{B}, \omega) = \widehat{G}_{0}(\mathbf{x}_{A}, \mathbf{x}_{B}, \omega) + \widehat{G}_{S}(\mathbf{x}_{A}, \mathbf{x}_{B}, \omega)$$

$$= \frac{-i}{4} H_{0}^{(1)}(k_{0}r_{AB}) + i \sum_{l=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} F_{lm}(\omega) H_{l}^{(1)}(k_{0}r_{Aj}) H_{m}^{(1)}(k_{0}r_{Bj}) e^{il\theta_{Aj} + im\theta_{Bj}},$$
(4)

where double series expansion coefficients  $F_{lm}$  are non-dimensional since  $\hat{G}$  is non-dimensional for the 2-D case. This is an extension of (Sánchez-Sesma *et al.* 2006, eq. 4) for a cylindrical obstacle to an arbitrary shape obstacle and an explicit representation of (Wapenaar *et al.* 2010, eq. 6).

According to the source–receiver reciprocity, the Green's function is symmetric with respect to the exchange of A and B, which leads to  $F_{lm} = F_{ml}$ . Replacing  $\mathbf{x}_{A}$  with  $\mathbf{x}$ , we see the Green's function in this representation  $\widehat{G}(\mathbf{x}, \mathbf{x}_{B}, \omega)$  explicitly satisfies the radiation condition  $\lim_{r_{x_{j}}\to\infty} \sqrt{r_{x_{j}}}(ik_{0}\widehat{G} - \partial_{r_{x_{j}}}\widehat{G}) = 0$  at a large distance from the source B and the scatterer *j* when  $r_{x_{j}} \gg k_{0}^{-1}$ ,  $r_{Bj}$  and *L*.

The coefficients  $F_{lm}$  can be determined by solving the appropriate boundary condition for the given obstacle  $L_i$ .

#### 3.2 Green's function retrieval from the CCF of random waves

We suppose a random and uniform distribution of uncorrelated noise sources on a large circle shell with radius *R* and thickness  $\Delta R$  surrounding the scattering obstacle *j* and receivers A and B as shown in Fig. 1, where  $R \gg 1/k_0$ ,  $r_{Aj}$ ,  $r_{Bj}$ , *L* and  $\Delta R \ll R$ . We imagine an ensemble of noise source distributions on the circle shell {*N*}. Each noise source generates stationary random signals in time (e.g. Sato 2009a; Sato *et al.* 2012). We first define the CCF of the source time function as the average over a long time window of length *T* and the average over the ensemble of source distributions as

$$\left\langle C_{N}(\mathbf{x},\mathbf{x}',\tau)\right\rangle \equiv \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} dt \left\langle N\left(\mathbf{x},t-\tau\right) N\left(\mathbf{x}',t\right)\right\rangle = \delta\left(\mathbf{x}-\mathbf{x}'\right) S_{N}(\tau),$$
(5)

where  $S_N(\tau)$  is the autocorrelation function (ACF) of the noise source signals and angular brackets mean the average over the ensemble and  $\delta(\mathbf{x})$  is a delta function in 2-D space. It means that noise sources at different locations are uncorrelated. In the angular frequency domain,

$$\lim_{T \to \infty} \frac{1}{T} \langle \hat{N}(\mathbf{x}, \omega)^* \, \hat{N}(\mathbf{x}', \omega) \rangle = \delta\left(\mathbf{x} - \mathbf{x}'\right) \widehat{S}_N(\omega),$$
(6)

where  $\hat{N}(\mathbf{x}, \omega) = \int_{-T/2}^{T/2} N(\mathbf{x}, t) e^{i\omega t} dt$ , and  $\hat{S}_N(\omega)$  is the power-spectral density function (PSDF) of the noise source signals. The CCF of random waves registered at receivers A and B is

$$\langle C_{u}(\mathbf{x}_{\mathrm{A}}, \mathbf{x}_{\mathrm{B}}, \tau) \rangle \equiv \lim_{T \to \infty} \frac{1}{T} \int_{-T/2}^{T/2} \langle u(\mathbf{x}_{\mathrm{A}}, t - \tau) u(\mathbf{x}_{\mathrm{B}}, t) \rangle \, dt,$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} d\omega e^{-i\omega\tau} \lim_{T \to \infty} \frac{1}{T} \langle \hat{u}(\mathbf{x}_{\mathrm{A}}, \omega)^{*} \hat{u}(\mathbf{x}_{\mathrm{B}}, \omega) \rangle.$$

$$(7)$$

By using a convolution integral of the Green's function and noise sources on the circle shell, random waves at the receiver location is

$$\hat{u}\left(\mathbf{x}_{A,B},\omega\right) = \int_{R,\Delta R} \widehat{G}\left(\mathbf{x}_{A,B},\mathbf{x},\omega\right) \hat{N}\left(\mathbf{x},\omega\right) d\mathbf{x},\tag{8}$$



Figure 1. Scattering obstacle *j* and receivers A and B are illuminated by random waves radiated from uncorrelated noise sources distributed on a circular shell with a large radius.

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where dx is an areal element. Substituting (8) into (7), and taking the average over the noise source ensemble using (6), we have

where the areal element  $d\mathbf{x}$  is written by a product of the thickness  $\Delta R$  and the line element  $dl(\mathbf{x})$ . This equation means that random waves radiated from different locations on the circle are incoherent.

If the relation

$$\oint_{R} \widehat{G}\left(\mathbf{x}_{\mathrm{A}}, \mathbf{x}, \omega\right)^{*} \widehat{G}\left(\mathbf{x}_{\mathrm{B}}, \mathbf{x}, \omega\right) dl(\mathbf{x}) = \frac{i}{2k_{0}} \left[\widehat{G}\left(\mathbf{x}_{\mathrm{A}}, \mathbf{x}_{\mathrm{B}}, \omega\right) - \widehat{G}\left(\mathbf{x}_{\mathrm{A}}, \mathbf{x}_{\mathrm{B}}, \omega\right)^{*}\right]$$
(10)

holds good, the lag-time derivative of the ensemble averaged CCF of random waves is given by

$$\frac{\mathrm{d}}{\mathrm{d}\tau} \left\langle C_{u}(\mathbf{x}_{\mathrm{A}}, \mathbf{x}_{\mathrm{B}}, \tau) \right\rangle = \Delta R \frac{V_{0}}{2} \int_{-\infty}^{\infty} \left[ G\left(\mathbf{x}_{\mathrm{A}}, \mathbf{x}_{\mathrm{B}}, \tau - \tau'\right) - G\left(\mathbf{x}_{\mathrm{A}}, \mathbf{x}_{\mathrm{B}}, -\tau - \tau'\right) \right] S_{N}\left(\tau'\right) \mathrm{d}\tau', \tag{11}$$

where we used that  $S(\tau)$  is symmetric and  $G(\mathbf{x}_A, \mathbf{x}_B, t)$  is real. The first term is the retarded Green's function and the second term is the advanced Green's function. Thus the derivative of the CCF with respect to lag time is a convolution of the antisymmetrized retarded Green's function and the ACF of the noise source signals. This equation means that the Green's function can be retrieved from the ensemble averaged CCF of random waves. Thus, the relation (10) is the key for the Green's function retrieval. In the following we seek the constraint for expansion coefficients  $F_{lm}$ 's in (4) to satisfy (10).

# 3.3 Constraint for the Hankel function expansion coefficients

In the following, we abbreviate  $\widehat{G}(\mathbf{x}_{A}, \mathbf{x}_{B}, \omega)$  to  $\widehat{G}(AB)$  and  $H_{l}^{(1,2)}(k_{0}r_{AB})$  to  $H_{l}^{(1,2)}(AB)$ . First we calculate the RHS of (10):

$$\widehat{G}(AB) - \widehat{G}(AB)^* = \frac{-i}{2}J_0(AB) + i\sum_{l,m=-\infty}^{\infty} F_{lm}H_l^{(1)}(Aj)H_m^{(1)}(Bj)e^{il\theta_{Aj} + im\theta_{Bj}} + i\sum_{l,m=-\infty}^{\infty} F_{lm}^*H_l^{(2)}(Aj)H_m^{(2)}(Bj)e^{-il\theta_{Aj} - im\theta_{Bj}}.$$
(12)

Next, we calculate the LHS of (10). According to the Graf's addition theorem,  $H_0^{(1)}(\sqrt{z^2 + y^2 - 2zy \cos \theta}) = \sum_{l=-\infty}^{\infty} H_l^{(1)}(z)J_l(y)e^{il\theta}$  for |z| > |y|, where included angle  $\theta = \angle_{zoy}$ . Using this theorem, we rewrite the direct propagation term  $\widehat{G}_0$  as if waves were radiated from the obstacle *j* since  $r_{xj} > r_{Aj}$ ,  $r_{Bj}$ :

$$\widehat{G}(Ax)^* = \frac{i}{4} \sum_{l=-\infty}^{\infty} H_l^{(2)}(xj) J_l(Aj) e^{-il(\theta_{Aj} - \theta_{xj})} - i \sum_{l,m=-\infty}^{\infty} F_{lm}^* H_l^{(2)}(Aj) H_m^{(2)}(xj) e^{-il\theta_{Aj} - im\theta_{xj}},$$
(13a)

$$\widehat{G}(Bx) = \frac{-i}{4} \sum_{l'=-\infty}^{\infty} H_{l'}^{(1)}(xj) J_{l'}(Bj) e^{il'(\theta_{Bj} - \theta_{xj})} + i \sum_{l',m'=-\infty}^{\infty} F_{l'm'} H_{l'}^{(1)}(Bj) H_{m'}^{(1)}(xj) e^{il'\theta_{Bj} + im'\theta_{xj}}.$$
(13b)

The line integral on the circle is written as the angular integral. Using the orthogonality  $\int_0^{2\pi} d\theta_{xj} e^{i(l-m)\theta_{xj}} = 2\pi \delta_{lm}$ , we perform the angular integral of the product  $\widehat{G}(Ax)^* \widehat{G}(Bx)$ :

$$\begin{split} \oint_{R} dl(\mathbf{x}) \,\widehat{G}(Ax)^{*} \widehat{G}(Bx) &= R \int_{0}^{2\pi} d\theta_{xj} \,\widehat{G}(Ax)^{*} \,\widehat{G}(Bx), \\ &= \frac{\pi R}{8} \sum_{l=-\infty}^{\infty} H_{l}^{(2)}(xj) J_{l}(Aj) H_{l}^{(1)}(xj) J_{l}(Bj) e^{il(\theta_{Aj} - \theta_{Bj})} \\ &- \frac{\pi R}{2} \sum_{l,m=-\infty}^{\infty} F_{lm}^{*} H_{l}^{(2)}(Aj) H_{m}^{(2)}(xj) H_{m}^{(1)}(xj) J_{m}(Bj) e^{-il\theta_{Aj} - im\theta_{Bj}} \\ &- \frac{\pi R}{2} \sum_{l',m'=-\infty}^{\infty} H_{m'}^{(2)}(xj) J_{m'}(Aj) F_{l'm'} H_{l'}^{(1)}(Bj) H_{m'}^{(1)}(xj) e^{im'\theta_{Aj} + il'\theta_{Bj}} \\ &+ 2\pi R \sum_{l,m,l'=-\infty}^{\infty} F_{lm}^{*} H_{l}^{(2)}(Aj) H_{m}^{(2)}(xj) F_{l'm} H_{l'}^{(1)}(Bj) H_{m'}^{(1)}(xj) e^{-il\theta_{Aj} + il'\theta_{Bj}}, \end{split}$$
(14)

where we used  $J_{-l}(z) = (-1)^l J_l(z)$  and  $H_{-l}^{(1,2)}(z) = (-1)^l H_l^{(1,2)}(z)$ . On the circle of radius R,  $H_l^{(1,2)}(x_j)$  means  $H_l^{(1,2)}(k_0 R)$ . We note that Hankel functions  $H_l^{(1)}(x_j)$  and  $H_l^{(2)}(x_j)$  appear as a pair having the same index l. Using the asymptote  $H_l^{(1)}(z) \approx \sqrt{2/(\pi z)} e^{+i[z-(2l+1)\pi/4]}$  and  $H_l^{(2)}(z) \approx \sqrt{2/(\pi z)} e^{-i[z-(2l+1)\pi/4]}$  for  $z \gg 1$ , we approximate the product  $H_l^{(2)}(k_0 R) H_l^{(1)}(k_0 R) \approx 2/(\pi k_0 R)$  since  $R \gg k_0^{-1}$ . Using this approximation, we obtain

$$\oint_{R} dl(\mathbf{x})\widehat{G}(Ax)^{*}\widehat{G}(Bx) = \frac{1}{4k_{0}}J_{0}(AB) - \frac{1}{k_{0}}\sum_{l,m=-\infty}^{\infty} F_{lm}^{*}e^{-il\theta_{Aj}-im\theta_{Bj}}H_{l}^{(2)}(Aj)J_{m}(Bj) - \frac{1}{k_{0}}\sum_{l,m=-\infty}^{\infty} F_{lm}e^{im\theta_{Aj}+il\theta_{Bj}}J_{m}(Aj)H_{l}^{(1)}(Bj) + \frac{4}{k_{0}}\sum_{l,m,n=-\infty}^{\infty} F_{ln}^{*}F_{mn}e^{-il\theta_{Aj}+im\theta_{Bj}}H_{l}^{(2)}(Aj)H_{m}^{(1)}(Bj),$$
(15)

where the Neumann's addition theorem  $J_0(AB) = \sum_{l=-\infty}^{\infty} J_l(Aj) J_l(Bj) e^{il(\theta_{Aj} - \theta_{Bj})}$  is also used in the first term.

Using the symmetry  $F_{lm} = F_{ml}$ , and writing Bessel functions by Hankel functions, we have

$$\oint_{R} dl(\mathbf{x})\widehat{G}(Ax)^{*}\widehat{G}(Bx) = \frac{i}{2k_{0}} \left[ -\frac{i}{2}J_{0}(AB) + i\sum_{l,m=-\infty}^{\infty} F_{lm}e^{il\theta_{Aj}+im\theta_{Bj}}H_{l}^{(1)}(Aj)H_{m}^{(1)}(Bj) + i\sum_{l,m=-\infty}^{\infty} F_{lm}^{*}e^{-il\theta_{Aj}-im\theta_{Bj}}H_{l}^{(2)}(Aj)H_{m}^{(2)}(Bj) \right] - \frac{1}{2k_{0}}\sum_{l,m=-\infty}^{\infty} F_{lm}^{*}e^{-il\theta_{Aj}-im\theta_{Bj}}H_{l}^{(2)}(Aj)H_{m}^{(1)}(Bj) - \frac{1}{2k_{0}}\sum_{l,m=-\infty}^{\infty} F_{lm}^{*}e^{-il\theta_{Aj}-im\theta_{Bj}}H_{l}^{(2)}(Aj)H_{m}^{(1)}(Bj) + \frac{4}{k_{0}}\sum_{l,m=-\infty}^{\infty} F_{lm}^{*}F_{mn}e^{-il\theta_{Aj}+im\theta_{Bj}}H_{l}^{(2)}(Aj)H_{m}^{(1)}(Bj).$$
(16)

Substituting the terms in angular braces with (12) and recovering the original arguments, we finally write the LHS of (10) as

$$\oint_{R} dl(\mathbf{x})\widehat{G}(\mathbf{x}_{A}, \mathbf{x}, \omega)^{*}\widehat{G}(\mathbf{x}_{B}, \mathbf{x}, \omega) = \frac{i}{2k_{0}} [\widehat{G}(\mathbf{x}_{A}, \mathbf{x}_{B}, \omega) - G(\mathbf{x}_{A}, \mathbf{x}_{B}, \omega)^{*}] \\
+ \frac{1}{2k_{0}} \sum_{l,m=-\infty}^{\infty} \left[ -(-1)^{m} F_{l,-m}^{*} - (-1)^{l} F_{m,-l} + 8 \sum_{n=-\infty}^{\infty} F_{ln}^{*} F_{mn} \right] e^{-il\theta_{Aj} + im\theta_{Bj}} H_{l}^{(2)}(k_{0}r_{Aj}) H_{m}^{(1)}(k_{0}r_{Bj}). \quad (17)$$

Thus the constraint for Hankel function expansion coefficients in (4) is to satisfy

$$(-1)^{m} F_{l,-m}^{*} + (-1)^{l} F_{m,-l} = 8 \sum_{n=-\infty}^{\infty} F_{ln}^{*} F_{mn} \quad \text{for any } l \text{ and } m.$$
(18)

This is the sufficient condition for the Green's function retrieval for the 2-D case (10) which we sought.

#### 3.4 Energy conservation

In the homogeneous medium outside of the obstacle *j*, the continuity equation of energy is given by

$$\partial_t \frac{1}{2} \left( \dot{u}^* \dot{u} + V_0^2 \nabla u^* \nabla u \right) + \nabla \left[ -\frac{V_0^2}{2} \left( \dot{u}^* \nabla u + \nabla u^* \dot{u} \right) \right] = 0, \tag{19}$$

where  $\mathbf{M}(\mathbf{x}, t) = -\frac{V_0^2}{2}(\dot{u}^* \nabla u + \nabla u^* \dot{u})$  is the energy flux density. For a plane wave of unit amplitude  $\hat{u} = e^{i\mathbf{k}\cdot\mathbf{x}}$ ,  $\mathbf{M} = V_0\omega^2\mathbf{k}/|\mathbf{k}|$ . In the angular frequency domain, the corresponding spectral density  $\widehat{\mathbf{M}}(\mathbf{x}, \omega) = -i\omega\frac{V_0^2}{2}(\hat{u}^*\nabla\hat{u} - \nabla\hat{u}^*\hat{u})$ . As illustrated in Fig. 2, the total radiated energy is given by the integral of the energy flux density over a closed circle of radius R' surrounding the obstacle j and the source B,  $\int_{-\infty}^{\infty} dt \oint dl(\mathbf{x}) \sum_{i=1}^{2} M_i(\mathbf{x}, t)n_i = \frac{1}{2\pi} \int_{-\infty}^{\infty} d\omega \oint dl(\mathbf{x}) \sum_{i=1}^{2} \widehat{M}_i(\mathbf{x}, \omega)n_i$ , where **n** is the outward normal unit vector of the circle. At angular frequency  $\omega$ , the line integral over the circle for a delta function source at B is

$$\oint_{R'} \mathrm{d}l(\mathbf{x}) \sum_{i=1}^{2} \widehat{M}_{i}(\widehat{G}(xB))n_{i} = -i\frac{\omega V_{0}^{2}}{2} \oint_{R'} \mathrm{d}l(\mathbf{x}) \sum_{i=1}^{2} n_{i}[\widehat{G}(xB)^{*}\partial_{i}\widehat{G}(xB) - \partial_{i}\widehat{G}(xB)^{*}\widehat{G}(xB)].$$
(20)



Figure 2. Line integral of  $\widehat{\mathbf{M}}\mathbf{n}$  over the circle of a large radius R' enclosing the source B and the scatterer *j* gives the total radiated energy at angular frequency  $\omega$ .

We write the line integral over the circle of radius R' into the areal integral by using the Gauss's divergence theorem. Then, using (3) and knowing  $L_j$  is real, we obtain

$$\begin{split} \oint_{R'} dl(\mathbf{x}) \sum_{i=1}^{2} \widehat{M}_{i}(\widehat{G}(xB))n_{i} &= -i \frac{\omega V_{0}^{2}}{2} \int_{R'} d\mathbf{x} \left[ \widehat{G}(xB)^{*} \Delta \widehat{G}(xB) - (\Delta \widehat{G}(xB)^{*}) \widehat{G}(xB) \right], \\ &= -i \frac{\omega V_{0}^{2}}{2} \int_{R'} d\mathbf{x} \left\{ \widehat{G}(xB)^{*} \left[ -k_{0}^{2} \widehat{G}(xB) - L_{j}(\mathbf{x}) \widehat{G}(xB) + \delta(\mathbf{x} - \mathbf{x}_{B}) \right] \right. \\ &- \left[ -k_{0}^{2} \widehat{G}(xB)^{*} - L_{j}(\mathbf{x}) \widehat{G}(xB)^{*} + \delta(\mathbf{x} - \mathbf{x}_{B}) \right] \widehat{G}(xB) \right\}, \\ &= -i \frac{\omega V_{0}^{2}}{2} \int_{R'} d\mathbf{x} \left[ \widehat{G}(xB)^{*} \delta(\mathbf{x} - \mathbf{x}_{B}) - \widehat{G}(xB) \delta(\mathbf{x} - \mathbf{x}_{B}) \right], \\ &= i \frac{\omega V_{0}^{2}}{2} \left[ \widehat{G}(BB) - \widehat{G}(BB)^{*} \right], \end{split}$$

$$\tag{21}$$

where the RHS is the imaginary part of the return Green's function  $\widehat{G}(BB)$ , of which the source and the receiver locations coincide with each other. This quantity is independent of radius R', which means the conservation of energy.

Then, we calculate the energy flux density integral (20) using the Hankel function expansion form of the Green's function (4). At a large distance  $R' \gg 1/k_0$ ,  $r_{Bj}$  and L, using the fact that the Green's function  $\widehat{G}$  satisfies the radiation condition, we may replace  $\sum_{i=1}^{2} n_i \partial_i \widehat{G} \rightarrow i k_0 \widehat{G}$  and  $\sum_{i=1}^{2} n_i \partial_i \widehat{G}^* \rightarrow -i k_0 \widehat{G}^*$  in (20) since  $\sum_{i=1}^{2} n_i \partial_i = \partial_r$  on the large circle. Then, this integral becomes the same form as the LHS of (10), where the source and the receiver are located at the same point. Using the source–receiver reciprocity and then putting A to B in (17), we get

$$\begin{split} \oint_{R'} \mathrm{d}l(\mathbf{x}) & \sum_{i=1}^{2} \widehat{M}_{i}(\widehat{G}(xB))n_{i} \approx \omega^{2} V_{0} \oint_{R'} \mathrm{d}l(\mathbf{x}) \, \widehat{G}(xB)^{*} \widehat{G}(xB), \\ &= i \frac{\omega V_{0}^{2}}{2} [\widehat{G}(BB) - \widehat{G}(BB)^{*}] \\ &+ \frac{\omega V_{0}^{2}}{2} \sum_{l,m=-\infty}^{\infty} \left[ -(-1)^{m} F_{l,-m}^{*} - (-1)^{l} F_{m,-l} + 8 \sum_{n=-\infty}^{\infty} F_{ln}^{*} F_{mn} \right] H_{l}^{(2)}(Bj) H_{m}^{(1)}(Bj) e^{i(-l+m)\theta_{Bj}}, \end{split}$$
(22)

which should be equal to (21). Thus, we derived the same constraint with (18) for the energy conservation. The constraint derived from the energy conservation principle is called the generalized optical theorem.



 $r_{xj}, r_{Bj} >> k_0^{-1}, L$ 

Figure 3. Source B and a receiver x are placed in the far field of the scattering obstacle *j*.

#### 3.5 Scattering amplitude and optical theorem

In order to clarify the physical meaning of the constraint (18), we place both source B and receiver **x** in the far field of the scattering obstacle *j* as illustrated in Fig. 3:  $r_{xj}$  and  $r_{Bj} \gg k_0^{-1}$  and *L*. Using the approximation  $H_l^{(1)}(k_0r_{xj}) \approx \sqrt{\frac{2}{\pi k_0r_{xj}}} e^{ik_0r_{xj}} e^{-i(2l+1)\pi/4}$  and  $H_m^{(1)}(k_0r_{Bj}) \approx e^{-im\pi/2} H_0^{(1)}(k_0r_{Bj})$ , we write the scattering term in (4) by a product of the incident wave  $\hat{G}_0$  and a cylindrically outgoing wave  $e^{ik_0r_{xj}}/\sqrt{r_{xj}}$  in the standard form of the scattering theory:

$$\widehat{G}(\mathbf{x}, \mathbf{x}_{\mathrm{B}}, \omega) \approx \widehat{G}_{0}(r_{xB}, \omega) + \sum_{l,m=-\infty}^{\infty} \sqrt{\frac{1}{r_{xj}}} e^{ik_{0}r_{xj}} \sqrt{\frac{32}{\pi k_{0}}} e^{i3\pi/4} (-i)^{l} i^{m} F_{lm}(\omega) e^{il\theta_{xj} + im(\theta_{Bj} + \pi)} \widehat{G}_{0}(r_{Bj}, \omega),$$

$$= \widehat{G}_{0}(r_{xB}, \omega) + \frac{e^{ik_{0}r_{xj}}}{\sqrt{r_{xj}}} \sum_{l,m=-\infty}^{\infty} f_{lm}(\omega) e^{il\theta_{xj} + im(\theta_{Bj} + \pi)} \widehat{G}_{0}(r_{Bj}, \omega),$$
(23)

where the functions  $f_{lm}$ 's are the double Fourier series coefficients of the scattering amplitude  $f(\theta_{xj}, \theta_{Bj} + \pi, \omega)$ . Angle  $\theta_{Bj} + \pi$  is for the incident ray direction from the source B to the obstacle *j* at the origin, and  $\theta_{xj}$  for the scattered wave ray direction from the obstacle *j* to the receiver at **x**. The scattering pattern depends on both angles in general. There is a relation

$$F_{lm} = \sqrt{\frac{\pi k_0}{32}} e^{-i3\pi/4} i^l (-i)^m f_{lm}, \tag{24}$$

where  $f_{lm} = (-1)^{l-m} f_{ml}$ . The Green's function retrieval constraint (18) for  $F_{lm}$  is written as the constraint for  $f_{lm}$ :

$$if_{l,-m}^{*}e^{i\pi/4} - if_{m,-l}e^{-i\pi/4} = \sqrt{2\pi k_0} \sum_{n=-\infty}^{\infty} f_{ln}^{*}f_{mn}.$$
(25)

For the special case of a cylindrically symmetric obstacle, the scattering amplitude *f* becomes a function of scattering angle  $\theta_{xj} - (\theta_{Bj} + \pi)$  measured from the incident ray direction. If we let

$$f_{lm} = \delta_{l,-m} f_l, \tag{26}$$

where  $f_l = f_{-l}$ , we rewrite (23) as

$$\widehat{G}(\mathbf{x}, \mathbf{x}_{\mathrm{B}}, \omega) \approx \widehat{G}_{0}(r_{xB}, \omega) + \frac{e^{ik_{0}r_{xj}}}{\sqrt{r_{xj}}} \sum_{l=-\infty}^{\infty} f_{l}(\omega) e^{il(\theta_{xj} - \theta_{Bj} - \pi)} \widehat{G}_{0}(r_{Bj}, \omega).$$
(27)

When the source B is on the *x*-axis left of the origin,  $\theta_{Bj} = \pi$ ,  $f_l$  is the Fourier expansion coefficient of the scattering amplitude. Substituting (26) into (25), we have

$$\operatorname{Im}\frac{f_{l}}{\sqrt{i}} = \sqrt{\frac{\pi k_{0}}{2}} |f_{l}|^{2}.$$
(28)

This is the conventional form of the optical theorem for scattering amplitude in the 2-D space (e.g. Sheng 2006, eq. 3.46). For example, in the special case of a cylindrical cavity of radius *a*, the free boundary condition is  $\partial_r \hat{G} = 0$  at  $r_{xj} = a$ . The Green's function is

$$\widehat{G}(\mathbf{x}_{\mathrm{A}}, \mathbf{x}_{\mathrm{B}}, \omega) = -\frac{i}{4} H_0^{(1)}(k_0 r_{\mathrm{AB}}) + \frac{i}{4} \sum_{l=-\infty}^{\infty} \frac{J_l'(k_0 a)}{H_l^{(1)}(k_0 a)} H_l^{(1)}(k_0 r_{Aj}) H_l^{(1)}(k_0 r_{Bj}) e^{il(\theta_{Aj} - \theta_{Bj})}$$
<sup>(29)</sup>

(e.g. Sánchez-Sesma et al. 2006, eq. 5), where

$$F_{lm}(\omega) = \delta_{l,-m} \frac{1}{4} (-1)^l \frac{J_l'(k_0 a)}{H_l^{(1)'}(k_0 a)},$$
(30a)

$$f_l(\omega) = \sqrt{\frac{2}{\pi k_0}} i e^{i\pi/4} \frac{J_l'(k_0 a)}{H_l^{(1)'}(k_0 a)}.$$
(30b)

We can easily confirm that this solution satisfies the optical theorem (28).

# 4 3-D CASE

Extending the theory developed in the previous section, we study the 3-D case using spherical Hankel functions and spherical harmonic functions. In order to avoid unnecessary duplication, we use mathematical equations developed in the previous section.

#### 4.1 Series expansion of the Green's function using spherical Hankel functions

A scattering obstacle i is placed around the origin as shown in Fig. 4. The Green's function  $\widehat{G}$  which satisfies the wave eq. (3) for a delta function source in 3-D space has a dimension of the reciprocal of length. Outside of the obstacle *j*, we may write the Green's function  $\widehat{G}$  by using spherical Hankel functions of the first kind representing outgoing waves and spherical harmonic functions:

 $\widehat{G}(\mathbf{x}_{\mathrm{A}},\mathbf{x}_{\mathrm{B}},\omega) = \widehat{G}_{0}(\mathbf{x}_{\mathrm{A}},\mathbf{x}_{\mathrm{B}},\omega) + \widehat{G}_{S}(\mathbf{x}_{\mathrm{A}},\mathbf{x}_{\mathrm{B}},\omega),$ 

$$= -i\frac{k_0}{4\pi}h_0^{(1)}(k_0r_{AB}) + ik_0\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\sum_{l'=0}^{\infty}\sum_{m'=-l'}^{l'}F_{lm,l'm'}(\omega)h_l^{(1)}(k_0r_{Aj})Y_{lm}(\theta_{Aj},\varphi_{Aj})h_{l'}^{(1)}(k_0r_{Bj})Y_{l'm'}(\theta_{Bj},\varphi_{Bj}),$$
(31)

where  $r_{Aj} = |\mathbf{x}_A - \mathbf{x}_j|$ , and  $\theta_{Aj}$  and  $\varphi_{Aj}$  are angles of A measured at j. The spherical harmonic function is

$$Y_{lm}(\theta,\varphi) = (-1)^{\frac{m+|m|}{2}} i^{l} \sqrt{\frac{(2l+1)}{4\pi} \frac{(l-|m|)!}{(l+|m|)!}} P_{l}^{|m|}(\cos\theta) e^{im\varphi},$$
(32)

where the phase factor is chosen as  $Y_{lm}^* = (-1)^{l-m} Y_{l,-m}$  (e.g. Landau & Lifshitz 2003, eqs 28.7 and 28.9). Double series expansion coefficients  $F_{lm,l'm'}$ 's are non-dimensional and the source-receiver reciprocity leads to  $F_{lm,l'm'} = F_{l'm',lm}$ . The Green's function  $\widehat{G}(\mathbf{x}, \mathbf{x}_{B}, \omega)$  explicitly satisfies the radiation condition  $\lim_{r_{x_j}\to\infty}r_{x_j}(ik_0\widehat{G}-\partial_{r_{x_j}}\widehat{G})=0$  at a large distance from the source B and the scatterer j when  $r_{x_j}\gg k_0^{-1}$ ,  $r_{Bj}$ and L.

#### 4.2 Green's function retrieval from the CCF of random waves

We suppose a random and uniform distribution of uncorrelated noise sources on a large closed spherical shell with radius R with a thickness  $\Delta R$  surrounding the scattering obstacle j and receivers A and B as shown in Fig. 4, where  $R \gg k_0^{-1}, r_{Aj}, r_{Bj}, L$  and  $\Delta R \ll R$ . Each noise source generates stationary random signals in time. We define the CCF of the source time function as the average over a long time window of length T and over the ensemble of noise source distributions  $\{N\}$  as given by (5), where  $\delta(\mathbf{x})$  is a delta function in 3-D space. The ensemble



Figure 4. Scattering obstacle *j* and receivers A and B are illuminated by random waves radiated from uncorrelated noise sources distributed on a spherical shell with a large radius.

averaged CCF of waves registered at two receivers A and B is

$$\langle C_{u}(\mathbf{x}_{\mathrm{A}}, \mathbf{x}_{\mathrm{B}}, \tau) \rangle = \frac{1}{2\pi} \int_{-\infty}^{\infty} \mathrm{d}\omega \, e^{-i\omega\tau} \int_{R,\Delta R} \widehat{G} \left( \mathbf{x}_{\mathrm{A}}, \mathbf{x}, \omega \right)^{*} \mathrm{d}\mathbf{x} \int_{R,\Delta R} \widehat{G} \left( \mathbf{x}_{\mathrm{B}}, \mathbf{x}', \omega \right) \delta \left( \mathbf{x} - \mathbf{x}' \right) \widehat{S}_{N} \left( \omega \right) \mathrm{d}\mathbf{x}',$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} \mathrm{d}\omega \, e^{-i\omega\tau} \widehat{S}_{N} \left( \omega \right) \Delta R \left[ \oint_{R} \widehat{G} \left( \mathbf{x}_{\mathrm{A}}, \mathbf{x}, \omega \right)^{*} \widehat{G} \left( \mathbf{x}_{\mathrm{B}}, \mathbf{x}, \omega \right) \mathrm{d}s(\mathbf{x}) \right],$$

$$(33)$$

where the volume element dx on the spherical shell is written by a product of the thickness  $\Delta R$  and the surface element ds(x). It means that random waves radiated from different locations on the spherical surface are incoherent.

If the relation

$$\oint_{R} \widehat{G} \left( \mathbf{x}_{\mathrm{A}}, \mathbf{x}, \omega \right)^{*} \widehat{G} \left( \mathbf{x}_{\mathrm{B}}, \mathbf{x}, \omega \right) \mathrm{d}s(\mathbf{x}) = \frac{i}{2k_{0}} \left[ \widehat{G} \left( \mathbf{x}_{\mathrm{A}}, \mathbf{x}_{\mathrm{B}}, \omega \right) - \widehat{G} \left( \mathbf{x}_{\mathrm{A}}, \mathbf{x}_{\mathrm{B}}, \omega \right)^{*} \right]$$
(34)

holds good, the relation (11) is established. Thus, the relation (34) is the key for the Green's function retrieval from the CCF of random waves. We seek the constraint for expansion coefficients  $F_{lm,l'm'}$ 's in (31) to satisfy (34).

#### 4.3 Constraint for the spherical Hankel function expansion coefficients

In the following, we abbreviate  $\widehat{G}(\mathbf{x}_{A}, \mathbf{x}_{B}, \omega)$  to  $\widehat{G}(AB)$ ,  $h_{l}^{(1)}(k_{0}r_{Aj})$  to  $h_{l}^{(1)}(Aj)$ , and  $Y_{lm}(\theta_{Aj}, \varphi_{Aj})$  to  $Y_{lm}(Aj)$ . We abbreviate  $\sum_{l=0}^{\infty} \sum_{m=-l}^{l}$  to  $\sum_{l,m}$  if not otherwise specified. The RHS of (34) is

$$\widehat{G}(AB) - \widehat{G}(AB)^{*} = -i\frac{k_{0}}{2\pi}j_{0}^{(1)}(AB) + ik_{0}\sum_{l,m}\sum_{l',m'}F_{lm,l'm'}h_{l}^{(1)}(Aj)h_{l'}^{(1)}(Bj)Y_{lm}(Aj)Y_{l'm'}(Bj) + ik_{0}\sum_{l,m}\sum_{l',m'}F_{lm,l'm'}^{*}h_{l}^{(2)}(Aj)h_{l'}^{(2)}(Bj)Y_{lm}^{*}(Aj)Y_{l'm'}^{*}(Bj)$$
(35)

since  $h_l^{(1)}(z)^* = h_l^{(2)}(z)$ .

Using the addition theorem  $h_0^{(1)}(\sqrt{z^2 + y^2 - 2zy\cos\theta}) = \sum_{l=0}^{\infty} (2l+1)h_l^{(1)}(z)j_l(y)P_l(\cos\theta)$  for |z| > |y|, where included angle  $\theta = \angle_{y_0z}$ (Abramowitz & Stegun 1970, eqs 10.1.45–46), we rewrite the direct propagation term  $\widehat{G}_0$  as if waves were radiated from the obstacle *j* since  $r_{xj} > r_{Aj}$ ,  $r_{Bj}$ :

$$\widehat{G}(Ax)^* = i \frac{k_0}{4\pi} \sum_{l=0}^{\infty} (2l+1) h_l^{(2)}(xj) j_l(Aj) P_l(\cos \angle_{Ajx}) - i k_0 \sum_{l,m} \sum_{l',m'} F_{lm,l',m'}^* h_l^{(2)}(Aj) h_{l'}^{(2)}(xj) Y_{lm}^*(Aj) Y_{l',m'}^*(xj),$$
(36a)

$$\widehat{G}(Bx) = -i\frac{k_0}{4\pi}\sum_{l''=0}^{\infty} (2l''+1)h_{l''}^{(1)}(xj)j_{l''}(Bj)P_{l''}(\cos \angle_{Bjx}) + ik_0\sum_{l'',m''}\sum_{l'',m'''}F_{l''m'',l''',m'''}h_{l''}^{(1)}(Bj)h_{l'''}^{(1)}(xj)Y_{l''m''}(Bj)Y_{l''',m'''}(xj).$$
(36b)

Using the addition theorem for Legendre polynomials  $P_l(\cos \angle_{Bjx}) = \frac{4\pi}{2l+1} \sum_{m=-l}^{l} Y_{lm}^*(Bj) Y_{lm}(xj) = \frac{4\pi}{2l+1} \sum_{m=-l}^{l} Y_{lm}(Bj) Y_{lm}^*(xj)$ , we may write the LHS of (34) as an angular integral of the product:

$$\oint_{R} ds(\mathbf{x})\widehat{G}(Ax)^{*}\widehat{G}(Bx) = \oint_{R} R^{2} d\Omega_{xj} \\
\times \left[ ik_{0} \sum_{l,m} h_{l}^{(2)}(xj)j_{l}(Aj)Y_{lm}(Aj)Y_{lm}^{*}(xj) - ik_{0} \sum_{l,m} \sum_{l',m'} F_{lm,l',m'}^{*} h_{l}^{(2)}(Aj)h_{l'}^{(2)}(xj)Y_{lm}^{*}(Aj)Y_{l',m'}^{*}(xj) \right] \\
\times \left[ -ik_{0} \sum_{l'',m''} h_{l''}^{(1)}(xj)j_{l''}(Bj)Y_{l''m''}^{*}(Bj)Y_{l''m''}(xj) + ik_{0} \sum_{l'',m''} \sum_{l''',m''} F_{l''m'',l''',m'''} h_{l''}^{(1)}(Bj)h_{l'''}^{(1)}(xj)Y_{l''m''}(Bj)Y_{l''m''}(xj) \right].$$
(37)

Then, we perform the angular integral by using the orthogonality of spherical harmonic functions  $\oint d\Omega Y_{lm} * Y_{l'm'} = \delta_{ll'} \delta_{mm'}$ :

$$\oint_{R} ds(\mathbf{x})\widehat{G}(Ax)^{*}\widehat{G}(Bx) = k_{0}^{2}R^{2} \sum_{l,m} h_{l}^{(2)}(xj)j_{l}(Aj)Y_{lm}(Aj)h_{l}^{(1)}(xj)j_{l}(Bj)Y_{lm}^{*}(Bj) 
- k_{0}^{2}R^{2} \sum_{l,m} \sum_{l',m'} h_{l}^{(2)}(xj)j_{l}(Aj)Y_{lm}(Aj)F_{l''m'',l,m}h_{l''}^{(1)}(Bj)h_{l}^{(1)}(xj)Y_{l''m''}(Bj) 
- k_{0}^{2}R^{2} \sum_{l,m} \sum_{l',m'} F_{lm,l',m'}^{*}h_{l}^{(2)}(Aj)h_{l'}^{(2)}(xj)Y_{lm}^{*}(Aj)h_{l'}^{(1)}(xj)j_{l'}(Bj)Y_{l'm'}^{*}(Bj) 
+ k_{0}^{2}R^{2} \sum_{l,m} \sum_{l',m'} \sum_{l',m'} F_{lm,l',m'}^{*}h_{l}^{(2)}(Aj)h_{l'}^{(2)}(xj)Y_{lm}^{*}(Aj)F_{l''m'',l'm'}h_{l''}^{(1)}(Bj)h_{l'}^{(1)}(xj)Y_{l''m''}(Bj).$$
(38)

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We note that spherical Hankel functions  $h_l^{(1)}(xj)$  and  $h_l^{(2)}(xj)$  appear as a pair having the same index *l*. On the spherical surface,  $h_l^{(1,2)}(xj)$  means  $h_l^{(1,2)}(k_0R)$ . Using the asymptotic behaviour  $h_l^{(1)}(z) \approx (-i)^{l+1}e^{iz}/z$  and  $h_l^{(2)}(z) \approx (i)^{l+1}e^{-iz}/z$  for  $z \gg 1$ , we approximate the product  $h_l^{(2)}(k_0R)$   $h_l^{(1)}(k_0R) \approx 1/(k_0R)$  for  $R \gg k_0^{-1}$ . Then the surface integral becomes

$$\oint_{R} ds(\mathbf{x})\widehat{G}(Ax)^{*}\widehat{G}(Bx) = \sum_{l,m} j_{l}(Aj)Y_{lm}(Aj)j_{l}(Bj)Y_{lm}^{*}(Bj) 
- \sum_{l,m} \sum_{l',m''} j_{l}(Aj)Y_{lm}(Aj)F_{l''m'',l,m}h_{l''}^{(1)}(Bj)Y_{l'm''}(Bj) 
- \sum_{l,m} \sum_{l',m'} F_{lm,l',m'}^{*}h_{l}^{(2)}(Aj)Y_{lm}^{*}(Aj)j_{l'}(Bj)Y_{l'm'}^{*}(Bj) 
+ \sum_{l,m} \sum_{l',m'} \sum_{l'',m''} F_{lm,l',m'}^{*}h_{l}^{(2)}(Aj)Y_{lm}^{*}(Aj)F_{l''m'',l',m'}h_{l''}^{(1)}(Bj)Y_{l'm''}(Bj).$$
(39)

Using relations  $2j_l(z) = h_l^{(1)}(z) + h_l^{(2)}(z)$ ,  $F_{l,m,l'm'} = F_{l',m',lm}$  and  $Y_{lm}^* = (-1)^{l+m}Y_{l,-m}$ , substituting (35) in the RHS, and then recovering original arguments, we obtain

$$\oint_{R} ds(\mathbf{x})\widehat{G}(\mathbf{x}_{A}, \mathbf{x}, \omega)^{*}\widehat{G}(\mathbf{x}_{B}, \mathbf{x}, \omega) = \frac{i}{2k_{0}} [\widehat{G}(\mathbf{x}_{A}, \mathbf{x}_{B}, \omega) - \widehat{G}(\mathbf{x}_{A}, \mathbf{x}_{B}, \omega)^{*}] \\
+ \frac{1}{2} \sum_{l,m} \sum_{l',m'} [-(-1)^{l-m} F_{l,-m,l',m'} - (-1)^{l'-m'} F_{lm,l',-m'}^{*} + 2 \sum_{l'',m''} F_{l,m,l'',m''}^{*} F_{l'm',l'',m''}] \\
\times h_{l}^{(2)}(k_{0}r_{Aj})h_{l'}^{(1)}(k_{0}r_{Bj})Y_{l,m}^{*}(\theta_{Aj}, \varphi_{Aj})Y_{l',m'}(\theta_{Bj}, \varphi_{Bj}).$$
(40)

Thus the Green's function retrieval condition (34) for spherical Hankel expansion coefficients in (31) is to satisfy

$$(-1)^{l-m}F_{l,-m,l',m'} + (-1)^{l'-m'}F^*_{lm,l',-m'} = 2\sum_{l'',m''}F^*_{l,m,l'',m''}F_{l'm',l'',m''} \quad \text{for any } l,m,l' \text{ and } m'.$$

$$(41)$$

This is the sufficient condition for the Green's function retrieval which we sought.

### 4.4 Energy conservation

As illustrated in Fig. 5, the total energy radiated from a delta function source at B is given by the integral of the energy flux density over a closed spherical surface of a large radius *R*' surrounding the scattering obstacle *j* and the source B,  $\frac{1}{2\pi} \int_{-\infty}^{\infty} d\omega \oint ds(\mathbf{x}) \sum_{i=1}^{3} \widehat{M}_i(\mathbf{x}, \omega) n_i$ , where **n** is the outward normal unit vector of the spherical surface. At angular frequency  $\omega$ , we valuate the following integral over the spherical



Figure 5. Surface integral of  $\widehat{\mathbf{M}}\mathbf{n}$  over the spherical surface of a large radius R' enclosing the source B and the scatterer j gives the total radiated energy at angular frequency  $\omega$ .

surface:

$$\oint_{R'} \mathrm{d}s(\mathbf{x}) \sum_{i=1}^{3} \widehat{M}_{i}(\widehat{G}(xB))n_{i} = -i \frac{\omega V_{0}^{2}}{2} \oint_{R'} \mathrm{d}s(\mathbf{x}) \sum_{i=1}^{3} n_{i}[\widehat{G}(xB)^{*}\partial_{i}\widehat{G}(xB) - \partial_{i}\widehat{G}(xB)^{*}\widehat{G}(xB)].$$

$$\tag{42}$$

We may write the surface integral on the sphere into a volumetric integral by using the Gauss's divergence theorem. Then, using the same procedure for the 2-D case (21), we obtain

$$\oint_{R'} ds(\mathbf{x}) \sum_{i=1}^{3} \widehat{M}_{i}(\widehat{G}(xB)) n_{i} = i \frac{\omega V_{0}^{2}}{2} [\widehat{G}(BB) - \widehat{G}(BB)^{*}].$$
(43)

It is independent of radius R', which means the conservation of energy.

We calculate the surface integral (42) using the spherical Hankel function expansion of the Green's function (31). At a large distance  $R' \gg 1/k_0$ ,  $r_{Bj}$  and L, using the fact that  $\widehat{G}$  satisfies the radiation condition, we may replace  $\sum_{i=1}^{3} n_i \partial_i \widehat{G} \rightarrow i k_0 \widehat{G}$  and  $\sum_{i=1}^{3} n_i \partial_i \widehat{G}^* \rightarrow -i k_0 \widehat{G}^*$  in (42) since  $\sum_{i=1}^{3} n_i \partial_i = \partial_r$  on the spherical surface. The surface integral becomes the same form as the LHS of (34) with A = B because of the source–receiver reciprocity. Putting A to B in (40), we have

$$\begin{split} \oint_{R'} ds(\mathbf{x}) \sum_{i=1}^{3} \widehat{M}_{i}(\widehat{G}(xB)) n_{i} &\approx \omega^{2} V_{0} \oint_{R'} ds(\mathbf{x}) \, \widehat{G}(Bx)^{*} \widehat{G}(Bx), \\ &= \frac{i\omega V_{0}^{2}}{2} [\widehat{G}(BB) - \widehat{G}(BB)^{*}] + \frac{\omega^{2} V_{0}}{2} \sum_{l,m} \sum_{l',m'} h_{l}^{(2)}(Bj) h_{l'}^{(1)}(Bj) Y_{l,m}^{*}(Bj) Y_{l',m'}(Bj) \\ &\times \big[ -(-1)^{l-m} F_{l,-m,l',m'} - (-1)^{l'-m'} F_{lm,l',-m'}^{*} + 2 \sum_{l'',m''} F_{l,m,l'',m''}^{*} F_{l'm',l'',m''} \big]. \end{split}$$

It should be the same as (43). Thus, we find the same constraint as (41) for  $F_{lm,l'm'}$ . This is the generalized optical theorem for the 3-D case.

## 4.5 Scattering amplitude and optical theorem

In order to clarify the physical meaning of the constraint (41), we place a source B and a receiver **x** in the far filed of the scattering obstacle *j* at the origin as illustrated in Fig. 6. We may approximate  $h_{l'}^{(1)}(k_0r_{Bj}) \approx (-i)^{l'+1} \frac{e^{ik_0r_{Bj}}}{k_0r_{Bj}} \approx (-i)^{l'} h_0^{(1)}(k_0r_{Bj})$  for  $k_0r_{Bj} \gg 1$ , and  $h_l^{(1)}(k_0r_{xj}) \approx (-i)^{l+1} \frac{e^{ik_0r_{xj}}}{k_0r_{xj}}$  for  $k_0r_{xj} \gg 1$ . Using these approximations, we rewrite the scattering term in (31) by a product of the incident wave  $\widehat{G}_0$  and spherically outgoing wave  $e^{ik_0r_{xj}}/r_{xj}$ :

$$\widehat{G}(\mathbf{x}, \mathbf{x}_{\mathrm{B}}, \omega) \approx \widehat{G}_{0}(r_{xB}, \omega) - \sum_{l,m} \sum_{l',m'} \frac{e^{ik_{0}r_{xj}}}{k_{0}r_{xj}} (-i)^{l+l'+1} 4\pi F_{lm,l'm'}(\omega) Y_{lm}(\theta_{xj}, \varphi_{xj}) Y_{l'm'}(\theta_{Bj}, \varphi_{Bj}) \widehat{G}_{0}(r_{Bj}, \omega).$$

$$(45)$$

Writing the RHS of this equation in the standard form of the scattering theory, we have

$$\widehat{G}(\mathbf{x}, \mathbf{x}_{\mathrm{B}}, \omega) \approx \widehat{G}_{0}(r_{xB}, \omega) + \sum_{l,m} \sum_{l',m'} \frac{e^{ik_{0}r_{xj}}}{r_{xj}} 4\pi f_{lm,l'm'}(\omega) Y_{lm}(\theta_{xj}, \varphi_{xj}) Y_{l'm'}^{*}(\pi - \theta_{Bj}, \varphi_{Bj} + \pi) \widehat{G}_{0}(r_{Bj}, \omega),$$

$$\tag{46}$$



Figure 6. Source B and a receiver x are placed in the far field of the scattering obstacle j.

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where  $f_{im,l'm'}$ 's are the spherical harmonic function expansion coefficients of the scattering amplitude f, where angles  $\pi - \theta_{Bj}$  and  $\varphi_{Bj} + \pi$  are for the incident ray direction from the source B to the obstacle j at the origin, and  $\theta_{xj}$  and  $\varphi_{xj}$  for the scattered wave ray direction from j to a receiver at  $\mathbf{x}$ . Their included angle is scattering angle  $\angle_{-Bix}$ . By using the relation

$$F_{lm,l'm'} = -(-1)^{m'} i^{l+l'+1} k_0 f_{lm,l',-m'},$$
(47)

where  $f_{lm,l',m'} = (-1)^{m+m'} f_{l',-m',l,-m}$ , the constraint (41) is written as the constraint for the scattering amplitude  $f_{lm}$ :

$$-if_{l',m',l,m} + if_{lm,l',m'}^* = 2k_0 \sum_{l'',m''} f_{lm,l'',m''}^* f_{l'm',l'',m''}^*.$$
(48)

For the case of a spherically symmetric obstacle, the scattering amplitude is a function of scattering angle  $\angle_{-Bjx}$ . In this case,

$$f_{lm,l'm'} = f_l \delta_{ll'} \delta_{mm'}, \tag{49a}$$

$$F_{lm,l'm'} = -i(-1)^{l-m}k_0 f_l \delta_{ll'} \delta_{m,-m'}.$$
(49b)

Then, (48) is written as the constraint for the scattering amplitude  $f_l$ :

Im 
$$f_l = k_0 |f_l|^2$$
. (50)

This is the standard form of the optical theorem for a spherical obstacle (e.g. Landau & Lifshitz 2003, eq. 125.14). Substitution of (49) into (31) leads to

$$\widehat{G}(\mathbf{x}, \mathbf{x}_{\mathrm{B}}, \omega) = \frac{-i}{4} h_0^{(1)}(k_0 r_{xB}) + k_0^2 \sum_{l,m} f_l(\omega) h_l^{(1)}(k_0 r_{xj}) h_l^{(1)}(k_0 r_{Bj}) Y_{lm}(\theta_{xj}, \varphi_{xj}) Y_{lm}^*(\theta_{Bj}, \varphi_{Bj}).$$
(51a)

In the far field, it is written as

$$\widehat{G}(\mathbf{x}, \mathbf{x}_{\mathrm{B}}, \omega) \approx \widehat{G}_{0}(r_{Bx}, \omega) + \frac{e^{ik_{0}r_{xj}}}{r_{xj}} \sum_{l=0}^{\infty} f_{l}(\omega)(2l+1)P_{l}(\cos \angle_{-Bjx})\widehat{G}_{0}(r_{Bj}, \omega),$$

$$= \widehat{G}_{0}(r_{Bx}, \omega) + \frac{e^{ik_{0}r_{xj}}}{r_{xj}} f(\angle_{-Bjx}, \omega)\widehat{G}_{0}(r_{Bj}, \omega),$$
(51b)

(e.g. Landau & Lifshitz 2003, eq. 123.14).

## **5** CONCLUSION

In the case that a single obstacle of arbitrary shape is placed in a 2-D homogeneous medium, the scattering part of the Green's function is written by a double series expansion using Hankel functions of the first kind which satisfy the radiation condition at a large distance from the source and the scattering obstacle. When two receivers and the scattering obstacle are illuminated by uncorrelated noise sources randomly and uniformly distributed on a closed circle of a large radius surrounding them, the lag-time derivative of the CCF of random waves at the two receivers can be written by a convolution of the antisymmetrized Green's function and the ACF of the noise source time function. We have explicitly derived the constraint for the Hankel function expansion coefficients as the sufficient condition for the Green's function retrieval. We have shown that the constraint is equal to the generalized optical theorem derived from the energy conservation principle. Physical meaning of the generalized optical theorem becomes clear when the Hankel function expansion coefficients are transformed into scattering amplitudes in the framework of the conventional scattering theory. Furthermore, we have studied the case that a single obstacle of arbitrary shape is placed in a 3-D homogeneous medium. The Green's function is written by a double series expansion using spherical Hankel functions of the first kind and spherical harmonic functions. When two receivers and a scattering obstacle are illuminated by noise sources randomly and uniformly distributed on a closed spherical surface of large radius surrounding them, we have explicitly derived the constraint for the spherical Hankel functions. The Green's function is written by a double series expansion using spherical Hankel functions of the first kind and spherical harmonic functions. When two receivers and a scattering obstacle are illuminated by noise sources randomly and uniformly distributed on a closed spherical surface of large radius surrounding them, we have explicit

The radiation condition, the source–receiver reciprocity and the energy conservation are key factors for the derivation of the constraint. The addition theorem is important for the mathematical derivation. We note that our derivation of the constraint does not assume that two receivers are in the far field of the scattering obstacle. The theory developed will be a mathematical foundation for imaging a single obstacle of arbitrary shape from the noise correlation measurements. We will study the constraint for the Green's function retrieval in the case that uncorrelated noise sources are uniformly and randomly distributed in a medium with intrinsic absorption.

# ACKNOWLEDGEMENTS

The author is grateful to Ludovic Margerin for his careful comments to the manuscript. Suggestions by Kees Wapenaar and an anonymous reviewer are helpful for the revision. This work is supported by the J-RAPID program of the Japan Science and Technology Agency and JSPS Grant-in-Aid for Scientific Research (C) No. 24540448.

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