James R. Wall

Green's relations for stochastic matrices

Czechoslovak Mathematical Journal, Vol. 25 (1975), No. 2, 247–260

Persistent URL: http://dml.cz/dmlcz/101315

Czechoslovak Mathematical Journal, 25 (100) 1975, Praha

GREEN'S RELATIONS FOR STOCHASTIC MATRICES

J. R. WALL, Auburn

(Received December 13, 1973)

A real nonnegative $m \times n$ matrix A is stochastic if the sum of all the entries in each row is 1. If A^T, the transpose of A, is also stochastic, then A is doubly stochastic. The set $S_n[D_n]$ of all $n \times n$ stochastic [doubly stochastic] matrices forms a compact Hausdorff semigroup under matrix multiplication with the $n \times n$ identity matrix I as identity element.

An element a of a semigroup S is regular if $axa = a$ for some $x \in S$. For A and B regular elements of $S_n[D_n]$, we obtain necessary and sufficient conditions for each of the systems

$$AX = B\,, \quad BY = A\,, \quad X, Y \in S_n[D_n] \tag{1}$$

and dually,

$$XA = B\,, \quad YB = A\,, \quad X, Y \in S_n[D_n] \tag{2}$$

to be consistent.

1. Preliminaries, idempotents. Certain definitions and results from the theory of semigroups will be useful. The algebraic facts summarized below may be found in [2, Chapter 2] and those requiring compactness in [6].

Let S be a semigroup with identity, and let $a, b \in S$. The relation $\mathscr{R}[\mathscr{L}, \mathscr{J}]$ is defined on S by $a \mathscr{R} b[a \mathscr{L} b, a \mathscr{J} b]$ if a and b generate the same principal right [left, two-sided] ideal of S. The relation $\mathscr{D}$ is defined as the join of $\mathscr{R}$ and $\mathscr{L}$, and the relation $\mathscr{H}$ as the intersection of $\mathscr{R}$ and $\mathscr{L}$. These relations are called Green's relations on S, and each is an equivalence relation on S.

Equivalently, $a \mathscr{R} b[a \mathscr{L} b]$ if and only if $ax = b$ and $by = a$ $[xa = b, yb = a]$ for some $x, y \in S$. Thus the problem of solving matrix equations (1) in S_n may be stated in two other ways: first, as the problem of characterizing $\mathscr{R}$ on S_n; and second, of finding all matrices B in S_n which lie in the same $\mathscr{R}$-equivalence class with a given A in S_n. Similar remarks hold for $\mathscr{L}$ and for D_n.

If a $\mathscr{D}$-class D of a semigroup S contains one regular element, then every element of D is regular. Each $\mathscr{R}$-class and each $\mathscr{L}$-class of S contained in D contains an idempotent. The $\mathcal{H}$-classes of S which contain an idempotent are just the maximal subgroups of S.

Since $S_n[D_n]$ is compact, $\mathcal{D} = \mathcal{J}$ on $S_n[D_n]$. Every compact semigroup S contains a unique minimal two-sided ideal called the kernel of S. The kernel of $S_n[D_n]$ is described in the following theorem, which also identifies the idempotents in $S_n[D_n]$ of rank 1.

Theorem 1.1. (Schwarz [10]) (a) *An idempotent element of $S_n[D_n]$ has rank 1 if and only if it has constant columns. Every such idempotent is a right zero element of S_n, and S_n contains no left zeros. The only rank 1 idempotent in D_n is the matrix having each entry equal to $1/n$, and it is the zero element of D_n.*

(b) *The kernel K of the semigroup $S_n[D_n]$ is the set of all rank 1 idempotents in $S_n[D_n]$. Each maximal subgroup contained in K contains only one element. Thus each $\mathcal{L}$-class of K is a singleton, and K contains only one $\mathcal{R}$-class.*

The next two results identify all idempotents in $S_n[D_n]$. In the sequel the term "canonical idempotent" in $S_n[D_n]$ will refer to an idempotent which is expressed exactly in the form and with the notation of these results.

Theorem 1.2. (Doob [3]) *Let E be an idempotent in S_n with rank k. Then there exists an $n \times n$ permutation matrix P such that*

$$PEP^T = \begin{pmatrix} E_1 & 0 & \dots & 0 & 0 \\ 0 & E_2 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & E_k & 0 \\ F_1 & F_2 & \dots & F_k & 0 \end{pmatrix}$$

where each E_i is a positive $n_i \times n_i$ stochastic matrix of rank 1; each F_i is an $n_{k+1} \times n_i$ matrix of the form $F_i = D_i E_i'$ with each F_i an $n_{k+1} \times n_{k+1}$ nonnegative diagonal matrix such that $D_1 + \dots + D_k = I$, and each E_i' consists of n_{k+1} rows each equal to a row of E_i; $n_1 + \dots + n_{k+1} = n$; $n_1 \geqq \dots \geqq n_k \geqq n_{k+1} \geqq 0$; and $n_k \geqq 1$.

It is understood here and in similar situations in what follows that the columns of zeros on the right of PEP^T or the blocks $F_1, \dots, F_k$ may not appear.

Corollary 1.3. (Doob [3], Schwarz [11]) *Let E be an idempotent in D_n with rank k. Then there exists an $n \times n$ permutation matrix P such that*

$$PEP^T = E_1 \oplus E_2 \oplus \dots \oplus E_k$$

where each E_i is the $n_i \times n_i$ matrix having each entry equal to $1/n_i$; $n_1 + \dots + n_k = n$; and $n_1 \geqq n_2 \geqq \dots \geqq n_k \geqq 1$.

2. Green's relation $\mathscr{R}$ for S_n. We begin by characterizing those elements of S_n which lie in the same $\mathscr{R}$-class with a canonical idempotent E. The form of such a matrix A depends upon the way E is partitioned into blocks of rows, that is, upon the numbers n_i, and upon the diagonal matrices D_i. In the course of the proof, a method for constructing solutions to the system (1) will be indicated for the case where B is idempotent.

Theorem 2.1. *Let E be a canonical idempotent in S_n with rank k, and let $A \in S_n$. Then the following statements are equivalent.*

(a) *$A \mathscr{R} E$; that is, the system $AX = E$, $EY = A$, $X, Y \in S_n$, is consistent.*

(b) *There exists an $n \times n$ permutation matrix P such that*

$$AP = \begin{pmatrix} A_1 & 0 & \dots & 0 & 0 \\ 0 & A_2 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & A_k & 0 \\ U_1 & U_2 & \dots & U_k & 0 \end{pmatrix} \tag{3}$$

where each A_i is an $n_i \times t_i$ stochastic matrix of rank 1, and $U_i = D_i A_i'$ with each row of A_i' equal to a row of A_i.

(c) *The rows of A can be partitioned into the form*

$$A = \begin{pmatrix} M_1 \\ \dots \\ M_k \\ M_{k+1} \end{pmatrix}$$

where each M_i is an $n_i \times n$ stochastic matrix of rank 1; the matrices $M_1, \dots, M_k$ are mutually orthogonal; and $M_{k+1} = D_1 M_1' + \dots + D_k M_k'$ where each row of M_i' is a row of M_i.

Proof. Assume (a) holds. Partition A by

$$A = \begin{pmatrix} M_1 \\ \dots \\ M_k \\ M_{k+1} \end{pmatrix}$$

where each M_i is $n_i \times n$. Since $EA = A$, $E_i M_i = M_i$ for $i = 1, \dots, k$, and $F_1 M_1 + \dots$ $\dots + F_k M_k = M_{k+1}$. Thus rank $(M_i) \leq$ rank $(E_i) = 1$, so rank $(M_i) = 1$ for $i = 1, \dots, k$. Since M_i is stochastic, the columns of M_i are constant for $i = 1, \dots, k$.

Partition X by

$$X = (X_1, \dots, X_k, X_{k+1})$$

where each X_i is $n \times n_i$. Then $AE = E$ implies

$$\begin{aligned} M_iX_i &= E_i \quad \text{for} \quad i = 1, \ldots, k\,; \\ M_iX_j &= 0 \quad \text{for} \quad i \neq j\,, \quad i, j = 1, \ldots, k\,; \\ M_iX_{k+1} &= 0 \quad \text{for} \quad i = 1, \ldots, k+1\,; \end{aligned}$$

and

$$M_{k+1}X_j = F_j \quad \text{for} \quad j = 1, \ldots, k\,.$$

For $k > 1$, suppose $M_i > 0$ for some $i = 1, \ldots, k$. Then, for any $j \neq i, j = 1, \ldots, k$, $M_iX_j = 0$, so $X_j = 0$. But then $E_j = M_jX_j = 0$, contrary to $E_j > 0$. Therefore M_i has at least one column of zeros for $i = 1, \ldots, k$. Let t_i be the number of nonzero columns of M_i, $i = 1, \ldots, k$. If $k = 1$, it may be that $t_1 = n$; in this case $A = M_1 = A_1$.

Now assume $t_1 < n$. Let P_1 be an $n \times n$ permutation matrix such that the first t_1 columns of M_1P_1 are nonzero. Then, for $j = 2, \ldots, k = 1$, $0 = M_1X_j = M_1P_1P_1^TX_j$, so the first t_1 rows of $P_1^TX_j$ are zero. Thus

$$AP_1 = \begin{pmatrix} A_1 & 0 \\ A_{21} & A_{22} \end{pmatrix}$$

where A_1 is an $n_1 \times t_1$ stochastic matrix of rank 1, and

$$P_1^TX = \begin{pmatrix} X_{11} & 0 \\ X_{21} & X_{22} \end{pmatrix}$$

where X_{11} is a $t_1 \times n_1$ stochastic matrix with

$$P_1^TX_1 = \begin{pmatrix} X_{11} \\ X_{21} \end{pmatrix}.$$

If $k = 1$, then AP_1 has the form

$$AP_1 = \begin{pmatrix} A_1 & 0 \\ U_1 & 0 \end{pmatrix}.$$

Assume now that $k > 1$, and, for $j = 2, \ldots, k$, let $M_jP_1 = (B_j, C_j)$ where B_j is $n_j \times t_1$. Then

$$0 = M_jX_1 = B_jX_{11} + C_jX_{21}$$

so $B_jX_{11} = 0$. Thus, if B_j has a positive column, then the corresponding row of X_{11} is a zero row, which is impossible since X_{11} is stochastic. Hence $B_j = 0$ for $j = 2, \ldots, k$.

By repeating the process k times, we obtain an $n \times n$ permutation matrix $P = P_1 P_2 \ldots P_k$ such that

$$AP = \begin{pmatrix} A_1 & 0 & \ldots & 0 & 0 \\ 0 & A_2 & \ldots & 0 & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & \ldots & A_k & 0 \\ U_1 & U_2 & \ldots & U_k & U_{k+1} \end{pmatrix}$$

where each A_i is an $n_i \times t_i$ stochastic matrix of rank 1. At each stage the matrix P_i is chosen so that it fixes the first $t_1 + \ldots + t_{i-1}$ columns of $AP_1 \ldots P_{i-1}$. Also $P^T X$ has the form

$$P^T X = \begin{pmatrix} X_{11} & \ldots & 0 & 0 \\ \ldots & \ldots & \ldots & \ldots \\ 0 & \ldots & X_{kk} & 0 \\ W_1 & \ldots & W_k & W_{k+1} \end{pmatrix}$$

where each X_{ii} is a $t_i \times n_i$ stochastic matrix.

Now consider the block

$$U = (U_1, \ldots, U_k, U_{k+1}) \,.$$

Now the rank of $A_1 \oplus \ldots \oplus A_k$ is $k = \operatorname{rank}(A)$, so each row of U is a linear combination of the first $n - n_{k+1}$ rows of AP. Hence, since AP is stochastic, $U_{k+1} = 0$, and $U_j = G_j A'_j$ for $j = 1, \ldots, k$, where each A'_j consists of n_{k+1} rows each equal to a row of A_j, each G_j is an $n_{k+1} \times n_{k+1}$ diagonal matrix, and $G_1 + \ldots + G_k = I$. Thus, for $j = 1, \ldots, k$,

$$D_j E'_j = F_j = M_{k+1} P P^T X_j = U_j X_{jj} = G_j A'_j X_{jj} = G_j M'_j X_j = G_j E'_j \,.$$

But $E'_j > 0$ and $D_j - G_j$ is diagonal, so $D_j = G_j$. Hence $U_j = D_j A'_j$ for $j = 1, \ldots, k$. This completes the proof that (a) implies (b).

Assume now that (b) holds. Let Y be the $n \times n$ stochastic matrix

$$Y = Y_1 \oplus \ldots \oplus Y_k \oplus Z$$

where each Y_i is the $t_i \times n_i$ stochastic matrix having each row equal to a row of E_i, and Z is any stochastic matrix of appropriate size. Then $APY = E$. But $EAP = AP$, so $AP \mathcal{R} E$. Now $AP \mathcal{R} A$ and $\mathcal{R}$ is transitive, so $A \mathcal{R} E$. Thus (a) holds.

That (b) and (c) are equivalent is clear, which completes the proof of the theorem.

The first of the following corollaries removes the restriction that the idempotent E be canonical, while the second gives necessary and sufficient conditions for two canonical idempotents to be $\mathcal{R}$-equivalent. For the proof, note that $A \mathcal{R} B$ if and only if $QAQ^T \mathcal{R} QBQ^T$ for Q any permutation matrix.

Corollary 2.2. *Let F be an idempotent in S_n and let $A \in S_n$. Then $A \mathcal{R} F$ if and only if for some $n \times n$ permutation matrices P and Q, $QAQ^T P$ has the form (3) and QFQ^T is a canonical idempotent.*

Corollary 2.3. *Two canonical idempotents in S_n are $\mathscr{R}$-equivalent if and only if their corresponding blocks have the same dimensions and the diagonal matrices which determine the last n_{k+1} rows of each are identical.*

Since there are uncountably many choices for the row in E_i whenever $n_i \geqq 2$, Corollary 2.3 shows that any $\mathscr{R}$-class of S_n which contains one canonical idempotent E must contain uncountably many, provided only that at least one block E_i of E has order 2 or greater.

As a result of the next theorem, if a stochastic matrix A itself has the form (3), then this is sufficient for A to be regular. The matrix

$$A = \begin{pmatrix} \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 1 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} \end{pmatrix}$$

is regular, since A is idempotent, but not of the form (3), so the condition is not necessary. Indeed, A cannot be put in the form (3) by any column permutation.

Theorem 2.4. *Let $A \in S_n$. If A has the form (3), then there exists a canonical idempotent E in S_n such that $A \mathscr{R} E$. Consequently, A is regular.*

Proof. Use the matrix Y introduced at the end of the proof of Theorem 2.1 with each Y_i having each entry equal to $1/n_i$.

The following corollary gives necessary and sufficient conditions for two regular stochastic matrices to lie in the same $\mathscr{R}$-class, and hence for the system (1) to be consistent. Regularity does not need to be assumed for the sufficiency.

Corollary 2.3. *Let $A, B \in S_n$. Then A and B are regular and $A \mathscr{R} B$ if and only if there exist $n \times n$ permutation matrices P_1, P_2, and Q such that QAQ^TP_1 and QBQ^TP_2 each have the form (3) with corresponding blocks having the same dimensions and with identical diagonal matrices determining the last n_{k+1} rows of each.*

Proof. Assume A and B are regular, and $A \mathscr{R} B$. Then there exists an $n \times n$ permutation matrix Q such that QAQ^T and QBQ^T are both in the same $\mathscr{R}$-class as some canonical idempotent E of S_n. Thus there exist $n \times n$ permutation matrices P_1 and P_2 such that QAQ^TP_1 and QBQ^TP_2 each have the form (3). The converse follows by using the canonical idempotent E of Theorem 2.4.

3. Green's relation $\mathscr{L}$ for S_n. We again begin by characterizing those elements A in S_n which lie in the same $\mathscr{L}$-class with a canonical idempotent E. The result is similar to that for $\mathscr{R}$, but here the form of A depends upon the matrices E_i as well as the way in which E is partitioned into blocks of columns.

Theorem 3.1. *Let E be a canonical idempotent in S_n with rank k, and let $A \in S_n$. Then the following statements are equivalent.*

(a) *$A \mathscr{L} E$; that is, the system $XA = E$, $YE = A$, $X, Y \in S_n$ is consistent.*

(b) *There exists an $n \times n$ permutation matrix P such that PA has the form*

$$\text{(4)} \qquad PA = \begin{pmatrix} A_1 & 0 & \dots & 0 & 0 \\ 0 & A_2 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & A_k & 0 \\ U_1 & U_2 & \dots & U_k & 0 \end{pmatrix}$$

where each A_i is a $t_i \times n_i$ stochastic matrix of rank 1 having each row equal to a row of E_i; and where $U_i = G_i A_i'$, where each A_i' consists of $t = n - t_1 - \dots - t_k$ rows of A_i, each G_i is a $t \times t$ diagonal matrix, and $G_1 + \dots + G_k = I$.

Proof. Assume (a) holds. Partition A by

$$A = (M_1, \dots, M_k, M_{k+1})$$

where each M_i is $n \times n_i$. Since $AE = A$, $M_{k+1} = 0$, and $M_i E_i = M_i$ for $i = 1, \dots, k$. Partition X by

$$X = \begin{pmatrix} X_1 \\ \dots \\ X_k \\ X_{k+1} \end{pmatrix}$$

where each X_i is $n_i \times n$. Then $XA = E$ implies

$$X_i M_i = E_i \quad \text{for} \quad i = 1, \dots, k\,;$$

$$X_i M_j = 0 \quad \text{for} \quad i \neq j\,, \quad i, j = 1, \dots, k\,;$$

and

$$X_{k+1} M_j = F_j \quad \text{for} \quad j = 1, \dots, k\,.$$

Now $M_i E_i = M_i$ and E_i positive imply that if M_i has a zero in any row, then that entire row must be a row of zeros. Thus a row of M_i must be either positive or a row of zeros. Now $M_i \neq 0$, so rank $(M_i) = 1$ for $i = 1, \dots, k$.

If $k = 1$, then $M_1 > 0$ since $A = (M_1, 0)$ is stochastic. In this case $M_1 E_1 = M_1$ implies each row of M_1 is a row of E_1. Thus A itself has the form (4) where $A_1 = M_1$ and the block U_1 is missing.

Now assume $k > 1$. If $M_i > 0$ for some $i = 1, \dots, k$, then, for any $j \neq i$, $X_j M_i = 0$ implies $X_j = 0$. Thus $E_j = X_j M_j = 0$, a contradiction. Hence each M_i has zero entries, thus rows of zeros. Let u_i be the number of positive rows in each M_i.

Let P_1 be an $n \times n$ permutation matrix such that the first u_1 rows of $P_1 M_1$ are nonzero. Then, for $i = 2, \dots, k$, $0 = X_i M_1 = X_i P_1^T P_1 M_1$, so the first u_1 columns of $X_i P_1^T$ are zero columns. Thus

$$P_1 A = \begin{pmatrix} A_1^* & A_{12} & 0 \\ 0 & A_{22}^* & 0 \end{pmatrix}$$

where A_1^* is $u_1 \times n_1$, A_{22}^* is $u_1 \times (n - n_1 - n_{k+1})$; and

$$XP_1^T = \begin{pmatrix} X_{11} & X_{12} \\ 0 & X_{22} \\ X_{31} & X_{32} \end{pmatrix}$$

where X_{11} is $n_1 \times u_1$, and X_{31} is $n_{k+1} \times u_1$. For $j = 2, \ldots, k$, let

$$P_1 M_j = \begin{pmatrix} B_j \\ C_j \end{pmatrix}$$

where B_j is $u_1 \times n_j$. Then

$$0 = X_1 M_j = X_{11} B_j + X_{12} C_j ,$$

so $X_{11} B_j = 0$. If B_j has a positive row, then the corresponding column of X_{11} is a column of zeros. If each of the u_1 rows of $A_{12} = (B_2, \ldots, B_k)$ has positive entries, then each column of X_{11} is a column of zeros. Thus $E_1 = X_{11} A_1^* = 0$, a contradiction. Hence A_{12} has at least one row of zeros. Let t_1 be the number of rows of zeros in A_{12}. Let Q_1 be an $n \times n$ permutation matrix such that the first t_1 rows of the matrix

$$Q_1 = \begin{pmatrix} A_{12} \\ A_{22}^* \end{pmatrix}$$

are rows of zeros, and the last $u_1 - t_1$ rows are the rows of A_{12} which contain positive entries. Then

$$Q_1 P_1 A = \begin{pmatrix} A_1 & 0 & 0 \\ 0 & A_{22} & 0 \\ U_1^* & A_{32} & 0 \end{pmatrix}$$

where A_1 is a $t_1 \times n_1$ stochastic matrix and U_1^* is $(u_1 - t_1) \times n_1$. If $t_1 = u_1$, then $U_1^* = 0$. Now

$$Q_1 P_1 M_1 = \begin{pmatrix} A_1 \\ 0 \\ U_1^* \end{pmatrix},$$

so, since $M_1 E_1 = M_1$, $A_1 E_1 = A_1$. Thus each row of A_1 is a row of E_1. Furthermore, rank $(Q_1 P_1 M_1) = \text{rank}\,(M_1) = 1$, so $U_1^* = G_1^* A_1''$ where G_1^* is a $(u_1 - t_1) \times \times (u_1 - t_1)$ nonnegative diagonal matrix and A_1'' consists of $(u_1 - t_1)$ rows each equal to a row of A_1.

By repeating this process we obtain $n \times n$ permutation matrices $P_1, \ldots, P_k$, $Q_1, \ldots, Q_k$ such that PA has the form (4) with $P = Q_k P_k, \ldots, Q_1 P_1$. For $i = 2, \ldots, k$, the number t_i is the number of rows of zeros in the submatrix at the i-th step which is analogous to A_{12}. If $t_i = u_i$ for some i, then $U_i = 0$ and so $G_i = 0$. Thus (b) holds.

Conversely, let Y be the $n \times n$ stochastic matrix

$$Y = \begin{pmatrix} Y_1 & \dots & 0 & 0 \\ \multicolumn{4}{c}{\dots\dots\dots\dots} \\ 0 & \dots & Y_k & 0 \\ Z_1 & \dots & Z_k & 0 \end{pmatrix}$$

where each Y_i is any $n_i \times t_i$ stochastic matrix of rank 1; and where $Z_i = D_i Y_i'$ with each Y_i' consisting of n_{k+1} rows each equal to a row of Y_i. Then, for $i = 1, \dots, k$, $Y_i E_i = A_i$ and

$$Z_i A_i = D_i Y_i' A_i = D_i A_i' = D_i E_i' = F_i$$

so $YPA = E$. But also $PAE = PA$, so $PA \mathscr{L} E$. Now $A \mathscr{L} PA$ and the relation $\mathscr{L}$ is transitive, so $A \mathscr{L} E$. This completes the proof.

The remaining results in this section are analogous to those at the end of the previous section. The proofs are similar to these previous results, and hence are omitted.

Corollary 3.2. *Let F be an idempotent in S_n, and let $A \in S_n$. Then $A \mathscr{L} F$ if and only if there exist $n \times n$ permutation matrices P and Q such that $PQAQ^T$ has the form* (4) *and QFQ^T is a canonical idempotent.*

Corollary 3.3. *Two canonical idempotents in S_n are $\mathscr{L}$-equivalent if and only if their corresponding blocks E_i are identical.*

Since there are uncountably many choices for the diagonal matrices D_i in the canonical form whenever the blocks F_i appear and $k \geqq 2$, Corollary 3.3 shows that an $\mathscr{L}$-class of S_n which contains one canonical idempotent must contain uncountably many canonical idempotents or else no other canonical idempotent except E.

The next theorem provides a sufficient condition for an element A of S_n to be regular. Again the matrix A given just before Theorem 2.4 is regular, but not of the form (4), so the condition is not necessary. In fact, no row permutation will bring A into the form (4).

Theorem 3.4. *Let $A \in S_n$. If A has the form* (4), *then there exists a canonical idempotent E in S_n such that $A \mathscr{L} E$. Consequently A is regular.*

As a corollary we obtain necessary and sufficient conditions for two regular elements of S_n to be in the same $\mathscr{L}$-class. Again, regularity is not required for the sufficiency.

Corollary 3.5. *Let $A, B \in S_n$. Then A and B are regular and $A \mathscr{L} B$ if and only if there exist $n \times n$ permutation matrices P_1, P_2, and Q such that P_1QAQ^T and P_2QBQ^T each have the form* (4), *where the corresponding blocks A_i on the main diagonals are identical.*

4. Maximal subgroups of S_n. We now apply the results of the last two sections to obtain a new proof of the theorem of Schwartz [10] that the maximal subgroups of S_n are isomorphic to full symmetric groups. Since the maximal subgroups containing the idempotents E and PEP^T, for any permutation matrix P, are isomorphic, it is sufficient to consider only those maximal subgroups H_E where E is a canonical idempotent.

Theorem 4.1. *Let E be a canonical idempotent in S_n with rank k, and let $A \in S_n$. Then $A \mathscr{H} E$ if and only if A has the form*

$$A = \begin{pmatrix} M_{11} & \dots & M_{1k} & 0 \\ \dots & \dots & \dots & \dots \\ M_{k1} & \dots & M_{kk} & 0 \\ W_1 & \dots & W_k & 0 \end{pmatrix} \tag{5}$$

where each M_{ij} is $n_i \times n_j$; for each i and each j there is exactly one M_{ij} which is nonzero, and each row of this M_{ij} is a row of E_j; and where $W_j = D_i M'_{ij}$, where M'_{ij} consists of n_{k+1} rows of the nonzero block M_{ij}.

Proof. Assume $A \mathscr{H} E$. Then $A \mathscr{L} E$ and $A \mathscr{R} E$, so there exist $n \times n$ permutation matrices P and Q such that AP has the form (3) and QA has the form (4). Partition A according to (5) where, for the moment, each M_{ij} is $n_i \times n_j$, and observe that the last n_{k+1} columns of A must be zero since $AE = A$. In the form (3), the block A_1 is positive, so

$$(M_{11}, \dots, M_{1k}, 0) = (A_1, 0, \dots, 0)\, P^T$$

has nonzero columns. Let M_{ij} be a block having a nonzero column. Then

$$QM_j = Q \begin{pmatrix} M_{1j} \\ \dots \\ M_{kj} \\ W_j \end{pmatrix}$$

is the j-th column of blocks in the form (4). Since $QM_jE_j = QM_j$, $M_{1j}E_j = M_{1j}$, so each row of M_{1j} is a row of E_j. Thus $M_{1i} = 0$ for all $i \neq j$.

Similarly, for each $i = 1, \dots, k$, exactly one block M_{ij} of $(M_{i1}, \dots, M_{ik}, 0)$ is positive, and all the rest are zero blocks. Further, each row of this M_{ij} is a row of E_j. If, for some $t \neq i$, M_{tj} is also positive, then $(M_{i1}, \dots, M_{ik}, 0)\, P$ and $(M_{t1}, \dots, M_{tk}, 0)\, P$ both have nonzero blocks in the j-th position of the block form (3), which is impossible. Hence, for each i and each j, exactly one M_{ij} is nonzero.

Moreover, since AP has the form (3), if M_{ij} is positive, then M_{ij} is the block A_i in AP, so

$$W_j = U_i = D_i A'_i = D_i M'_{ij}$$

where M'_{ij} consists of n_{k+1} rows of M_{ij}.

Conversely suppose A has the form (5). Let $P = (P_{ij})$, $i, j = 1, \ldots, k+1$, where the size of the block P_{ij} is determined as follows: P_{ij} is $n_i \times n_i$ whenever $M_{ji} \neq 0$, for $i = 1, \ldots, k$, and $P_{k+1,j}$ is $n_{k+1} \times n_j$ for $j = 1, \ldots, k+1$. Let $P_{ij} = I$ if P_{ij} is square. Otherwise set $P_{ij} = 0$. Then P is an $n \times n$ permutation matrix such that AP has the form (3). Thus $A \mathscr{R} E$. Similarly there exists an $n \times n$ permutation matrix Q such that QA has the form (4), so $A \mathscr{L} E$. Thus $A \mathscr{H} E$, which completes the proof.

For each $i = 1, \ldots, k$, the mapping $i \to j$ if $M_{ij} \neq 0$ is a permutation of the set $\{1, \ldots, k\}$. Such a mapping may be defined for each $A \in H_E$, and there are precisely $k!$ such mappings. Thus we obtain the following corollary.

Corollary 4.2. (Schwarz) *Let E be an idempotent in S_n with rank k. Then the maximal subgroup H_E of S_n having E as identity element is isomorphic to the full symmetric group on k letters.*

5. Green's relations for regular elements of D_n. In this section Theorems 2.1 and 3.1 are used to characterize Green's relations for regular elements of D_n. These results have been obtained in another way by MONTAGUE and PLEMMONS [7] who were able to remove the restriction of regularity. The section concludes with several necessary and sufficient conditions for an element of D_n to be regular.

The first two lemmas, which are dual, characterize $\mathscr{R}$-classes and $\mathscr{L}$-classes of D_n containing canonical idempotents.

Lemma 5.1. *Let E be a canonical idempotent in D_n with rank k, and let $A \in D_n$. Then $A \mathscr{R} E$ if and only if there exists an $n \times n$ permutation matrix P such that $AP = E$.*

Proof. Assume $A \mathscr{R} E$. Since E is a canonical idempotent in D_n, E is also a canonical idempotent in S_n. Thus, by Theorem 2.1, there exists an $n \times n$ permutation matrix P such that

$$AP = A_1 \oplus \ldots \oplus A_k$$

where each A_i is an $n_i \times t_i$ stochastic matrix of rank 1. But each A_i is doubly stochastic, so $n_i = t_i$ and $A_i = E_i$. Thus $AP = E$. The converse is obvious.

Lemma 5.2. *Let E be a canonical idempotent in D_n with rank k, and let $A \in D_n$. Then $A \mathscr{L} E$ if and only if there exists an $n \times n$ permutation matrix Q such that $QA = E$.*

As a result of the next three theorems, we obtain a characterization of the relations $\mathscr{R}$ and $\mathscr{L}$ for regular elements of D_n. The characterizations of the relations $\mathscr{H}$ and $\mathscr{D}$ follow as corollaries.

Theorem 5.3. *Let A be a regular element of D_n. Then there exists a unique idempotent E in D_n such that $AP = E$ for some $n \times n$ permutation matrix P.*

Proof. Since A is regular, $A \mathscr{R} E$ for some idempotent E in D_n. If also $A \mathscr{R} F$ and $F^2 = F$, then $EF = F$ and $FE = E$ since an idempotent is a left identity for its $\mathscr{R}$-class. But then $E = E^T = (FE)^T = E^T F^T = EF = F$, so E is unique. Now $G = QEQ^T$ is a canonical idempotent for some $n \times n$ permutation matrix Q. Also $QAQ^T \mathscr{R} G$, so, by Lemma 5.1, there exists an $n \times n$ permutation matrix P_1 such that $QAQ^T P_1 = G = QEQ^T$. Thus $AQ^T P_1 Q = E$, so $AP = E$ where $P = Q^T P_1 Q$.

Theorem 5.4. *Let A be a regular element of D_n. Then there exists a unique idempotent F in D_n such that $QA = F$ for some $n \times n$ permutation matrix Q.*

Theorem 5.5. *Let A, B be regular elements of D_n. Then $A \mathscr{R} B[A \mathscr{L} B]$ if and only if $B = AP[B = QA]$ for some $n \times n$ permutation matrix $P[Q]$.*

Proof. Assume $A \mathscr{R} B$. By Theorem 5.3 there exists a unique idempotent E in D_n such that $AP_1 = E$ for some $n \times n$ permutation matrix P_1. Now $B \mathscr{R} E$, so, by Lemma 5.1, there exists an $n \times n$ permutation matrix P_2 such that $BP_2 = E$. Thus $B = EP_2^T = AP_1P_2^T = AP$, where $P = P_1P_2^T$. The converse is obvious.

Corollary 5.6. *Let A, B be regular elements of D_n. Then $A \mathscr{H} B$ if and only if $B = QA = AP$ for some $n \times n$ permutation matrices P and Q.*

Proof. The corollary is an immediate consequence of Theorem 5.5 and the definition of $\mathscr{H}$.

Corollary 5.7. *Let A, B be regular elements of D_n. Then $A \mathscr{D} B$ if and only if $B = QAP$ for some $n \times n$ permutation matrices P and Q.*

Proof. Assume $A \mathscr{D} B$. Then there exists X in D_n such that $A \mathscr{R} X$ and $X \mathscr{L} B$ [2, Lemma 2.1]. Now $X = AP = Q^T B$ for some $n \times n$ permutation matrices P and Q, so $B = QAP$. The converse is obtained by reversing steps.

We conclude this section with several necessary and sufficient conditions for an element of D_n to be regular. A number of other conditions may be found in [7].

Theorem 5.8. *Let $A \in D_n$. Then the following statements are equivalent.*

(1) *A is regular.*

(2) *QAP is idempotent for some $n \times n$ permutation matrices P and Q.*

(3) *$Q_1 A$ is idempotent for some $n \times n$ permutation matrix Q_1.*

(4) *AP_1 is idempotent for some $n \times n$ permutation matrix P_1.*

(5) *$AA^T \mathscr{R} A$; that is, $AA^T = AP$ for some $n \times n$ permutation matrix P.*

(6) *$A^T A \mathscr{L} A$; that is, $A^T A = QA$ for some $n \times n$ permutation matrix Q.*

Proof. The equivalence of the first four statements follows from Theorems 5.3 and 5.4 and Corollary 5.7. If (4) holds, then $A = EP_1^T$, where $E^2 = E$, so $AA^T = EP_1^T P_1 E = E = AP_1$, so $AA^T \mathscr{R} A$, proving (5). We now show (5) implies (1), which will complete the proof since (6) is dual to (5). Assume $AA^T = AP$. Now $AA^T = (AA^T)^T = (AP)^T = P^T A^T$, so $A = AA^T P^T = P^T A^T P^T$. Thus

$$AA^T A = APA = APP^T A^T P^T = AA^T P^T = A\,,$$

so A is regular.

6. Maximal subgroups of D_n. In this section we give a new proof of the characterization of the maximal subgroups of D_n which was first found independently by Schwarz [11] and FARAHAT [4]. Every regular $\mathscr{D}$-class of D_n contains a canonical idempotent, and every idempotent element of D_n is permutationally similar to a canonical idempotent. Thus it suffices to consider only those maximal subgroups H_E of D_n having a canonical idempotent E as identity element, since all other maximal subgroups will be isomorphic to one of these.

Theorem 6.1. (Schwarz; Farahat) *Let E be the canonical idempotent in D_n corresponding to $n = n_1 + \ldots + n_k$, with $n_1 \geqq n_2 \geqq \ldots \geqq n_k \geqq 1$. Let $m_1, \ldots, m_t$ be the distinct members of the sequence $n_1, \ldots, n_k$, with $m_1 = n_1$, $m_t = n_k$, and $m_1 > m_2 > \ldots > m_t$. For $i = 1, \ldots, t$, let p_i be the multiplicity of m_i in the sequence $n_1, \ldots, n_k$. Then the maximal subgroup H_E of D_n is isomorphic to the group $S(p_1) \oplus S(p_2) \oplus \ldots \oplus S(p_t)$ where, for $i = 1, \ldots, t$, $S(p_i)$ is the full symmetric group on p_i letters.*

Proof. Let $A \in D_n$. Then, by Corollary 5.6, $A \in H_E$ if and only if there exist $n \times n$ permutation matrices P and Q such that $A = QE = EP$; that is, $E = Q^T EP$. Thus $A \in H_E$ if and only if A can be obtained from E by a permutation of rows (or columns) which permutes the p_i blocks of E which have each entry equal to $1/m_i$, $i = 1, \ldots, t$, and which does not interchange blocks with entries $1/m_i$ and $1/m_j$ whenever $i \neq j$. For each $i = 1, \ldots, t$, the set of all such permutations of blocks of E is isomorphic to the group $S(p_i)$. Thus H_E is isomorphic to the group $S(p)_1 \oplus \ldots \oplus S(p_t)$.

7. Generalized matrix inverses in D_n. Given any matrix A over the complex field, PENROSE [8] shows that there exists a unique complex matrix X such that $AXA = A$, $XAX = X$, and each of AX and XA is Hermitian. The matrix X is called the Moore-Penrose inverse of A, and is denoted by A^+. It has applications in approximation theory and in numerical analysis; see, e.g., [1], [5]. A semi-inverse of A is a matrix X satisfying the equations $AXA = A$ and $XAX = X$.

In this section we give a new proof of the result of PLEMMONS and CLINE [9] concerning the semi-inverse and the Moore-Penrose inverse of a regular doubly stochastic matrix. The stochastic case is considered in [12].

Theorem 7.1. (PLEMMONS and CLINE) *A regular element A of D_n has A^T as its unique semi-inverse in D_n. Moreover, A^T is the Moore-Penrose inverse of A.*

Proof. Let A be a regular element of D_n. By Theorem 5.3 there exists a unique idempotent E in D_n such that $A \mathscr{R} E$, so $A = EP$ for some $n \times n$ permutation matrix P. Now $A^T = P^T E$, so

$$AA^TA = EPP^TEEP = EP = A$$

and

$$A^TAA^T = P^TEEPP^TE = P^TE = A^T,$$

so A^T is a semi-inverse of A in D_n. Since E is the only idempotent such that $A \mathscr{R} E$ it follows [2, Theorem 2.18] that A^T is the unique semi-inverse of A in D_n. Since doubly stochastic idempotents are symmetric, A^T must be the Moore-Penrose inverse of A.

References

[1] *A. Ben-Israel* and *A. Charnes*, "Contributions to the theory of generalized inverses", J. Soc. Indust. Appl. Math., *XI* (1963), 667—699.

[2] *A. H. Clifford* and *G. B. Preston*, The Algebraic Theory of Semigroups, Vol. *I*, Math. Surveys No. 7, Amer. Math. Soc., Providence, 1961.

[3] *J. L. Doob*, "Topics in the theory of Markov chains", Trans. Amer. Math. Soc., *52* (1942), 37—64.

[4] *H. K. Farahat*, "The semigroup of doubly-stochastic matrices", Proc. Glasgow Math. Assoc., *7* (1966), 178—183.

[5] *T. N. E. Greville*, "Some applications of the pseudoinverse of a matrix", SIAM Rev., *II* (1960), 15—22.

[6] *K. H. Hofmann* and *P. S. Mostert*, Elements of Compact Semigroups, Charles E. Merrill, Columbus, 1966.

[7] *J. S. Montague* and *R. J. Plemmons*, "Convex matrix equations", Bull. Amer. Math. Soc., *78* (1972), 965—968.

[8] *R. Penrose*, "A generalized inverse for matrices", Proc. Cambridge Phil. Soc., *51* (1955), 406—413.

[9] *R. J. Plemmons* and *R. E. Cline*, "The generalized inverse of a nonnegative matrix", Proc. Amer. Math. Soc., *31* (1972), 46—50.

[10] *Š. Schwarz*, "On the structure of the semigroup of stochastic matrices", Magyar Tud. Akad. Kutato Int. Kozl., (A) *9* (1964), 297—311.

[11] *Š. Schwarz*, "A note on the structure of the semigroup of doubly stochastic matrices", Mat. Časopis Sloven. Akad. Vied., *17* (1967), 308—316.

[12] *J. R. Wall*, "Generalized inverses of stochastic matrices", Lin. Alg. Appl., to appear.

Author's address: Mathematics Department, Auburn University, Auburn, Alabama 36830, U.S.A.