Green's generic syzygy conjecture for curves of even genus lying on a $K3$ surface

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1 Introduction

If C is a smooth projective curve of genus g and K_C is its canonical bundle, the theorem of Noether asserts that the multiplication map

$$\mu_0 : H^0(C, K_C) \otimes H^0(C, K_C) \to H^0(C, K_C^{\otimes 2})$$

is surjective when C is non hyperelliptic.

The theorem of Petri concerns then the ideal I of C in its canonical embedding, assuming C is not hyperelliptic. It says that I is generated by its elements of degree 2 if C is neither trigonal nor a plane quintic.

In [7], M. Green introduced and studied the Koszul complexes

$$\bigwedge^{p+1} H^0(X,L)\otimes H^0(X,L^{q-1}) \xrightarrow{\delta} \bigwedge^{p} H^0(X,L)\otimes H^0(X,L^{q}) \xrightarrow{\delta} \bigwedge^{p-1} H^0(X,L)\otimes H^0(X,L^{q+1})$$

for X a variety and L a line bundle on X. Denoting by $K_{p,q}(X, L)$ the cohomology at the middle of the sequence above, one sees immediately that the surjectivity of the map μ_0 is equivalent to $K_{0,2}(C, K_C) = 0$, and that if this is the case, the ideal I is generated by quadrics if and only if $K_{1,2}(C, K_C) = 0$. On the other hand, C being non hyperelliptic is equivalent to the fact that the Clifford index $Cliff\, C$ is strictly positive, where

$$Cliff C := Min\{d - 2r,\ \exists L \in Pic\, C, d^0L = d,\ h^0(L) = r + 1 \geq 2,\ h^1(L) \geq 2\}.$$

Similarly, C is neither hyperelliptic, nor trigonal nor a plane quintic if and only if $Cliff\, C > 1$.

Green's conjecture on syzygies of canonical curves generalizes then the theorems of Noether and Petri as follows

Conjecture 1 *[7] For a smooth projective curve C in characteristic 0, the condition $Cliff\, C > l$ is equivalent to the fact that $K_{l',2}(C, K_C) = 0$, $\forall l' \leq l$.*

The interest of this formulation of Noether and Petri's theorems is already illustrated in [9], where these theorems are given a modern proof, using geometric technics of computation of syzygies.

For our purpose, and as is done in [7], it is convenient to use the duality (cf [7])

$$K_{p,2}(C, K_C) \cong K_{g-p-2,1}(C, K_C)^*$$

to reformulate the conjecture as follows

Conjecture 2 *[7] For a smooth projective curve C of genus g in characteristic 0, the condition $Cliff\, C > l$ is equivalent to the fact that $K_{g-l'-2,1}(C, K_C) = 0,\ \forall l' \leq l$.*

If C is now a generic curve, the theorem of Brill-Noether (cf [2], [11]) implies that

$$Cliff\, C = gon\, C - 2$$

where the gonality $gon\, C := Min\,\{d,\ \exists L \in Pic\, C,\ d^0 L = d,\ h^0(L) \geq 2\}$, and that

$$gon\, C = \frac{g+3}{2},\ if\ g\ is\ odd,$$

$$gon\, C = \frac{g+2}{2},\ if\ g\ is\ even.$$

Hence we arrive at the following conjecture (the generic Green conjecture on syzygies of a canonical curve) :

Conjecture 3 *Let C be a generic curve of genus g. Then if $g = 2k+1$ or $g = 2k$, we have $K_{k,1}(C, K_C) = 0$.*

Remark 1 *The actual conjecture is $K_{l,1}(C, K_C) = 0,\ \forall l \geq k$; but it is easy to prove that*

$$K_{k,1}(C, K_C) = 0 \Rightarrow K_{l,1}(C, K_C) = 0,\ \forall l \geq k.$$

Notice that in the appendix to [7], Green and Lazarsfeld prove the conjecture 1 in the direction $\Leftarrow$ (i. e. they produce non zero syzygies from special linear systems.) Hence the conjecture above cannot be improved, namely, under the assumptions above, we have $K_{k-1,1}(C, K_C) \neq 0$.

Teixidor [16] has recently proposed an approach to the conjecture 3. Her method uses a degeneration to a tree of elliptic curves and the theory of limit linear series of Eisenbud and Harris [6], adapted to vector bundles of higher rank. It is very likely that her method will lead to a proof of the generic Green conjecture.

We propose here a completely different approach, which at the moment works only for curves of even genus, but provides further evidence for Green's conjecture 1 (cf Corollaries 1 and 2).

Recall from [11] that if S is a $K3$ surface endowed with a ample line bundle L such that L generates $Pic\, S$ and $L^2 = 2g-2$, the smooth members $C \in \mid L \mid$ are generic in the sense of Brill-Noether, so that in particular they have the same Clifford index as a generic curve. Hence conjecture 1 predicts that their syzygies vanish as stated in conjecture 3. This is indeed what we prove here, in the case where the genus is even.

Theorem 1 *The pair (S, L) being as above, with $g = 2k$, we have*

$$K_{k,1}(C, K_C) = 0$$

for $C \in \mid L \mid$.

Remark 2 *The hyperplane restriction theorem [7] says that*

$$K_{k,1}(C, K_C) = K_{k,1}(S, L) \tag{1.1}$$

whenever C is a hyperplane section of a $K3$ surface S (note that $K_C = L_{|C}$ in this case). What we prove in fact is the equality

$$K_{k,1}(S, L) = 0. \tag{1.2}$$

The body of the paper will be devoted to the proof of (1.2). We state and prove here the following corollaries.

Corollary 1 *Let C be a generic curve of genus $g = 2k - 1$; then*

$$K_{k,1}(C, K_C) = 0.$$

Notice that the generic Green conjecture predicts in fact that $K_{k-1,1}(C, K_C) = 0$.

Proof of Corollary 1. The $K3$ surface S being as above, let X be a member of $\mid L \mid$ with exactly one node as singularity. Let C be the normalization of X. Then the genus of C is equal to $2k - 1$. Let p, $q \in C$ be the two points which are identified in X via the normalization map : $n : C \to X$. Then we have

$$n^* K_X = K_C(p + q)$$

and an isomorphism

$$H^0(X, K_X) = H^0(C, K_C(p + q)). \tag{1.3}$$

The hyperplane restriction theorem can be applied to $X \subset S$, and together with (1.2), it gives

$$K_{k,1}(X, K_X) = 0.$$

But the isomorphism (1.3) shows that this implies

$$K_{k,1}(C, K_C(p + q)) = 0.$$

Now one shows that the natural map

$$K_{k,1}(C, K_C) \to K_{k,1}(C, K_C(p + q))$$

is injective. Indeed in general consider the Koszul differential

$$\delta : \bigwedge^{l} H^0(Y, \mathcal{L}) \to H^0(Y, \mathcal{L}) \otimes \bigwedge^{l-1} H^0(Y, \mathcal{L}).$$

Then if
$$\wedge : H^0(Y,\mathcal{L}) \otimes \bigwedge^{l-1} H^0(Y,\mathcal{L}) \to \bigwedge^{l} H^0(Y,\mathcal{L})$$
is the wedge product map, one has
$$\wedge \circ \delta = \pm l Id. \tag{1.4}$$
Consider now the inclusion
$$j : H^0(C,K_C) \otimes \bigwedge^{k} H^0(C,K_C) \to H^0(C,K_C(p+q)) \otimes \bigwedge^{k} H^0(C,K_C(p+q)).$$
Let $\alpha \in H^0(C,K_C) \otimes \bigwedge^k H^0(C,K_C)$ such that $\delta\alpha = 0$ and $j(\alpha) = \delta\beta$. Then (1.4) gives
$$\begin{aligned} j(\alpha) = \delta\beta &= \pm\frac{1}{k+1}\delta(\wedge\circ\delta\beta) \\ &= \pm\frac{1}{k+1}\delta(\wedge(j(\alpha))). \end{aligned}$$
But $\wedge(j(\alpha)) \in \wedge^{k+1} H^0(C,K_C)$, so that α is in fact exact. ■

Corollary 2 *For any $\delta \leq \frac{k}{2}$, the generic curve of genus $2k-\delta$ which is $k+1-\delta$-gonal satisfies*
$$K_{k,1}(C,K_C) = 0.$$

Notice that this result is optimal and exactly predicted by Green's conjecture 1, since the Clifford index of such curve is equal to $k-1-\delta$.

Proof of Corollary 2. Again let (S,L) be as above. A generic member X of $\mid L \mid$ is $k+1$-gonal. As in section 2, and following [11], it follows that there is a stable vector bundle E on S with $det\, E = L$, $c_2(E) = k+1$, and $h^0(E) = k+2$. The zero set of a generic section of E is a generic member of a g^1_{k+1} of a generic curve $X \in \mid L \mid$.

Now let $x_1, \dots, x_\delta$ be generic points of S. The space
$$H_{x.} = H^0(S, E \otimes \mathcal{I}_{x_1} \otimes \ldots \otimes \mathcal{I}_{x_\delta})$$
has rank at least 2. One checks that for α, β generic in this space, the curve X defined by the equation
$$det\,(\alpha \wedge \beta) \in H^0(S, det\, E) = H^0(S,L)$$
is nodal with nodes exactly as the x_i's. On the other hand, the two sections α, β generate a rank 1 subsheaf of the restriction $E_{|X}$. Let now
$$n : C \to X$$
be the normalization. The rank 1 subsheaf introduced above induces a subline bundle
$$D \subset n^*E$$

with two sections, and it is obvious that the moving part of this linear system on C is of degree $k+1-\delta$, since the sections $\lambda\alpha+\mu\beta$ of E vanish at the x_i's, so that the moving part of their zero sets is of degree $k+1-\delta$. Hence C is $k+1-\delta$-gonal. It remains to show that

$$K_{k,1}(C,K_C)=0. \tag{1.5}$$

This is proven exactly as in the proof of Corollary 1, using the fact that

$$K_{k,1}(X,K_X)=0. \tag{1.6}$$

Notice that it is not true for $\delta\geq 2$ that

$$n^*:H^0(X,K_X)\to H^0(C,K_C(\sum_i p_i+q_i))$$

is an isomorphism, but it is injective onto a subspace which contains $H^0(C,K_C)$, and this is enough to deduce (1.5) from (1.6).

■

We conclude this introduction with a sketch of the main ideas in the proof of (1.2). The very starting point is the following observation : denote by $S^{[l]}$ the Hilbert scheme parametrizing 0-dimensional length l subschemes of S. Let $I_l\subset S\times S^{[l]}$ be the incidence subscheme and

$$\begin{array}{ccc} I_l & \stackrel{\pi_l}{\to} & S^{[l]} \\ q\downarrow & & \\ S & & \end{array}$$

be the incidence correspondence. Let

$$\mathcal{E}_L:=R^0\pi_{l*}q^*L$$

and $L_l:=det\,\mathcal{E}_L$. Then we have

Fact. $K_{l-1,1}(S,L)=0$ *if and only if*

$$H^0(I_l,\pi_l^*L_l)=\pi_l^*H^0(S^{[l]},L_l).$$

Our strategy will be then to construct a subvariety Z of $S^{[k+1]}$, such that

$$H^0(\tilde{Z},\pi_l^*L_l)=\pi_l^*H^0(Z,L_l)$$

where $\tilde{Z}:=\pi_l^{-1}(Z)$, and the restriction map

$$H^0(I_l,\pi_l^*L_l)\to H^0(\tilde{Z},\pi_l^*L_l)$$

is injective.

As in the papers [11], [8], the key role in constructing our variety Z and verifying the conditions above will be played by the vector bundles on S associated with base-point free linear systems on smooth members of $\mid L\mid$.

Terminology. In this paper, we shall say that a Zariski open subset $U \subset X$ is large if the complementary closed subset $Z = X - U$ has codimension non smaller that 2 in X. In the considered cases, the variety X will be normal, and we will use freely the fact that for a line bundle $\mathcal{L}$ on X

$$H^0(X, \mathcal{L}) \cong H^0(U, \mathcal{L}_{|U})$$

for U a large open subset of X.

2 Strategy of the proof

We start with the following observation : Let X be a smooth projective variety. Denote by $X^{[k]}_{curv}$ the Hilbert scheme parametrizing curvilinear 0-dimensional subschemes of X of length k. $X^{[k]}_{curv}$ is smooth, and if X is a curve or a surface, it is a large open set in the Hilbert scheme $X^{[k]}$ which is smooth.

Let

$$\begin{array}{ccc} I_k & \stackrel{\pi_k}{\to} & X^{[k]}_{curv} \\ q \downarrow & & \\ X & & \end{array}$$

be the incidence correspondence. For a line bundle L on X denote by $\mathcal{E}_L$ the vector bundle on $X^{[k]}_{curv}$ defined by $\mathcal{E}_L = R^0\pi_{k*}q^*L$, and let

$$L_k := det\, \mathcal{E}_L.$$

We have

Lemma 1 *There is a natural isomorphism*

$$K_{k,1}(X, L) \cong H^0(I_{k+1}, \pi^*_{k+1}L_{k+1})/\pi^*_{k+1}H^0(X^{[k+1]}_{curv}, L_{k+1}).$$

In particular, $K_{k,1}(X, L) = 0$ *is equivalent to*

$$H^0(I_{k+1}, \pi^*_{k+1}L_{k+1}) = \pi^*_{k+1}H^0(X^{[k+1]}_{curv}, L_{k+1}).$$

Proof. Recall that $K_{k,1}(X, L)$ is the cohomology at the middle of the sequence

$$\bigwedge^{k+1} H^0(X, L) \stackrel{\delta}{\to} H^0(X, L) \otimes \bigwedge^{k} H^0(X, L) \stackrel{\delta}{\to} H^0(X, L^{\otimes 2}) \otimes \bigwedge^{k-1} H^0(X, L). \quad (2.7)$$

Now note that there is a natural morphism

$$\tau : I_{k+1} \to X \times X^{[k]}_{curv} \quad (2.8)$$

which to (x, z), $x \in Supp\, z$ associates (x, z'), where z' is the residual scheme of x in z. This morphism is well defined because we are working with curvilinear schemes.

One shows easily that τ identifies I_{k+1} to a large open subset of the blow-up of $X \times X^{[k]}_{curv}$ along the incidence subscheme I_k. Furthermore, if $D \subset I_{k+1}$ is the exceptional divisor one has

$$\pi^*_{k+1} L_{k+1} = \tau^*(L \boxtimes L_k)(-D). \tag{2.9}$$

It follows that

$$H^0(I_{k+1}, \pi^*_{k+1} L_{k+1}) = Ker\,(H^0(X,L) \otimes H^0(X^{[k]}_{curv}, L_k) \stackrel{rest}{\to} H^0(I_k, L \boxtimes L_{k|I_k})) \tag{2.10}$$

On the other hand one checks easily that the natural map

$$\bigwedge^{l} H^0(X,L) \to H^0(X^{[l]}_{curv}, L_l) \tag{2.11}$$

induced by the evaluation map

$$H^0(X,L) \otimes \mathcal{O}_{X^{[k]}_{curv}} \to \mathcal{E}_L$$

are isomorphisms for any l.

We now apply the description above to $I_k :$ we note that denoting by p_i, $i = 1,\,2$, the compositions of the projections with the inclusion $I_k \hookrightarrow X \times X^{[k]}_{curv}$, we have

$$p_2 = \pi_k,\ p_1 = pr_1 \circ \tau,$$

where

$$\tau : I_k \to X \times X^{[k-1]}_{curv}$$

is defined as in (2.8). Hence applying formula (2.9), we get

$$L \boxtimes L_{k|I_k} = \tau^*(L^2 \boxtimes L_{k-1})(-D).$$

So we conclude that there is a natural inclusion

$$i : H^0(I_k, L \boxtimes L_{k|I_k}) \subset H^0(X, L^{\otimes 2}) \otimes \bigwedge^{k-1} H^0(X,L).$$

Hence we have an exact sequence

$$0 \to H^0(I_{k+1}, \pi^*_{k+1} L_{k+1}) \stackrel{j}{\to} H^0(X,L) \otimes H^0(X^{[k]}_{curv}, L_k) \stackrel{i\circ rest}{\to} H^0(X, L^{\otimes 2}) \otimes H^0(X^{[k-1]}_{curv}, L_{k-1}).$$

To conclude, it remains to show that the maps $j \circ \pi^*_{k+1}$ and $i \circ rest$ identify via the isomorphisms (2.11) for $l = k+1,\ k,\ k-1$ to the differentials δ of the sequence (2.7). This is quite easy for the first one, working on the open set U of $X^{[k+1]}$ parametrizing reduced subschemes. The second follows similarly. ∎

We consider now a $K3$ surface S endowed with an ample line bundle L generating $Pic\,S$ and satisfying

$$L^2 = 2g-2,\ g = 2k,\ k > 1.$$

As mentioned in the introduction, Green's conjecture 1 together with Lazarsfeld's work [11] implies that

$$K_{k,1}(C,K_C)=0$$

for a smooth member $C \in\mid L \mid$ or equivalently that

$$K_{k,1}(S,L)=0.$$

We now explain our strategy to prove this. Assume we have a subscheme $T\subset S^{[k+1]}$ such that, if $\tilde{T}$ denotes the subvariety $\pi_{k+1}^{-1}(T)$ of I_{k+1}, the following conditions are satisfied : (Here we use the notation π for π_{k+1}.)

1. We have an isomorphism

$$H^0(\tilde{T},\pi^*L_{k+1})=\pi^*H^0(T,L_{k+1}).$$

2. The restriction map

$$H^0(I_{k+1},\pi^*L_{k+1})\rightarrow H^0(\tilde{T},\pi^*L_{k+1})$$

   is injective.

Then we claim that $K_{k,1}(S,L)=0$.

Indeed we have the trace maps

$$tr:H^0(I_{k+1},\pi^*L_{k+1})\rightarrow H^0(S^{[k+1]}_{curv},L_{k+1})$$

$$tr_T:H^0(\tilde{T},\pi^*_{k+1}L_{k+1})\rightarrow H^0(T,L_{k+1})$$

which commute with the restriction maps and which compose to $(k+1)\,Id$ with the pull-back maps. If $\sigma\in H^0(I_{k+1},\pi^*L_{k+1})$, the section

$$\sigma'=\sigma-\pi^*(\frac{1}{k+1}Tr\,\sigma)$$

vanishes on $\tilde{T}$ by property 1, hence it is zero by property 2. Hence

$$H^0(I_{k+1},\pi^*L_{k+1})=\pi^*H^0(S^{[k+1]}_{curv},L_{k+1})$$

and this proves our claim, using lemma 1.

We will have to weaken the assumptions above as follows : Suppose we have a normal scheme Z together with a morphism

$$j:Z\rightarrow I_{k+1}$$

such that $\pi\circ j$ is generically one to one on its image, which is not contained in the branch locus of π. Suppose also that we have a normal scheme Z' together with a proper degree k morphism $\pi' : Z' \to Z$ and a morphism $j' : Z' \to I_{k+1}$ satisfying the conditions that

$$\pi \circ j' = j \circ \pi'$$

and the union $j(Z) \cup j'(Z')$ is equal set theoretically to $\pi^{-1}(\pi \circ j(Z))$. Finally assume there are subschemes $Z'_1 \subset Z'$, $Z_1 \subset Z$ such that

$$\pi'_{|Z'_1} =: \phi : Z'_1 \to Z_1$$

is a birational isomorphism and $j \circ \phi = j'_{|Z'_1}$.

(Hence roughly speaking, and up to birational maps, $\pi^{-1}(\pi \circ j(Z))$ is the scheme obtained by gluing Z' and Z along $Z'_1 \cong Z_1$.)

Assume now that they satisfy the following set (H) of hypotheses

1. The map

$$\pi'^* : H^0(Z, (\pi \circ j)^* L_{k+1}) \to H^0(Z', (\pi \circ j')^* L_{k+1})$$

is an isomorphism.

2. The restriction map

$$H^0(Z, (\pi \circ j)^* L_{k+1}) \to H^0(Z_1, (\pi \circ j)^* L_{k+1|Z_1})$$

is injective.

3. The restriction map

$$j^* : H^0(I_{k+1}, \pi^* L_{k+1}) \to H^0(Z, (\pi \circ j)^* L_{k+1})$$

is injective.

Then we claim that $K_{k,1}(S, L) = 0$.

Indeed by Lemma 1 we have to show that

$$H^0(I_{k+1}, \pi^* L_{k+1}) = \pi^* H^0(S^{[k+1]}_{curv}, L_{k+1}).$$

Now if $\sigma \in H^0(I_{k+1}, \pi^* L_{k+1})$, by hypothesis H1, $j'^* \sigma = \pi'^* \alpha$ for some $\alpha \in H^0(Z, (\pi \circ j)^* L_{k+1})$. We show now that $j^* \sigma = \alpha$. Indeed, by property H2, it suffices to show that this is true on Z_1, and since $\phi : Z'_1 \to Z_1$ is dominating, it suffices to show that

$$\phi^*(\alpha_{|Z_1}) = \phi^*(j^* \sigma_{|Z_1}).$$

But this follows from $j \circ \phi = j'_{|Z'_1}$ and from $j'^* \sigma = \pi'^* \alpha$, with $\phi = \pi'_{|Z'_1}$.

Finally it follows from the equality $\alpha = j^* \sigma$ that

$$\sigma' = \sigma - \pi^*(\frac{1}{k+1} Tr\, \sigma)$$

satisfies the condition $j^* \sigma' = 0$. Hence it vanishes by hypothesis H3. This concludes the proof of our claim. ∎

We conclude this section with the description of the schemes Z, Z' we will be considering.

Recall from [8], [11], [12], that there is a unique stable bundle E of rank 2 on S which satisfies the following properties:

$$det\, E = L,\, c_2(E) = k+1,\, h^0(E) = k+2.$$

Such vector bundle is obtained by choosing a line bundle D on a generic member C of $\mid L \mid$, such that $h^0(D) = 2$ and $d^0 D = k+1$. Such a line bundle exists by Brill-Noether theory, and it is generated by global sections since C does not carry a g^1_k by Lazarsfeld [11]. Then we have a vector bundle F on S defined by the exact sequence

$$0 \to F \to H^0(D) \otimes \mathcal{O}_S \to D \to 0 \tag{2.12}$$

and E is defined as the dual of F. The stability of E follows from the fact that $Pic\, S = \mathbb{Z}L$ and $H^0(S, E(-L)) = 0$. The uniqueness of such E follows then from the fact that $\chi(E, E') = 2$ for any other vector bundle E' with the same numerical properties, so that either $Hom\,(E, E') \neq 0$ or $Hom\,(E', E) \neq 0$. But then by stability, $E = E'$.

The property $h^0(S, E) = k+2$ follows from the sequence dual to (2.12)

$$0 \to H^0(D)^* \otimes \mathcal{O}_S \to E \to K_C - D \to 0, \tag{2.13}$$

and from Riemann-Roch which gives $h^0(K_C - D) = k$.

Another way to construct the bundle E is via Serre's construction. By Riemann-Roch the divisors D of degree $k+1$ on smooth members C of $\mid L \mid$ which satisfy $h^0(C, D) = 2$ are exactly the subschemes z of degree $k+1$ on S contained in a smooth member C of $\mid L \mid$ and satisfying the condition that the restriction map

$$H^0(S, L) \to H^0(L_{|z})$$

is not surjective. Note that since the curves C are general in the sense of Brill-Noether, the corank of this map is exactly 1 and furthermore for any $z' \subsetneq z$ the restriction map

$$H^0(S, L) \to H^0(L_{|z'})$$

is surjective. Hence, since K_S is trivial, to such z corresponds a vector bundle E together with a section σ_z vanishing on z. This E is an extension

$$0 \to \mathcal{O}_S \overset{\sigma_z}{\to} E \to\overset{\wedge\sigma_z}{\to} \mathcal{I}_z(L) \to 0. \tag{2.14}$$

Computing the numerical invariants of this bundle E, and arguing as before by stability, we see that this bundle is isomorphic to the one constructed above. Notice that each g^1_{k+1}, D on a smooth member $C \in\mid L \mid$ provides by (2.13) a rank 2 subspace of sections of E, and that the zero sets of these sections identify to the members of $\mid D \mid$, as subschemes of S.

It follows from the exact sequence (2.14) twisted by E that $h^0(S, E\otimes\mathcal{I}_z) = 1$ for any z as above. Hence the morphism

$$\mathbb{P}(H^0(S,E)) \rightarrow S^{[k+1]}$$

which to σ associates its zero set, is in fact an embedding. One sees easily that the open set $\mathbb{P}(H^0(S,E))_{curv}$ corresponding to curvilinear subschemes is large in $\mathbb{P}(H^0(S,E))$.

Let now $W := \pi^{-1}(\mathbb{P}(H^0(S,E))_{curv}) \subset I_{k+1}$. W is easily shown to be smooth. There is a natural morphism

$$\psi : W \rightarrow S^{[k]}_{curv}$$

defined as the restriction of $pr_2 \circ \tau$ to W. This ψ can be shown to be generically of degree one on its image.

Consider the blow-up $\widetilde{S \times W}$ of $S \times W$ along $K := (Id, \psi)^{-1}(I_k)$. It admits a morphism $\widetilde{(Id, \psi)}$ to the blow-up of $S \times S^{[k]}_{curv}$ along I_k, and the later contains I_{k+1} as a large open set. One verifies that $\widetilde{(Id, \psi)}^{-1}(I_{k+1})$ is a large open set of $\widetilde{S \times W}$. This will be our scheme Z. The morphism $j : Z \rightarrow I_{k+1}$ will be simply the restriction to Z of $\widetilde{(Id, \psi)}$.

Again one can show (using now the assumption that $k > 1$) that the morphism $\pi \circ j : Z \rightarrow S^{[k+1]}$ is generically of degree one on its image.

Next let $\pi'' : \tilde{W} \rightarrow W$ be the degree k cover obtained by completing the Cartesian diagram

$$\begin{array}{ccc} \tilde{W} & \rightarrow & I_k \\ \pi'' \downarrow & & \pi_k \downarrow \\ W & \stackrel{\psi}{\rightarrow} & S^{[k]}_{curv} \end{array} .$$

Consider the rational map

$$j' : S \times \tilde{W} - - - - > I_{k+1}$$

which to (s, s_1, w) $s_1 \in Supp\,\psi(w)$ associates $(s_1, s \cup \psi(w))$. This morphism becomes well defined after blowing-up $K' := (Id, \pi'')^{-1}(K)$ and restricting to a large open subset. Our scheme Z' will be this large open set. The morphism $\pi' : Z' \rightarrow Z$ is the restriction to Z' of the morphism $Bl_{K'}(S\times\tilde{W}) \rightarrow Bl_K(S\times Z)$ induced by (Id, π''). The morphism $j' : Z' \rightarrow I_{k+1}$ is induced by the rational map j' above. It is obvious that $\pi^{-1}(\pi \circ j(Z))$ is equal to $j(Z) \cup j'(Z')$. Indeed, the fiber over $s \cup \psi(w) \in \pi \circ j(Z)$ consists in choosing one point in the scheme $s \cup \psi(w)$. This point may be s in which case we are in $j(Z)$, or may be contained in $\psi(w)$ in which case it determines a point of $\tilde{W}$ over w, and we are then in $j'(Z')$.

Remark 3 *The scheme Z is non necessarily smooth, but one can show that K is reduced, so that its singular locus is of codimension at least two in $S \times W$. The same thing is true for Z' and K'. If one wants to work with smooth* *schemes Z_0 and Z'_0 (so as to be exactly in the conditions (H) described above), it suffices to restrict to the blowing-ups of $S \times W - K_{sing}$ along $K - K_{sing}$ and $S \times W - K'_{sing}$ along $K' - K'_{sing}$. All what follows will be true for these subschemes.*

To conclude, it remains now to construct Z_1 and Z'_1. Z_1 will be the exceptional divisor of Z (recall that Z is a large open set in $Bl_K(S \times W)$). Hence Z_1 is the inverse image under the blow-up map $Z \to S \times W$ of $K = \{(s, w) \in S \times W,\, s \in Supp\, \psi(w)\}$.

We now construct a generic lifting of Z_1 in Z', the closure of the image of which will be Z'_1. By definition of Z' as a large open set of $Bl_{K'}(S \times \tilde{W})$, it suffices to construct a lifting of K to a component of K' in $S \times \tilde{W}$. But if $(s, w) \in K$, we have $s \in Supp\, \psi(w)$ so that (s, w) identifies to an element $\tilde{w}$ of $\tilde{W}$. Our lifting sends simply (s, w) to $(s, \tilde{w})$.

It remains finally to see that the morphisms j' and $j \circ \pi'$ agree on Z'_1. Since I_{k+1} is contained in $S \times S^{[k+1]}_{curv}$, it suffices to prove that $pr_1 \circ j'$ and $pr_1 \circ j \circ \pi'$ agree on Z'_1 and that $pr_2 \circ j'$ and $pr_2 \circ j \circ \pi'$ agree on Z'_1, with $pr_2 = \pi$ on I_{k+1}. For the first one this is obvious since both maps factor through the contraction $Z'_1 \to K'$, and are equal on $K' \subset S \times \tilde{W}$ to the first projection on S, as follows from the definition of the lifting $K \to K'$.

As for the second one, it follows from the fact that, by construction, $\pi \circ j'$ and $\pi \circ j \circ \pi'$ agree on Z'. ∎

3 Proof of the assumptions H2 and H3

We start with the proof of hypothesis H2.

Proposition 1 *Let*

$$Z_1 \subset Z \stackrel{\pi \circ j}{\to} S^{[k+1]}$$

be as in the previous section. Then the restriction map

$$H^0(Z, (\pi \circ j)^* L_{k+1}) \to H^0(Z_1, (\pi \circ j)^* L_{k+1|Z_1})$$

is injective.

The proof will be obtained by restricting the construction to a generic smooth member $C \in \mid L \mid$. Indeed, recall that Z is a large open set in the blow-up of $S \times W$ along the incidence subscheme $K = (id, \psi)^{-1}(I_k)$, where

$$W = \{(x, \sigma) \in S \times \mathbb{P}(H^0(S, E))_{curv},\, \sigma(x) = 0\},$$

and $\psi : W \to S^{[k]}$ sends (x, σ) to the residual scheme of x in $V(\sigma)$. Now since $k \geq 1$, the generic element $z = V(\sigma)$ is supported in a pencil of elements of $\mid L \mid$, the generic member being smooth. It follows that a generic element of $S \times W$ is of the form (s_1, s_2, z), $z = V(\sigma)$, $\sigma(s_2) = 0$ and there exists a smooth member $C \in\mid L \mid$ such that s_1, s_2, z are supported on C. Hence it suffices to prove the analogue of proposition 1 with Z replaced by Z_C, the proper transform of $C \times W_C$ in $Z \subset Bl_K(S \times W)$, where

$$W_C := \{(c, \sigma) \in C \times \mathbb{P}(H^0(S, E)),\, \sigma(c) = 0,\, V(\sigma) \subset C\},$$

and Z_1 is replaced by $Z_{1,C} := Z_1 \cap Z_C$.

Proposition 2 *The restriction map*

$$H^0(Z_C, (\pi \circ j)^* L_{k+1|Z_C}) \to H^0(Z_1, (\pi \circ j)^* L_{k+1|Z_{1,C}})$$

is injective.

Proof. By the description of the bundle E given in the previous section, we note that the set

$$\{\sigma \in \mathbb{P}(H^0(S, E)),\ V(\sigma) \subset C\},$$

identifies by the map $\sigma \mapsto V(\sigma)$ to the disjoint union of the $\mathbb{P}^1 \subset C^{(k+1)}$ corresponding to g^1_{k+1}'s on C. If D is such a g^1_{k+1} on C, D gives a morphism of degree $k+1$

$$\phi_D : C \to \mathbb{P}^1$$

or a line bundle L_D on C of degree $k+1$ with two sections. By definition, W_C identifies (via ψ) to the disjoint union of copies C_D of C contained in $C^{(k)}$, where the map

$$\psi_D : C \cong C_D \to C^{(k)}$$

is given by $c \mapsto$ *the unique effective divisor equivalent to* $D - c$.

Finally Z_C identifies to a disjoint union of surfaces $Z_{C,D}$ isomorphic to $C \times C$, since the pull-back Δ_D to $C \times C_D$ of the incidence scheme in $C \times C^{(k)}$ is of pure codimension 1.

Recall now that

$$(\pi \circ j)^* L_{k+1} = L \boxtimes \psi^* L_k(-Z_1).$$

We have $L_{|C} = K_C$ and in the sequel we will use the notation H_D for the line bundle $K_{C^{(k)}|C_D}$. (It will be shown that $H_D \equiv kL_D$ but this will not be used now.) We have to show that for each D the restriction map

$$H^0(C \times C, K_C \boxtimes H_D(-\Delta_D)) \to H^0(\Delta_D, K_C \boxtimes H_D(-\Delta_D)_{|\Delta_D})$$

is injective, where $\Delta_D := Z_1 \cap (C \times C_D)$. In other words we want to show that

$$H^0(C \times C, K_C \boxtimes H_D(-2\Delta_D) = 0. \tag{3.15}$$

Now, since Δ_D is the restriction to $C \times C_D$ of the incidence scheme, and since C_D parametrizes the effective divisors of the form $L_D - x$, $x \in C$, it is clear that

$$\Delta_D = (\phi_D, \phi_D)^{-1}(diag\,(\mathbb{P}^1)) - diag\,(C).$$

Hence we have

$$\Delta_D \equiv L_D \boxtimes L_D - diag\,(C)$$

in $C \times C$. Hence we have

$$K_C \boxtimes H_D(-2\Delta_D) \equiv (K_C - 2L_D) \boxtimes (H_D - 2L_D) + 2diag\,C.$$

Now we have the equality

$$H^0(C, K_C - 2L_D) = 0, \tag{3.16}$$

which is proven in [11], since C is generic in S. (Indeed for a base point free pencil, $\mid L_D \mid$, the condition that the μ_0-map

$$H^0(C, L_D) \otimes H^0(C, K_C - L_D) \to H^0(C, K_C)$$

is injective is equivalent by the base-point free pencil trick to the condition

$$H^0(C, K_C - 2L_D) = 0.$$

The equality (3.15) follows now from (3.16) and from the fact that the map $H^0(C, 2L_D) \to H^0(2L_{D|2x})$ is surjective for generic x in C. Hence by Riemann-Roch, $H^0(C, K_C - 2L_D) = 0$ implies $H^0(C, K_C - 2L_D + 2x) = 0$ for generic $x \in C$. It follows that

$$H^0(C \times C, (K_C - 2L_D) \boxtimes (H_D - 2L_D) + 2diag\,C) = 0,$$

which proves the proposition 2, and hence proposition 1 is proven. ∎

We turn now to the proof of hypothesis H3.

Proposition 3 *The morphism $Z \xrightarrow{j} I_{k+1}$ being defined as in the previous section, the pull-back map*

$$j^* : H^0(I_{k+1}, \pi^* L_{k+1}) \to H^0(Z, (\pi \circ j)^* L_{k+1})$$

is injective.

The proof proceeds in several steps, and occupies the remainder of this section. Recall that I_{k+1} is a large open set in the blow-up of $S \times S^{[k]}$ along the incidence subscheme I_k and that we have the following formula

$$\pi^* L_{k+1} = \tau^*(L \boxtimes L_k)(-D),$$

where D is the exceptional divisor and τ is the blowing-up map. Since Z is a large open set in the proper transform of this blowing-up under the morphism $(Id, \psi) : S \times W \to S \times S^{[k]}$, it suffices to prove

Proposition 4 *The restriction map*

$$\psi^* : H^0(S^{[k]}, L_k) \to H^0(W, \psi^* L_k)$$

is injective.

In order to prove this proposition, we first show

Lemma 2 *Denoting by* $\pi : W \to \mathbb{P}(H^0(S,E))$ *the restriction of the morphism* $\pi_{k+1} : I_{k+1} \to S^{[k+1]}$, *we have the formula*

$$\psi^* L_k = \pi^* \mathcal{O}_{\mathbb{P}(H^0(S,E))}(k).$$

Proof. By definition, $\psi^* L_k = det\, \psi^* \mathcal{E}_{L,k}$, where the bundle $\mathcal{E}_{L,k}$ has for fiber $H^0(L_{|z})$ at a point $z \in S^{[k]}$. Now, if $z \in W$, the scheme $z' = \psi(z)$ has length k, hence the restriction map

$$H^0(S,L) \to H^0(L_{|z'})$$

is surjective. On the other hand if $z'' = \pi(z)$, we have $z' \subset z''$ and the restriction map

$$H^0(S,L) \to H^0(L_{|z''})$$

is not surjective. Hence we have

$$H^0(S, L \otimes \mathcal{I}_{z'}) = H^0(S, L \otimes \mathcal{I}_{z''}),$$

and the fiber of $\psi^* \mathcal{E}_{L,k}$ at z is canonically isomorphic to $H^0(S,L)/H^0(S, L \otimes \mathcal{I}_{\pi(z)})$. Hence we have

$$\psi^* L_k = -\pi^* det \mathcal{F},$$

where the bundle $\mathcal{F}$ on $\mathbb{P}(H^0(S,E))$ is the bundle with fiber $H^0(S, L \otimes \mathcal{I}_z)$ at σ, $z = V(\sigma)$. Now recall that for each σ we have the exact sequence

$$0 \to \mathcal{O}_S \stackrel{\sigma}{\to} E \stackrel{\wedge\sigma}{\to} \mathcal{I}_z(L) \to 0.$$

This induces the exact sequence

$$0 \to < \sigma > \to H^0(S,E) \stackrel{\wedge\sigma}{\to} H^0(S, \mathcal{I}_z(L)) \to 0.$$

We conclude immediately from this that $\mathcal{F}$ fits into the exact sequence

$$0 \to \mathcal{O}_{\mathbb{P}(H^0(S,E))}(-2) \to H^0(S,E) \otimes \mathcal{O}_{\mathbb{P}(H^0(S,E))}(-1) \to \mathcal{F} \to 0.$$

Since $rank\ H^0(S,E) = k+2$, it follows that $det\, \mathcal{F} = \mathcal{O}_{\mathbb{P}(H^0(S,E))}(-k)$. ∎

It follows from this lemma that we have a natural inclusion

$$S^k H^0(S,E)^* \hookrightarrow H^0(W, \psi^* L_k). \tag{3.17}$$

(It will be proven in the next section that this inclusion is in fact an isomorphism, but we shall not need this here.)

Our strategy to prove proposition 4 will be first to construct an isomorphism

$$H^0(S^{[k]}, L_k) = \wedge^k H^0(S,L) \cong S^k H^0(S,E)^* \tag{3.18}$$

and then to show that composed with the inclusion (3.17), it is equal, up to a coefficient, to the pull-back map ψ^*.

Construction of the isomorphism (3.18).

We note first that the determinant map

$$det : \bigwedge^2 H^0(S,E) \to H^0(S, det\, E) = H^0(S,L)$$

does not vanish on any element of rank 2. Indeed, such element of rank 2 is given by a subspace W of rank 2 of $H^0(S,E)$, and if its determinant would vanish this would imply that W generates a rank 1 subsheaf of E with at least two sections. But since $Pic\, S$ is generated by L and $H^0(S, E(-L)) = 0$ this is impossible.Hence det provides a morphism

$$d : G_2 \to \mathbb{P}(H^0(S,L)),$$

where G_2 is the Grassmannian of rank two vector subspaces of $H^0(S,E)$, or dually a base-point free linear system

$$K := H^0(S,L)^* \stackrel{d^*}{\hookrightarrow} H^0(G_2, \mathcal{L}) = \wedge^2 H^0(S,E)^*,$$

where $\mathcal{L}$ is the Plücker polarization on G_2. Notice that $rank\, K = 2k+1$. Since K is base-point free, we have the exact Koszul complex on G_2

$$0 \to \bigwedge^{2k+1} K \otimes \mathcal{L}^{-(2k+1)} \to \ldots \to K \otimes \mathcal{L}^{-1} \to \mathcal{O}_{G_2} \to 0.$$

We can now tensor this sequence with $S^k\mathcal{E}$, where the rank 2 vector bundle $\mathcal{E}$ on G_2 is dual to the tautological rank two subbundle and satisfies $H^0(G_2, S^k\mathcal{E}) = S^k H^0(S,E)^*$.

This provides the exact complex

$$0 \to \bigwedge^{2k+1} K \otimes \mathcal{L}^{-(2k+1)} \otimes S^k\mathcal{E} \to \ldots \to K \otimes \mathcal{L}^{-1} \otimes S^k\mathcal{E} \to S^k\mathcal{E} \to 0. \quad (3.19)$$

In this complex $\mathcal{K}^\cdot$, the term $S^k\mathcal{E}$ is put in degree 0. The hypercohomology $\mathbb{H}^0(G_2, \mathcal{K}^\cdot)$ vanishes. Now we have a spectral sequence

$$E_1^{p,q} = H^q(G_2, \mathcal{K}^p) \Rightarrow \mathbb{H}^{p+q}(G_2, \mathcal{K}^\cdot).$$

It is obvious for degree reasons that all differentials d_r starting from the term $E_r^{0,0}$ vanish. On the other hand the terms $E_1^{p,q}$ with $p+q = -1$ are of the form

$$H^q(G_2, \bigwedge^{q+1} K \otimes \mathcal{L}^{-q-1} \otimes S^k\mathcal{E}).$$

Using the proposition 9 proven in the appendix, we see that these terms are all 0, except for

$$E_1^{-k-1,k} = H^k(G_2, \bigwedge^{k+1} K \otimes \mathcal{L}^{-k-1} \otimes S^k\mathcal{E}),$$

which is equal to $\bigwedge^{k+1} K$. It follows that there is only one non zero differential which arrives in some $E_r^{0,0}$, namely

$$d_{k+1} : E_{k+1}^{-k-1,k} \to E_{k+1}^{0,0}.$$

This implies that

$$E_{k+1}^{0,0} = E_1^{0,0} = H^0(G_2, S^k\mathcal{E}) = S^k H^0(S,E)^*$$

and that the differential d_{k+1} above is surjective, since the spectral sequence abuts to 0. Hence we have build a surjective map d_{k+1} from a subquotient of $E_1^{-k-1,k} = \bigwedge^{k+1} K$ to $S^k H^0(S,E)^*$. Since $dim \bigwedge^{k+1} K = dim\, S^k H^0(S,E)^*$ this subquotient must in fact be equal to $\bigwedge^{k+1} K$ and the map d_{k+1} has to be an isomorphism. Finally, since $rank\, K = 2k+1$,

$$\bigwedge^{k+1} K = (\bigwedge^{k} K)^* = \bigwedge^{k} H^0(S,L).$$

Hence we have constructed our isomorphism

$$d_{k+1} : \bigwedge^{k} H^0(S,L) \to S^k H^0(S,E)^*.$$

■

To conclude the proof of proposition 4, it remains only to show :

Proposition 5 *The map d_{k+1} constructed above identifies up to a coefficient to the map*

$$\psi^* : H^0(S^{[k]}, L_k) \to H^0(W, \psi^* L_k),$$

which takes values in $S^k H^0(S,E)^* \subset H^0(W, \psi^* L_k)$.

Proof. First of all it is clear that ψ^* takes values in $\pi^* H^0(\mathbb{P}(H^0(S,E)), \mathcal{O}(k)) = S^k H^0(S,E)^*$. Indeed, this map is the pull-back map associated to the morphism

$$W \to Grass(k+1, H^0(S,L))$$

$$z \mapsto H^0(S, L \otimes \mathcal{I}_{z'}),\ z' = \psi(z).$$

But as mentioned in the proof of lemma 2, this morphism factors through $\pi : W \to \mathbb{P}(H^0(S,E))$.

Next, we note that, with the same spectral sequence argument, and replacing $K = H^0(S,L)^* \subset \bigwedge^2 H^0(S,E)^*$ by the base point free linear system $K' = \bigwedge^2 H^0(S,E)^*$ on G_2, we could have constructed more generally a surjective map

$$D_{k+1} : \bigwedge^{k+1}(\bigwedge^{2} H^0(S,E)^*) \to S^k H^0(S,E)^*,$$

whose restriction to $\bigwedge^{k+1} K$ is equal to d_{k+1}.

On the other hand, we already noticed that the restriction map

$$\psi^* : \wedge^k H^0(S,L) \to S^k H^0(S,E)^*$$

corresponds to the morphism

$$\mathbb{P}(H^0(S,E)) \to Grass(k+1, H^0(S,L))$$
$$\sigma \mapsto det(\sigma \wedge H^0(S,E)).$$

But this morphism is the composition of the morphism

$$\beta : \mathbb{P}(H^0(S,E)) \to Grass(k+1, \bigwedge^2 H^0(S,E))$$
$$\sigma \mapsto \sigma \wedge H^0(S,E).$$

and of the rational map induced by the determinant

$$det : Grass(k+1, \bigwedge^2 H^0(S,E)) \to Grass(k+1, H^0(S,L)).$$

Hence proposition 5 will follow from the following

Lemma 3 *The maps D_{k+1} and β^* from $\bigwedge^{k+1}(\bigwedge^2 H^0(S,E)^*)$ to $S^k H^0(S,E)^*$ coincide up to a coefficient.*

Proof. We could argue by $Sl(k+2)$-equivariance. A more direct way to prove this is to note the following : If $W \subset \bigwedge^2 H^0(S,E)^*$ is a rank $k+1$ vector subspace in general position, it defines a codimension $k+1$ subvariety G_W of G_2. Consider the incidence correspondence

$$\begin{array}{ccc} P & \xrightarrow{\pi} & \mathbb{P}(H^0(S,E)) \\ p\downarrow & & \\ G_2 & & \end{array}$$

Then we have an hypersurface $X_W = \pi(p^{-1})(G_W)$ of $\mathbb{P}(H^0(S,E))$, which is easily proven to be of degree k. It is clear that

$$H^0(G_2, S^k\mathcal{E} \otimes \mathcal{I}_{G_W}) = H^0(\mathbb{P}(H^0(S,E)), \mathcal{O}_{\mathbb{P}(H^0(S,E))}(k)(-X_W)).$$

On the other hand, from the linear system W we can construct a Koszul complex which is a resolution of $\mathcal{I}_{G_W}$. Hence it is clear that

$$D_{k+1}(\bigwedge^{k+1} W) \subset H^0(G_2, S^k\mathcal{E} \otimes \mathcal{I}_{G_W}).$$

In other words, if η is a generator of $\bigwedge^{k+1} W$, $D_{k+1}(\eta)$ is a defining equation of X_W or 0. It remains then only to prove that $\beta^*\eta$ also vanishes on X_W. But by definition

$$X_W = \{x \in \mathbb{P}(H^0(S,E)), \exists 0 \neq \gamma \in \mathbb{P}(H^0(S,E)/ < x >), x \wedge \gamma \perp W\}.$$

This means that for $x \in X_W$, the composed map

$$W \hookrightarrow \bigwedge^2 H^0(S,E)^* \to (x \wedge H^0(S,E))^*$$

is not an isomorphism, hence its determinant vanishes. But this determinant is equal to $\beta^*\eta(x)$. ∎

4 Proof of the assumption H1

Recall that we have a Cartesian diagram

$$\begin{array}{ccc} Z' & \stackrel{\pi'}{\rightarrow} & Z \\ \tau' \downarrow & & \tau \downarrow \\ S \times \tilde{W} & \stackrel{\pi''}{\rightarrow} & S \times W \end{array}$$

where the vertical maps τ, τ' are blow-ups and the degree k morphism π'' fits into the Cartesian diagram

$$\begin{array}{ccc} \tilde{W} & \rightarrow & I_k \\ \pi'' \downarrow & & \pi_k \downarrow \\ W & \stackrel{\psi}{\rightarrow} & S^{[k]}_{curv} \end{array} .$$

We have the morphisms

$$j' : Z' \rightarrow I_{k+1},\ j : Z \rightarrow I_{k+1}$$

such that $\pi \circ j' = \pi \circ j \circ \pi'$ and the formula

$$(\pi \circ j)^* L_{k+1} = \tau^*(L \boxtimes \psi^* L_k)(-D)$$

where D is the exceptional divisor of τ. Similarly we have

$$(\pi \circ j')^* L_{k+1} = \tau'^*(L \boxtimes (\psi \circ r)^* L_k)(-D').$$

Since $D' = \pi'^{-1}(D)$ and π' is surjective, we conclude that in order to prove H1, that is the fact that the pull-back map

$$\pi'^* : H^0(Z, (\pi \circ j)^* L_{k+1}) \rightarrow H^0(Z', (\pi \circ j')^* L_{k+1})$$

is surjective, it suffices to show that the pull-back map

$$\pi''^* : H^0(W, \psi^* L_k) \rightarrow H^0(\tilde{W}, (\psi \circ \pi'')^* L_k)$$

is surjective.

Now recall that we have a morphism

$$\pi : W \rightarrow \mathbb{P}(H^0(S, E))$$

such that (cf lemma 2)

$$\psi^* L_k = \pi^* \mathcal{O}_{\mathbb{P}(H^0(S,E))}(k).$$

Denoting by $\beta := \pi \circ \pi'' : \tilde{W} \rightarrow \mathbb{P}((H^0(S, E))$, we shall prove the following stronger statement

Theorem 2 *The pull-back map*

$$\beta^* : H^0(\mathbb{P}(H^0(S,E)), \mathcal{O}_{\mathbb{P}(H^0(S,E))}(k)) \to H^0(\tilde{W}, (\psi \circ r)^* L_k) \tag{4.20}$$

is surjective.

The end of this section will be devoted to the proof of this theorem, which proceeds in several steps. In what follows, we shall use the notation $H^0(E)$ for $H^0(S,E)$.

Notice to begin with that $\tilde{W}$ is a large open set in the subscheme

$$W' \subset \widetilde{S \times S} \times \mathbb{P}(H^0(E)),$$

where $\widetilde{S \times S}$ is the blow-up of $S \times S$ along the diagonal, defined as

$$W' := \{(x, y, \eta, \sigma),\ \sigma_{|\eta=0},\ \{x\},\ \{y\} \subset \eta\}.$$

(Here η is a subscheme of length 2 of S, and we see elements of $\widetilde{S \times S}$ as elements (x, y) of $S \times S$ together with a schematic structure η of length 2 on $\{x\} \cup \{y\}$.)

The map β is just the restriction to W' of the second projection. Hence we have

$$H^0(\tilde{W}, (\psi \circ r)^* L_k) = H^0(W', pr_2^* \mathcal{O}_{\mathbb{P}(H^0(E))}(k))$$

and the surjectivity of (4.20) is equivalent to the condition

$$H^1(\widetilde{S \times S} \times \mathbb{P}(H^0(E)), pr_2^* \mathcal{O}(k) \otimes \mathcal{I}_{W'}) = 0. \tag{4.21}$$

Now notice that there is a vector bundle $\tilde{E}_2$ on $\widetilde{S \times S}$ such that W' is the zero set of a section σ of $\tilde{E}_2 \boxtimes \mathcal{O}_{\mathbb{P}(H^0(E))}(1)$. Indeed it suffices to take for $\tilde{E}_2$ the vector bundle with fiber $H^0(E_{|\eta})$ at the point (x, y, η) of $\widetilde{S \times S}$. Then the section σ takes the value $\tau_{|\eta}$ at the point (x, y, η, τ) of $\widetilde{S \times S} \times \mathbb{P}(H^0(E))$. One checks easily that W' is reduced of codimension 4. Hence we have a Koszul resolution of $\mathcal{I}_{W'}$

$$0 \to \bigwedge^4 \tilde{E}_2^* \boxtimes \mathcal{O}(-4) \to \ldots \to \tilde{E}_2^* \boxtimes \mathcal{O}(-1) \to \mathcal{I}_{W'} \to 0. \tag{4.22}$$

Our first goal will be to compute the cohomology groups of $\widetilde{S \times S} \times \mathbb{P}(H^0(E))$ with value in $\bigwedge^i \tilde{E}_2^* \boxtimes \mathcal{O}(k-i)$. Since $k \geq 2$, $i \leq 4$, $\mathcal{O}(k-i)$ has no higher cohomology on $\mathbb{P}(H^0(E)) = \mathbb{P}^{k+1}$. Hence we have

$$H^l(\widetilde{S \times S} \times \mathbb{P}(H^0(E)), \bigwedge^i \tilde{E}_2^* \boxtimes \mathcal{O}(k-i)) = H^l(\widetilde{S \times S}, \bigwedge^i \tilde{E}_2^*) \otimes S^{k-i} H^0(S,E)^*.$$

We have now the following proposition

Proposition 6 *1.* $H^2(\widetilde{S\times S},\tilde{E}_2^*)=pr_1^*H^2(S,E^*)\oplus pr_2^*H^2(S,E^*)$ *and*

$$H^1(\widetilde{S\times S},\tilde{E}_2^*))=0.$$

2. $H^2(\widetilde{S\times S},\bigwedge^2\tilde{E}_2^*)=pr_1^*H^2(S,-L)\oplus pr_2^*H^2(S,E-L)$.

3. $H^4(\widetilde{S\times S},\bigwedge^4\tilde{E}_2^*)$ *is dual to* $Ker\,(H^0(S,L)\otimes H^0(S,L)\to H^0(S,2L))$.

4. $H^3(\widetilde{S\times S},\bigwedge^3\tilde{E}_2^*)=0$ *and* $H^4(\widetilde{S\times S},\bigwedge^3\tilde{E}_2^*)$ *admits as a quotient*

$$H^4(\widetilde{S\times S},\tau^*(pr_1^*E^*\otimes pr_2^*(-L))(2\Delta))\oplus H^4(\widetilde{S\times S},\tau^*(pr_1^*(-L)\otimes pr_1^*E^*)(2\Delta)),$$

which is dual to the direct sum of two copies of

$$Ker\,H^0(S,E)\otimes H^0(S,L)\to H^0(S,E\otimes L).$$

(Here $\Delta\subset\widetilde{S\times S}$ *is the exceptional divisor.)*

Proof.

1. The bundle $\tilde{E}_2$ fits into the exact sequence

$$0\to\tilde{E}_2\to\tau^*(pr_1^*E\oplus pr_2^*E)\to\tau'^*E\to 0 \qquad (4.23)$$

where $\tau:\widetilde{S\times S}\to S\times S$ is the contraction, and where $\tau':\Delta\to Diag\,S$ is its restriction to the exceptional divisor.

Dualizing, we get the following exact sequence

$$0\to\tau^*(pr_1^*E^*\oplus pr_2^*E^*)\to\tilde{E}_2^*\to\tau'^*E^*\otimes\mathcal{O}_\Delta(\Delta)\to 0. \qquad (4.24)$$

Now $R^0\tau'_*\mathcal{O}_\Delta(\Delta)=R^1\tau'_*\mathcal{O}_\Delta(\Delta)=0$ hence the sheaf on the right has no cohomology. It follows that

$$H^i(\widetilde{S\times S},\tilde{E}_2^*)=H^i(\widetilde{S\times S},\tau^*(pr_1^*E^*\oplus pr_2^*E))$$

$$=H^i(S\times S,pr_1^*E^*\oplus pr_2^*E).$$

Since E^* has no odd dimensional cohomology, nor $\mathcal{O}_S$, it follows from Künneth formula that the same is true for $pr_1^*E^*\oplus pr_2^*E$ on $S\times S$. Finally we have

$$H^2(S\times S,pr_1^*E^*)=H^2(S,E^*)$$

since $H^0(S,E^*)=0$. This proves 1.

2. From (4.24) we deduce that $\bigwedge^2\tilde{E}_2^*$ has a filtration whose successive terms are

$$\bigwedge^2\tau^*(pr_1^*E^*\oplus pr_2^*E^*),\ \tau^*(pr_1^*E^*\oplus pr_2^*E^*)\otimes\tau'^*E^*\otimes\mathcal{O}_\Delta(\Delta),\ \bigwedge^2\tau'^*E^*\otimes\mathcal{O}_\Delta(2\Delta).$$

The sheaf $(pr_1^*E^*\oplus pr_2^*E^*)\otimes\tau'^*E^*\otimes\mathcal{O}_\Delta(\Delta)$ has no cohomology, since $\mathcal{O}_\Delta(\Delta)$ has no cohomology along the fibers of τ'. Hence we have an exact sequence

$$H^1(\Delta,\bigwedge^2\tau'^*E^*\otimes\mathcal{O}_\Delta(2\Delta))\to H^2(\widetilde{S\times S},\tau^*\bigwedge^2(pr_1^*E^*\oplus pr_2^*E^*))\to H^2(\widetilde{S\times S},\bigwedge^2\tilde{E}_2^*)$$

$$\to H^2(\Delta,\bigwedge^2\tau'^*E^*\otimes\mathcal{O}_\Delta(2\Delta))\ldots$$

But since

$$R^1\tau'_*(2\Delta_{|\Delta})=\mathcal{O}_S,\ R^0\tau'_*(2\Delta_{|\Delta})=0,$$

the term on the left is equal to $H^0(S,\bigwedge^2E^*)=0$ and the term on the right is equal to $H^1(S,\bigwedge^2E^*)=0$. Hence we have

$$H^2(\widetilde{S\times S},\bigwedge^2\tilde{E}_2^*)=H^2(\widetilde{S\times S},\tau^*\bigwedge^2(pr_1^*E^*\oplus pr_2^*E^*))=H^2(S\times S,\bigwedge^2(pr_1^*E^*\oplus pr_2^*E^*))$$

Finally

$$\bigwedge^2(pr_1^*E^*\oplus pr_2^*E^*)=pr_1^*\bigwedge^2E^*\oplus E^*\boxtimes E^*\oplus pr_2^*\bigwedge^2E^*.$$

The central term has no cohomology in degree 2 by Künneth formula, because $H^1(S,E^*)=H^0(S,E^*)=0$, and we have

$$H^2(S\times S,pr_1^*\bigwedge^2E^*)=H^2(S,\bigwedge^2E^*)=H^2(S,-L).$$

This proves 2.

3. We have $det\,\tilde{E}_2^*=\tau^*((-L)\boxtimes(-L))(2\Delta)$ by the exact sequence (4.23). Hence

$$\bigwedge^3\tilde{E}_2^*=\tilde{E}_2\otimes det\,\tilde{E}_2^*=\tilde{E}_2\otimes\tau^*((-L)\boxtimes(-L))(2\Delta).\tag{4.25}$$

The exact sequence (4.23) gives now the long exact sequence

$$H^2(\Delta,\tau'^*(E(-2L))(2\Delta_{|\Delta}))\to H^3(\widetilde{S\times S},\bigwedge^3\tilde{E}_2^*)$$

$$\to H^3(\widetilde{S\times S},\tau^*((pr_1^*E\oplus pr_2^*E)\otimes\tau^*((-L)\boxtimes(-L))(2\Delta)).$$

Since $R^0\tau'_*\mathcal{O}_\Delta(2\Delta)=0$, $R^1\tau'_*\mathcal{O}_\Delta(2\Delta)=\mathcal{O}_S$, the left hand side is equal to $H^1(S,E(-2L))$, which is easily seen to be 0.

Next we have $K_{\widetilde{S\times S}}=\mathcal{O}_{\widetilde{S\times S}}(\Delta)$, hence $H^3(\widetilde{S\times S},\tau^*(pr_1^*E\oplus pr_2^*E)\otimes\tau^*((-L)\boxtimes(-L))(2\Delta))$ is dual to

$$H^1(\widetilde{S\times S},\tau^*(pr_1^*E^*\oplus pr_2^*E^*)\otimes\tau^*(L\boxtimes L)(-\Delta)).\tag{4.26}$$

But one checks easily that the multiplication map

$$H^0(S,E)\otimes H^0(S,L)\to H^0(S,E\otimes L)$$

is surjective, and it follows that the group (4.26) is 0 since $H^1(S\times S,(pr_1^*E^*\oplus pr_2^*E^*)\otimes L\boxtimes L)=0$. (We use here the equality $E^*\otimes L=E$.)

Finally the equality (4.25) and the exact sequence (4.23) also show that $H^4(\widetilde{S\times S},\bigwedge^3\tilde{E}_2^*)$ admits $H^4(\widetilde{S\times S},\tau^*((pr_1^*E\oplus pr_2^*E)\otimes((-L)\boxtimes(-L)))(2\Delta))$ as a quotient. By Serre's duality this space is dual to

$$H^0(\widetilde{S\times S},\tau^*((pr_1^*E^*\oplus pr_2^*E^*)\otimes(L\boxtimes L))(-\Delta)). \tag{4.27}$$

But this is equal to

$$H^0(S\times S,(pr_1^*E^*\oplus pr_2^*E^*)\otimes(L\boxtimes L)\otimes\mathcal{I}_{Diag}).$$

We use then the fact that

$$pr_1^*E^*\otimes(L\boxtimes L)=E\boxtimes L$$

to conclude that (4.27) is equal to the sum of two copies of

$$Ker\,H^0(S,E)\otimes H^0(S,L)\to H^0(S,E\otimes L).$$

4. We already noticed that

$$\bigwedge^4\tilde{E}_2^*=det\,\tilde{E}_2^*=\tau^*((-L)\boxtimes(-L))(2\Delta).$$

It follows then from Serre's duality and $K_{\widetilde{S\times S}}=\mathcal{O}_{\widetilde{S\times S}}(\Delta)$ that $H^4(\widetilde{S\times S},\bigwedge^4\tilde{E}_2^*)$ is dual to

$$H^0(\widetilde{S\times S},\tau^*(L\boxtimes L)(-\Delta))=Ker\,H^0(S,L)\otimes H^0(S,L)\to H^0(S,2L)).$$

Hence 4 is proven.

∎

Coming back to the Koszul resolution of $\mathcal{I}_{W'}\otimes pr_2^*\mathcal{O}(k)$ induced by (4.22), we see that in order to prove the vanishing (4.21), it suffices to show :

a) $H^1(\widetilde{S\times S}\times\mathbb{P}(H^0(E)),\tilde{E}_2^*\boxtimes\mathcal{O}(k-1))=0$.

b) The interior product with σ

$$int(\sigma):H^2(\widetilde{S\times S}\times\mathbb{P}(H^0(E)),\bigwedge^2\tilde{E}_2^*\boxtimes\mathcal{O}(k-2))$$

$$\to H^2(\widetilde{S\times S}\times\mathbb{P}(H^0(E)),\tilde{E}_2^*\boxtimes\mathcal{O}(k-1))$$

is injective.

c) $H^3(\widetilde{S\times S}\times\mathbb{P}(H^0(E)),\bigwedge^3\tilde{E}_2^*\boxtimes\mathcal{O}(k-3))=0$.

d) The interior product with σ

$$int(\sigma) : H^4(\widetilde{S \times S} \times \mathbb{P}(H^0(E)), \bigwedge^4 \tilde{E}_2^* \boxtimes \mathcal{O}(k-4))$$

$$\to H^4(\widetilde{S \times S} \times \mathbb{P}(H^0(E)), \bigwedge^3 \tilde{E}_2^* \boxtimes \mathcal{O}(k-3))$$

is injective.

The conditions a) and c) have been established in proposition 6. We now dualize property b) as follows : by proposition 6 we have

$$H^2(\widetilde{S \times S} \times \mathbb{P}(H^0(E)), \bigwedge^2 \tilde{E}_2^* \boxtimes \mathcal{O}(k-2)) = (pr_1^* H^2(S, -L) \oplus pr_2^* H^2(S, -L)) \otimes S^{k-2} H^0(S, E)^*,$$

and

$$H^2(\widetilde{S \times S} \times \mathbb{P}(H^0(E)), \tilde{E}_2^* \boxtimes \mathcal{O}(k-1)) = (pr_1^* H^2(S, E^*) \oplus pr_2^* H^2(S, E^*)) \otimes S^{k-1} H^0(S, E)^*.$$

Dualizing, we get

$$H^2(\widetilde{S \times S} \times \mathbb{P}(H^0(E)), \bigwedge^2 \tilde{E}_2^* \boxtimes \mathcal{O}(k-2))^* = (H^0(S, L) \oplus H^0(S, L)) \otimes S^{k-2} H^0(S, E),$$

and

$$H^2(\widetilde{S \times S} \times \mathbb{P}(H^0(E)), \tilde{E}_2^* \boxtimes \mathcal{O}(k-1))^* = (H^0(S, E) \oplus H^0(S, E)) \otimes S^{k-1} H^0(S, E).$$

It is then immediate to check that the transpose of the map $int(\sigma)$ is the map $\wedge\sigma$, so that b) translates into the condition that

$$\wedge\sigma : (H^0(S, E) \oplus H^0(S, E)) \otimes S^{k-1} H^0(S, E) \to (H^0(S, L) \oplus H^0(S, L)) \otimes S^{k-2} H^0(S, E)$$

is surjective.

Now retracing through the isomorphisms given by proposition 6, one checks that the map $\wedge\sigma$ is up to sign equal to the direct sum of two copies of the composed map

$$\mu : H^0(S, E) \otimes S^{k-1} H^0(S, E) \to H^0(S, E) \otimes H^0(S, E) \otimes S^{k-2} H^0(S, E)$$

$$\stackrel{det \otimes id}{\to} H^0(S, L) \otimes S^{k-2} H^0(S, E).$$

Similarly statement d) dualizes as follows : by proposition 6, the space

$$H^4(\widetilde{S \times S}, \bigwedge^4 \tilde{E}_2^* \boxtimes \mathcal{O}(k-4)) \cong H^4(\widetilde{S \times S}, \bigwedge^4 \tilde{E}_2^*) \otimes S^{k-4} H^0(S, E)^*$$

is dual to

$$Ker\,(H^0(S, L) \otimes H^0(S, L) \to H^0(S, 2L)) \otimes S^{k-4} H^0(S, E).$$

Next, we know by proposition 6, 4, that

$$H^4(\widetilde{S\times S},\bigwedge^3\tilde{E}_2^*\boxtimes\mathcal{O}(k-3))\cong H^4(\widetilde{S\times S},\bigwedge^3\tilde{E}_2^*)\otimes S^{k-3}H^0(S,E)^*$$

admits a quotient which is dual to the direct sum of two copies of

$$Ker\,(H^0(S,E)\otimes H^0(S,L)\to H^0(S,E\otimes L))\otimes S^{k-3}H^0(S,E).$$

Denoting by $Q_{E,L}:=Ker\,(H^0(S,E)\otimes H^0(S,L)\to H^0(S,E\otimes L))$, $Q_{L,E}:=Ker\,(H^0(S,L)\otimes H^0(S,E)\to H^0(S,E\otimes L))$ and $Q_{L,L}=Ker\,(H^0(S,L)\otimes H^0(S,L)\to H^0(S,2L))$, we have an inclusion

$$(Q_{L,E}\oplus Q_{E,L})\otimes S^{k-3}H^0(S,E)\subset H^4(\widetilde{S\times S},\bigwedge^3\tilde{E}_2^*\boxtimes\mathcal{O}(k-3))^*$$

and to prove d) it suffices to show that the map dual to $int(\sigma)$ restricts on this subspace to a surjection

$$\wedge\sigma:(Q_{L,E}\oplus Q_{E,L})\otimes S^{k-3}H^0(S,E)\to Q_{L,L}\otimes S^{k-4}H^0(S,E).$$

But retracing through the isomorphisms of proposition 6 and recalling the definition of σ, one checks easily that the first component

$$\wedge\sigma_1:Q_{L,E}\otimes S^{k-3}H^0(S,E)\to Q_{L,L}\otimes S^{k-4}H^0(S,E)$$

of the map above is the following composite

$$\mu':Q_{L,E}\otimes S^{k-3}H^0(S,E)\subset H^0(S,L)\otimes H^0(S,E)\otimes S^{k-3}H^0(S,E)\to$$

$$H^0(S,L)\otimes H^0(S,E)\otimes H^0(S,E)\otimes S^{k-4}H^0(S,E)\stackrel{id\otimes det\otimes id}{\to}H^0(S,L)\otimes H^0(S,L)\otimes S^{k-4}H^0(S,E),$$

which takes obviously value in $Q_{L,L}\otimes S^{k-4}H^0(S,E)$, while the second component is equal to the first composed with the permutation exchanging factors on both sides.

To conclude then that

$$\wedge\sigma:(Q_{L,E}\oplus Q_{E,L})\otimes S^{k-3}H^0(S,E)\to Q_{L,L}\otimes S^{k-4}H^0(S,E)$$

is surjective, it suffices to show that

$$\mu'_-:Q_{L,E}\otimes S^{k-3}H^0(S,E)\to Q^-_{L,L}\otimes S^{k-4}H^0(S,E)$$

and

$$\mu'_+:Q_{L,E}\otimes S^{k-3}H^0(S,E)\to Q^+_{L,L}\otimes S^{k-4}H^0(S,E)$$

are surjective, where $Q^+_{L,L}$, (resp. $Q^-_{L,L}$) are the symmetric, resp. antisymmetric part of $Q_{L,L}$ and μ'_+ (resp. μ'_-) are the composition of μ' with the projections on the symmetric (resp. antisymmetric) part of $Q_{L,L}$.

In conclusion, the theorem 2 will be a consequence of the following propositions

Proposition 7 *The composed map*

$$\mu : H^0(S,E)\otimes S^{k-1}H^0(S,E) \to H^0(S,E)\otimes H^0(S,E)\otimes S^{k-2}H^0(S,E)$$

$$\stackrel{det}{\to} H^0(S,L)\otimes S^{k-2}H^0(S,E)$$

is surjective.

Proposition 8 *a) The map*

$$\mu'_- : Q_{L,E}\otimes S^{k-3}H^0(S,E)\to Q^-_{L,L}\otimes S^{k-4}H^0(S,E)$$

defined above is surjective.

b) The map

$$\mu'_+ : Q_{L,E}\otimes S^{k-3}H^0(S,E)\to Q^+_{L,L}\otimes S^{k-4}H^0(S,E)$$

defined above is surjective.

Proof of proposition 7. Let $\alpha,\,\beta\in H^0(S,E)$ and $\gamma\in H^0(S,L)$ such that

$$\gamma = det(\alpha\wedge\beta).$$

Then we observe first that if $D\subset H^0(S,E)$ is the rank 2 vector subspace generated by α and β, we have

$$\gamma\otimes S^{k-2}D\subset Im\,\mu$$

since the composite

$$D\otimes S^{k-1}D\to D\otimes D\otimes S^{k-2}D\to \bigwedge^2 D\otimes S^{k-2}D$$

is surjective.

Recall now that the map det determines a morphism

$$d : G_2\to \mathbb{P}H^0(S,L)$$

which is surjective and finite since both spaces are of the same dimension $2k$. The fiber $d^{-1}(\gamma)$ is then a finite subscheme $Z_\gamma\subset G_2$ which is the complete intersection of a space W of hyperplane sections of the Grassmannian G_2.

Now by the above observation, and since d is surjective, it suffices to show that the subspaces $S^{k-2}D$ for $D\in Z_\gamma$ generate $S^{k-2}H^0(S,E)$. If we dualize, this is equivalent to say that the dual map

$$S^{k-2}H^0(S,E)^*\to \oplus_{D\in Z_\gamma}S^{k-2}D^*$$

is injective. But this map identifies to the restriction

$$H^0(G_2,S^{k-2}\mathcal{E})\to H^0(Z_\gamma,S^{k-2}\mathcal{E}_{|Z_\gamma}),$$

at least for a reduced Z_γ, which will be the case for a generic γ.

Hence it suffices to show that

$$H^0(G_2, S^{k-2}\mathcal{E} \otimes \mathcal{I}_{Z_\gamma}) = 0. \tag{4.28}$$

Now we use the Koszul resolution

$$0 \to \bigwedge^{2k} W \otimes \mathcal{L}^{-2k} \to \ldots \to W \otimes \mathcal{L}^{-1} \to \mathcal{I}_{Z_\gamma} \to 0$$

The vanishing (4.28) will then follow from the vanishing

$$H^i(G_2, S^{k-2}\mathcal{E} \otimes \mathcal{L}^{-i-1}),\ i = 0,\ 2k-1$$

which is proved in proposition 9 of the appendix. ■

Proof of proposition 8, a). Notice first that the natural composed map

$$\bigwedge^3 H^0(S,E) \to \bigwedge^2 H^0(S,E) \otimes H^0(S,E)$$
$$\stackrel{det\otimes id}{\to} H^0(S,L) \otimes H^0(S,E)$$

has its image contained in $Q_{L,E}$. Hence it suffices to show that the following composite

$$\mu'' : \bigwedge^3 H^0(S,E) \otimes S^{k-3}H^0(S,E) \to \bigwedge^2 H^0(S,E) \otimes H^0(S,E) \otimes S^{k-3}H^0(S,E)$$

$$\stackrel{det\otimes\mu}{\to} H^0(S,L) \otimes H^0(S,L) \otimes S^{k-4}H^0(S,E) \to \bigwedge^2 H^0(S,L) \otimes S^{k-4}H^0(S,E)$$

is surjective.

Now note that for $\alpha_1,\ \alpha_2,\ \alpha_3 \in H^0(S,E)$

$$\mu''(\alpha_1 \wedge \alpha_2 \wedge \alpha_3 \otimes \alpha_3^{k-3}) = 2(k-3)det(\alpha_2 \wedge \alpha_3) \wedge det(\alpha_1 \wedge \alpha_3) \otimes \alpha_3^{k-4}. \tag{4.29}$$

Fix now $\gamma \in H^0(S,L)$ and consider the set of couples (α_1, α_3) such that $det(\alpha_1 \wedge \alpha_3) = \gamma$. For any α_2 and any such (α_1, α_3), we have

$$\mu''(\alpha_1 \wedge \alpha_2 \wedge \alpha_3 \otimes \alpha_3^{k-3}) = 2(k-3)det(\alpha_2 \wedge \alpha_3) \wedge \gamma \otimes \alpha_3^{k-4}.$$

Note that the vector α_3 for such pairs takes arbitrary value in some of the lines $D \in Z_\gamma$, where the notations are as in the previous proposition.

Now we have the map

$$\mu''' : H^0(S,E) \otimes S^{k-3}H^0(S,E) \to H^0(S,L) \otimes S^{k-4}H^0(S,E)$$

analogous to μ and the formula above shows that

$$\mu''(\alpha_1 \wedge \alpha_2 \wedge \alpha_3 \otimes \alpha_3^{k-3}) = 2\gamma \wedge \mu'''(\alpha_2 \otimes \alpha_3^{k-3}).$$

With the same proof as in the previous proposition, one shows now that the $S^{k-3}D$, $D \in Z_\gamma$ generate $S^{k-3}H^0(S,E)$ and that μ''' is surjective. Hence the $\alpha_2 \otimes \alpha_3^{k-3}$, $\alpha_3 \in D$, $D \in Z_\gamma$ generate $H^0(S,E) \otimes S^{k-3}H^0(S,E)$ and the $\mu'''(\alpha_2 \otimes \alpha_3^{k-3})$, $\alpha_3 \in D$, $D \in Z_\gamma$ generate by the surjectivity of μ''' the space $H^0(S,L) \otimes S^{k-4}H^0(S,E)$. Hence $Im\,\mu''$ contains $\gamma \wedge H^0(S,L) \otimes S^{k-4}H^0(S,E)$, and since γ was generic, we conclude that μ'' is surjective.

■

Proof of proposition 8, b). We want to prove that

$$\mu'_+ : Q_{L,E} \otimes S^{k-3}H^0(S,E) \to Q^+_{L,L} \otimes S^{k-4}H^0(S,E)$$

is surjective. Denote similarly, for C a generic member of $\mid L \mid$,

$$Q_{K_C,E} := Ker\,(H^0(C,K_C) \otimes H^0(C,E_{|C}) \to H^0(C, E \otimes K_C)),$$

$$Q^+_{K_C,K_C} := Ker\,(S^2H^0(C,K_C) \to H^0(C,K_C^{\otimes 2})).$$

Then we can define similarly

$$\mu'_{+,C} : Q_{K_C,E} \otimes S^{k-3}H^0(C,E_{|C}) \to Q^+_{K_C,K_C} \otimes S^{k-4}H^0(C,E_{|C})$$

as the composite

$$Q_{K_C,E} \otimes S^{k-3}H^0(C,E_{|C}) \subset H^0(C,K_C) \otimes H^0(C,E_{|C}) \otimes S^{k-3}H^0(C,E_{|C})$$

$$\to H^0(C,K_C) \otimes H^0(C,E_{|C}) \otimes H^0(C,E_{|C}) \otimes S^{k-4}H^0(C,E_{|C}) \stackrel{id\otimes det\otimes id}{\to}$$

$$H^0(C,K_C) \otimes H^0(C,K_C) \otimes S^{k-4}H^0(C,E_{|C}) \to S^2H^0(C,K_C) \otimes S^{k-4}H^0(C,E_{|C}).$$

Now the restriction map $H^0(S,E) \to H^0(C,E_{|C})$ is an isomorphism, and the restriction map $H^0(S,L) \to H^0(C,K_C)$ is surjective with kernel σ_C. Hence the restrictions induce a surjection

$$Q_{L,E} \to Q_{K_C,E}$$

and an isomorphism

$$Q^+_{L,L} \cong Q^+_{K_C,K_C},$$

and it suffices to show that $\mu'_{+,C}$ is surjective. A fortiori it suffices to show that the composite

$$\mu'_C : Q_{K_C,E} \otimes S^{k-3}H^0(C,E_{|C}) \subset H^0(C,K_C) \otimes H^0(C,E_{|C}) \otimes S^{k-3}H^0(C,E_{|C})$$

$$\to H^0(C,K_C) \otimes H^0(C,E_{|C}) \otimes H^0(C,E_{|C}) \otimes S^{k-4}H^0(C,E_{|C})$$

$$\stackrel{id\otimes det\otimes id}{\to} H^0(C,K_C) \otimes H^0(C,K_C) \otimes S^{k-4}H^0(C,E_{|C})$$

which takes value in $Q_{K_C,K_C} := Ker\,(H^0(C,K_C)^{\otimes 2} \to H^0(C,K_C^{\otimes 2}))$, is surjective on this last space.

Let us now consider the following diagram of exact sequences

$$\begin{array}{ccccc}
0 & \to & Q_{K_C,E} \otimes S^{k-3}H^0(C,E_{|C}) & \to & H^0(C,K_C) \otimes H^0(C,E_{|C}) \otimes S^{k-3}H^0(C,E_{|C}) \\
 & & \mu'_C \downarrow & & id \otimes \mu_C \downarrow \\
0 & \to & Q_{K_C,K_C} \otimes S^{k-4}H^0(C,E_{|C}) & \to & H^0(C,K_C) \otimes H^0(C,K_C) \otimes S^{k-4}H^0(C,E_{|C})
\end{array}$$

$$\begin{array}{cccc}
\to & H^0(C,E \otimes K_C) \otimes S^{k-3}H^0(C,E_{|C}) & \to & 0 \\
 & \mu_{C,K_C} \downarrow & & \\
\to & H^0(C,K_C^{\otimes 2}) \otimes S^{k-4}H^0(C,E_{|C}) & \to & 0
\end{array}$$

One checks easily the surjectivity of the multiplication maps on the left. The vertical maps μ_C and μ_{C,K_C} are defined in a way similar to μ e.g μ_C is the composite

$$H^0(C, E_{|C}) \otimes S^{k-3}H^0(C, E_{|C}) \subset H^0(C, E_{|C}) \otimes H^0(C, E_{|C}) \otimes S^{k-4}H^0(C, E_{|C})$$

$$\stackrel{det\otimes id}{\rightarrow} H^0(C, K_C) \otimes S^{k-4}H^0(C, E_{|C}),$$

and μ_{C,K_C} is defined similarly with a twist by K_C.

The proof of proposition 7 shows as well that μ_C is surjective, as is μ_{C,K_C} by the commutativity of the diagram above. Hence the surjectivity of μ'_C will follow by diagram chasing from the surjectivity of the induced multiplication map

$$H^0(C, K_C) \otimes Ker\,\mu_C \rightarrow Ker\,\mu_{C,K_C}. \tag{4.30}$$

In what follows we will use again the notation $H^0(E)$ for $H^0(S, E) = H^0(C, E_{|C})$. Define the vector bundle $\mathcal{Q}$ on C as the kernel of the surjective composite morphism of vector bundles

$$S^{k-3}H^0(E) \otimes E \subset S^{k-4}H^0(E) \otimes H^0(E) \otimes E \stackrel{id\otimes det}{\rightarrow} S^{k-4}H^0(E) \otimes K_C.$$

Then we clearly have

$$Ker\,\mu_C = H^0(C, \mathcal{Q}),\ Ker\,\mu_{C,K_C} = H^0(C, \mathcal{Q} \otimes K_C)$$

so that the surjectivity of the map (4.30) is equivalent to the surjectivity of the multiplication map

$$H^0(C, \mathcal{Q}) \otimes H^0(C, K_C) \rightarrow H^0(C, \mathcal{Q} \otimes K_C). \tag{4.31}$$

Now we proceed as follows : let $\sigma \in H^0(S, L)$ be the defining equation for C. Recall the finite reduced subscheme $Z_\sigma = d^{-1}(\sigma) \subset G_2$ made of the rank 2 vector subspaces D of $H^0(S, E)$ such that $det\, D = \sigma$. For each such D there is a subline bundle L_D of E on C, of degree $k+1$ with two sections without common zeroes (see section 2). The space D identifies naturally to $H^0(C, L_D)$.

Clearly the image of the inclusion

$$S^{k-3}H^0(C, L_D) \otimes L_D \subset S^{k-3}H^0(E) \otimes E$$

is contained in $\mathcal{Q}$.

Let now

$$\mathcal{N} := \oplus_{D\in Z_\sigma} S^{k-3}D \otimes L_D.$$

Then by the observation above we have a morphism

$$\alpha : \mathcal{N} \rightarrow \mathcal{Q}.$$

The surjectivity of 4.31 will follow from the following three lemmas :

Lemma 4 *The morphism α is surjective.*

Denoting $\mathcal{M} := Ker\,\alpha$ we also prove

Lemma 5 *The vector bundle $\mathcal{M}$ is generated by its sections.*

Lemma 6 *The space $H^0(C, \mathcal{M})$ is generated by the subspaces $H^0(C, \mathcal{M}(-x))$, $x \in C$.*

We explain first how these three lemmas imply our result. Using the exact sequence

$$0 \to \mathcal{M} \to \mathcal{N} \to \mathcal{Q} \to 0$$

given by lemma 4, we see that the map (4.31) will be surjective if the multiplication map

$$H^0(C, \mathcal{N}) \otimes H^0(C, K_C) \to H^0(C, \mathcal{N} \otimes K_C)$$

is surjective, and $H^1(C, \mathcal{M} \otimes K_C) = 0$.

The first condition is easy to check. Indeed $\mathcal{N}$ is a direct sum of line bundles L_D corresponding to g^1_{k+1}'s on C, and the result is easy to prove for each of them. As for the second condition, it is equivalent to $H^0(C, \mathcal{M}^*) = 0$ by Serre's duality. But since $\mathcal{M}$ is generated by sections by lemma 5, we have an inclusion

$$H^0(C, \mathcal{M}^*) \subset H^0(C, \mathcal{M})^*.$$

The image of this inclusion obviously vanishes on each subspace $H^0(C, \mathcal{M}(-x))$, hence it must be 0 since we know by lemma 6 that these subspaces generate $H^0(C, \mathcal{M})$. ■

To conclude the proof of 8,b) it remains only to prove these three lemmas.

Proof of lemma 4. First of all we note that the bundle $\mathcal{Q}$ is generated by its sections, since there is a natural surjection

$$S^{k-2}H^0(E) \otimes \mathcal{O}_C \to \mathcal{Q} \to 0.$$

Hence it suffices to show that the map

$$H^0(C, \mathcal{N}) \to H^0(C, \mathcal{Q})$$

is surjective.

But by definition

$$H^0(C, \mathcal{Q}) = Ker\,(H^0(E) \otimes S^{k-3}H^0(E) \stackrel{\mu_C}{\to} H^0(C, K_C) \otimes S^{k-4}H^0(E))$$

and

$$H^0(C, \mathcal{N}) = \oplus_{D \in Z_\sigma} D \otimes S^{k-3}D.$$

Hence we need to show that the sequence

$$\oplus_{D \in Z_\sigma} D \otimes S^{k-3}D \to H^0(E) \otimes S^{k-3}H^0(E) \stackrel{\mu_C}{\to} H^0(C, K_C) \otimes S^{k-4}H^0(E)$$

is exact at the middle. Again this will follow from a cohomological computation on the Grassmannian G_2. Indeed, the notations being as in the proof of Propositions 4 and 7, the sequence above dualizes as

$$I_{Z_\sigma}(\mathcal{L}) \otimes S^{k-4}H^0(G_2,\mathcal{E}) \to H^0(G_2,\mathcal{E}) \otimes S^{k-3}H^0(G_2,\mathcal{E}) \\ \to H^0(\mathcal{E} \otimes S^{k-3}\mathcal{E}_{|Z_\sigma}), \tag{4.32}$$

where the map

$$I_{Z_\sigma}(\mathcal{L}) \otimes S^{k-4}H^0(G_2,\mathcal{E}) \to H^0(G_2,\mathcal{E}) \otimes S^{k-3}H^0(G_2,\mathcal{E})$$

is composed of the inclusion

$$I_{Z_\sigma}(\mathcal{L}) \otimes S^{k-4}H^0(G_2,\mathcal{E}) \subset H^0(G_2,\mathcal{L}) \otimes S^{k-4}H^0(G_2,\mathcal{E}) \cong \wedge^2 H^0(G_2,\mathcal{E}) \otimes S^{k-4}H^0(G_2,\mathcal{E})$$

and of the (Koszul) map

$$\wedge^2 H^0(G_2,\mathcal{E}) \otimes S^{k-4}H^0(G_2,\mathcal{E}) \to H^0(G_2,\mathcal{E}) \otimes S^{k-3}H^0(G_2,\mathcal{E}).$$

One checks easily that $H^0(G_2,\mathcal{E}) \otimes S^{k-3}H^0(G_2,\mathcal{E}) \cong H^0(G_2,\mathcal{E} \otimes S^{k-3}\mathcal{E})$. Hence the kernel in the middle identifies to $H^0(G_2,\mathcal{E} \otimes S^{k-3}\mathcal{E} \otimes \mathcal{I}_{Z_\sigma})$. Furthermore $S^{k-4}H^0(G_2,\mathcal{E}) \cong H^0(G_2,S^{k-4}\mathcal{E})$ identifies to $H^0(G_2,\mathcal{E} \otimes S^{k-3}\mathcal{E} \otimes \mathcal{L}^{-1})$ via the (Koszul) inclusion

$$S^{k-4}\mathcal{E} \otimes \mathcal{L} = S^{k-4}\mathcal{E} \otimes \bigwedge^2 \mathcal{E} \subset \mathcal{E} \otimes S^{k-3}\mathcal{E}.$$

Hence the exactness at the middle of the sequence 4.32 will follow from the equality

$$H^0(G_2,\mathcal{E} \otimes S^{k-3}\mathcal{E} \otimes \mathcal{I}_{Z_\sigma}) = H^0(G_2,\mathcal{E} \otimes S^{k-3}\mathcal{E} \otimes \mathcal{L}^{-1}) \otimes I_{Z_\sigma}(\mathcal{L}). \tag{4.33}$$

Now let $W := I_{Z_\sigma}(\mathcal{L})$. The Koszul resolution of $\mathcal{I}_{Z_\sigma}$

$$0 \to \bigwedge^{2k} W \otimes \mathcal{L}^{-2k} \to \ldots \to W \otimes \mathcal{L}^{-1} \to \mathcal{I}_{Z_\sigma} \to 0$$

twisted by $\mathcal{E} \otimes S^{k-3}\mathcal{E}$ shows that the equality (4.33) will hold if we know that

$$H^i(G_2,\mathcal{E} \otimes S^{k-3}\mathcal{E} \otimes \mathcal{L}^{-i-1}) = 0,\ 1 \le i < 2k.$$

Since we have the exact sequence

$$0 \to S^{k-4}\mathcal{E} \otimes \mathcal{L} \to \mathcal{E} \otimes S^{k-3}\mathcal{E} \to S^{k-2}\mathcal{E} \to 0,$$

it suffices to know that

$$H^i(G_2,S^{k-4}\mathcal{E} \otimes \mathcal{L}^{-i}) = 0,\ 1 \le i < 2k,$$

and

$$H^i(G_2,S^{k-2}\mathcal{E} \otimes \mathcal{L}^{-i-1}) = 0,\ 1 \le i < 2k.$$

This is proved in Proposition 9. ∎

Proof of lemma 5. The bundles $\mathcal{N}$ and $\mathcal{Q}$ are generated by global sections. To prove that $\mathcal{M}$ is generated by global sections, it suffices to prove that for any $x \in C$, the restriction map $H^0(C, \mathcal{N}(-x)) \to H^0(C, \mathcal{Q}(-x))$ is surjective. For each g^1_{k+1} L_D on C, denote by $\sigma_{D,x} \in H^0(C, L_D) \cong D$ a generator for $H^0(C, L_D(-x))$. We need to show the exactness of the sequence

$$\begin{aligned} \oplus_{D \in Z_\sigma} \sigma_{D,x} \otimes S^{k-3}D \to H^0(C, E(-x)) \otimes S^{k-3}H^0(E) \\ \stackrel{\mu_C}{\to} H^0(C, K_C(-x)) \otimes S^{k-4}H^0(E) \end{aligned} \tag{4.34}$$

Denote by $K_x \subset H^0(E)$ the subspace $H^0(C, E(-x))$. Note that via the identification $H^0(C, L_D) = D$, $\sigma_{D,x}$ becomes a generator of the one-dimensional vector space $D \cap K_x$. Furthermore, K_x determines a section $\tau_x \in \bigwedge^2 H^0(E)^*$ up to a coefficient. Clearly $\tau_x \in H^0(C, K_C)^* \subset \bigwedge^2 H^0(E)^*$ identifies also to the linear form on $H^0(C, K_C)$ defining $H^0(C, K_C(-x))$. Let $G_x \subset G_2$ be the hyperplane section defined by τ_x. The scheme Z_σ is a complete intersection of hyperplane sections of G_x. The variety G_x admits a desingularization $P_x \stackrel{p}{\to} G_x$ defined as

$$P_x = \{(u, \Delta) \in \mathbb{P}(K_x) \times G_2,\, u \in \Delta \cap K_x\}.$$

Note that if

$$\begin{array}{ccc} P & \stackrel{p}{\to} & G_2 \\ \pi \downarrow & & \\ \mathbb{P}(H^0(E)) & & \end{array}$$

is the incidence variety, P_x can also be defined as $\pi^{-1}(\mathbb{P}(K_x)) \subset P$.

Since each line D parametrized by Z_σ meets K_x along a one dimensional vector space, the scheme Z_σ can also be seen as the complete intersection in P_x of hypersurfaces in $\mid p^*\mathcal{L} \mid$.

We now dualize the sequence (4.34). The space $H^0(C, K_C(-x))$ admits for dual the space $W \subset H^0(P_x, p^*\mathcal{L})$ defining $Z_\sigma \subset P_x$. The vector space $< \sigma_{D,x} >^*$ identifies clearly to the fiber of the line bundle $\pi^*\mathcal{O}_{\mathbb{P}(K_x)}(1)$ at the point $D \in Z_\sigma$. Hence our sequence dualizes as

$$\begin{aligned} W \otimes S^{k-4}H^0(E)^* \to \pi^* H^0(\mathbb{P}(K_x), \mathcal{O}(1)) \otimes H^0(P_x, p^*S^{k-3}\mathcal{E}) \\ \to H^0(S^{k-3}\mathcal{E} \otimes H_x \mid Z_\sigma), \end{aligned} \tag{4.35}$$

where $H_x := p^*\mathcal{O}_{\mathbb{P}(K_x)}(1)$. The second space in this sequence identifies to $H^0(P_x, p^*S^{k-3}\mathcal{E} \otimes H_x)$ so that the kernel at the middle is equal to $H^0(P_x, p^*S^{k-3}\mathcal{E} \otimes H_x \otimes \mathcal{I}_{Z_\sigma})$. The first map in (4.35)is induced by the isomorphism

$$S^{k-4}H^0(E)^* \cong H^0(P_x, p^*S^{k-4}\mathcal{E}),$$

the multiplication

$$W \otimes H^0(P_x, p^*S^{k-4}\mathcal{E}) \to H^0(P_x, p^*(S^{k-4}\mathcal{E} \otimes \mathcal{L}) \otimes \mathcal{I}_{Z_\sigma})$$

and by the composed bundle map

$$p^*S^{k-4}\mathcal{E} \otimes \mathcal{L} \to p^*S^{k-3}\mathcal{E} \otimes \mathcal{E} \to p^*S^{k-3}\mathcal{E} \otimes H_x,$$

where the last map is induced by the natural surjective map $p^*\mathcal{E} \to H_x$.

The exactness of (4.35) will then follow from the surjectivity of

$$W \otimes H^0(P_x, p^*S^{k-3}\mathcal{E} \otimes H_x \otimes \mathcal{L}^{-1}) \to H^0(P_x, p^*S^{k-3}\mathcal{E} \otimes H_x \otimes \mathcal{I}_{Z_\sigma}) \quad (4.36)$$

and from the equality

$$H^0(P_x, p^*S^{k-3}\mathcal{E} \otimes H_x \otimes \mathcal{L}^{-1}) = H^0(P_x, p^*S^{k-4}\mathcal{E}). \quad (4.37)$$

This last equality is proved as follows : on P_x we have the exact sequence

$$0 \to p^*\mathcal{L} \otimes H_x^{-1} \to p^*\mathcal{E} \to H_x \to 0,$$

which gives

$$0 \to p^*S^{k-4}\mathcal{E} \otimes p^*\mathcal{L} \otimes H_x^{-1} \to p^*S^{k-3}\mathcal{E} \to H_x^{k-3} \to 0.$$

Tensoring this with $H_x \otimes \mathcal{L}^{-1}$ we get

$$0 \to p^*S^{k-4}\mathcal{E} \to p^*S^{k-3}\mathcal{E} \otimes H_x \otimes \mathcal{L}^{-1} \to H_x^{k-2} \otimes p^*\mathcal{L}^{-1} \to 0.$$

But the right hand side has no non-zero sections since it is of negative degree on the fibers of π. Hence the equality (4.37).

Since $Z_\sigma \subset P_x$ is the complete intersection of the space W of sections of $p^*\mathcal{L}$, we have a Koszul resolution of $\mathcal{I}_{Z_\sigma}$, which takes the form

$$0 \to \bigwedge^{2k-1} W \otimes p^*\mathcal{L}^{-2k-1} \to \ldots \to W \otimes p^*\mathcal{L}^{-1} \to \mathcal{I}_{Z_\sigma} \to 0.$$

We can tensor it with $p^*S^{k-3}\mathcal{E} \otimes H_x$, and the surjectivity of the map (4.36) will follow from the following vanishing

$$H^i(P_x, p^*S^{k-3}\mathcal{E} \otimes H_x \otimes p^*\mathcal{L}^{-i-1}) = 0,\ 1 \le i < 2k-1 = dim\, P_x. \quad (4.38)$$

Recall now that $P_x \subset P$ is the complete intersection of two sections of $H = \pi^*\mathcal{O}_{\mathbb{P}(H^0(E))}(1)$, with $H_x = H_{|P_x}$. The vanishing (4.38) will then follow from

$$H^i(P, p^*S^{k-3}\mathcal{E} \otimes H \otimes p^*\mathcal{L}^{-i-1}) = 0,\ 1 \le i < 2k-1$$

$$H^{i+1}(P, p^*S^{k-3}\mathcal{E} \otimes p^*\mathcal{L}^{-i-1}) = 0,\ 1 \le i < 2k-1$$

$$H^{i+2}(P, p^*S^{k-3}\mathcal{E} \otimes \otimes p^*\mathcal{L}^{-i-1} \otimes H^{-1}) = 0,\ 1 \le i < 2k-1.$$

The second equality follows immediately from the proposition 9, and the third is obvious since H^{-1} has no cohomology on the fibers of p. The first equality is proven as follows : we have

$$H^i(P, p^*S^{k-3}\mathcal{E} \otimes H \otimes p^*\mathcal{L}^{-i-1}) = H^i(G_2, S^{k-3}\mathcal{E} \otimes \mathcal{E} \otimes \mathcal{L}^{-i-1}),$$

since $R^0p_*H = \mathcal{E}$. Now we have the exact sequence on G_2

$$0 \to S^{k-4}\mathcal{E} \otimes \mathcal{L} \to S^{k-3}\mathcal{E} \otimes \mathcal{E} \to S^{k-2}\mathcal{E} \to 0.$$

Hence the needed equality will follow from the vanishings

$$H^i(G_2, S^{k-4}\mathcal{E} \otimes \mathcal{L}^{-i}) = 0,$$

$$H^i(G_2, S^{k-2}\mathcal{E} \otimes \mathcal{L}^{-i-1}) = 0,$$

for $1 \leq i < 2k-1$, which are proved in proposition (9). Hence lemma 5 is proven.

∎

Proof of lemma 6. Let $x_1, \ldots, x_{2k-1}$ be points of C in general position. We will show that the natural map

$$\oplus_i H^0(C, \mathcal{M}(-x_i)) \to H^0(C, \mathcal{M}) \tag{4.39}$$

is surjective.

Recall that

$$H^0(C, \mathcal{M}) = Ker\ \oplus_{D \in Z_\sigma} S^{k-3}D \otimes D \to S^{k-3}H^0(E) \otimes H^0(E).$$

It follows from this that

$$H^0(C, \mathcal{M})^* = Coker\, H^0(G_2, S^{k-3}\mathcal{E} \otimes \mathcal{E}) \to H^0(S^{k-3}\mathcal{E} \otimes \mathcal{E}_{|Z_\sigma})$$

$$= H^1(G_2, S^{k-3}\mathcal{E} \otimes \mathcal{E} \otimes \mathcal{I}_{Z_\sigma}).$$

Similarly

$$H^0(C, \mathcal{M}(-x_i)) = Ker\ \oplus_{D \in Z_\sigma} S^{k-3}D \otimes \sigma_{D,x_i} \to S^{k-3}H^0(E) \otimes K_{x_i}$$

which, with the notations of the previous proof, dualizes to

$$H^0(C, \mathcal{M}(-x_i))^* = Coker\,(H^0(P_{x_i}, p^*S^{k-3}\mathcal{E} \otimes H_{x_i}) \to H^0(Z_\sigma, S^{k-3}\mathcal{E} \otimes H_{x_i}))$$

$$= H^1(P_{x_i}, p^*S^{k-3}\mathcal{E} \otimes H_{x_i} \otimes \mathcal{I}_{Z_\sigma}),$$

where we view Z_σ as a subscheme of P_{x_i} as well. Hence we have to show that the natural map (induced by the morphism $p^*\mathcal{E} \to H_{x_i}$ on P_{x_i})

$$H^1(G_2, S^{k-3}\mathcal{E} \otimes \mathcal{E} \otimes \mathcal{I}_{Z_\sigma}) \to \oplus_i H^1(P_{x_i}, p^*S^{k-3}\mathcal{E} \otimes H_{x_i} \otimes \mathcal{I}_{Z_\sigma}) \tag{4.40}$$

is injective.

Let $D \subset G_2$ be the curve complete intersection of the sections $\sigma_{x_i} \in H^0(G_2, \mathcal{L})$. We have first

Fact. *The restriction map*

$$H^0(G_2, S^{k-3}\mathcal{E} \otimes \mathcal{E}) \to H^0(D, S^{k-3}\mathcal{E} \otimes \mathcal{E})_{|D})$$

is surjective.

Using the Koszul resolution of $\mathcal{I}_D$ this is obtained by application of the proposition 9. ∎

From this we conclude that the restriction map

$$H^1(G_2, S^{k-3}\mathcal{E}\otimes\mathcal{E}\otimes\mathcal{I}_{Z_\sigma}) \to H^1(D, S^{k-3}\mathcal{E}\otimes\mathcal{E}_{|D}\otimes\mathcal{I}_{Z_\sigma})$$

is injective.

Consider now the inverse image $\tilde{D}$ of D in the fibered product

$$P\times_{G_2}\times\ldots\times_{G_2}P.$$

Denote by $\tilde{p}:\tilde{D}\to D\subset G_2$ the natural morphism. One shows easily that the curve $\tilde{D}$ is isomorphic to D excepted over the intersection of D with a Grassmannian of lines in $\mathbb{P}(K_{x_i})$ for some i. Here D has nodes, which are replaced in $\tilde{D}$ by lines.

This fact is obviously true set theoretically, and is proved scheme theoretically by the computation of the canonical bundles, which gives :

$$K_{\tilde{D}}=\tilde{p}^*K_D.$$

The zero set Z_λ is supported away of this singular locus. For each i we have a natural restriction map

$$H^1(P_{x_i}, p^*S^{k-3}\mathcal{E}\otimes H_{x_i}\otimes\mathcal{I}_{Z_\sigma}) \to H^1(\tilde{D}, \tilde{p}^*S^{k-3}\mathcal{E}\otimes H_{x_i}\otimes\mathcal{I}_{Z_\sigma}),$$

since

$$\tilde{D}=P_{x_1}\times_{G_2}\ldots\times_{G_2}P_{x_{2k-1}}$$

admits a natural morphism to P_{x_i}. Next we have by the above description of $\tilde{D}$ an isomorphism

$$H^1(D, S^{k-3}\mathcal{E}\otimes\mathcal{E}_{|D}\otimes\mathcal{I}_{Z_\sigma}) \cong H^1(\tilde{D}, \tilde{p}^*S^{k-3}\mathcal{E}\otimes\mathcal{E}\otimes\mathcal{I}_{Z_\sigma})$$

and it follows that the injectivity of the map (4.40) will be a consequence of the injectivity of the map

$$H^1(\tilde{D}, \tilde{p}^*S^{k-3}\mathcal{E}\otimes\mathcal{E}\otimes\mathcal{I}_{Z_\sigma}) \to \oplus_i H^1(\tilde{D}, S^{k-3}\mathcal{E}\otimes H_{x_i}\otimes\mathcal{I}_{Z_\sigma}) \qquad (4.41)$$

induced by the morphisms $\tilde{p}^*\mathcal{E}\to H_{x_i}$ on $\tilde{D}$. Recall now that $Z_\sigma\subset D$ is defined by a section of $\mathcal{L}$ so that similarly $Z_\sigma\subset\tilde{D}$ is defined by a section of $p^*\mathcal{L}$. Hence we have

$$\mathcal{I}_{Z_\sigma}\cong\tilde{p}^*\mathcal{L}^{-1}.$$

Furthermore

$$K_{\tilde{D}}=\tilde{p}^*K_D=\tilde{p}^*(K_{G_2|D}\otimes\mathcal{L}^{2k-1})=\tilde{p}^*\mathcal{L}^{k-3}.$$

Hence the map (4.41) dualizes by Serre's duality as the map

$$\oplus_i H^0(\tilde{D}, \tilde{p}^*S^{k-3}\mathcal{E}^*\otimes H^*_{x_i}\otimes\tilde{p}^*\mathcal{L}\otimes\tilde{p}^*\mathcal{L}^{k-3}) \to H^0(\tilde{D}, \tilde{p}^*S^{k-3}\mathcal{E}^*\otimes\mathcal{E}^*\otimes\tilde{p}^*\mathcal{L}\otimes\tilde{p}^*\mathcal{L}^{k-3}) \qquad (4.42)$$

given by the inclusions $H^*_{x_i} \subset \mathcal{E}^*$ on $\tilde{D}$. Since $det\,\mathcal{E} = \mathcal{L}$, we have

$$\mathcal{E}^* \otimes \mathcal{L} \cong \mathcal{E},$$

Hence this rewrites as

$$\oplus_i H^0(\tilde{D}, p^* S^{k-3}\mathcal{E} \otimes H^*_{x_i} \otimes \mathcal{L}) \to H^0(\tilde{D}, p^* S^{k-3}\mathcal{E} \otimes \mathcal{E}) \tag{4.43}$$

given by the inclusions

$$H^*_{x_i} \otimes \mathcal{L} \subset p^*\mathcal{E}.$$

We want to show that (4.41) is injective, or that (4.43) is surjective. We already noticed that the restriction map

$$H^0(G_2, S^{k-3}\mathcal{E} \otimes \mathcal{E}) = S^{k-3}H^0(E)^* \otimes H^0(E)^*$$

$$\to H^0(D, S^{k-3}\mathcal{E} \otimes \mathcal{E}) = H^0(\tilde{D}, \tilde{p}^* S^{k-3}\mathcal{E} \otimes \mathcal{E})$$

is surjective. On the other hand, consider the subspace $H_{x_i} = K^\perp_{x_i} \subset H^0(E)^*$. It is obvious that it gives a section of

$$Ker\,(H^0(P_{x_i}, \mathcal{E}) \to H^0(P_{x_i}, H_{x_i})) = H^0(P_{x_i}, p^*\mathcal{L} \otimes H^*_{x_i}).$$

Hence the surjective map

$$S^{k-3}H^0(E)^* \otimes H^0(E)^* \to H^0(\tilde{D}, \tilde{p}^* S^{k-3}\mathcal{E} \otimes \mathcal{E})$$

sends $S^{k-3}H^0(E)^* \otimes H_{x_i}$ in $H^0(\tilde{D}, p^* S^{k-3}\mathcal{E} \otimes H^*_{x_i} \otimes \mathcal{L})$.

Now since the x_i's are generic, the spaces H_{x_i} generate $H^0(E)^*$, hence the $S^{k-3}H^0(E)^* \otimes H_{x_i}$'s generate $S^{k-3}H^0(E)^* \otimes H^0(E)^*$. Hence we have shown that (4.43) is surjective.

∎

5 Appendix

We consider the Grassmannian G_2 of rank 2 vector subspaces of a $k+2$-dimensional vector space V. Let $\mathcal{L}$ be the line bundle on G_2 whose sections give the Plücker embedding. If $\mathcal{E}$ is the dual of the tautological subbundle $\mathcal{S} \subset V \otimes \mathcal{O}_{G_2}$, we have $\mathcal{L} = det\,\mathcal{E}$. The cohomology groups $H^p(G_2, \mathcal{L}^{-q} \otimes S^{q'}\mathcal{E})$ are described in the following proposition.

Proposition 9 *For $q > 0$, $q' > 0$, we have $H^p(G_2, \mathcal{L}^{-q} \otimes S^{q'}\mathcal{E}) = 0$ if $p \neq k$, $2k$. Furthermore, for $p = k$, we have $H^p(G_2, \mathcal{L}^{-q} \otimes S^{q'}\mathcal{E}) = 0$ if $-q+q'+1 < 0$, and for $p = 2k$, we have $H^p(G_2, \mathcal{L}^{-q} \otimes S^{q'}\mathcal{E}) = 0$ if $-q+q' \geq -k-1$.*

Proof. Let

$$\begin{array}{ccc} P & \stackrel{p}{\to} & G_2 \\ \pi \downarrow & & \\ \mathbb{P}(V) & & \end{array}$$

be the incidence variety. P is a $\mathbb{P}^1$-bundle over G_2 and a $\mathbb{P}^k$-bundle over $\mathbb{P}(V)$. Let $H := \pi^*\mathcal{O}_{\mathbb{P}(V)}(1)$ and let $L' = p^*\mathcal{L}$. Then $\mathcal{E} = R^0 p_* H$ and $S^{q'}\mathcal{E} = R^0 p_*(q'H)$. It follows that we have

$$H^p(G_2, \mathcal{L}^{-q} \otimes S^{q'}\mathcal{E}) = H^p(P, -qL' + q'H).$$

Next the line bundle L' restricts to $\mathcal{O}(1)$ on the fibers of π. It follows from this that

$$K_P = -(k+1)L' - 2H,$$

and $K_{P/\mathbb{P}(V)} = -(k+1)L' + kH$.

Now since $q > 0$ we have $R^l\pi_*(-qL' + q'H) = 0$ for $l < k$ and hence

$$H^p(P, -qL' + q'H) = H^{p-k}R^k\pi_*(L' + q'H).$$

By Serre's duality, we have

$$R^k\pi_*(-qL'+q'H) = (R^0\pi_*(qL'-q'H-(k+1)L'+kH))^* = (R^0\pi_*((q-(k+1))L'+(k-q')H))^*.$$

Now we have $R^0\pi_*((q-(k+1))L') = 0$ if $q < k+1$, and

$$R^0\pi_*((q-(k+1))L') \cong S^{q-k-1}(\Omega_{\mathbb{P}(V)}(2)) \tag{5.44}$$

for $q \geq k+1$. The isomorphism (5.44) follows from the isomorphism

$$H^0(P, L') = H^0(G_2, \mathcal{L}) = \bigwedge^2 V^* = H^0(\mathbb{P}(V), \Omega_{\mathbb{P}(V)}(2))$$

and from the comparison of the kernels of the surjective evaluation maps

$$H^0(P, L') \to H^0(\pi^{-1}(x), L')$$

and

$$H^0(\mathbb{P}(V), \Omega_{\mathbb{P}(V)}(2)) \to \Omega_{\mathbb{P}(V)}(2)_x.$$

Finally we conclude that

1. $H^p(P, -qL' + q'H) = 0$ for $p < k$.
2. $H^p(P, -qL' + q'H) = 0$ for $q < k+1$.
3. For $p \geq k$, $q \geq k+1$,
$$H^p(P, -qL' + q'H) = H^{p-k}(\mathbb{P}(V), S^{q-k-1}(T_{\mathbb{P}(V)}(-2))(q'-k)).$$

To conclude, consider the Euler exact sequence

$$0 \to \mathcal{O}_{\mathbb{P}(V)}(-1) \to V \otimes \mathcal{O}_{\mathbb{P}(V)} \to T_{\mathbb{P}(V)}(-1) \to 0.$$

It induces the exact sequences

$$\begin{aligned}0 \to S^{q-k-2}V \otimes \mathcal{O}_{\mathbb{P}(V)}(-q+q') \to S^{q-k-1}V \otimes \mathcal{O}_{\mathbb{P}(V)}(-q+q'+1)\\ \to S^{q-k-1}(T_{\mathbb{P}(V)}(-1))(-q+q'+1) \to 0.\end{aligned}$$

Hence we conclude that

$$H^{p-k}(\mathbb{P}(V), S^{q-k-1}(T_{\mathbb{P}(V)}(-2))(q'-k)) = H^{p-k}(\mathbb{P}(V), S^{q-k-1}(T_{\mathbb{P}(V)}(-1))(-q+q'+1))$$

is equal to 0 for $p-k \neq 0$, k (since $p \leq 2k$), and that : for $p-k = 0$ it is 0 if $-q+q'+1 < 0$; for $p-k = k$ it is 0 if $-q+q' \geq -k-1$. ∎

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